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THE NUMBER OF PROOFS FOR A BCK-FORMULA

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In this note, we give a necessary and sufficient condition for a BCK-formula to have the unique normal form proof.

We call implicational propositional formulas formulas for short. BCK-formulas are the formulas which are derivable from axioms $B = (a \rightarrow b) \rightarrow (c \rightarrow a) \rightarrow c \rightarrow b$, $C = (a \rightarrow b \rightarrow c) \rightarrow b \rightarrow a \rightarrow c$, and $K = a \rightarrow b \rightarrow a$ by substitution and modus ponens. It is known that the property of being a BCK-formula is decidable (Jaskowski [11, Theorem 6.5], Ben-Yelles [3, Chapter 3, Theorem 3.22], Komori [12, Corollary 6]). The set of BCK-formulas is identical to the set of provable formulas in the natural deduction system with the following two inference rules.

$$\begin{array}{c} [\gamma] \\ \vdots \\ \delta \\ \hline \gamma \rightarrow \delta \end{array} (\rightarrow I) \qquad \frac{\gamma \rightarrow \delta \quad \gamma}{\delta} (\rightarrow E).$$

Here γ occurs at most once in $(\rightarrow I)$. By the formulae-as-types correspondence [10], this set is identical to the set of type-schemes of closed BCK- λ -terms. (See [5].) A BCK- λ -term is a λ -term in which no variable occurs twice. Basic notions concerning the type assignment system can be found [4]. Uniqueness of normal form proofs has been known for balanced formulas. (See [2, 14].) It is related to the coherence theorem in cartesian closed categories. A formula is balanced when no variable occurs more than twice in it. It was shown in [8] that the proofs of balanced formulas are BCK-proofs. Relevantly balanced formulas were defined in [9], and it was proved that such formulas have unique normal form proofs. Balanced formulas are included in the set of relevantly balanced formulas. We show a necessary and sufficient condition for a BCK-formula to have a unique

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normal form proof using the following notion of minimality. The notion of BCK-minimality was introduced by Komori [13]. A formula α is called a *trivial substitution instance* of β iff α is a substitution instance of β and β is a substitution instance of α .

DEFINITION 1. A formula is *BCK-minimal* iff it is a BCK-formula and it is not a nontrivial substitution instance of another BCK-formula. A BCK-formula β is a *minimal formula* of α iff β is BCK-minimal and α is a substitution instance of β .

It is clear that a BCK-minimal formula is a principal type-scheme of a closed BCK- λ -term.

We identify two λ -terms when they are α -convertible. Similarly, two types are identified when one is a trivial substitution instance of the other.

LEMMA 1 ([7]). *If two closed BCK- λ -terms in β -normal form have the same principal type, then they are identical.*

LEMMA 2 ([8]). *A BCK-formula is BCK-minimal iff it is a principal type-scheme of a closed BCK- λ -term in $\beta\eta$ -normal form.*

THEOREM 1. *Given a BCK-formula α , the number of closed BCK- λ -terms in $\beta\eta$ -normal form which has type α is identical to the number of minimal formulas of α .*

PROOF. Let α be a BCK-formula. We denote by $\text{proof}(\alpha)$ the set of closed BCK- λ -terms in $\beta\eta$ -normal form which have type α and we denote by $\text{min}(\alpha)$ the set of minimal formulas of α . We define a function from $\text{proof}(\alpha)$ to $\text{min}(\alpha)$ and show that it is surjective and injective. Let $M \in \text{proof}(\alpha)$. Then M has type α . By the principal type-scheme theorem (Theorem 15.26 of [4]), M has a principal type-scheme. We denote it by $\text{pts}(M)$. Since M is in $\beta\eta$ -normal form, $\text{pts}(M)$ is minimal by Lemma 2. So we have $\text{pts}(M) \in \text{min}(\alpha)$. Thus pts is a function from $\text{proof}(\alpha)$ to $\text{min}(\alpha)$. Injectivity of pts is immediate from Lemma 1. To prove the surjectivity, let $\beta \in \text{min}(\alpha)$ and apply Lemma 2 to β . Then there is a closed BCK- λ -term N in $\beta\eta$ -normal form whose principal type-scheme is β . Therefore pts is surjective. \square

One consequence of the theorem is that a BCK-formula α has only a finite number of normal form proofs. In fact, we can enumerate all the minimal formulas instead of λ -terms. Given a formula γ , we denote by $s_0(\gamma)$ the set of formulas β such that γ is a substitution instance of β . Since we identify trivial substitution instances, the set $s_0(\gamma)$ is finite. Next we denote by $s(\gamma)$ the set of BCK-formulas in $s_0(\gamma)$. Since BCK-provability is decidable, we can enumerate the elements of $s(\gamma)$ from $s_0(\gamma)$. Finally note that β is BCK-minimal iff $s(\beta) = \{\beta\}$. Therefore we have

$$\text{min}(\alpha) = \{\beta \in s_0(\alpha) \mid s(\beta) = \{\beta\}\}.$$

Thus we can enumerate all the elements of $\text{min}(\alpha)$.

Akama [1] showed that the number of cut-free proof (in sequent calculus) for a BCK-formula is finite.

COROLLARY 1. *A BCK-formula has a unique proof in $\beta\eta$ -normal form iff it has a unique minimal formula.*

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