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A construction method for discrete constant negative Gaussian curvature surfaces

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Abstract This is a survey based on the author's paper [9] about a construction method for discrete constant negative Gaussian curvature surfaces, the nonlinear d'Alembert formula. The heart of this formula is the Birkhoff decomposition and we give a simple algorithm for the Birkhoff decomposition.

Keywords: Discrete differential geometry, pseudospherical surface, loop groups, integrable systems

1 Introduction

The study of smooth constant negative Gaussian curvature surfaces (PS surfaces¹ in this survey) is a classical subject of differential geometry. It is known that the Gauss-Codazzi equations (nonlinear partial differential equations) for a PS surface become a famous integrable system, *sine-Gordon* equation:

$$\partial_y \partial_x u - \sin u = 0.$$

One of the prominent features of integrable systems is that they can be obtained by compatibility conditions for certain linear partial differential equations, the so-called *Lax pairs*. Moreover, the Lax pair contains an additional parameter, the *spectral parameter*, and it is a fundamental tool to study the integrable systems. On a PS surface, the spectral parameter induces a family of PS surfaces, which will be called the *associated family*, and the Lax pair is a family of moving frames (Darboux frames) and it will be called the *extended frame* of a PS surface. The extended frame can be thought as an element of the set of maps from the unit circle S^1 in the complex plane into a Lie group, the *loop group*, see Appendix A.

In [10, 13], it was shown that loop group decompositions (Birkhoff decompositions, see Theorem A.1) of the extended frame F of a PS surface induced a pair of 1-forms (ξ_+, ξ_-) , that is, $F = F_+ F_- = G_- G_+$ with $\xi_+ = F_+^{-1} dF_+$ and $\xi_- = G_-^{-1} dG_-$. Then it was proved that ξ_+ and ξ_- depended only on x and y , respectively. Conversely it was shown that solving the pair of ordinary differential equations $dF_+ = F_+ \xi_+$ and $dG_+ = G_+ \xi_-$ and using the loop group decomposition,

¹A constant negative Gaussian curvature surface is sometimes called a pseudospherical surface, thus we use "PS" for the shortened name.

the extended frame could be recovered. This construction is called the *nonlinear d'Alembert formula* for PS surfaces.

On the one hand a discrete analogue of smooth PS surfaces was defined in [1] and the nonlinear d'Alembert formula for discrete PS surfaces was recently shown in [9]. In this survey we show basic results for smooth/discrete PS surfaces and the nonlinear d'Alembert formula for smooth/discrete PS surfaces according to [5, 1, 9]. The heart of the formula is the Birkhoff decomposition and we give a simple algorithm (Lemma 3.1) for the Birkhoff decomposition in case of discrete PS surfaces in Section 3.

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2 Preliminaries

We briefly recall basic notation and results about smooth and discrete PS surfaces in the Euclidean three space \mathbb{E}^3 , that is \mathbb{R}^3 with the standard inner product $\langle \cdot, \cdot \rangle$, see for examples [11, 1, 13, 5, 10]. Moreover, we recall the nonlinear d'Alembert formula for discrete PS surfaces [9].

2.1 Pseudospherical surfaces

We first identify \mathbb{E}^3 with the Lie algebra of the special unitary group SU_2 , which will be denoted by \mathfrak{su}_2 :

$${}^t(x, y, z) \in \mathbb{E}^3 \longleftrightarrow \frac{i}{2}x\sigma_1 - \frac{i}{2}y\sigma_2 + \frac{i}{2}z\sigma_3 \in \mathfrak{su}_2, \quad (2.1)$$

where σ_j ($j = 1, 2, 3$) are the Pauli matrices as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the inner product of \mathbb{E}^3 can be computed as $\langle \mathbf{x}, \mathbf{y} \rangle = -2 \operatorname{trace}(XY)$, where $\mathbf{x}, \mathbf{y} \in \mathbb{E}^3$ and $X, Y \in \mathfrak{su}_2$ are the corresponding matrices in (2.1). Let f be a PS surface in \mathbb{E}^3 with Gaussian curvature $K = -1$. It is known that there exist the *Chebyshev coordinates* $(x, y) \in \mathbb{R}^2$ for f , that is, they are asymptotic coordinates normalized by $|f_x| = |f_y| = 1$. Here the subscripts x and y denote the x - and y -derivatives ∂_x and ∂_y , respectively. Then the first and second fundamental forms for f can be computed as

$$\mathrm{I} = dx^2 + 2 \cos u \, dx dy + dy^2, \quad \mathrm{II} = 2 \sin u \, dx dy,$$

where $0 < u < \pi/2$ is the angle between two asymptotic lines. Let

$$e_1 = \frac{1}{2} \sec(u/2)(f_x + f_y), \quad e_2 = \frac{1}{2} \csc(u/2)(f_x - f_y) \quad \text{and} \quad e_3 = e_1 \times e_2$$

be the Darboux frame rotating on the tangent plane clockwise angle u . Note that it is easy to see that $\{e_1, e_2, e_3\}$ is an orthonormal basis of \mathbb{E}^3 . Under the identification (2.1), $\{-\frac{i}{2}\sigma_1, -\frac{i}{2}\sigma_2, -\frac{i}{2}\sigma_3\}$ is an orthonormal basis of \mathbb{E}^3 , and for given $F \in SU_2$ and $x \in \mathfrak{su}_2$, $\operatorname{Ad}(F)(x) := Fx F^{-1}$ denotes the rotation of x . Thus there exists a \tilde{F} taking values in SU_2 such that

$$e_1 = -\frac{i}{2} \tilde{F} \sigma_1 \tilde{F}^{-1}, \quad e_2 = -\frac{i}{2} \tilde{F} \sigma_2 \tilde{F}^{-1} \quad \text{and} \quad e_3 = -\frac{i}{2} \tilde{F} \sigma_3 \tilde{F}^{-1}. \quad (2.2)$$

Without loss of generality, at some base point $(x_*, y_*) \in \mathbb{R}^2$, we have $\tilde{F}(x_*, y_*) = \text{Id}$. Then there exists a family of frames F parametrized by $\lambda \in \mathbb{R}_+ := \{r \in \mathbb{R} \mid r > 0\}$ satisfying the following system of partial differential equations, see [5] in detail:

$$F_x = FU \quad \text{and} \quad F_y = FV, \quad (2.3)$$

where

$$U = \frac{i}{2} \begin{pmatrix} -u_x & \lambda \\ \lambda & u_x \end{pmatrix}, \quad V = -\frac{i}{2} \begin{pmatrix} 0 & \lambda^{-1} e^{iu} \\ \lambda^{-1} e^{-iu} & 0 \end{pmatrix}. \quad (2.4)$$

The parameter $\lambda \in \mathbb{R}_+$ will be called the *spectral parameter*. We choose F such that

$$F|_{\lambda=1} = \tilde{F} \quad \text{and} \quad F|_{(x_*, y_*)} = \text{Id}.$$

The compatibility condition of the system in (2.3), that is $U_y - V_x + [V, U] = 0$, becomes a version of the sine-Gordon equation:

$$u_{xy} - \sin u = 0. \quad (2.5)$$

It turns out that the sine-Gordon equation is the Gauss-Codazzi equations for PS surfaces. Thus from the fundamental theorem of surface theory there exists a family of PS surfaces parametrized by the spectral parameter $\lambda \in \mathbb{R}_+$. Then the family of frames F will be called the *extended frame* for f .

From the extended frame F , a family of PS surfaces f^λ , ($\lambda \in \mathbb{R}_+$) are given by the so-called *Sym formula*, [12]:

$$f^\lambda = \lambda \left. \frac{\partial F}{\partial \lambda} F^{-1} \right|_{\lambda \in \mathbb{R}_+}. \quad (2.6)$$

The immersion $f^\lambda|_{\lambda=1}$ is the original PS surface f up to rigid motion. The one-parameter family $\{f^\lambda\}_{\lambda \in \mathbb{R}_+}$ will be called the *associated family* of f .

2.2 Nonlinear d'Alembert formula

Firstly, we note that the extended frame F of a PS surface f is an element of the *loop group* for SU_2 , that is, it is a set of smooth maps from S^1 into SU_2 , see Appendix A for the definition. In fact the extended frame is defined on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and it can be thought as a map from S^1 into SU_2 . Then it becomes a Banach Lie group with suitable topology, which is an infinite dimensional Lie group and thus it will be called the loop group. Then the Birkhoff decomposition of the loop group is fundamental, which will be now explained. The loop group of SU_2 will be denoted by ΛSU_2 and we consider two subgroups $\Lambda^+ \text{SU}_2$ and $\Lambda^- \text{SU}_2$ of ΛSU_2 as sets of maps which can be extended inside the unit disk and outside the unit disk, respectively. In other words, maps $F \in \Lambda \text{SU}_2$, $F_+ \in \Lambda^+ \text{SU}_2$ and $F_- \in \Lambda^- \text{SU}_2$ have the following Fourier expansions:

$$F = \sum_{j=-\infty}^{\infty} F_j \lambda^j, \quad F_+ = \sum_{j=0}^{\infty} F_j^+ \lambda^j \quad \text{and} \quad F_- = \sum_{j=-\infty}^0 F_j^- \lambda^j.$$

Then we consider the following problem: for a given map $F \in \Lambda \text{SU}_2$, does there exist F_\pm or G_\pm taking values in $\Lambda^\pm \text{SU}_2$ such that

$$F = F_+ F_- \quad \text{or} \quad F = G_- G_+$$

holds? The Birkhoff decomposition theorem assures that this decomposition always holds in case of the loop group of SU_2 , see Theorem A.1 in detail. Using the Birkhoff decomposition theorem, we give a construction method for PS surfaces, the so-called *the nonlinear d'Alembert formula*.

From now on, for simplicity, we assume that the base point is $(x_*, y_*) = (0, 0)$ and the extended frame F at the base point is identity:

$$F(0, 0, \lambda) = \text{Id}.$$

The nonlinear d'Alembert formula for smooth PS surfaces is summarized as follows, [10, 13, 5].

Theorem 2.1 ([13, 5]). *Let F be the extended frame for a PS surface f in \mathbb{E}^3 . Moreover, let $F = F_+ F_-$ and $F = G_- G_+$ be the Birkhoff decompositions given in Theorem A.1, respectively. Then F_+ and G_- do not depend on y and x , respectively, and the Maurer-Cartan forms of F_+ and G_- are given as follows:*

$$\begin{cases} \xi_+ = F_+^{-1} dF_+ = \frac{i}{2} \lambda \begin{pmatrix} 0 & e^{-i\alpha(x)} \\ e^{i\alpha(x)} & 0 \end{pmatrix} dx, \\ \xi_- = G_-^{-1} dG_- = -\frac{i}{2} \lambda^{-1} \begin{pmatrix} 0 & e^{i\beta(y)} \\ e^{-i\beta(y)} & 0 \end{pmatrix} dy, \end{cases} \quad (2.7)$$

where, using the angle function $u(x, y)$, α and β are given by

$$\alpha(x) = u(x, 0) - u(0, 0) \quad \text{and} \quad \beta(y) = u(0, y).$$

Conversely, let ξ_{\pm} be a pair of 1-forms defined in (2.7) with functions $\alpha(x)$ and $\beta(y)$ satisfying $\alpha(0) = 0$. Moreover, let F_+ and G_- be solutions of the pair of following ordinary differential equations:

$$\begin{cases} dF_+ = F_+ \xi_+, \\ dG_- = G_- \xi_-, \end{cases}$$

with $F_+(x = 0, \lambda) = G_-(y = 0, \lambda) = \text{Id}$. Moreover let $D = \text{diag}(e^{-\frac{i}{2}\alpha}, e^{\frac{i}{2}\alpha})$ and decompose $(F_+ D)^{-1} G_-$ by the Birkhoff decomposition in Theorem A.1:

$$(F_+ D)^{-1} G_- = V_- V_+^{-1},$$

where $V_- \in \Lambda_*^- \text{SU}_2$ and $V_+ \in \Lambda^+ \text{SU}_2$. Then $F = G_- V_+ = F_+ D V_-$ is the extended frame of some PS surface in \mathbb{E}^3 .

Definition 1. The pair of 1-forms (ξ_+, ξ_-) in (2.7) will be called the *pair of normalized potentials*.

Definition 2. In [5], it was shown that the extended frames of PS surfaces can be also constructed from the following pair of 1-forms:

$$\eta^x = \sum_{j=-\infty}^1 \eta_j^x \lambda^j dx \quad \text{and} \quad \eta^y = \sum_{j=-1}^{\infty} \eta_j^y \lambda^j dy, \quad (2.8)$$

where η_j^x and η_j^y take values in \mathfrak{su}_2 , and each entry of η_j^x (resp. η_j^y) is smooth on x (resp. y), and $\det \eta_1^x \neq 0, \det \eta_{-1}^y \neq 0$. Moreover η_j^x and η_j^y are diagonal (resp. off-diagonal) if j is even (resp. odd). This pair of 1-forms (η^x, η^y) is a generalization of the normalized potentials (ξ_+, ξ_-) in (2.7) and will be called the *pair of generalized potentials*, see also [4].

2.3 Discrete pseudospherical surfaces

Discrete PS surfaces were first defined in [1]. Instead of the smooth coordinates $(x, y) \in \mathbb{R}^2$, we use the quadrilateral lattice $(n, m) \in \mathbb{Z}^2$, that is, all functions depend on the lattice $(n, m) \in \mathbb{Z}^2$. The subscripts 1 and 2 (resp. $\bar{1}$ and $\bar{2}$) denote the forward (resp. backward) lattice points with respect to n and m : For a function $f(n, m)$ of the lattice $(n, m) \in \mathbb{Z}^2$, we define $f_1, f_2, f_{\bar{1}}$ and $f_{\bar{2}}$ by

$$f_1 = f(n+1, m), \quad f_{\bar{1}} = f(n-1, m), \quad f_2 = f(n, m+1) \quad \text{and} \quad f_{\bar{2}} = f(n, m-1).$$

Then a discrete PS surface f was defined by the following two conditions:

1. For each point f , there is a plane P such that

$$f, f_1, f_{\bar{1}}, f_2, f_{\bar{2}} \in P.$$

2. The length of the opposite edge of an elementary quadrilateral are equal:

$$|f_1 - f| = |f_{12} - f_2| = a(n) \neq 0, \quad |f_2 - f| = |f_{12} - f_1| = b(m) \neq 0.$$

Then the *discrete extended frame* F of a discrete PS surface can be defined by the following partial difference system, see [2, Section 3.2] and [1]:

$$F_1 = FU \quad \text{and} \quad F_2 = FV, \quad (2.9)$$

where

$$U = \frac{1}{\Delta_+} \begin{pmatrix} e^{-\frac{i}{2}(u_1-u)} & \frac{i}{2}p\lambda \\ \frac{i}{2}p\lambda & e^{\frac{i}{2}(u_1-u)} \end{pmatrix}, \quad V = \frac{1}{\Delta_-} \begin{pmatrix} 1 & -\frac{i}{2}qe^{\frac{i}{2}(u_2+u)}\lambda^{-1} \\ -\frac{i}{2}qe^{-\frac{i}{2}(u_2+u)}\lambda^{-1} & 1 \end{pmatrix}, \quad (2.10)$$

with $\Delta_+ = \sqrt{1 + (p/2)^2 \lambda^2}$ and $\Delta_- = \sqrt{1 + (q/2)^2 \lambda^{-2}}$. Here u is a real function depending on both n and m , and $p \neq 0$ and $q \neq 0$ are real functions depending only on n and m , respectively:

$$u = u(n, m), \quad p = p(n) \quad \text{and} \quad q = q(m).$$

The compatibility condition of the system in (2.9), that is $VU_2 = UV_1$, gives the so-called *discrete sine-Gordon equation*:

$$\sin\left(\frac{u_{12} - u_1 - u_2 + u}{4}\right) = \frac{pq}{4} \sin\left(\frac{u_{12} + u_1 + u_2 + u}{4}\right). \quad (2.11)$$

The equation (2.11) was first found by Hirota in [7] and also called the *Hirota equation*.

Remark 2.2. Strictly speaking, length of the edges for a discrete PS surface should be small (less than 1). If the length is big (greater than or equal to 1), then the compatibility condition $VU_2 = UV_1$ gives a discrete analogue of mKdV equation, see [8] in detail. To consider the discrete nonlinear d'Alembert formula, this restriction is fundamental and thus we assume the conditions in (2.13).

Then a discrete PS surface f can be given by the so-called Sym formula, [1]:

$$f^\lambda = \lambda \frac{\partial F}{\partial \lambda} F^{-1} \Big|_{\lambda \in \mathbb{R}_+}. \quad (2.12)$$

The family of frames F defined by (2.9) with $F|_{(n_*, m_*)} = \text{Id}$ will be called the *discrete extended frame* for a discrete PS surface. It is easy to see that the map f^λ defined in (2.12) has these properties and f^λ gives a family of discrete PS surfaces, see [2, Theorem 3].

2.4 Nonlinear d'Alembert formula for discrete PS surfaces

In this subsection we assume that the base point is $(n_*, m_*) = (0, 0)$ and the discrete extended frame F at the base point is identity:

$$F(0, 0, \lambda) = \text{Id}.$$

Moreover, we also assume that the functions p and q in (2.10) satisfy the inequalities

$$0 < \left| \frac{p}{2} \right| < 1 \quad \text{and} \quad 0 < \left| \frac{q}{2} \right| < 1. \quad (2.13)$$

Then the discrete nonlinear d'Alembert formula can be summarized as follows.

Theorem 2.3 ([9]). *Let f be a discrete PS surface and F the corresponding discrete extended frame. Decompose F according to the Birkhoff decomposition in Theorem A.1:*

$$F = F_+ F_- = G_- G_+,$$

where $F_+ \in \Lambda^+ \text{SU}_2$, $F_- \in \Lambda^- \text{SU}_2$, $G_- \in \Lambda_*^- \text{SU}_2$ and $G_+ \in \Lambda^+ \text{SU}_2$. Then F_+ and G_- do not depend on $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, respectively, and the discrete Maurer-Cartan forms of F_+ and G_- are given as follows:

$$\begin{cases} \xi_+ = F_+^{-1}(F_+)_1 = \frac{1}{\Delta_+} \begin{pmatrix} 1 & \frac{i}{2} p e^{-i\alpha} \lambda \\ \frac{i}{2} p e^{i\alpha} \lambda & 1 \end{pmatrix}, \\ \xi_- = G_-^{-1}(G_-)_2 = \frac{1}{\Delta_-} \begin{pmatrix} 1 & -\frac{i}{2} q e^{i\beta} \lambda^{-1} \\ -\frac{i}{2} q e^{-i\beta} \lambda^{-1} & 1 \end{pmatrix}, \end{cases} \quad (2.14)$$

where $\Delta_+ = \sqrt{1 + (p/2)^2 \lambda^2}$ and $\Delta_- = \sqrt{1 + (q/2)^2 \lambda^{-2}}$, the functions p and q are given in (2.10), and α and β are functions of $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, respectively. Moreover using the function $u(n, m)$ in (2.10), $\alpha(n)$ and $\beta(m)$ are given by

$$\begin{cases} \alpha(n) = \frac{1}{2} u(n+1, 0) + \frac{1}{2} u(n, 0) - u(0, 0), \\ \beta(m) = \frac{1}{2} u(0, m+1) + \frac{1}{2} u(0, m). \end{cases} \quad (2.15)$$

Conversely, Let ξ_{\pm} be a pair of matrices defined in (2.14) with arbitrary functions $\alpha = \alpha(n)$, $\beta = \beta(m)$ with $\alpha(0) = 0$ and $p = p(n)$, $q = q(m)$ satisfying the conditions (2.13). Moreover let $F_+ = F_+(n, \lambda)$ and $G_- = G_-(m, \lambda)$ be the solutions of the ordinary difference equations

$$(F_+)_1 = F_+ \xi_+ \quad \text{and} \quad (G_-)_2 = G_- \xi_-, \quad (2.16)$$

with $F_+(n=0, \lambda) = G_-(m=0, \lambda) = \text{Id}$ and set a matrix $D = \text{diag}(e^{\frac{i}{2}k}, e^{-\frac{i}{2}k}) \in \text{U}_1$, where $k(0) = 0$ and $k(n) = 2 \sum_{j=0}^{n-1} (-1)^{j+n} \alpha(j)$ for $n \geq 1$. Decompose $(F_+ D)^{-1} G_-$ by the Birkhoff decomposition in Theorem A.1:

$$(F_+ D)^{-1} G_- = V_- V_+^{-1}, \quad (2.17)$$

where $V_- \in \Lambda_*^- \text{SU}_2$, $V_+ \in \Lambda^+ \text{SU}_2$. Then $F = G_- V_+ = F_+ D V_-$ is the discrete extended frame of some discrete PS surface in \mathbb{E}^3 . Moreover the solution $u = u(n, m)$ of the discrete sine-Gordon for the discrete PS surface satisfies the relations in (2.15).

Definition 3. The pair of matrices (ξ_-, ξ_+) given in (2.14) will be called the *pair of discrete normalized potentials*.

Similar to the smooth case, we generalize the pair of discrete normalized potentials:

Definition 4. Let (ξ_-, ξ_+) be a pair of discrete normalized potentials and let η_m and η_n be

$$\eta_n = P_-^l \xi_+ P_-^r, \quad \eta_m = P_+^l \xi_- P_+^r. \quad (2.18)$$

Here we assume that P_{\pm}^{\star} ($\star = l$ or r) take values in $\Lambda^{\pm} \text{SU}_2$ and do not depend on m and n , respectively, that is, $P_-^{\star} = P_-^{\star}(n, \lambda)$ and $P_+^{\star} = P_+^{\star}(m, \lambda)$. Thus the η_n and η_m do not depend on m and n , respectively:

$$\eta_n = \eta_n(n, \lambda), \quad \eta_m = \eta_m(m, \lambda).$$

The pair (η_n, η_m) given in (2.18) will be called the *pair of discrete generalized potentials*.

Remark 2.4. The pair of normalized potentials (ξ_+, ξ_-) and the corresponding pair of discrete generalized potentials (η_n, η_m) in (2.18) give in general different discrete PS surfaces.

3 Algorithm for Birkhoff decomposition

In this section, we give a simple algorithm performing the Birkhoff decomposition used in Theorem 2.3.

When one looks at the discrete extended frame F defined in (2.9), F_+ and G_- defined in (2.16), one notices that they are given by products of two types of matrices:

$$e_+ = \frac{1}{\sqrt{1 + |a|^2 \lambda^2}} \begin{pmatrix} e^{i\theta} & a\lambda \\ -\bar{a}\lambda & e^{-i\theta} \end{pmatrix} \quad \text{and} \quad e_- = \frac{1}{\sqrt{1 + |b|^2 \lambda^{-2}}} \begin{pmatrix} e^{i\kappa} & b\lambda^{-1} \\ -\bar{b}\lambda^{-1} & e^{-i\kappa} \end{pmatrix}, \quad (3.1)$$

where $\theta, \kappa \in \mathbb{R}$, $a, b \in \mathbb{C}$ and $|a|, |b| < 1$. It is easy to see that e_{\pm} take values in $\Lambda^{\pm} \text{SU}_2$, respectively. Two matrices e_+ and e_- do not commute in general, however, the following lemma holds.

Lemma 3.1. *Let e_{\pm} be matrices in (3.1). Then there exist matrices \tilde{e}_{\pm} which takes values in $\Lambda^{\pm} \text{SU}_2$ such that*

$$e_+ e_- = \tilde{e}_- \tilde{e}_+$$

holds. In particular \tilde{e}_{\pm} can be explicitly computed as follows:

$$\tilde{e}_+ = \frac{1}{\sqrt{1 + |\tilde{a}|^2 \lambda^2}} \begin{pmatrix} e^{i\tilde{\theta}} & \tilde{a}\lambda \\ -\tilde{a}\lambda & e^{-i\tilde{\theta}} \end{pmatrix} \quad \text{and} \quad \tilde{e}_- = \frac{1}{\sqrt{1 + |\tilde{b}|^2 \lambda^{-2}}} \begin{pmatrix} e^{i\tilde{\kappa}} & \tilde{b}\lambda^{-1} \\ -\tilde{b}\lambda^{-1} & e^{-i\tilde{\kappa}} \end{pmatrix},$$

where $\tilde{a}, \tilde{b}, \tilde{\theta}$ and $\tilde{\kappa}$ are explicitly chosen by the following equations:

$$\tilde{a} = a e^{-i(\kappa + \tilde{\kappa})}, \quad \tilde{b} = b e^{i(\theta + \tilde{\theta})} \quad \text{and} \quad \tilde{\theta} + \tilde{\kappa} = \theta + \kappa + 2 \arg(1 - a \bar{b} e^{-i(\theta + \kappa)}).$$

Note that \tilde{e}_{\pm} are not unique and one can always choose $\tilde{\theta} = 0$ or $\tilde{\kappa} = 0$.

Proof. It is just a consequence of a direct computation of $e_+ e_-$ and $\tilde{e}_- \tilde{e}_+$, respectively. \square

Using Lemma 3.1 iteratively, we obtain the following algorithm for the Birkhoff decomposition.

Theorem 3.2. *Let F be the discrete extended frame of a PS surface. Moreover let F_+, G_- and D be the matrices defined in (2.16). Then the Birkhoff decompositions for F and $(F_+ D)^{-1} G_-$ can be explicitly computed.*

A Loop groups and the Birkhoff decomposition

In this appendix we give a definition of the loop group of SU_2 and its subgroups $\Lambda^{\pm} \text{SU}_2$. Moreover the Birkhoff decomposition will be stated.

It is easy to see that F defined in (2.3) together with the condition $F|_{(x_*, y_*)} = \text{Id}$ is an element in the twisted SU_2 -loop group:

$$\Lambda \text{SU}_2 := \left\{ g : \mathbb{R}^{\times} \cup S^1 \rightarrow \text{SL}_2 \mathbb{C} \mid g \text{ is smooth, } g(\lambda) = \left(\overline{g(\bar{\lambda})}^{-1} \right)^T \text{ and } \sigma g(\lambda) = g(-\lambda) \right\}, \quad (\text{A.1})$$

where $\mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}$, $\sigma X = \text{Ad}(\sigma_3)X = \sigma_3 X \sigma_3^{-1}$, ($X \in \text{SL}_2 \mathbb{C}$) is an involution on $\text{SL}_2 \mathbb{C}$. In order to make the above group a Banach Lie group, we restrict the occurring matrix coefficients to the Wiener algebra $\mathcal{A} = \{f(\lambda) = \sum_{n \in \mathbb{Z}} f_n \lambda^n : S^1 \rightarrow \mathbb{C} \mid \sum_{n \in \mathbb{Z}} |f_n| < \infty\}$, where we denote

the Fourier expansion of f on S^1 by $f(\lambda) = \sum_{n \in \mathbb{Z}} f_n \lambda^n$. Then the Wiener algebra is a Banach algebra relative to the norm $\|f\| = \sum |f_n|$ and the loop group ΛSU_2 is a Banach Lie group, [6].

Let \mathbb{D}^+ and \mathbb{D}^- be the interior of the unit disk in the complex plane and the union of the exterior of the unit disk in the complex plane and infinity, respectively. We first define two subgroups of ΛSU_2 :

$$\Lambda^+ \text{SU}_2 = \{g \in \Lambda \text{SU}_2 \mid g \text{ can be analytically extend to } \mathbb{D}^+\}, \quad (\text{A.2})$$

$$\Lambda^- \text{SU}_2 = \{g \in \Lambda \text{SU}_2 \mid g \text{ can be analytically extend to } \mathbb{D}^-\}. \quad (\text{A.3})$$

Then $\Lambda_*^+ \text{SU}_2$ and $\Lambda_*^- \text{SU}_2$ denote subgroups of $\Lambda^+ \text{SU}_2$ and $\Lambda^- \text{SU}_2$ normalized at $\lambda = 0$ and $\lambda = \infty$, respectively:

$$\Lambda_*^+ \text{SU}_2 = \{g \in \Lambda^+ \text{SU}_2 \mid g(\lambda = 0) = \text{Id}\} \quad \text{and} \quad \Lambda_*^- \text{SU}_2 = \{g \in \Lambda^- \text{SU}_2 \mid g(\lambda = \infty) = \text{Id}\}.$$

The following decomposition theorem is fundamental.

Theorem A.1 (Birkhoff decomposition, [6, 3]). *The multiplication maps*

$$\Lambda_*^+ \text{SU}_2 \times \Lambda^- \text{SU}_2 \rightarrow \Lambda \text{SU}_2 \quad \text{and} \quad \Lambda_*^- \text{SU}_2 \times \Lambda^+ \text{SU}_2 \rightarrow \Lambda \text{SU}_2$$

are diffeomorphisms onto ΛSU_2 , respectively.

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