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出版情報：MI lecture note series. 64, pp.23-30, 2015-09-18. 九州大学マス・フォア・インダストリ研究所

MATHEMATICAL PROGRESS IN EXPRESSIVE IMAGE SYNTHESIS 2015
Sep 25-27, 2015, Nishijin Plaza Kyushu University, Fukuoka

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Abstract We present a few computationally tractable models of finite-dimensional Reproducing Kernel Hilbert Spaces (RKHS) that give a theoretical foundation of the techniques we have developed to solve several problems in computer graphics. The problems we deal with in this paper are signal and geometry interpolation/extrapolation as well as solving an inverse problem in animation.

Keywords: computer graphics, interpolation, extrapolation, superresolution, Reproducing Kernel Hilbert Space

1 Introduction

This section briefly describes a general regularization problem on a Reproducing Kernel Hilbert Space (RKHS). In later sections, this general approach will give an alternative formulation to solve several problems that we have encountered in computer graphics, such as [6, 12, 1, 3, 8].

The RKHS itself is a real-valued function space, and is denoted by $\boldsymbol{H}(\Omega)$ or by $\boldsymbol{H}$ for short, throughout this article. Typically Ω then means a domain in $\mathbf{R}^k$ or simply a finite set of numbers $\{1, 2, \ldots, N\}$. In the case of Ω being the finite set, an element of $\boldsymbol{H}(\Omega)$ simply means a vector in $\mathbf{R}^N$. The RKHS $\boldsymbol{H}(\Omega)$ is associated with its kernel function K. The kernel function K is a real-valued function defined on $\Omega \times \Omega$, and is a symmetric, positive semi-definite function. An RKHS is prescribed completely with the kernel K, which in particular defines the norm of this special Hilbert space.

Now we consider the following regularization problem in [13]:

$$\min_{\mathbf{f} \in \boldsymbol{H}} \left\{ \sum_{i=1}^{n} (L_i(\mathbf{f}) - y_i)^2 + \alpha \|\mathbf{f}\|_K^2 \right\} \tag{1}$$

where, for $1 \leq i \leq n$, a real number $y_i \in \mathbf{R}$ and a continuous linear functional $L_i : \boldsymbol{H} \to \mathbf{R}$ are given, and $\|\cdot\|_K$ denotes the norm of $\boldsymbol{H}$ induced by its kernel K. It was proved in [5] that there exists a minimizer for the regularization problem in (1). In solving this problem for our practical purposes, it is key to choose a good kernel function. This means to employ an appropriate RKHS, because, for a given symmetric, positive semi-definite function K, we can construct an RKHS associated with K.

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As a typical case, where $\Omega = \mathbf{R}^k$ and $L_i(\mathbf{f}) = \mathbf{f}(\mathbf{x}_i)$ for given $\mathbf{x}_i \in \Omega$ $(1 \leq i \leq n)$, we can derive several types of splines from the above RKHS formulation. For $k = 2$, for example, the thin plate spline is connected with the kernel K such that: $K(\mathbf{x}, \mathbf{y}) = \phi(\|\mathbf{x} - \mathbf{y}\|)$, where $\phi(r) = r^2 \log r$ is called thin plate spline. More generally a rich class of radial basis functions (RBF) can also be derived from the RKHS formulation, employing the Green's function for various differential operators.

Our focus in this article is on a finite-dimensional real-valued RKHS. We first deal with the case where $\Omega = \{1, 2, \ldots, N\}$. Then the kernel function simply means a symmetric, positive semi-definite matrix A and the RKHS is therefore the image of A, which is a linear subspace in $\mathbf{R}^N$: $A(\mathbf{R}^N) := \{A\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^N\}$. In our context, such as for direct manipulation blendshapes for facial animation [3], the matrix A is the covariance matrix of given prior data. We will show that the original formulation in [3] is reduced to solving the above regularization problem in the RKHS.

Another RKHS discussed in this article is a finite-dimensional function space, rather than a numerical vector space. This will be used to explain a superresolution technique which is derived from RBF regression in [8].

2 Mathematical background

In this section we briefly review the definitions and fundamental properties of RKHS (for reference see [4, 11] or [1]).

Let Ω be an abstract set, and $\boldsymbol{H}(\Omega)$ ($\boldsymbol{H}$, for short) be a Hilbert space consisting of the real-valued functions defined on Ω, with the inner product $\langle\ ,\ \rangle$.

Definition 1 If the function $K : \Omega \times \Omega \to \mathbf{R}$ satisfies the following conditions[1], K is called a *reproducing kernel* of $\boldsymbol{H}$:

1. For any fixed $y \in \Omega$, $K(x, y)$ belongs to $\boldsymbol{H}$ as a function of x.
2. For any $f \in \boldsymbol{H}$, we have $f(y) = \langle f(x), K(x, y)\rangle_x$.

Definition 2 If Hilbert space $\boldsymbol{H}$ has the kernel K in Definition 1, then $\boldsymbol{H}$ is referred to as a *reproducing kernel Hilbert space* (RKHS).

Proposition 1. For the reproducing kernel K, we have:

$$K(y, z) = \langle K(x, y), K(x, z)\rangle_x, \tag{2}$$

for any $y, z \in \Omega$.

Considering that the inner product in $\boldsymbol{H}(\Omega)$ is a symmetric, bilinear form, we get the fundamental property of the kernel K from the above proposition:

Proposition 2. Let $K : \Omega \times \Omega \to \mathbf{R}$ be the kernel function of RKHS $\boldsymbol{H}(\Omega)$. Then K satisfies the following properties:

1. K is symmetric: $K(x, y) = K(y, x)$ for any $x, y \in \Omega$.

[1]In condition 2, the inner product $\langle\ ,\ \rangle_x$ means that we get the inner product value of the two functions with variable x.

2. K is positive semi-definite [2]: For any $n \in \mathbf{N}$, $(x_1, \ldots, x_n)^T \in \Omega^n$ and $(a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$, we always have $\sum_{i,j=1}^{n} a_i a_j K(x_i, x_j) \geq 0$.

Conversely, if we are given a symmetric, positive semi-definite function K defined on $\Omega \times \Omega$, the following theorem holds:

Theorem 1. Suppose that K is a symmetric, positive semi-definite function on $\Omega \times \Omega$. Then there exists an RKHS $\boldsymbol{H}$ that has K as its reproducing kernel.

3 RKHS and Bayesian estimates

In this section we first describe the numerical vector space $A(\mathbf{R}^N)$ in section 1 as an RKHS. We will thereafter see how the regularization problem for $A(\mathbf{R}^N)$ in section 1 is solved for learning the prior data of facial animations in [3], where A is the covariance matrix. In this section we set $\boldsymbol{H}_A := A(\mathbf{R}^N) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^N\}$.

3.1 H_A as RKHS

We consider the case where $\Omega = \{1, 2, \ldots, N\}$. Any mapping: $\Omega \times \Omega \to \mathbf{R}$ can then be represented as a matrix. Let $A : \Omega \times \Omega \to \mathbf{R}$ be a mapping which is symmetric, positive semi-definite in the sense of Proposition 2 in section 2. This simply means that A is an N-th order positive semi-definite symmetric matrix. Considering Theorem in section 2, we can make $\boldsymbol{H}_A$ a computationally tractable RKHS with the given matrix A as its kernel in the following way.

We first note that $\boldsymbol{H}_A$ is endowed with an inner product $\langle\ ,\ \rangle$, which is given by

$$\langle f, g \rangle = (\mathbf{x}, A\mathbf{y}), \tag{3}$$

where $f = A\mathbf{x}, g = A\mathbf{y} \in \boldsymbol{H}_A$, and $(\ ,\ )$ denotes the usual inner product in $\mathbf{R}^N$. Actually $\boldsymbol{H}_A$ is a finite-dimensional linear subspace of $\mathbf{R}^N$. Let us then check the well-definedness of the above inner product. Suppose that $f = A\mathbf{x}_1 = A\mathbf{x}_2$ and $g = A\mathbf{y}_1 = A\mathbf{y}_2 \in \boldsymbol{H}_A$. We then have:

$$\begin{aligned}(\mathbf{x}_1, A\mathbf{y}_1) &= (\mathbf{x}_1, A\mathbf{y}_2) = (A^T\mathbf{x}_1, \mathbf{y}_2) \\ &= (A\mathbf{x}_1, \mathbf{y}_2) = (A\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}_2, A\mathbf{y}_2).\end{aligned}$$

This means that the mapping $\langle\ ,\ \rangle : \boldsymbol{H}_A \times \boldsymbol{H}_A \to \mathbf{R}$ is well-defined by (3). It is easy to see that this mapping is symmetric, and bilinear. Further, since A is positive semi-definite, it follows that $\langle f, f \rangle \geq 0$, for any $f \in \boldsymbol{H}_A$. We now denote $\sqrt{\langle f, f \rangle}$ by $\|f\|_A$. Next we show that $\|f\|_A = 0$, if and only if $f = 0$. Denoting the rank of A by r, we first note that $\boldsymbol{H}_A$ is a linear space spanned by the eigenvectors u_i of A, whose eigen values λ_i are positive:

$$f \in \boldsymbol{H}_A \Leftrightarrow f = \sum_{i=1}^{r} c_i u_i, \tag{4}$$

where $\lambda_j > 0$ for $1 \leq j \leq r$, and $\lambda_k = 0$ for $r < k \leq N$. We can then rewrite f by introducing λ_i,

$$f = \sum_{i=1}^{r} c_i u_i = \sum_{i=1}^{r} \frac{c_i}{\lambda_i} A u_i = A(\sum_{i=1}^{r} \frac{c_i}{\lambda_i} u_i).$$

[2] In section 4 we also consider RKHS for a symmetric, positive definite function.

According to (3), we thus have

$$\|f\|_A^2 = (\sum_{i=1}^{r} \frac{c_i}{\lambda_i} u_i \, , \, A(\sum_{j=1}^{r} \frac{c_j}{\lambda_j} u_j))$$
$$= (\sum_{i=1}^{r} \frac{c_i}{\lambda_i} u_i \, , \, \sum_{j=1}^{r} c_j u_j) = \sum_{i=1}^{r} \frac{c_i^2}{\lambda_i}. \tag{5}$$

We note that equation (5) can be understood as the finite dimensional version of the Mercer's theorem. So any coefficient c_i must be 0 ($1 \leq i \leq r$), if $\|f\|_A = 0$. We have thus confirmed that the mapping $\langle \, , \, \rangle$ is the inner product of $\boldsymbol{H}_A$, and that $\|\cdot\|_A$ is its norm.

Since $\boldsymbol{H}_A$ is finite-dimensional, it is easy to see that $\boldsymbol{H}_A$ is a Hilbert space with the inner product (3). Finally we see that the given matrix A is the kernel of $\boldsymbol{H}_A$. Let $\mathbf{e}_i (1 \leq i \leq N)$ be the canonical basis in $\mathbf{R}^N$, such that $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$ for $1 \leq i, j \leq N$. The i-th column vector of A, denoted also by $A(\cdot, i)$, is then given by $A\mathbf{e}_i$, which means that A satisfies condition 1 in Definition 1. Next suppose that $f = A\mathbf{x} \in \boldsymbol{H}_A$, denoted as $f = (f_1, \ldots, f_N)^T$. According to the definition of the inner product (3), it follows that

$$f_i = (A\mathbf{x}, \mathbf{e}_i) = \langle f, A\mathbf{e}_i \rangle = \langle f, A(\cdot, i) \rangle$$

This holds for $1 \leq i \leq N$, which means that A satisfies condition 2 in Definition 1.

3.2 Regularization for direct manipulation blendshapes

Our blendshape facial model will be described with:

$$\mathbf{f} = \mathbf{B}\mathbf{w} + \mathbf{f}_0 \tag{6}$$

where $\mathbf{f}$ is a $3n$-dimensional vector containing the components of each of the n vertices or control points on the face vectorized in some arbitrary order such as xyzxyzxyz... The matrix $\mathbf{B} \in \mathbf{R}^{3n \times m}$ contains the m delta blendshape targets $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ in its columns using the same component ordering (see [9] for more details). $\mathbf{w} \in \mathbf{R}^m$ are the blendshape weights, and $\mathbf{f}_0$ is the neutral shape in similarly vectorized form. In the direct manipulation problem, the positions of one or several vertices from $\mathbf{f}$ are constrained or partially constrained with a pin-and-drag operation by the artist. We then denote a vector consisting of the position-constrained vertices by $\overline{\mathbf{f}}$ for $\mathbf{f}$ in (6). From now on, for an arbitrary $3n$ dimentional vector $\mathbf{v}$, $\overline{\mathbf{v}}$ means a vector consisting of the components of $\mathbf{v}$ that have the same indices as those of the position-contrained vertices of $\mathbf{f}$. More precisely, considering (6), let $\mathbf{f}_b = \mathbf{B}\mathbf{w}_b + \mathbf{f}_0$ and $\mathbf{f}_a = \mathbf{B}\mathbf{w}_a + \mathbf{f}_0$, where $\mathbf{f}_b$ is the face model before the constraints and $\mathbf{f}_a$ is the constrained result. In concept, we would like to solve the following problem:

$$\min_{\Delta \mathbf{f}} \; \|L(\Delta \mathbf{f}) - \Delta \overline{\mathbf{m}}\|^2 \tag{7}$$

where $\Delta \mathbf{f} = \mathbf{f}_a - \mathbf{f}_b$, $\Delta \overline{\mathbf{m}}$ is the differences of the vertex positions that are constrained by the artist, and L is the *projection* that is defined as the diagonal matrix taking 1 as the components corresponding to these constrained vertices. It can be recognized that this is a severely underconstrained problem – for example, the artist may constrain only one or several vertices (with one vertex corresponding to three diagonal elements in L), whereas most diagonal elements in L may take 0. As a solution to the direct manipulation problem, what we want to find is the weight $\mathbf{w}_a$, whereas the weight $\mathbf{w}_b$ is known in advance. To make it, we first solve the regularization problem regarding (7) in the RKHS framework, where we find the minimizer $\mathbf{f}$, rather than the weights. Once we find the solution to this regularization problem, we can easily get the weights, as is described in [7, 3].

3.2.1 Learned direct manipulation

A strong and flexible prior would be desired to regularize problem (7). Let us assume that the prior training data are given, such as the face capture data. We then select the RKHS prior developed in section 3.1, by choosing the kernel K as the covariance matrix A of the prior data:

$$A = E[(\mathbf{x} - \mathbf{e}_0)(\mathbf{x} - \mathbf{e}_0)^T],$$

where each $\mathbf{x}$ is a face vector of the training data, and $\mathbf{e}_0$ means the mean shape of the face data. Since a representation of the unknown function $\mathbf{f}$ in terms of the eigenvectors of A was made in section 1, we will also explore this representation for the data-fitting term in equation (1). We will see that this choice leads to a particularly simple solution to the direct manipulation problem.

3.2.2 RKHS for data + prior algorithm

To develop this data+prior approach to the direct manipulation problem, we start with a principal component model. The PCA model will then be denoted

$$\mathbf{f} = \mathbf{U}\mathbf{c} + \mathbf{e}_0 \tag{8}$$

where $\mathbf{U}$ contains eigenvectors of the data covariance matrix A. The vector $\mathbf{c}$ in (8) will be referred to as the coefficient vector of the eigenvectors (or the coefficients, for short) throughout this report. In our context, we may assume that $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r)$ is a $3n \times r$ matrix containing only the eigenvectors $\mathbf{u}_i$ whose eigenvalue λ_i are all positive for $1 \leq i \leq r$ (thus r would be much smaller than $3n$ in practice). In the following discussion we put $\overline{\mathbf{U}} := (\overline{\mathbf{u}}_1, \overline{\mathbf{u}}_2, \ldots, \overline{\mathbf{u}}_r)$.

Now let us denote the expansions of $\mathbf{f}_a$ and $\mathbf{f}_b$ in terms of the eigenvectors of A as $\mathbf{f}_a = \mathbf{U}\mathbf{c}_a + \mathbf{e}_0$; $\mathbf{f}_b = \mathbf{U}\mathbf{c}_b + \mathbf{e}_0$. Unlike [3], we do not assume that the coefficients $\mathbf{c}_a$ and $\mathbf{c}_b$ are zero-mean Gaussian. Setting $\Delta\mathbf{f} = \mathbf{f}_a - \mathbf{f}_b = \mathbf{U}(\mathbf{c}_a - \mathbf{c}_b) = \mathbf{U}\Delta\mathbf{c}$, we therefore formulate the problem for unknown $\Delta\mathbf{f}$ (or $\Delta\mathbf{c}$), rather than for $\mathbf{c}_a$.

The direct manipulation problem regarding (8) can thus be interpreted through RKHS formulation. Consider $H_A = A(\mathbf{R}^{3n})$, where we wish to solve the following regularization problem:

$$\min_{\Delta\mathbf{f} \in H_A} \|L(\Delta\mathbf{f}) - \Delta\overline{\mathbf{m}}\|^2 + \beta\|\Delta\mathbf{f}\|_A^2, \tag{9}$$

where $L(\Delta\mathbf{f}) = \overline{\mathbf{U}}\Delta\mathbf{c}$ in our context and $\|\cdot\|_A$ denotes the RKHS norm (5) for H_A. This is therefore equivalent to

$$\min_{\Delta\mathbf{c} \in \mathbf{R}^r} \|\overline{\mathbf{U}}\Delta\mathbf{c} - \Delta\overline{\mathbf{m}}\|^2 + \beta\|\Delta\mathbf{c}\|_\Lambda^2 \tag{10}$$

where $\|\cdot\|$ is the usual Euclidian norm and the second norm $\|\cdot\|_\Lambda$ is defined for $\mathbf{y} \in \mathbf{R}^r$ as

$$\|\mathbf{y}\|_\Lambda^2 = \mathbf{y}^T \mathbf{\Lambda}^{-1} \mathbf{y}, \tag{11}$$

having $\mathbf{\Lambda}^{-1}$ as the $r \times r$ diagonal matrix whose diagonal element is λ_i^{-1} for $1 \leq i \leq r$. According to section 1, the problem (9) is theoretically solvable and we can make it numerically regarding the least square problem (10), as shown in [3].

4 RKHS for superresolution

In [8] we discussed RBF (radial basis function) regression of an exponential type. With the known n training data points $\mathbf{p}_k$, we denote the data to be interpolated by a vector $\mathbf{f}_0 = (f_1, f_2, \ldots, f_n)^T$.

We then assume that the RBF regression at a location $\mathbf{p}$ has the form $f(\mathbf{p}) = \sum_{k=1}^{n} w_k G(\|\mathbf{p} - \mathbf{p}_k\|)$, where $G()$ is a radial function of exponential type. So we want to decide the weights $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ such that $f(\mathbf{p}_k) = f_k$ for $1 \leq k \leq n$. The $n \times n$ matrix $\mathbf{G}_0$, whose (i,j) component given by $G_{ij} \equiv G(\|\mathbf{p}_i - \mathbf{p}_j\|)$, is positive definite, symmetric, and therefore invertible. In matrix-vector notation, the regression can thus take the following form:

$$f(\mathbf{p}) = \mathbf{r}^T \mathbf{w} = \mathbf{r}^T \mathbf{G}_0^{-1} \mathbf{f}_0, \tag{12}$$

where $\mathbf{r} = (G(\|\mathbf{p} - \mathbf{p}_1\|), G(\|\mathbf{p} - \mathbf{p}_2\|), \ldots, G(\|\mathbf{p} - \mathbf{p}_n\|))^T$.

The superresolution technique in [8] is motivated by the above expression (12). Having a stationary stochastic process in mind, we now consider $\mathbf{r}$ as a vector of cross-covariances indexed by the difference between the location $\mathbf{p}$ and the locations of the data points $\mathbf{p}_i$, instead of the $\mathbf{r}$ defined above in (12), $\mathbf{r} = (C(\|\mathbf{p} - \mathbf{p}_1\|), C(\|\mathbf{p} - \mathbf{p}_2\|), \ldots, C(\|\mathbf{p} - \mathbf{p}_n\|))^T$. We also replace $\mathbf{G}_0$ by a covariance matrix $\mathbf{C}_0$, whose (i,j) component is given by $C(\|\mathbf{p}_i - \mathbf{p}_j\|)$. Assuming that the covariance function $C(\Delta)$ is known for all offsets Δ and that $\mathbf{C}_0$ is invertible, we thus have the following superresolution scheme:

$$f(\mathbf{p}) = \mathbf{r}^T \mathbf{C}_0^{-1} \mathbf{f}_0. \tag{13}$$

This is an extension of the RBF regression (12) in that we can treat a wider class of the "kernel" functions (or the covariance functions) with (13). See [8] for numerical illustrations about this extension.

In this section we will give an RKHS framework for this scheme.

4.1 Finite dimensional function space having a reproducing kernel

Let $\boldsymbol{H}_n(\Omega)$ ($\boldsymbol{H}_n$, for short) be a finite-dimensional real-valued function space, having $\{\mathbf{h}_i(\mathbf{p})\}_{1 \leq i \leq n}$ as its basis:

$$\mathbf{f} \in \boldsymbol{H}_n \Leftrightarrow \mathbf{f}(\mathbf{p}) = \sum_{i=1}^{n} w_i \mathbf{h}_i(\mathbf{p}). \tag{14}$$

where the coefficients $\{w_i\}_{1 \leq i \leq n}$ are uniquely determined according to the choice of the basis.

Suppose that an $n \times n$ positive definite symmetric matrix $\mathbf{S}$ is given, while denoting its (i,j) component by s_{ij}: $\mathbf{S} \equiv [s_{ij}]$. Then, for any element $\mathbf{f}(\mathbf{p}) = \sum_{i=1}^{n} w_i \mathbf{h}_i(\mathbf{p})$ in $\boldsymbol{H}_n$, we define the norm $\|\cdot\|_{\boldsymbol{H}_n}$ with:

$$\begin{aligned} \|\mathbf{f}\|_{\boldsymbol{H}_n}^2 &= \sum_{i,j} s_{ij} w_i w_j \\ &\equiv (\mathbf{w}, \mathbf{S}\mathbf{w}) \end{aligned} \tag{15}$$

where we put $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$. We then note that putting $\langle \mathbf{h}_i, \mathbf{h}_j \rangle_{\boldsymbol{H}_n} := s_{ij}$ induces the inner product in $\boldsymbol{H}_n$, which is denoted by $\langle \ , \ \rangle_{\boldsymbol{H}_n}$. We thus get the following (see [11]):

Proposition 3. Denoting $\mathbf{S}^{-1}$ by $[t_{ij}]$, let us define $K : \Omega \times \Omega \to \mathbf{R}$ as

$$K(\mathbf{p}, \mathbf{q}) := \sum_{i,j} t_{ij} \mathbf{h}_i(\mathbf{p}) \mathbf{h}_j(\mathbf{q}). \tag{16}$$

Then K is the reproducing kernel of $\boldsymbol{H}_n(\Omega)$.

Actually it is easy to see the above K in (16) satisfies conditions 1 and 2 in the definition of the reproducing kernel (Definition 1 in section 2). For instance, we have, for any $\mathbf{f} \in \boldsymbol{H}_n$,

$$\begin{aligned}
\langle \mathbf{f}, K(\cdot, \mathbf{q}) \rangle_{H_n} &= \langle \sum_i w_i \mathbf{h}_i, \sum_k (\sum_j t_{kj} \mathbf{h}_j(\mathbf{q})) \mathbf{h}_k \rangle_{H_n} \\
&= \sum_i \sum_j \sum_k w_i t_{kj} \langle \mathbf{h}_i, \mathbf{h}_k \rangle_{H_n} \mathbf{h}_j(\mathbf{q}) \\
&= \sum_{i,j,k} w_i t_{jk} s_{ki} \mathbf{h}_j(\mathbf{q}) \\
&= \sum_{i,j} w_i (\sum_k t_{jk} s_{ki}) \mathbf{h}_j(\mathbf{q}) = \sum_{i,j} w_i \delta_{ji} \mathbf{h}_j(\mathbf{q}) \\
&= \sum_i w_i \mathbf{h}_i(\mathbf{q}) \\
&= \mathbf{f}(\mathbf{q}).
\end{aligned}$$

This means that the above K satisfies condition 2 in Definition 1.

Finally it should be noted that we can select an arbitrary positive definite symmetric matrix $\mathbf{S}$ for the definition of $\| \cdot \|_{H_n}$ in (15) and therefore the kernel K in (16), independent of the choice of the basis $\{\mathbf{h}_i(\mathbf{p})\}_{1 \leq i \leq n}$.

4.2 The finite dimensional RKHS for superresolution

We have shown that $\boldsymbol{H}_n(\Omega)$ is a finite dimensional RKHS. A very nice feature of this formulation is that, once we can specify the basis functions $\{\mathbf{h}_i(\mathbf{p})\}_{1 \leq i \leq n}$ for a practical situation, the RKHS is computationally tractable, so that we can numerically solve the regularization problem in section 1.

We now go back to the superresolution scheme (13), where we consider $\mathbf{h}_i(\mathbf{p}) := C(\mathbf{p} - \mathbf{p}_i)$ for $1 \leq i \leq n$, having $\Omega = \mathbf{R}^k$. We may assume that $\{\mathbf{h}_i(\mathbf{p})\}_{1 \leq i \leq n}$ is a linearly independent system with the covariance matrix $\mathbf{C}_0$ being invertible. This assumption is actually quite reasonable for our practical situations in computer graphics (see [2, 8], for instance). We can therefore deal with a variety of regularization problems on $\boldsymbol{H}_n$. We may then have $\mathbf{C}_0$ as a choice of $\mathbf{S}$ in (15).

On one hand, when we consider only the regression problem in this section, it is easily solved in $\boldsymbol{H}_n$. This is because the condition

$$\mathbf{f}(\mathbf{p}_i) \equiv \sum_{j=1}^{n} w_j \mathbf{h}_j(\mathbf{p}_i) = f_i \text{ , for } 1 \leq i \leq n$$

simply means that $\mathbf{f}_0 = \mathbf{C}_0 \mathbf{w}$, which leads to

$$\begin{aligned}
\mathbf{f}(\mathbf{p}) &= \sum_{j=1}^{n} w_j \mathbf{h}_j(\mathbf{p}) \equiv \sum_{j=1}^{n} w_j C(\mathbf{p} - \mathbf{p}_j) \\
&= \mathbf{r}^T \mathbf{w} = \mathbf{r}^T \mathbf{C}_0^{-1} \mathbf{f}_0.
\end{aligned}$$

This is equal to the superresolution form (13).

5 Concluding remark

The general idea of formulating inverse problems as a sum of data and prior, or data and smoothness terms, has been independently discovered in several different fields. There are both probabilistic and deterministic formulations [10]. This paper describes a finite-dimensional RKHS framework that can encompass both probabilistic (section 3.2.2) and deterministic (section 4.2) formulations for regularizing an inverse problem.

The generality of this framework is illustrated by defining finite-dimensional RKHS formulations for several example problems in computer graphics: signal and geometry interpolation/extrapolation, and solving an inverse problem in animation. We are currently exploring further applications of RKHS, such as those for rendering and texturing problems.

Acknowledgement

The author would like to thank J.P. Lewis, Hiroyuki Ochiai and Ayumi Kimura for their valuable comments and warm words of encouragement.

References

[1] K. Anjyo and J.P. Lewis: RBF interpolation and Gaussian process regression through an RKHS formulation, *Journal of Math-for-Industry*, 3: 63-71, 2011.

[2] K. Anjyo and J.P. Lewis: Scattered Data Interpolation for Computer Graphics. In *SIGGRAPH 2014 Course*: `http://portal.acm.org`, 2014.

[3] K. Anjyo, J.P. Lewis, and H. Todo: A practical approach to direct manipulation blendshapes. *Journal of Graphics Tools*, 16(3): 160-176, 2012.

[4] N. Aronszajn: Theory of reproducing kernels, *Trans.Amer. Math. Soc.* 68 (1950), 337404.

[5] G. Kimeldorf and G. Wahba: Some results on Tchebycheffian spline functions, *Journal of Mathematical Analysis and Applications*, 33 (1): 82-95, 1971.

[6] J.P. Lewis: Generalized Stochastic Subdivision, *ACM Trans. Graph.* 6(3): 168-190, 1987.

[7] J.P. Lewis and K. Anjyo: Direct Manipulation Blendshapes, *IEEE Comput. Graph. Appl.* 30(4): 42-50, 1987.

[8] J.P. Lewis, K. Anjyo and T. Rhee: Superresolution from Principal Component Models by RKHS Sampling, In: Proc. MEIS2014, *MI Lecture Note 58, Kyushu University*: 99-106, 2014.

[9] J.P. Lewis, K. Anjyo, T. Rhee, M. Zhang, F.H. Pighin and Z. Deng: Practice and Theory of Blendshape Facial Models. In: *Eurographics 2014 (State of the Art Reports)*: 199-218, 2014.

[10] D. Mumford and J. Shah: Optimal approximations by piecewise smooth functions and associated variational problems, *Comm. Pure. Appl. Math.* 42(5):577-685, 1989.

[11] S. Saito, *An introduction to the theory of reproducing kernel Hilbert spaces* (in Japanese), Makino-shoten, 2002.

[12] H. Todo, K. Anjyo, W. Baxter, T. Igarashi: Locally Controllable Stylized Shading, *ACM Trans. Graph.* 26(3): Article 17, 2007.

[13] G. Wahba, *Spline models for observational data*, SIAM, 1990.