

The infinite base change lifting associated to an APF extension of a p -adic field

高田, 芽味

<https://doi.org/10.15017/1543931>

出版情報 : 九州大学, 2015, 博士 (数理学) , 課程博士
バージョン :
権利関係 : 全文ファイル公表済

The infinite base change lifting associated to an APF extension of a p -adic field

Megumi Takata

September 3, 2015

Abstract

In this paper, we construct a base change lifting for an APF extension of a mixed characteristic local field.

1 Introduction

Let p be a prime number. In this paper, we shall construct a local base change lifting for an almost pro- p cyclic extension of infinite degree. The point is that the local base change lifting for a totally ramified extension coincides with an operation coming from the close local fields theory of Kazhdan under some conditions.

We state the result more precisely. For a local field L with a finite residue field, we denote by $\mathcal{A}(\mathrm{GL}_N(L))$ the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_N(L)$ over \mathbb{C} . We denote the Weil group of L by W_L . We recall that an L-parameter of $\mathrm{GL}_N(L)$ is a group homomorphism $\phi: W_L \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_N(\mathbb{C})$ such that $\phi|_{W_L}$ is semi-simple and smooth and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic. Let $\Phi(\mathrm{GL}_N(L))$ denote the set of isomorphism classes of L-parameters of $\mathrm{GL}_N(L)$. We note that $\Phi(\mathrm{GL}_1(L))$ is equal to the set $\mathrm{Hom}(L^\times, \mathbb{C}^\times)$ of smooth characters of L^\times . We denote by LLC_L the local Langlands correspondence of GL_N over L , whose existence was firstly proven by [17] for L of positive characteristic and by [13] for L of characteristic zero. Let F be a finite extension of \mathbb{Q}_p and E an APF extension of infinite degree, in particular an almost pro- p extension. Let F_∞ be the field of norms associated with E/F . We denote by $\mathrm{Res}_{\infty/0}$ the restriction map $\Phi(\mathrm{GL}_N(F)) \rightarrow \Phi(\mathrm{GL}_N(F_\infty))$ with respect to the natural injection $W_{F_\infty} \hookrightarrow W_F$.

Theorem 1.1. *Suppose that the extension E/F is cyclic. Then we can construct a map $\mathrm{BC}_{\infty/0}: \mathcal{A}(\mathrm{GL}_N(F)) \rightarrow \mathcal{A}(\mathrm{GL}_N(F_\infty))$ such that the following*

diagram is commutative:

$$\begin{array}{ccc}
\mathcal{A}(\mathrm{GL}_N(F_\infty)) & \xrightarrow{\mathrm{LLC}_{F_\infty}} & \Phi(\mathrm{GL}_N(F_\infty)) \\
\mathrm{BC}_{\infty/0} \uparrow & & \uparrow \mathrm{Res}_{\infty/0} \\
\mathcal{A}(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{LLC}_F} & \Phi(\mathrm{GL}_N(F)).
\end{array}$$

We shall call $\mathrm{BC}_{\infty/0}$ *the base change lifting of infinite degree*. We construct $\mathrm{BC}_{\infty/0}$ by using Arthur and Clozel's result [1] and close fields theory of Kazhdan [15]. Hence our construction is basically on the representation theory of p -adic groups, that is to say, the automorphic side. However, we use LLC when we prove Theorem 1.3 by showing the corresponding statement in the terms of L-parameters, that is, the Galois side. The author expects that in the future we will be able to avoid such arguments.

To construct the lifting, we shall adapt Kazhdan's theory to our setting. Let L be a local field with a finite residue field, $\mathcal{O} \subset L$ the ring of integers, and $\mathfrak{p} \subset \mathcal{O}$ the maximal ideal. Let $\mathbb{K}_l(L)$ denote the principal congruence subgroup of level l of $\mathrm{GL}_N(L)$:

$$\mathbb{K}_l(L) = \mathrm{Ker}(\mathrm{GL}_N(\mathcal{O}) \rightarrow \mathrm{GL}_N(\mathcal{O}/\mathfrak{p}^l)).$$

We denote by $\mathrm{Rep}(\mathrm{GL}_N(L))$ the category of admissible smooth representations of $\mathrm{GL}_N(L)$ and by $\mathrm{Rep}_l(\mathrm{GL}_N(L))$ the full subcategory of $\mathrm{Rep}(\mathrm{GL}_N(L))$ consisting of representations generated by their $\mathbb{K}_l(L)$ -fixed vectors.

We fix an algebraic closure \overline{F} of E . For any real number $v \geq -1$, we denote by $\mathrm{Gal}(\overline{F}/F)^v$ the v -th ramification group in upper numbering. Let $b_1 < b_2 < \dots$ be the ramification breaks of E/F . We put

$$F_n = \overline{F}^{\mathrm{Gal}(\overline{F}/E) \mathrm{Gal}(\overline{F}/F)^{b_n}}.$$

For a real number $u \geq 0$, we define

$$\psi_{E/F}(u) = \int_0^u (\mathrm{Gal}(\overline{F}/F) : \mathrm{Gal}(\overline{F}/E) \mathrm{Gal}(\overline{F}/F)^v) dv.$$

We take a non-decreasing sequence of non-negative integers $\{l_n\}_{n=1}^\infty$ satisfying the following:

Condition (L) . $l_n \rightarrow \infty$ ($n \rightarrow \infty$) and $l_n \leq p^{-1}(p-1)\psi_{E/F}(b_n)$.

Then we have a theorem that $\mathrm{Rep}(\mathrm{GL}_N(F_\infty))$ can be obtained by taking the limit of certain subcategories of $\mathrm{Rep}(\mathrm{GL}_N(F_n))$:

Theorem 1.2. *For any indices $1 \leq n < m \leq \infty$, there exists a natural equivalence of categories*

$$A_{m/n} : \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_m)).$$

This $\{A_{m/n} \mid 0 \leq n < m \leq \infty\}$ makes the diagram

$$\begin{array}{ccc}
\mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_n)) & \xrightarrow{A_{m'/n}} & \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_{m'})) \\
\downarrow A_{m/n} & & \downarrow \\
\mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_m)) & & \\
\downarrow & & \downarrow \\
\mathrm{Rep}_{l_m}(\mathrm{GL}_N(F_m)) & \xrightarrow{A_{m'/m}} & \mathrm{Rep}_{l_m}(\mathrm{GL}_N(F_{m'}))
\end{array} \tag{1}$$

commute for any $n \leq m \leq m'$. We also denote by $A_{m/n}$ the bijection

$$\mathcal{A}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathcal{A}_{l_n}(\mathrm{GL}_N(F_m))$$

induced by the equivalence $A_{m/n}$. Then we can take the direct limit of $\{A_{\infty/n}\}_n$:

$$\varinjlim_n A_{\infty/n} : \varinjlim_n \mathcal{A}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathcal{A}(\mathrm{GL}_N(F_\infty)),$$

which is also bijective.

Next, we shall prove that $A_{m/n}$ coincides with the base change lifting. For a cyclic extension F'/F of prime degree, let

$$\mathrm{BC}_{F'/F} : \mathcal{A}(\mathrm{GL}_N(F)) \rightarrow \mathcal{A}(\mathrm{GL}_N(F'))$$

be the base change lifting in the sense of [1, Chapter 1, Section 6]. For a general cyclic extension F'/F of finite degree, we define $\mathrm{BC}_{F'/F}$ as the composite of the base changes attached to intermediate extensions of F'/F of prime degree. In particular, we write $\mathrm{BC}_{F_m/F_n} = \mathrm{BC}_{m/n}$. We denote by $\mathcal{A}_l(\mathrm{GL}_N(F))$ the subset of $\mathcal{A}(\mathrm{GL}_N(F))$ consisting of representations which have a non-trivial $\mathbb{K}_l(F)$ -fixed vector.

In the rest of this section, we suppose that E/F is cyclic. We put $\Gamma = \mathrm{Gal}(E/F)$ and denote by $\widehat{\Gamma}$ the group of smooth characters of Γ with valued in \mathbb{C}^\times . By local class field theory, we identify an element of $\widehat{\Gamma}$ with a character $F^\times \rightarrow \mathbb{C}^\times$ which factors through $F^\times/N_{F_n/F}(F_n^\times)$ for some n .

Theorem 1.3. *We take a sequence $\{l'_n\}_{n=1}^\infty$ satisfying the condition (L) and such that there exists a positive integer n_0 such that $l'_n < 2^{-N} \lfloor p^{-1}(p-1)\psi_{E/F}(b_n) \rfloor$ for any $n \geq n_0$.*

(i) *For any indices $n_0 \leq n \leq m < \infty$, the bijection*

$$A_{m/n} : \mathcal{A}_{l'_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathcal{A}_{l'_n}(\mathrm{GL}_N(F_m))$$

coincides with the base change lifting $\mathrm{BC}_{m/n} = \mathrm{BC}_{F_m/F_n}$.

(ii) For any $\pi \in \mathcal{A}(\mathrm{GL}_N(F))$, there exists an integer $n \geq 0$ such that

$$\mathrm{BC}_{n/0}(\pi) \in \mathcal{A}_{l'_n}(\mathrm{GL}_N(F_n)).$$

Now we can construct the base change lifting $\mathrm{BC}_{\infty/0}$ of infinite degree.

Definition 1.4. We define

$$\mathrm{BC}_{\infty/0}: \mathcal{A}(\mathrm{GL}_N(F)) \rightarrow \mathcal{A}(\mathrm{GL}_N(F_\infty))$$

by mapping π to $A_{\infty/n} \circ \mathrm{BC}_{n/0}(\pi)$, where the n is as in Theorem 1.3 (ii).

Remark 1.5. (a) By Theorem 1.3 (i), the definition of $\mathrm{BC}_{\infty/0}$ is independent of the choice of n .

(b) As noted above, at present, we can not avoid appealing to the local Langlands correspondence for GL_N over F to prove Theorem 1.3.

(c) The commutativity of the diagram in Theorem 1.1 follows from [2, Theorem 6.1] and the compatibility of BC with Res via LLC .

Furthermore, we study the structure of the fibers of $\mathrm{BC}_{\infty/0}$. Now we recall the Langlands sum following the exposition of [13, Chapter 1]. We take a partition (N_1, \dots, N_r) of N . Let $\pi_i \in \mathcal{A}(\mathrm{GL}_{N_i}(F))$ be an essentially square-integrable representation for each $1 \leq i \leq r$. Let s_i be the real number such that $|\cdot|^{s_i}$ is the absolute value of the central character of π_i . We reorder π_1, \dots, π_r so that $N_1^{-1}s_1 \geq \dots \geq N_r^{-1}s_r$. We denote by $P(N_1, \dots, N_r)$ the standard parabolic subgroup of $\mathrm{GL}_N(F)$ whose Levi component is $\mathrm{GL}_{N_1}(F) \times \dots \times \mathrm{GL}_{N_r}(F)$. Then the normalized induction

$$\mathrm{n}\text{-Ind}_{P(N_1, \dots, N_r)}^{\mathrm{GL}_N(F)}(\pi_1 \boxtimes \dots \boxtimes \pi_r)$$

has a unique irreducible quotient, which we denote by $\pi_1 \boxplus \dots \boxplus \pi_r$ and call the Langlands sum of π_1, \dots, π_r . Each $\pi \in \mathcal{A}(\mathrm{GL}_N(F))$ can be written as a Langlands sum and the π_1, \dots, π_r are uniquely determined up to a permutation.

Theorem 1.6. Let the notations and assumptions be as in Theorem 1.3. We suppose that $(p, N) = 1$.

(i) Let $\pi \in \mathcal{A}(\mathrm{GL}_N(F))$ be an essentially square-integrable representation. We put $\pi_\infty = \mathrm{BC}_{\infty/0}(\pi)$. Let ω_∞ denote the central character of π_∞ . Then $\mathrm{BC}_{\infty/0}^{-1}(\pi_\infty)$ has a natural $\widehat{\Gamma}$ -torsor structure and the map

$$\omega: \mathrm{BC}_{\infty/0}^{-1}(\pi_\infty) \rightarrow \mathrm{BC}_{\infty/0}^{-1}(\omega_\infty)$$

which maps π' to its central character $\omega_{\pi'}$ is bijective.

(ii) Let π be any element of $\mathcal{A}(\mathrm{GL}_N(F))$. We suppose that $p > N$. We can write

$$\begin{aligned} \pi = & \pi_1 \boxplus (\pi_1 \otimes \eta_{1,2}) \cdots \boxplus (\pi_1 \otimes \eta_{1,\mu_1}) \\ & \boxplus \cdots \\ & \boxplus \pi_r \boxplus (\pi_r \otimes \eta_{r,2}) \cdots \boxplus (\pi_r \otimes \eta_{r,\mu_r}), \end{aligned}$$

where μ_i is an integer, $\pi_i \in \mathcal{A}(\mathrm{GL}_{N_i}(F))$ is an essentially square-integrable representation for each $1 \leq i \leq r$, and $\eta_{i,j}$ is an element of $\widehat{\Gamma}$ for each $1 \leq i \leq r$ and $2 \leq j \leq \mu_i$ such that $\mu_1 N_1 + \cdots + \mu_r N_r = N$ and the lifts $\mathrm{BC}_{\infty/0}(\pi_1), \dots, \mathrm{BC}_{\infty/0}(\pi_r)$ are all distinct. Then the group $\widehat{\Gamma}(\pi) = \widehat{\Gamma}^{\mu_1} \times \cdots \times \widehat{\Gamma}^{\mu_r}$ transitively acts on $\mathrm{BC}_{\infty/0}^{-1}(\pi_\infty)$. As a homogeneous space of $\widehat{\Gamma}(\pi)$, this is isomorphic to

$$\widehat{\Gamma}(1, \eta_{1,2}, \dots, \eta_{1,\mu_1}) \times \cdots \times \widehat{\Gamma}(1, \eta_{r,2}, \dots, \eta_{r,\mu_r}).$$

Here, for $(\eta_1, \dots, \eta_\mu) \in \widehat{\Gamma}^\mu$, we denote by $\widehat{\Gamma}(\eta_1, \dots, \eta_r)$ the quotient of $\widehat{\Gamma}^\mu$ by the following equivalence relation: Two elements (ξ_1, \dots, ξ_μ) and $(\theta_1, \dots, \theta_\mu)$ in $\widehat{\Gamma}^\mu$ are equivalent if there exists a permutation σ of $\{1, \dots, \mu\}$ such that $\eta_j \xi_j = \eta_{\sigma(j)} \theta_{\sigma(j)}$ for each j .

Remark 1.7. We denote the local reciprocity map of F by $\mathrm{rec}_F: W_F \rightarrow F^\times$. For $\phi \in \Phi(\mathrm{GL}_N(F))$, let χ_ϕ denote the determinant character of ϕ . If $p > N$, then Theorem 1.6 shows that, using LLC_{F_∞} , we can characterize LLC_F as a map which makes the diagram

$$\begin{array}{ccccc} \mathrm{Hom}(F^\times, \mathbb{C}^\times) & \xleftarrow{\omega} & \mathcal{A}(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{BC}_{\infty/0}} & \mathcal{A}(\mathrm{GL}_N(F_\infty)) \\ \downarrow \mathrm{rec}_F^* & & \downarrow \mathrm{LLC}_F & & \downarrow \mathrm{LLC}_{F_\infty} \\ \mathrm{Hom}(W_F, \mathbb{C}^\times) & \xleftarrow{\chi} & \Phi(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{Res}_{\infty/0}} & \Phi(\mathrm{GL}_N(F_\infty)) \end{array}$$

commute and has the following properties:

- a Steinberg representation $\mathrm{St}_m(\sigma)$ maps to the outer tensor product

$$\mathrm{LLC}_F(\sigma) \boxtimes \mathrm{Sym}^{m-1} \mathbf{Std},$$

where \mathbf{Std} is the standard representation of $\mathrm{SL}_2(\mathbb{C})$, and

- a Langlands sum maps to the corresponding direct sum.

2 Key lemmas

In this section, we prove an important lemma, which is a statement in Galois side corresponding to Theorem 1.3 (i) in the automorphic side. This is a

compatibility of the restriction functor of Galois groups with respect to a finite totally ramified extension and Deligne's theory of close fields ([10]). We recall Deligne's theory. Let K be a local field with a finite residue field and l a positive integer. We denote the ring of integers of K by \mathcal{O} and the maximal ideal of \mathcal{O} by \mathfrak{p} . We denote by $\text{Tr}_l(K)$ the triple $(\mathcal{O}/\mathfrak{p}^l, \mathfrak{p}/\mathfrak{p}^{l+1}, \varepsilon)$ attached to K , where ε is the composite of the natural maps $\mathfrak{p}/\mathfrak{p}^{l+1} \rightarrow \mathfrak{p}/\mathfrak{p}^l \rightarrow \mathcal{O}/\mathfrak{p}^l$. We fix a separable closure \bar{K} of K . Let $\text{Ext}(K)^l$ denote the category of finite separable field extensions K' of K contained in \bar{K} such that $\text{Gal}(\bar{K}/K') \supset \text{Gal}(\bar{K}/K)^l$. Then we can construct a natural equivalence of categories

$$T_K^l: \text{Ext}(K)^l \xrightarrow{\sim} \text{Ext}(\text{Tr}_l(K))^l,$$

where $\text{Ext}(\text{Tr}_l(K))^l$ is the category whose objects are extensions of $\text{Tr}_l(K)$ which satisfy the condition C^l in [10, 1.5.4] and morphisms are $R(l)$ -equivalence classes ([10, 2.3]) of morphisms of $\text{Ext}(\text{Tr}_l(K))$. For an object K' of $\text{Ext}(K)^l$, $T_K^l(K')$ is defined to be the extension of triples $\text{Tr}_l(K) \rightarrow \text{Tr}_{lr}(K')$ attached to the field extension K'/K , where r is the ramification index of K'/K .

We take another local field K_1 with finite residue field and denote the ring of integers of K_1 by \mathcal{O}_1 and the maximal ideal of \mathcal{O}_1 by \mathfrak{p}_1 . Recall that K and K_1 are called l -close if there exists an isomorphism of rings $\mathcal{O}_1/\mathfrak{p}_1^l \xrightarrow{\sim} \mathcal{O}/\mathfrak{p}^l$. Then we can construct an isomorphism of triples $\gamma: \text{Tr}_l(K_1) \xrightarrow{\sim} \text{Tr}_l(K)$. By mapping an extension $\text{Tr}_l(K) \rightarrow X$ to $\text{Tr}_l(K_1) \xrightarrow{\gamma} \text{Tr}_l(K) \rightarrow X$ of $\text{Tr}_l(K_1)$, we obtain an equivalence of categories

$$\gamma^*: \text{Ext}(\text{Tr}_l(K))^l \rightarrow \text{Ext}(\text{Tr}_l(K_1))^l.$$

Now let $L \subset \bar{K}$ be a finite totally ramified extension of K . We have

$$\text{Gal}(\bar{K}/L) \cap \text{Gal}(\bar{K}/K)^u = W_L \cap \text{Gal}(\bar{K}/K)^u = \text{Gal}(\bar{K}/L)^{\psi_{L/K}(u)} \quad (2)$$

[20, 1.1.2]. We denote by $i(L/K)$ the largest i satisfying

$$\text{Gal}(\bar{K}/L) \text{Gal}(\bar{K}/K)^i = \text{Gal}(\bar{K}/K).$$

Then for any integer $l \leq p^{-1}(p-1)i(L/K)$, the norm map $N_{L/K}$ induces an isomorphism of rings $\mathcal{O}_L/\mathfrak{p}_L^l \xrightarrow{\sim} \mathcal{O}_K/\mathfrak{p}_K^l$ (see [20, Proposition 2.2.1]). In particular, K and L are l -close. Hence there is a canonical isomorphism $\text{Tr}_l(L) \xrightarrow{\sim} \text{Tr}_l(K)$ which sends the image of a uniformizer ϖ_L of L in $\mathfrak{p}_L/\mathfrak{p}_L^{l+1}$ to that of $N_{L/K}(\varpi_L)$ in $\mathfrak{p}_K/\mathfrak{p}_K^{l+1}$. We denote the isomorphism of the triples by $\mathfrak{N}_{L/K}$.

Now we assume $l \leq p^{-1}(p-1)i(L/K)$. Then we have an equivalence of categories

$$\mathfrak{N}_{L/K}^*: \text{Ext}(\text{Tr}_l(K))^l \xrightarrow{\sim} \text{Ext}(\text{Tr}_l(L))^l.$$

On the other hand, we have a functor $\text{Ext}(K) \rightarrow \text{Ext}(L)$ which maps an extension K' of K to the composite $K'L$. If K' is an object of $\text{Ext}(K)^l$, then by the equalities (2), we have

$$\begin{aligned} \text{Gal}(\bar{K}/K'L) &= \text{Gal}(\bar{K}/K') \cap \text{Gal}(\bar{K}/L) \\ &\supset \text{Gal}(\bar{K}/K)^l \cap \text{Gal}(\bar{K}/L) \\ &= \text{Gal}(\bar{K}/L)^{\psi_{L/K}(l)} = \text{Gal}(\bar{K}/L)^l. \end{aligned}$$

Thus $K'L$ is in $\text{Ext}(L)^l$.

Now we can prove the following lemma:

Lemma 2.1. *Suppose $l \leq (2p)^{-1}(p-1)i(L/K)$. Then the group isomorphism*

$$\mathfrak{N}_{L/K*}: \text{Gal}(\overline{K}/L)/\text{Gal}(\overline{K}/L)^l \rightarrow \text{Gal}(\overline{K}/K)\text{Gal}(\overline{K}/K)^l$$

induced by $\mathfrak{N}_{L/K}$ coincides with the homomorphism which comes from the natural injection $\text{Gal}(\overline{K}/L) \hookrightarrow \text{Gal}(\overline{K}/K)$.

Proof. We take a Galois object K' of $\text{Ext}(K)^l$. We put $L' = K'L$. We shall construct an isomorphism

$$\mathfrak{N}': T_L^l(L') \xrightarrow{\sim} \mathfrak{N}_{L/K}^* T_K^l(K')$$

in $\text{Ext}(\text{Tr}_l(L))^l$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{Gal}(L'/L) & \xrightarrow{\cdot|_{K'}} & \text{Gal}(K'/K) \\ \downarrow T_L^l & & \downarrow T_K^l \\ \text{Aut}_{\text{Tr}_l(L)}(T_L^l(L')) & & \text{Aut}_{\text{Tr}_l(K)}(T_K^l(K')) \\ & \searrow \text{ad}(\mathfrak{N}') & \parallel \\ & & \text{Aut}_{\text{Tr}_l(L)}(\mathfrak{N}_{L/K}^* T_K^l(K')). \end{array} \quad (3)$$

Let r denote the ramification index of K'/K . We have $l \leq 2^{-1}i(L/K)$ and

$$\text{Gal}(\overline{K}/K') \supset \text{Gal}(\overline{K}/K)^l \supset \text{Gal}(\overline{K}/K)^{2^{-1}i(L/K)}.$$

Hence we obtain inequalities

$$\psi_{K'/K}^{-1} \left(\frac{1}{2} i(L/K) r \right) \leq \frac{1}{2} i(L/K) + \frac{r-1}{r} \cdot \frac{1}{2} i(L/K) \leq i(L/K).$$

Taking account of

$$\text{Gal}(\overline{K}/L) \text{Gal}(\overline{K}/K)^{i(L/K)} = \text{Gal}(\overline{K}/K),$$

we have

$$\begin{aligned} & \text{Gal}(\overline{K}/L') \text{Gal}(\overline{K}/K')^{2^{-1}i(L/K)r} \\ &= \text{Gal}(\overline{K}/L') (\text{Gal}(\overline{K}/K)^{\psi_{K'/K}^{-1}(2^{-1}i(L/K)r)} \cap \text{Gal}(\overline{K}/K')) \\ &\supset \text{Gal}(\overline{K}/L') (\text{Gal}(\overline{K}/K)^{i(L/K)} \cap \text{Gal}(\overline{K}/K')) \\ &= \text{Gal}(\overline{K}/L') \cap \text{Gal}(\overline{K}/K)^{i(L/K)} \\ &= \text{Gal}(\overline{K}/K') \cap (\text{Gal}(\overline{K}/K)^{i(L/K)} \text{Gal}(\overline{K}/L')) \\ &= \text{Gal}(\overline{K}/K'). \end{aligned}$$

Hence we obtain $\text{Gal}(\overline{K}/L') \text{Gal}(\overline{K}/K')^{2^{-1}i(L/K)r} = \text{Gal}(\overline{K}/K')$. Thus we have $2^{-1}i(L/K)r \leq i(L'/K')$ and the norm map $N_{L'/K'}$ provides an isomorphism $\mathfrak{N}_{L'/K'}: \text{Tr}_{lr}(L') \xrightarrow{\sim} \text{Tr}_{lr}(K')$, which makes the diagram

$$\begin{array}{ccc} \text{Tr}_l(K) & \longrightarrow & \text{Tr}_{lr}(K') \\ \downarrow \mathfrak{N}_{L/K} & & \downarrow \mathfrak{N}_{L'/K'} \\ \text{Tr}_l(L) & \longrightarrow & \text{Tr}_{lr}(L') \end{array}$$

commute. Thus $\mathfrak{N}_{L'/K'}$ is in fact an isomorphism $T_L(L') \xrightarrow{\sim} \mathfrak{N}_{L/K}^* T_K(K')$ in $\text{Ext}(\text{Tr}_l(L))^l$. We put $\mathfrak{N}' = \mathfrak{N}_{L'/K'}$.

The commutativity of the diagram (3) follows from the equality $N_{L'/K'} \circ \sigma = \sigma \circ N_{L'/K'}$ for any $\sigma \in \text{Gal}(L'/L)$. Lemma 2.1 follows from the diagram (3). \square

For any real number $l \geq 0$, we define

$$\Phi_l(\text{GL}_N(K)) = \{\phi \in \Phi(\text{GL}_N(K)) \mid \text{Gal}(\overline{K}/K)^l \subset \text{Ker } \phi\}.$$

By Lemma 2.1, we obtain the first key lemma:

Lemma 2.2. *Let K be a local field with a finite residue field and L a finite totally ramified extension of K . Then, for any $l < (2p)^{-1}(p-1)i(L/K)$, the restriction of L -parameters*

$$\begin{array}{ccc} \Phi_l(\text{GL}_n(K)) & \rightarrow & \Phi_l(\text{GL}_n(L)) \\ \phi & \mapsto & \phi|_{W_L \times \text{SL}_2(\mathbb{C})} \end{array}$$

coincides with the map

$$\begin{array}{ccc} \mathfrak{N}_{L/K}^*: \Phi_l(\text{GL}_n(K)) & \rightarrow & \Phi_l(\text{GL}_n(L)) \\ \phi & \mapsto & \phi \circ (\mathfrak{N}_{L/K*} \times \text{id}_{\text{SL}_2(\mathbb{C})}). \end{array}$$

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. For this, we recall two ingredients. One is an equivalence of $\text{Rep}_l(\text{GL}_N(L))$ and the category of representations of some Hecke algebra, where L is a local field with a finite residue field. The other is Kazhdan's theory of close local fields [15].

We denote by $\mathcal{H}_l(\text{GL}_N(L))$ the algebra of compactly supported $\mathbb{K}_l(L)$ -bi-invariant functions on $\text{GL}_N(L)$ with values in \mathbb{C} whose product is the convolution $*_l$ with respect to the Haar measure $\mu_{\text{GL}_N(L),l}$ on $\text{GL}_N(L)$ normalized by

$$\mu_{\text{GL}_N(L),l}(\mathbb{K}_l(L)) = 1.$$

The characteristic function $e_{\mathbb{K}_l(L)}$ of $\mathbb{K}_l(L)$ is the unity of $\mathcal{H}_l(\text{GL}_N(L))$. The category of $\mathcal{H}_l(\text{GL}_N(L))$ -modules is denoted by $\text{Mod}(\mathcal{H}_l(\text{GL}_N(L)))$.

Lemma 3.1 ([4, Corollaire 3.9 (ii)]). *The functor $V \mapsto V^{\mathbb{K}_l(L)}$ gives an equivalence of categories*

$$\text{Rep}_l(\text{GL}_N(L)) \rightarrow \text{Mod}(\mathcal{H}_l(\text{GL}_N(L))).$$

By using this, we can prove the following:

Lemma 3.2. *For $l \leq m$, the functor*

$$\begin{aligned} \text{Mod}(\mathcal{H}_l(\text{GL}_N(L))) &\rightarrow \text{Mod}(\mathcal{H}_m(\text{GL}_N(L))) \\ W &\mapsto (\mathcal{H}_m(\text{GL}_N(L)) *_m e_{\mathbb{K}_l(L)})_{\mathcal{H}_l(\text{GL}_N(L))} \otimes_{\mathcal{H}_l(\text{GL}_N(L))} W \end{aligned}$$

makes the diagram

$$\begin{array}{ccc} \text{Rep}_l(\text{GL}_N(L)) & \hookrightarrow & \text{Rep}_m(\text{GL}_N(L)) \\ \downarrow & & \downarrow \\ \text{Mod}(\mathcal{H}_l(\text{GL}_N(L))) & \longrightarrow & \text{Mod}(\mathcal{H}_m(\text{GL}_N(L))) \end{array}$$

commute, where the two vertical arrows are the equivalences in Lemma 3.1 and the top horizontal arrow is a natural injection.

Proof. The following proof is similar to that of [4, Corollaire 3.9 (ii)]. Throughout this proof, we put $G = \text{GL}_N(L)$, $K_l = \mathbb{K}_l(L)$, $\mathcal{H}_l = \mathcal{H}_l(G)$ and $e_l = e_{K_l}$. Note that the \mathbb{C} -vector space $\mathcal{H}_m *_m e_l$ has an \mathcal{H}_m - \mathcal{H}_l -bimodule structure via $(h_m, h'_m *_m e_l, h_l) \mapsto h_m *_m h'_m *_m h_l$ for any $h_m, h'_m \in \mathcal{H}_m$ and $h_l \in \mathcal{H}_l$. Let (π, V) be any object of $\text{Rep}_l(G)$. The map

$$(\mathcal{H}_m *_m e_l) \otimes_{\mathcal{H}_l} V^{K_l} \rightarrow V^{K_m}$$

defined by

$$(h *_m e_l) \otimes v \mapsto \int_G (h *_m e_l)(g) \pi(g) v d\mu_{G,m}(g)$$

is a well-defined left \mathcal{H}_m -module homomorphism. It suffices to show that this is an isomorphism. This is surjective since π is an object of $\text{Rep}_l(G)$. We denote by \mathcal{N} the kernel of the above homomorphism. Now let $\text{Mod}_l(\mathcal{H}_m)$ denote the full subcategory of $\text{Mod}(\mathcal{H}_m)$ consisting of objects W which are generated by $e_l * W$. Then the equivalence of categories of Lemma 3.1 induces that of $\text{Rep}_l(G)$ and $\text{Mod}_l(\mathcal{H}_m)$. This equivalence and Lemma 3.1 imply that the latter is equivalent to $\text{Mod}(\mathcal{H}_l)$ and stable by sub-quotient. Since $\mathcal{H}_m *_m e_l$ and V^{K_m} are object of $\text{Mod}_l(\mathcal{H}_m)$, so is \mathcal{N} . In addition, there is no non-trivial vectors on \mathcal{N} which is fixed by the left action of e_l . Therefore $\mathcal{N} = 0$ and the above homomorphism is an isomorphism. \square

Next we recall Kazhdan's theory. Let F_1 and F_2 be local fields of residual characteristic p which are l -close. Let \mathcal{O}_i and \mathfrak{p}_i denote the ring of integers and

the maximal ideal of F_i ($i = 1, 2$). Let $\alpha: \mathcal{O}_2/\mathfrak{p}_2^l \xrightarrow{\sim} \mathcal{O}_1/\mathfrak{p}_1^l$ be an isomorphism of rings. We fix a uniformizer ϖ_2 of F_2 and choose a lift $\varpi_1 \in \mathfrak{p}_1$ of $\alpha(\varpi_2 \bmod \mathfrak{p}_2)$. By the Cartan decomposition, the datum $(\alpha, \varpi_2, \varpi_1)$ gives a \mathbb{C} -linear isomorphism

$$(\alpha, \varpi_2, \varpi_1)^*: \mathcal{H}_l(\mathrm{GL}_N(F_1)) \xrightarrow{\sim} \mathcal{H}_l(\mathrm{GL}_N(F_2))$$

(see [15]). In Kazhdan's original paper, he showed that if F_1 and F_2 are sufficiently close then $(\alpha, \varpi_2, \varpi_1)^*$ is compatible with the convolution products. Lemaire showed a more precise result for GL_N :

Lemma 3.3 ([18, Proposition 3.1.1]). *If F_1 and F_2 are l -close, the isomorphism $(\alpha, \varpi_2, \varpi_1)^*$ is compatible with the convolution products. Hence it is a \mathbb{C} -algebra isomorphism.*

Now we prove Theorem 1.2. Let E/F be an infinite APF extension. For any indices $1 \leq n < m \leq \infty$, we have

$$l_n \leq \frac{p-1}{p} \psi_{E/F}(b_n) = \frac{p-1}{p} i(F_m/F_n).$$

Here, we use equalities $\psi_{E/F}(b_n) = i(F_{n+1}/F_n) = i(E/F_n)$ (see [20, 1.4.1 (b)]) and inequalities $i(E/F_n) \leq i(F_m/F_n) \leq i(F_{n+1}/F_n)$ (see [20, Proposition 1.2.3]). Thus the norm map with respect to F_m/F_n induces an isomorphism of rings

$$\alpha_{m/n}: \mathcal{O}_{F_m}/\mathfrak{p}_{F_m}^{l_n} \xrightarrow{\sim} \mathcal{O}_{F_n}/\mathfrak{p}_{F_n}^{l_n}$$

[20, Proposition 2.2.1]. We fix a uniformizer ϖ_m of F_m . By Lemma 3.3, we obtain an isomorphism of \mathbb{C} -algebras

$$\beta_{m/n}^* = (\alpha_{m/n}, \varpi_m, N_{F_m/F_n}(\varpi_m))^*: \mathcal{H}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathcal{H}_{l_n}(\mathrm{GL}_N(F_m)).$$

By Lemma 3.1, this induces an equivalence of categories

$$A_{m/n}: \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_m)).$$

The transitivity of norm maps implies that the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{H}_{l_n}(\mathrm{GL}_N(F_n)) & \xrightarrow{\beta_{m/n}^*} & \mathcal{H}_{l_n}(\mathrm{GL}_N(F_m)) \\ & \searrow \beta_{m'/n}^* & \downarrow \beta_{m'/m}^* \\ & & \mathcal{H}_{l_n}(\mathrm{GL}_N(F_{m'})). \end{array}$$

This and Corollary 3.2 show the commutativity of the diagram (1). This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Let F be a mixed characteristic local field. We denote the local Langlands correspondence for GL_N over F by

$$\mathrm{LLC}_F: \mathcal{A}(\mathrm{GL}_N(F)) \rightarrow \Phi(\mathrm{GL}_N(F)).$$

By [2, Proposition 4.2], we have

$$\mathrm{LLC}_F(\mathcal{A}_l(\mathrm{GL}_N(F))) \subset \Phi_l(\mathrm{GL}_N(F)).$$

Let F_1 and F_2 be local fields with finite residue fields which are l -close. We choose a datum $\beta = (\alpha, \varpi_2, \varpi_1)$ as in Kazhdan's theory. Then we obtain Kazhdan's correspondence $\beta^*: \mathcal{A}_l(\mathrm{GL}_N(F_1)) \xrightarrow{\sim} \mathcal{A}_l(\mathrm{GL}_N(F_2))$. Moreover, from β we can canonically define an isomorphism of triples $\gamma: \mathrm{Tr}_l(F_2) \rightarrow \mathrm{Tr}_l(F_1)$. The following compatibility of β^* with γ^* via the local Langlands correspondence was proved by Aubert, Baum, Plymen and Solleveld in their preprint [2].

Theorem 4.1 ([2, Theorem 6.1]). *Let l' be any integer such that $0 \leq l' < 2^{-N}l$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{A}_{l'}(\mathrm{GL}_N(F_1)) & \xrightarrow{\beta^*} & \mathcal{A}_{l'}(\mathrm{GL}_N(F_2)) \\ \downarrow \mathrm{LLC} & & \downarrow \mathrm{LLC} \\ \Phi_{l'}(\mathrm{GL}_N(F_1)) & \xrightarrow{\gamma^*} & \Phi_{l'}(\mathrm{GL}_N(F_2)). \end{array}$$

Now let us prove Theorem 1.3 (i). By Theorem 4.1, the map $\beta_{m/n}^*$ in Section 3 is compatible with the map \mathfrak{N}_{F_m/F_n}^* in Lemma 2.2 via LLC. Now we have inequalities $l'_n \leq 2^{-N}p^{-1}(p-1)i(F_m/F_n) \leq (2p)^{-1}(p-1)i(F_m/F_n)$. Hence, by Lemma 2.2, the map \mathfrak{N}_{F_m/F_n}^* coincides with the map induced by the restriction $W_{F_m} \hookrightarrow W_{F_n}$. Since the latter map is compatible with $\mathrm{BC}_{m/n}$ via LLC, we have completed the proof.

Next we show Theorem 1.3 (ii). By the local Langlands correspondence and Theorem 4.1, this is also reduced to showing the corresponding assertion on Galois representations. Thus we shall show that for any $\phi \in \Phi(\mathrm{GL}_N(F))$ there exists n such that $\phi|_{W_{F_n}} \in \Phi_{l'_n}(\mathrm{GL}_N(F_n))$. Take any $\phi \in \Phi(\mathrm{GL}_N(F))$. Then we have $\phi \in \Phi_l(\mathrm{GL}_N(F))$ for some l . By the equality (2) in Section 2, we have $W_{F_n} \cap \mathrm{Gal}(\overline{F}/F)^l = \mathrm{Gal}(\overline{F}/F_n)^{\psi_{E/F}(l)}$ for any n . Since $l'_n \rightarrow \infty$ as $n \rightarrow \infty$, there exists an integer n such that $\psi_{E/F}(l) \leq l'_n$. Thus $\phi|_{W_{F_n}}$ is trivial on $\mathrm{Gal}(\overline{F}/F_n)^{l'_n}$ i.e. $\phi|_{W_{F_n}} \in \Phi_{l'_n}(\mathrm{GL}_N(F_n))$, as claimed.

5 Proof of Theorem 1.6

Finally, we prove Theorem 1.6. First, we show (i) for a supercuspidal π . We put $\pi_\infty = \mathrm{BC}_{\infty/0}(\pi)$. The fiber $\mathrm{BC}_{\infty/0}^{-1}(\pi_\infty)$ has a $\widehat{\Gamma}$ -set structure via $\pi' \mapsto \pi' \otimes \eta$,

where $\pi' \in \mathrm{BC}_{\infty/0}^{-1}(\pi_\infty)$ and $\eta \in \widehat{\Gamma}$. We shall show that this action is simply transitive. The assumption $(p, N) = 1$ shows that the $\widehat{\Gamma}$ -action is simple. Let us prove the transitivity. We take any $\pi' \in \mathrm{BC}_{\infty/0}^{-1}(\pi_\infty)$. By Theorem 1.3 (ii), we can take an integer n such that both $\mathrm{BC}_{n/0}(\pi)$ and $\mathrm{BC}_{n/0}(\pi')$ belong to $\mathcal{A}_n(\mathrm{GL}_N(F_n))$. Since $\mathrm{BC}_{\infty/0} = A_{\infty/n} \circ \mathrm{BC}_{n/0}$ and $A_{\infty/n}$ is injective, we have $\mathrm{BC}_{n/0}(\pi) = \mathrm{BC}_{n/0}(\pi')$. It suffices to show that there exists a smooth character $\eta: F^\times \rightarrow \mathbb{C}^\times$ which factors through $F^\times/N_{F_n/F}(F_n^\times)$ such that $\pi' \simeq \pi \otimes \eta$. We show this by induction on n . The case $n = 1$ is [1, Chapter 1, Proposition 6.7]. We assume that the assertion holds for $n - 1$. By the case $n = 1$, we can find a smooth character $\eta_1: F_{n-1}^\times \rightarrow \mathbb{C}^\times$ which factors through $F_{n-1}^\times/N_{F_n/F_{n-1}}(F_n^\times)$ and satisfies $\mathrm{BC}_{n-1/0}(\pi') \simeq \mathrm{BC}_{n-1/0}(\pi) \otimes \eta_1$. Let ω_π (resp. $\omega_{\pi'}$) denote the central character of π (resp. π'). Then we have

$$\omega_{\pi'} \circ N_{F_{n-1}/F} = (\omega_\pi \circ N_{F_{n-1}/F})\eta_1^N.$$

Thus we obtain

$$\eta_1^N = (\omega_{\pi'}\omega_\pi^{-1}) \circ N_{F_{n-1}/F}.$$

By the assumption that $(p, N) = 1$, we find a character η'_1 on F^\times such that

$$\eta_1 = \eta'_1 \circ N_{F_{n-1}/F}.$$

Hence we have $\mathrm{BC}_{n-1/0}(\pi') \simeq \mathrm{BC}_{n-1/0}(\pi \otimes \eta'_1)$ and by the induction hypothesis there exists a smooth character η_{n-1} on F^\times which is trivial on $N_{F_{n-1}/F}(F_{n-1}^\times)$ and satisfies $\pi' \simeq \pi \otimes (\eta'_1 \eta_{n-1})$. Then $\eta = \eta'_1 \eta_{n-1}$ is the requested character.

Taking central character maps $\pi \otimes \eta$ to the character $\omega_\pi \eta^N$. By the assumption $(p, N) = 1$, this gives a bijection $\mathrm{BC}_{\infty/0}^{-1}(\pi_\infty) \rightarrow \mathrm{BC}_{\infty/0}^{-1}(\omega_\infty)$.

Now we show (i) for any essentially square-integrable π . Then there exists a unique divisor m of N and a unique supercuspidal representation $\sigma \in \mathcal{A}(\mathrm{GL}_{N/m}(F))$ such that π is equivalent to the unique irreducible quotient $\mathrm{St}_m(\sigma)$ of

$$\mathrm{n}\text{-Ind}_{P(N/m, \dots, N/m)}^{\mathrm{GL}_N(F)}(\sigma \otimes |\det|^{(1-m)/2} \boxtimes \dots \boxtimes \sigma \otimes |\det|^{(m-1)/2})$$

([21, Theorem 9.3]). We put $\sigma_\infty = \mathrm{BC}_{\infty/0}(\sigma)$. Let us show that the map

$$\begin{aligned} \mathrm{BC}_{\infty/0}^{-1}(\sigma_\infty) &\rightarrow \mathrm{BC}_{\infty/0}^{-1}(\pi_\infty) \\ \sigma' &\mapsto \mathrm{St}_m(\sigma') \end{aligned}$$

is bijective. Its well-definedness follows from [1, Lemma 6.12], [3, Théorème 2.17 (c)] and [14, Proposition A.4.1]. Its injectivity follows from the uniqueness of the expression $\mathrm{St}_m(\sigma')$. We show its surjectivity. Take any $\pi' \in \mathrm{BC}_{\infty/0}^{-1}(\pi_\infty)$. Then by [3, Théorème 2.17 (c)] and [14, Proposition A.4.1], we have $\mathrm{BC}_{n/0}(\pi') = \mathrm{St}_m(\mathrm{BC}_{n/0}(\sigma))$ for some n . Since π' is essentially square-integrable, there exists a divisor m' of N and a supercuspidal representation $\sigma' \in \mathcal{A}(\mathrm{GL}_{N/m'}(F))$ such that $\pi' = \mathrm{St}_{m'}(\sigma')$. The assumption $(p, N) = 1$ and [1, Lemma 6.12]

show that $\mathrm{BC}_{n/0}(\pi') = \mathrm{St}_{m'}(\mathrm{BC}_{n/0}(\sigma'))$. Hence we have $m' = m$ and $\sigma' \in \mathrm{BC}_{n/0}^{-1}(\mathrm{BC}_{n/0}(\sigma)) \subset \mathrm{BC}_{\infty/0}^{-1}(\sigma_\infty)$. Therefore the surjectivity follows, as claimed.

The statement (ii) follows from the uniqueness of the Langlands sum and the fact that the functor $A_{\infty/n}$ preserves the Langlands sum ([14, Proposition A.4.1]).

6 Prospects for the future

6.1 Fundamental lemma: totally ramified case

We would like to find a proof of Theorem 1.3 (i) within the automorphic side. Let F be a finite extension of \mathbb{Q}_p and F' a totally ramified cyclic extension of F . Take an integer $l \leq p^{-1}(p-1)i(F'/F)$ and $\pi \in \mathcal{A}_l(\mathrm{GL}_N(F))$. Then our problem is to prove

$$A_{F'/F}(\pi) = \mathrm{BC}_{F'/F}(\pi) \quad (4)$$

without using LLC. Here we recall that $A_{F'/F}: \mathcal{A}_l(\mathrm{GL}_N(F)) \xrightarrow{\sim} \mathcal{A}_l(\mathrm{GL}_N(F'))$ is the bijection coming from the \mathbb{C} -algebra isomorphism

$$\mathfrak{N}: \mathcal{H}_l(\mathrm{GL}_N(F)) \xrightarrow{\sim} \mathcal{H}_l(\mathrm{GL}_N(F'))$$

induced by the norm map $N_{F'/F}: (F')^\times \rightarrow F^\times$. Put $\Pi = \mathrm{BC}_{F'/F}(\pi)$. Then the equality (4) is equivalent to saying

$$\mathrm{Tr} \pi(f) = \mathrm{Tr} \Pi(\mathfrak{N}^* f) \quad (5)$$

for any $f \in \mathcal{H}_l(\mathrm{GL}_N(F))$.

Now we recall the Shintani character relation with respect to $\mathrm{BC}_{F'/F}$ ([1, Chapter 1, Definition 6.1]). By Harish-Chandra's theorem [12], the distribution

$$C_c^\infty(\mathrm{GL}_N(F)) \rightarrow \mathbb{C}: f \mapsto \mathrm{Tr} \pi(f)$$

is given by a locally integrable function Θ_π on $\mathrm{GL}_N(F)$, which is locally constant on the regular locus $\mathrm{GL}_N(F)^{\mathrm{reg}}$. By Clozel's theorem [8], which is an extension of Harish-Chandra's theorem to non-connected algebraic groups, the distribution

$$C_c^\infty(\mathrm{GL}_N(F')) \rightarrow \mathbb{C}: f' \mapsto \mathrm{Tr}(\Pi(f') \circ I_\sigma)$$

is also given by a locally integrable function $\Theta_{\Pi, \sigma}$ on $\mathrm{GL}_N(F')$, which is locally constant on the σ -regular locus $\mathrm{GL}_N(F')^{\sigma\text{-reg}}$. Here $I_\sigma: \Pi^\sigma \rightarrow \Pi$ is a $\mathrm{GL}_N(F')$ -equivariant homomorphism normalized suitably (see for example [1, Chapter 1, §2]).

The Shintani character relation is an identity of Θ_π and $\Theta_{\Pi, \sigma}$. To state it, we need the norm map \mathcal{N} , which maps σ -conjugacy classes in $\mathrm{GL}(F')$ to conjugacy classes in $\mathrm{GL}(F)$. For $g' \in \mathrm{GL}_N(F')$, the element

$$g' \sigma(g') \cdots \sigma^{[F':F]-1}(g') \in \mathrm{GL}_N(F)$$

is conjugate to an element $g \in \mathrm{GL}_N(F)$, which is unique upto $\mathrm{GL}_N(F)$ -conjugation. We denote the $\mathrm{GL}_N(F)$ -conjugacy class of g by $\mathcal{N}g'$. We call g a norm of g' . The Shintani character relation is as follows: For any $g' \in \mathrm{GL}_N(F')$ such that $\mathcal{N}(g)$ is regular,

$$\Theta_{\Pi,\sigma}(g) = \Theta_{\pi}(\mathcal{N}g'). \quad (6)$$

Our aim is to deduce the equality (5) from (6). For this, we shall recall the notion of orbital integrals. Choose Haar measures dg of $\mathrm{GL}_N(F)$ and dg' of $\mathrm{GL}_N(F')$ such that the volumes of $\mathrm{GL}_N(\mathcal{O}_F)$ and $\mathrm{GL}_N(\mathcal{O}_{F'})$ are both equal to 1. Let $\tau \in \mathrm{GL}_N(F)$, $\tau' \in \mathrm{GL}_N(F')$ and suppose that $\mathcal{N}\tau'$ is the $\mathrm{GL}_N(F)$ -conjugacy class of τ . Then the σ -centralizer $\mathrm{GL}_{N,\tau',\sigma}$ of τ' is an inner form of the centralizer $\mathrm{GL}_{N,\tau}$ of τ . Hence we can choose Haar measures dt of $\mathrm{GL}_{N,\tau}(F)$ and dt' of $\mathrm{GL}_{N,\tau',\sigma}(F')$ such that dt and dt' are compatible. For $f \in C_c^\infty(\mathrm{GL}_N(F))$ and $f' \in C_c^\infty(\mathrm{GL}_N(F'))$, we define the orbital integral

$$\Phi_f(\tau) = \int_{\mathrm{GL}_{N,\tau}(F) \backslash \mathrm{GL}_N(F)} f(g^{-1}\tau g) \frac{dg}{dt}$$

and the twisted orbital integral

$$\Phi_{f',\sigma}(\tau') = \int_{\mathrm{GL}_{N,\tau',\sigma}(F') \backslash \mathrm{GL}_N(F')} f'(g'^{-1}\tau' \sigma(g')) \frac{dg'}{dt'}.$$

Let T denote the algebraic subgroup of GL_N consisting of diagonal matrices. Applying Weyl's integral formula to the left-hand side of (5), we obtain

$$\begin{aligned} \mathrm{Tr} \pi(f) &= \int_{\mathrm{GL}_N(F)} f(g) \Theta_{\pi}(g) dg \\ &= \frac{1}{\#W(\mathrm{GL}_N, T)} \int_{T(F)} |D_{\mathrm{GL}_N}(t)| \int_{T(F) \backslash \mathrm{GL}_N(F)} f(g^{-1}tg) \frac{dg}{dt} dt \\ &= \frac{1}{N!} \int_{T(F)} |D_{\mathrm{GL}_N}(t)| \Theta_{\pi}(t) \Phi_f(t) dt. \end{aligned}$$

On the other hand, calculation similar to [1, p.93] shows

$$\begin{aligned} &\mathrm{Tr} \Pi(\mathfrak{N}^* f) \\ &= \mathrm{Tr}(\Pi(\mathfrak{N}^* f) \circ I_{\sigma}) \\ &= \int_{\mathrm{GL}_N(F)} (\mathfrak{N}^* f)(g') \Phi_{\Pi,\sigma}(g') dg' \\ &= \frac{1}{\#W(\mathrm{GL}_N, T)} \int_{T(F')^{1-\sigma} \backslash T(F')} |D_{\mathrm{GL}_N}(\mathcal{N}(t'))| \Theta_{\Pi,\sigma}(t') \Phi_{\mathfrak{N}^* f, \sigma}(t') dt'. \end{aligned}$$

Here, $T(F')^{1-\sigma} = \{t' \sigma(t'^{-1}) \mid t' \in T(F')\}$. Remark that, if t' is not σ -regular, then $D_{\mathrm{GL}_N}(\mathcal{N}(t')) = 0$. Hence by the equality (6), we can rewrite the right-hand side of (5) as

$$\frac{1}{N!} \int_{T(F')^{1-\sigma} \backslash T(F')} |D_{\mathrm{GL}_N}(\mathcal{N}(t'))| \Theta_{\pi}(\mathcal{N}t') \Phi_{\mathfrak{N}^* f, \sigma}(t') dt'.$$

Since T is commutative, the map \mathcal{N} is given by the group homomorphism

$$T(F') \rightarrow T(F): t' \mapsto t' \sigma(t') \cdots \sigma^{[F':F]-1}(t'),$$

whose kernel is $T(E)^{1-\sigma}$. Therefore the equality (5) is reduced to the following:

Conjecture 6.1. *For any $t \in T(F)$ and $f \in \mathcal{H}_l(\mathrm{GL}_N(F))$,*

$$\Phi_f(t) = \begin{cases} \Phi_{\mathfrak{N}^* f, \sigma}(t') & \text{if } t \text{ is a norm of some } t', \\ 0 & \text{otherwise.} \end{cases}$$

Such a statement is called a *fundamental lemma* or a *matching problem*. If the field extension is unramified, then such a statement was proved by Arthur-Clozel for GL_N [1, Chapter 1, §4] and by Clozel for any unramified connected reductive group over F [9]. For the unit elements of the Hecke algebras, a proof was earlier given by Kottwitz [16]. However in present the extension F'/F is totally ramified, and the author does not know whether Conjecture 6.1 have been proved or not even for the unit elements of $\mathcal{H}_l(\mathrm{GL}_N(F))$ and $\mathcal{H}_l(\mathrm{GL}_N(F'))$.

6.2 Effective LLC

We try to apply $\mathrm{BC}_{\infty/0}$ to the problem so-called the effective local Langlands correspondence. Its aim is to describe LLC in terms of Bushnell-Kutzko's theory [7], which classifies $\mathcal{A}(\mathrm{GL}_N(F))$ by using data called “types”. We would like to overview a part of Bushnell-Henniart's work on the effective LLC [6] and present some questions, which should be first considered when we analyze the effective LLC by using $\mathrm{BC}_{\infty/0}$.

Here we only deal with supercuspidal representations of $\mathrm{GL}_N(F)$. We denote the subset of $\mathcal{A}(\mathrm{GL}_N(F))$ consisting of supercuspidal representations by $\mathcal{A}^0(\mathrm{GL}_N(F))$. The image $\mathrm{LLC}_F(\mathcal{A}^0(\mathrm{GL}_N(F)))$ consists of $\phi \in \Phi(\mathrm{GL}_N(F))$ such that $\phi|_{W_F}$ is irreducible and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial. Hence we can identify it with the set of isomorphism classes of N -dimensional smooth irreducible representations of W_F , which is denoted by $\mathcal{G}_N^0(F)$. We put $\mathcal{G}^0(F) = \cup_N \mathcal{G}_N^0(F)$. For $\sigma \in \mathcal{G}(F)$, we write ${}^L\sigma$ for $\mathrm{LLC}_F^{-1}(\sigma)$.

Let P_F be the wild inertia subgroup of W_F and $\widehat{P_F}$ the set of smooth irreducible representations of P_F over \mathbb{C} . Then W_F acts on P_F by conjugation. We recall Bushnell-Henniart's partition of $\mathcal{G}^0(F)$ by using pairs (m, \mathcal{O}) , where m is a positive integer and \mathcal{O} is an element of $W_F \setminus \widehat{P_F}$. We define $\mathcal{G}_m^0(F, \mathcal{O})$ as the subset of $\mathcal{G}^0(F)$ consisting of σ such that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_F}(\alpha, \sigma) = m$$

for some $\alpha \in \mathcal{O}$. Remark that the above definition is independent of the choice of α and $\mathcal{G}_m^0(F, \mathcal{O})$ is a subset of $\mathcal{G}_{m[Z_F(\alpha):F] \dim \alpha}^0(F)$, not of $\mathcal{G}_m^0(F)$ in general. Here $Z_F(\alpha)$ is a finite tamely ramified extension of F defined as follows: Consider

$$N_F(\alpha) = \{g \in W_F \mid \alpha^g \simeq \alpha\}.$$

Then there exists a finite tamely ramified extension of F whose Weil group equals $N_F(\alpha)$. We denote the field by $Z_F(\alpha)$ and call the F -centralizer field of α . We note that the F -isomorphism class of $Z_F(\alpha)$ depends only on \mathcal{O} and is independent of the choice of α . We denote by $r_F^1(\sigma)$ a unique element of $W_F \backslash \widehat{P}_F$ such that $\sigma \in \mathcal{G}_m^0(F, r_F^1(\sigma))$.

We put $\mathcal{A}^0(F) = \cup_N \mathcal{A}^0(\mathrm{GL}_N(F))$. In parallel with the Galois side, we also have a similar partition of $\mathcal{A}^0(F)$ by using Bushnell-Kutzko theory. We consider a simple stratum of the form $[\mathfrak{A}, l, 0, \beta]$. Thus \mathfrak{A} is a hereditary \mathcal{O}_F -order in $M_N(F)$ and β is an element of \mathfrak{P}^{-l} , where \mathfrak{P} is the Jacobson radical of \mathfrak{A} . Then we can define a compact open subgroup $H^1(\beta, \mathfrak{A})$ of $1 + \mathfrak{P}$ and a set of characters of $H^1(\beta, \mathfrak{A})$, which are called simple characters. In [5], Bushnell-Henniart defined an equivalence relation on the set of the all simple characters varying \mathfrak{A} and β , which they called the endo-equivalence. For any $\pi \in \mathcal{A}^0(F)$, there exists an m-simple character θ_π contained in π . Here, a simple character θ is said to be m-simple if \mathfrak{A} is maximal among hereditary orders which are stable under the conjugation of $F[\beta]^\times$. Then the endo-equivalence class of θ_π depends only on π . We denote it by $\vartheta(\pi)$.

Now we shall recall main theorems of Bushnell-Henniart's work. Let $\mathcal{E}(F)$ denote the set of endo-equivalence classes of simple characters of F .

Theorem 6.2 ([6, Ramification Theorem]). *There exists a unique bijection*

$$\Phi_F: W_F \backslash \widehat{P}_F \xrightarrow{\sim} \mathcal{E}(F)$$

such that $\vartheta({}^L\sigma) = \Phi_F(r_F^1(\sigma))$ for all $\sigma \in \mathcal{G}^0(F)$.

As in the Galois side, for any endo-equivalence class Θ , we can define a finite tamely ramified extension T/F such that the tame lift Θ_T [5, §9] is totally ramified and T/F is minimal among such finite tamely ramified extensions. The T is called the tame parameter field of Θ .

Theorem 6.3 ([6, Tame Parameter Theorem]). *For $\alpha \in \widehat{P}_F$, we denote the W_F -conjugacy class of α by $\mathcal{O}(\alpha)$.*

- (i) *The F -centralizer field $Z_F(\alpha)$ is F -isomorphic to the tame parameter field of $\Phi_F(\mathcal{O}(\alpha))$.*
- (ii) *For each integer $m \leq 1$ and each $\mathcal{O} \in W_F \backslash \widehat{P}_F$, the local Langlands correspondence for F induces a bijection*

$$\mathcal{G}_m^0(F, \mathcal{O}) \xrightarrow{\sim} \mathcal{A}_m^0(F, \Phi_F(\mathcal{O})).$$

Now we fix a positive integer m and $\alpha \in \widehat{P}_F$. Bushnell-Henniart constructed a bijection

$$\mathcal{G}_m^0(F, \mathcal{O}(\alpha)) \xrightarrow{\sim} \mathcal{A}_m^0(F, \Phi_F(\mathcal{O}(\alpha))): \sigma \mapsto {}^N\sigma,$$

which is constructed by using Bushnell-Kutzko's classification and have effective description modulo the totally wildly ramified case. Moreover they studied the gap of ${}^L\sigma$ and ${}^N\sigma$:

Theorem 6.4 ([6, Comparison Theorem]). *We put $C = Z_F(\alpha)$ and $\Phi_C(\alpha) = \Phi_C(\mathcal{O}(\alpha))$. There exists a tamely ramified character $\mu = \mu_{m, \mathcal{O}(\alpha)}^F$ of C^\times depending only on F , m and $\mathcal{O}(\alpha)$ such that, for any $\sigma \in \mathcal{G}_m^0(F, \mathcal{O}(\alpha))$,*

$${}^L\sigma = \mu \odot_{\Phi_C(\alpha)} {}^N\sigma.$$

Moreover the character μ is uniquely determined modulo $X_0(C)_m$.

Here $X_0(C)_m$ is the group of unramified characters of C^\times such that $\chi^m = 1$. For the definition of “the twist $\mu \odot_{\Phi_C(\alpha)} \bullet$ ”, see [6, Chapter 4]. We call μ the discrepancy character.

Now let E/F be a totally ramified \mathbb{Z}_p -extension. Then we obtain the norm field $X(E/F)$, which we denote by F_∞ . Then Bushnell-Henniart’s theory can be also applied to F_∞ . Fix an integer $m \leq 1$ and $\mathcal{O} \in W_F \setminus \widehat{P}_F$. We take

$$\sigma \in \mathcal{G}_m^0(F, \mathcal{O})$$

such that $\sigma_\infty = \sigma|_{W_{F_\infty}}$ is also irreducible. Then there exist a positive integer m_∞ and $\mathcal{O}_\infty \in W_{F_\infty} \setminus \widehat{P}_{F_\infty}$ such that

$$\sigma_\infty \in \mathcal{G}_{m_\infty}^0(F_\infty, \mathcal{O}_\infty).$$

Question 6.5. (i) Is $\text{BC}_{\infty/0} {}^N\sigma$ equal to ${}^N\sigma_\infty$?

(ii) We fix an F -centralizer (resp. F_∞ -centralizer) field C of \mathcal{O} (resp. C_∞ of \mathcal{O}_∞). Then we can construct the norm field $X(CE/C)$ associated to the APF extension CE/C . Is it F_∞ -isomorphic to C_∞ ?

(iii) If (ii) is true, then we can construct a homomorphism $N_{C_\infty/C}: C_\infty^\times \rightarrow C^\times$. Is the character $N_{C_\infty/C} \circ \mu_{m, \mathcal{O}}^F$ congruent to $\mu_{m_\infty, \mathcal{O}_\infty}^{F_\infty}$ modulo $X_0(C_\infty)_m$?

If these questions are solved affirmatively, then we can calculate the restriction of $\mu_{m, \mathcal{O}}^F$ to $N_{C_\infty/C}(C_\infty^\times)$ by using $\mu_{m_\infty, \mathcal{O}_\infty}^{F_\infty}$. In addition, Bushnell-Henniart give a behavior of $\mu_{m, \mathcal{O}}^F$ on the unit group of C . Hence we can transfer the study of the discrepancy character for mixed characteristic to that for equal characteristic.

Acknowledgments

The author would like to thank his supervisor Professor Yuichiro Taguchi for suggesting the problem and giving many stimulating comments. He is grateful to Professor Yoichi Mieda for useful discussion, in particular pointing out the importance of Theorem 1.3 (i). He would also like to express his gratitude to Professor Noriyuki Abe, who suggest me that Conjecture 6.1 might yield Theorem 1.3.

References

- [1] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Ann. of Math. Stud. **120** (1989).
- [2] A. M. Aubert, P. Baum, R. Plymen, M. Solleveld, *The local Langlands correspondence for inner forms of SL_n* , preprint (available at arXiv:1305.2638v3).
- [3] A. I. Badulescu, *Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle*, Ann. Scient. Éc. Norm. Sup. 4^e série, **35** (2002), 695–747.
- [4] J. N. Bernstein, *Le “centre” de Bernstein*, Representations of reductive groups over a local field (edited by P. Deligne), Travaux en Cours, 1–32 (Hermann, Paris, 1984).
- [5] C. J. Bushnell, G. Henniart, *Local tame lifting for $\mathrm{GL}(N)$. I: Simple characters*, Publ. Math. de l’I.H.É.S., **83** (1996), 105–233.
- [6] C. J. Bushnell, G. Henniart, *To an Effective Local Langlands Correspondence*, Mem. Amer. Math. Soc. **231** (2014) no. 1087.
- [7] C. J. Bushnell, P. C. Kutzko, *The admissible dual of $\mathrm{GL}(N)$ via compact open subgroups*, Ann. of Math. Stud. **129** (1993).
- [8] L. Clozel, *Characters of non-connected, reductive p -adic groups*, Can. J. Math. **39** (1987), 149–167.
- [9] L. Clozel, *The fundamental lemma for stable base change*, Duke Math. J. **61**, no. 1 (1990), 255–302.
- [10] P. Deligne, *Les corps locaux de caractéristique p , limites de corps locaux de caractéristique 0*, Representations of reductive groups over a local field (edited by P. Deligne), Travaux en Cours, 119–157 (Hermann, Paris, 1984).
- [11] R. Ganapathy, *The local Langlands correspondence for GSp_4 over local function fields*, preprint (available at arXiv:1305.6088v2).
- [12] Harish-Chandra, *Admissible invariant distributions on reductive p -adic groups*, Queen’s Papers in Pure and Applied Math. **48** (1978), 281–347.
- [13] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Ann. of Math. Stud. **151** (2001), Princeton Univ. Press.
- [14] G. Henniart, B. Lemaire, *Intégrales orbitales tordues sur $\mathrm{GL}(n, F)$ et corps locaux proches: applications*, Canad. J. Math. **58** (6) (2006), 1229–1267.

- [15] D. Kazhdan, *Representations of groups over close local fields*, J. Anal. Math. **47** (1986), 175–179.
- [16] R. E. Kottwitz, *Base change for unit elements of Hecke algebras*, Compos. Math. **60**, no. 2 (1986), 237–250.
- [17] G. Laumon, M. Rapoport, U. Stuhler, *\mathcal{D} -elliptic sheaves and the Langlands correspondence*, Invent. Math. **113** (1993), 217–338.
- [18] B. Lemaire, *Représentations de GL_N et corps locaux proches*, J. Algebra **236** (2001), 549–574.
- [19] J. P. Serre, *Corps Locaux*, Hermann, Paris, 1962.
- [20] J. P. Wintenberger, *Le corps des norms de certaines extensions infinies de corps locaux; applications*, Ann. Sci. Éc. Norm. Supér. (4) **16**, no. 1 (1983), 59–89.
- [21] A. V. Zelevinsky, *Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$* , Ann. Sci. Éc. Norm. Supér. (4) **13**, no. 2 (1980), 165–210.