Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

Kagei, Yoshiyuki
Faculty of Mathematics, Kyushu University

Maekawa, Yasunori
Department of Mathematics, Graduate School of Science, Kobe University

出版情報：MI Preprint Series. 2009-29, 2011-05-15. Academic Press
バージョン：

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

Y. Kagei & Y. Maekawa

MI 2009-29

(Received August 19, 2009)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

ASYMPTOTIC BEHAVIORS OF SOLUTIONS TO EVOLUTION EQUATIONS IN THE PRESENCE OF TRANSLATION AND SCALING INVARIANCE

YOSHIYUKI KAGEI[1] AND YASUNORI MAEKAWA[2]

Abstract. There are wide classes of nonlinear evolution equations which possess invariant properties with respect to a scaling and translations. If a solution is invariant under the scaling then it is called a self-similar solution, which is a candidate for the asymptotic profile of general solutions at large time. In this paper we establish an abstract framework to find more precise asymptotic profiles by shifting self-similar solutions suitably.

1. Introduction

In [8] Escobedo and Zuazua studied large time behaviors of solutions to convection di usion equations such as

$$\text{(n-B)} \qquad \partial_t \Omega - \Delta \Omega + a \cdot \nabla(|\Omega|^p \Omega) = 0, \qquad t > 0, \quad x \in \mathbb{R}^n,$$

where $n \geq 1$; $a \in \mathbb{R}^n$ is a given constant vector; $p > 0$ is a given number; $\Delta = \sum_{i=1}^n \partial_{x_i}^2$; and $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})^\top$. When $p > \frac{1}{n}$ in (n-B), which is called the weakly nonlinear case, the nonlinear term can be regarded as a perturbation from the scaling point of view. It is proved in [8] that solutions asymptotically behaves like the Gauss kernel which is the self-similar solution to the linear heat equations

$$\text{(H)} \qquad \partial_t \Omega - \Delta \Omega = 0, \qquad t > 0, \quad x \in \mathbb{R}^n.$$

On the other hand, when $p = \frac{1}{n}$, which is called the critical case, the nonlinear term is not a perturbation and the Gauss kernel no longer describes the large time behaviors of solutions to (n-B). Instead, in this case the self-similar solutions to the nonlinear equation (n-B) itself describe large time behaviors of solutions to (n-B). The existence and the uniqueness of

[1]Faculty of Mathematics, Kyushu University
Address: 6-10-1, Hakozaki, Higashi-ku, Fukuoka, 812-8581, Japan
E-mail: kagei@math.kyushu-u.ac.jp
[2]Department of Mathematics, Graduate School of Science, Kobe University
Address: 1-1, Rokkodai, Nada-ku, Kobe, 657-8501, Japan
E-mail: yasunori@math.kobe-u.ac.jp

self-similar solutions are shown in [1], and it is proved in [8] that these self-similar solutions give the large time asymptotics of solutions in the following sense:

$$\|\Omega(t)-t^{-\frac{n}{2}}U_\delta(\frac{\cdot}{\sqrt{t}})\|_{L^p(\mathbb{R}^n)} = o(t^{-\frac{n}{2}(1-\frac{1}{p})}), \qquad t\to\infty, \quad 1\le p\le\infty. \tag{1.1}$$

Here $\Omega(t)$ is a solution to (n-B) with initial data Ω_0 and U_δ is the pro le function of the self-similar solution satisfying $\int_{\mathbb{R}^n} U_\delta(x)dx = \delta := \int_{\mathbb{R}^n}\Omega_0(x)dx$.

Eq. (n-B) is considered as a generalization of the well-known viscous Burgers equations

$$\partial_t\Omega - \partial_x^2\Omega + \frac{1}{2}\partial_x(\Omega^2) = 0, \qquad t>0, \quad x\in\mathbb{R}. \tag{1-B}$$

For (1-B) a more precise asymptotic pro le was given by [21]. Indeed, when the initial data has a suitable decay at spatial in nity the next estimate holds with some $y^*,\ t^*\in\mathbb{R}$:

$$\|\Omega(t)-(t+t^*)^{-\frac{1}{2}}U_\delta(\frac{\cdot+y^*}{\sqrt{t+t^*}})\|_{L^p(\mathbb{R})} = o(t^{-2+\frac{1}{2p}}), \qquad t\to\infty, \quad 1\le p\le\infty. \tag{1.2}$$

This result is extended by [27], which established further improvements of the rate of convergence for solutions to (1-B).

Roughly speaking, the above result implies that large time behaviors of solutions are described more precisely by suitably shifted self-similar solutions. The key idea in [21] and [27] is to reduce (1-B) to the linear heat equations by using the Hopf-Cole transformation. Hence, if we try to obtain analogous results with (1.2) for (n-B) or other nonlinear equations, we can not use the arguments in [21, 27].

Recently analogous observations with (1.2) were achieved in [24] for a one dimensional parabolic system modelling chemotaxis, usually called a Keller-Segel system. In [24] it is proved that a suitably shifted Gaussian gives a more precise asymptotic pro le of solutions than the Gaussian itself, which improves the results of [22, 16]. The shift is determined by solving some ODEs. The one dimensional Keller-Segel system treated in [24] is classi ed in the category of the weakly nonlinear case. As is remarked in [24], it is expected that an analogous result to that in [24] also holds for the multi-dimensional Keller-Segel systems which are in the category of the weakly nonlinear case. On the other hand, there is a two dimensional Keller-Segel system which is classi ed in the critical case. For such a system it is known [2, 23] that self-similar solutions describe large time behavior as in (1.1), while an estimate like (1.2) seems to have not yet been achieved for this case.

The aim of this paper is to provide an abstract framework which clari es the structure leading to the estimates like (1.2). This enables us to capture more precise asymptotic behaviors of solutions at time infinity for equations other than (1-B) in a systematic way. Our approach is based on two symmetries of the equations; translation invariance and scaling invariance. If an evolution equation is invariant with respect to some scaling, we can expect the existence of self-similar solutions and they are candidates for the asymptotic profiles of general solutions. It is well-known that, by introducing the self-similar variables, this problem can be reduced to the stability problem of stationary solutions for the equations in the new variables. The key observation here is that if translations and the scaling are suitably connected, then the linearized operators around the stationary solutions are shown to possess some point spectra which reflect the symmetries but also correspond to the second order asymptotic profiles of solutions. Then we determine the (time-dependent) shifts of stationary solutions so that the difference of solutions and shifted stationary solutions always belongs to a complement subspace of the eigenspace for these eigenvalues, by which we obtain more precise asymptotic profiles of solutions. The time dependent shifts are constructed by solving suitable ODEs as in [24].

By focusing on the relations between the symmetries and the spectrum of the linearized operators we can develop systematic and abstract arguments to this problem. Especially, our method can be applied for wider classes of equations including multi-dimensional convection diffusion equations, two dimensional vorticity equations for viscous incompressible flows, two dimensional Keller-Segel systems. We further introduce a family of translations. This enables us to treat the Vlasov-Poisson-Fokker-Planck equations within our abstract framework, which is a degenerate parabolic equation and is not invariant with respect to a translation of a spatial variable. We also remark that our approach can be applied not only for the case when nonlinear equations possess self-similar solutions but also for the weakly nonlinear case, i.e., large time behaviors of solutions are described by self-similar solutions to the linear equations.

This paper is organized as follows. From Section 2 to Section 6 we discuss in rather abstract settings, by regarding translations and scaling as the actions of the additive groups and the multiplicative group, respectively. In Section 2.1 we introduce the idea of translation and scaling in the Banach spaces and define the self-similarity of functions. We also fix the idea of solutions for the abstract Cauchy problems in this section. Especially, we always deal with mild solutions which are solutions to the integral equations defined through strongly continuous semigroups. In Section 2.2 we give several assumptions for operators. The main results in this paper are stated in Section 2.3. In Section 3 we derive the equations in self-similar variables in the abstract settings. Then the existence and the stability of self-similar solutions to the original equations are shown to be equivalent with those of stationary solutions to the new equations. The existence and the uniqueness of stationary solutions are proved in Section 4. In Section 5 we establish the local stability of the stationary solutions with a "rough" rate of convergence. Section 6 is the main contribution of this paper, in which we determine a suitable shift of self-similar solutions so as to get the more precise asymptotic pro les of solutions. For this purpose in Section 6.1 we study the spectrum of the linearized operator around stationary solutions by applying the general perturbation theory and semigroup theory of linear operators. In Section 6.2 we construct a time dependent shift by solving the nonlinear ODEs. Finally, in Section 6.3 we calculate the rate of convergence to shifted stationary solutions (or equivalently, shifted self-similar solutions), which completes the proofs of the main results in Section 2.3. In Section 7 we apply our abstract results obtained in Section 2- Section 6 to several nonlinear PDEs.

2. Preliminaries and main results

2.1. **Scaling and translations in abstract settings.** We consider an evolution equations in a Banach space X:

$$\frac{d}{dt}\Omega - \mathcal{A}\Omega + \mathcal{N}(\Omega) = 0, \qquad t > 0. \tag{E}$$

Here $\mathcal{A}$ is a closed linear operator in X and $\mathcal{N}$ is a nonlinear operator. We are interested in large time behaviors of solutions to (E) in the presence of several invariant properties with respect to actions of groups, which we call *scaling* or *translation* below. First of all, we x the idea of scaling and translation in the abstract setting. We denote by $\mathbb{R}^\times$ the multiplicative group $\{\lambda \in \mathbb{R} \mid \lambda > 0\}$ and by $\mathbb{R}^+$ the additive group $\mathbb{R}$. Both groups are endowed with the usual Eucledian topology. Let $\mathcal{B}(X)$ be the Banach space of all bounded linear operators in X.

Definition 2.1. (1) *We say $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times} \subset \mathcal{B}(X)$ a scaling if $\mathcal{R}$ is a strongly continuous action of $\mathbb{R}^\times$ on X, i.e.,*

$$R_{\lambda_1\lambda_2} = R_{\lambda_1}R_{\lambda_2}, \qquad \lambda_1,\ \lambda_2 \in \mathbb{R}^\times \tag{2.1}$$

$$R_1 = I, \tag{2.2}$$

$$R_{\lambda'}u \to R_\lambda u \ \text{ in } X \text{ when } \lambda' \to \lambda \quad \text{for each } u \in X. \tag{2.3}$$

(2) *We say $\mathcal{T} = \{\tau_a\}_{a\in\mathbb{R}^+} \subset \mathcal{B}(X)$ a translation if $\mathcal{T}$ is a strongly continuous group acting on X.*

For one-parameter family of translations $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ with $\mathcal{T}_\theta = \{\tau_{a,\theta}\}_{a\in\mathbb{R}^+}$, we say that it is strongly continuous if $\tau_{a,\theta'}(f) \to \tau_{a,\theta}(f)$ in X as $\theta' \to \theta$ for each $a \in \mathbb{R}^+$ and $f \in X$. When there are n one-parameter families of translations $\{\mathcal{T}^{(j)}_\theta\}_{\theta\in\mathbb{R}}$, $j = 1, \cdots, n$, we say that they are independent if for *all* $a, a', \theta \in \mathbb{R}^+$ *it follows that*

$$\tau^{(i)}_{a,\theta}\tau^{(j)}_{a',\theta} = \tau^{(j)}_{a',\theta}\tau^{(i)}_{a,\theta}, \qquad 1 \le i, j \le n. \tag{2.4}$$

The generator of $\{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ is the operator B given by

$$\begin{aligned} \mathrm{Dom}(B) &= \{f \in X \mid \lim_{h\to 0} \frac{R_{1+h}f - f}{h} \text{ exists }\}, \\ Bf &= \lim_{h\to 0} \frac{R_{1+h}f - f}{h}, \qquad f \in \mathrm{Dom}(B). \end{aligned} \tag{2.5}$$

Note that if $f \in \mathrm{Dom}(B)$ then $R_\lambda f \in \mathrm{Dom}(B)$ and

$$BR_\lambda f = R_\lambda Bf. \tag{2.6}$$

Moreover, if $f \in \mathrm{Dom}(B)$ then $R_\lambda f$ is di erentiable in X at each $\lambda \in \mathbb{R}^\times$, and we have

$$\frac{d}{d\lambda} R_\lambda f|_{\lambda=\lambda_0} = \frac{1}{\lambda_0} R_{\lambda_0} Bf. \tag{2.7}$$

A scaling $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ naturally induces an action on $C((0,\infty); X)$ as follows. For $f \in C((0,\infty); X)$ we set

$$\Theta_\lambda(f)(t) = R_\lambda(f(\lambda t)), \qquad \lambda \in \mathbb{R}^\times. \tag{2.8}$$

Then it is not difficult to see

Proposition 2.1. (1) $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *is an action of* $\mathbb{R}^\times$ *on* $C((0,\infty); X)$ *and* Θ_λ *is linear for each* $\lambda \in \mathbb{R}^\times$.

(2) $\Theta_{\lambda'}(f)(t) \to \Theta_\lambda(f)(t)$ *in* X *as* $\lambda' \to \lambda$ *for each* $t > 0$ *and* $f \in C((0,\infty); X)$.

We call $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ the scaling induced by $\mathcal{R}$.

Let $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ be a strongly continuous one-parameter family of translations. The generator of $\mathcal{T}_\theta$ for each $\theta \in \mathbb{R}$ is the operator D_θ given by

$$\begin{aligned} \mathrm{Dom}(D_\theta) &= \{f \in X \mid \lim_{a\to 0} \frac{\tau_{a,\theta}(f) - f}{a} \text{ exists }\}, \\ D_\theta f &= \lim_{a\to 0} \frac{\tau_{a,\theta}(f) - f}{a}, \qquad f \in \mathrm{Dom}(D_\theta). \end{aligned} \tag{2.9}$$

We can also consider a linear operator $\Gamma_{a,\theta}$ which is a derivative of $\tau_{a,\theta}$ with respect to θ:

$$\begin{aligned} \mathrm{Dom}(\Gamma_{a,\theta}) &= \{f \in X \mid \lim_{h\to 0} \frac{\tau_{a,\theta+h}(f) - \tau_{a,\theta}(f)}{h} \text{ exists }\}, \\ \Gamma_{a,\theta}(f) &= \lim_{h\to 0} \frac{\tau_{a,\theta+h}(f) - \tau_{a,\theta}(f)}{h}, \qquad f \in \mathrm{Dom}(\Gamma_{a,\theta}). \end{aligned} \tag{2.10}$$

Definition 2.2. *Let* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *be the scaling induced by* $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$. *We say that* $f \in C((0,\infty);X)$ *is self-similar with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *iff*

$$\Theta_\lambda(f) = f, \qquad \lambda \in \mathbb{R}^\times. \tag{2.11}$$

Then we easily see that

Proposition 2.2. *The function* $f \in C((0,\infty);X)$ *is self-similar with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *if and only if* f *can be expressed as*

$$f(t) = R_{\frac{1}{t}}(h) \tag{2.12}$$

with a function $h \in X$.

Next let us x the idea of solutions to (E). Throughout of this paper we consider only mild solutions, so we assume that $\mathcal{A}$ generates a strongly continuous (C_0) semigroup $e^{t\mathcal{A}}$ in X, which gives a mild solution to the linear equation

$$\frac{d}{dt}\Omega - \mathcal{A}\Omega = 0, \quad t > 0. \tag{E$_0$}$$

Definition 2.3. *We say that* $\Omega(t) \in C((0,\infty);X)$ *is a mild solution to (E) if* $\int_0^t \|e^{(t-s)\mathcal{A}}\mathcal{N}(\Omega(s))\|_X ds < \infty$ *for any* $t > 0$ *and* $\Omega(t)$ *satisfies the equality*

$$\Omega(t) = e^{(t-s)\mathcal{A}}\Omega(s) - \int_s^t e^{(t-\tau)\mathcal{A}}\mathcal{N}(\Omega(\tau))d\tau, \qquad \text{for all } t > s > 0. \tag{2.13}$$

Moreover, if $\Omega(t)$ *satisfies in addition*

$$\lim_{t\to 0}\Omega(t) = \Omega_0 \in X, \qquad \lim_{t\to 0}\int_0^t \|e^{(t-s)\mathcal{A}}\mathcal{N}(\Omega(s))\|_X ds = 0, \tag{2.14}$$

then we say that $\Omega(t)$ *is a mild solution to (E) with initial data* Ω_0.

Remark 2.1. *If* $\Omega(t)$ *is a mild solution to (E) with initial data* Ω_0, *then* $\Omega \in C([0,\infty);X)$ *and it satisfies*

$$\Omega(t) = e^{t\mathcal{A}}\Omega_0 - \int_0^t e^{(t-s)\mathcal{A}}\mathcal{N}(\Omega(s))ds, \qquad \text{for all } t \geq 0. \tag{2.15}$$

Definition 2.4. *We call* $\Omega(t) \in C((0,\infty);X)$ *a self-similar solution to (E) with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *if* $\Omega(t)$ *is a mild solution to (E) and is self-similar with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$.

The existence of self-similar solutions and its asymptotic stability are related with the invariant properties of (E) for the associated scaling. Moreover, if (E) also possesses a translation invariance, we can expect more precise informations on the asymptotic pro les of general solutions. The main purpose of this paper is to verify this idea under some assumptions on the operators.

Definition 2.5. *Let $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ be a scaling and let $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ be a strongly continuous one-parameter family of translations.*

(i) We say that (E) is invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ if $\Theta_\lambda(\Omega)(t)$ is a mild solution to (E) with initial data $R_\lambda(\Omega_0)$ for each $\lambda \in \mathbb{R}^\times$ whenever $\Omega(t)$ is a mild solution to (E) with initial data Ω_0.

(ii) We say that (E) is invariant with respect to $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ if $\tau_{a,t+\theta}(\Omega(t))$ is a mild solution to (E) with initial data $\tau_{a,\theta}(\Omega_0)$ for each $a \in \mathbb{R}^+$ and $\theta \geq 0$ whenever $\Omega(t)$ is a mild solution to (E) with initial data Ω_0.

Let (E_0) be invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ and $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$. Then the above de nition is expressed by the equalities

$$R_\lambda(e^{\lambda t\mathcal{A}}\Omega_0) = \Theta_\lambda(e^{t\mathcal{A}}\Omega_0) = \Theta_\lambda(\Omega)(t) = e^{t\mathcal{A}}R_\lambda(\Omega_0),$$

$$\tau_{a,t+\theta}e^{t\mathcal{A}}\Omega_0 = e^{t\mathcal{A}}\tau_{a,\theta}\Omega_0,$$

which holds for any $\Omega_0 \in X$, $\lambda \in \mathbb{R}^\times$, $a \in \mathbb{R}^+$, and $t, \theta \geq 0$. Hence in this case we have key relations, which are equivalent representations of the above de nitions, such as

$$R_\lambda e^{\lambda t\mathcal{A}} = e^{t\mathcal{A}}R_\lambda, \tag{2.16}$$

$$\tau_{a,t+\theta}e^{t\mathcal{A}} = e^{t\mathcal{A}}\tau_{a,\theta}. \tag{2.17}$$

This relations formally imply that

$$\mathcal{A}R_\lambda f = \lambda R_\lambda \mathcal{A}f, \qquad [\mathcal{A}, \tau_{a,\theta}]f = \Gamma_{a,\theta}f,$$

where $[\mathcal{A}, \tau_{a,\theta}] = \mathcal{A}\tau_{a,\theta} - \tau_{a,\theta}\mathcal{A}$. When (2.16) and (2.17) are satis ed, we will also see that the condition

$$\lambda R_\lambda \mathcal{N}(f) = \mathcal{N}(R_\lambda f), \qquad \tau_{a,\theta}\mathcal{N}(f) = \mathcal{N}(\tau_{a,\theta}f),$$

are natural sufficient conditions in order for (E) to be also invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ and $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$.

Next we introduce the "similarity transform" in the abstract setting. When (E_0) is invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ induced by $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ we set

$$\Theta(t) = R_{e^t}e^{(e^t-1)\mathcal{A}} = e^{(1-e^{-t})\mathcal{A}}R_{e^t}, \quad t \geq 0. \tag{2.18}$$

Then we have

Lemma 2.1. *The one parameter family $\{\Theta(t)\}_{t\geq 0}$ defined by (2.18) is a strongly continuous semigroup in X.*

Proof. It is easy to see that $\Theta(t)$ is strongly continuous and $\Theta(0) = I$. Hence we check the relation $\Theta(t+s) = \Theta(t)\Theta(s)$. From (2.18) and (2.16) we have

$$\begin{aligned}\Theta(t)\Theta(s) = e^{(1-e^{-t})\mathcal{A}}R_{e^t}\Theta(s) &= e^{(1-e^{-t})\mathcal{A}}R_{e^t}R_{e^s}e^{(e^s-1)\mathcal{A}}\\ &= e^{(1-e^{-t})\mathcal{A}}R_{e^{(t+s)}}e^{e^{t+s}(e^{-t}-e^{-t-s})\mathcal{A}}\\ &= e^{(1-e^{-t})\mathcal{A}}e^{(e^{-t}-e^{-t-s})\mathcal{A}}R_{e^{(t+s)}}\\ &= e^{(1-e^{-t-s})\mathcal{A}}R_{e^{(t+s)}} = \Theta(t+s).\end{aligned}$$

This completes the proof.

Let A be the generator of $\Theta(t)$, i.e.,

$$\begin{aligned}\mathrm{Dom}(A) &= \{f \in X \mid \lim_{t\to 0}\frac{\Theta(t)f-f}{t} \text{ exists }\},\\ Af &= \lim_{t\to 0}\frac{\Theta(t)f-f}{t}, \qquad f\in \mathrm{Dom}(A).\end{aligned} \tag{2.19}$$

Then it follows that $\mathrm{Dom}(\mathcal{A})\cap\mathrm{Dom}(B)\subset\mathrm{Dom}(A)$ and

$$Af = \mathcal{A}f + Bf, \quad \text{for} \quad f\in\mathrm{Dom}(\mathcal{A})\cap\mathrm{Dom}(B). \tag{2.20}$$

The proof is omitted. In general, we can not expect that B is relatively bounded with respect to $\mathcal{A}$. Especially, the spectral properties of A can quite di er from those of $\mathcal{A}$. In the next section we give several assumptions on the operators $\mathcal{A}$ and $\mathcal{N}$. Typical examples of $\mathcal{A}$, $\mathcal{R}$, and $\mathcal{T}^{(j)}$ are Δ, $(R_\lambda f)(x) = \lambda^{\frac{n}{2}}f(\lambda^{\frac12}x)$, and $(\tau_a^{(j)}f)(x) = f(x_1,\cdots,x_j+a,\cdots,x_n)$. Then the generators of $\mathcal{R}$ and $\mathcal{T}^{(j)}$ are given by $B=\frac{x}{2}\cdot\nabla+\frac{n}{2}$ and $D^{(j)}=\partial_{x_j}$. Especially, in this case we have $\mathrm{Dom}(A)=\mathrm{Dom}(\mathcal{A})\cap\mathrm{Dom}(B)$ if they are considered in polynomial weighted L^2 spaces. For details, see Section 7.

2.2. **Several assumptions.** In this section we collect several assumptions on (E) and operators which we deal with.

2.2.1. *Assumptions on (E_0).* We rst state the assumptions on (E_0). As stated in the previous section, the operator $\mathcal{A}$ is assumed to generate a strongly continuous semigroup $e^{t\mathcal{A}}$ in X.

(E1) *There is a scaling $\mathcal{R}=\{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ such that (E_0) is invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$.*

(E2) *There are finite numbers of strongly continuous one-parameter families of translations $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $1\le j\le n$, such that they are independent and (E_0) is invariant with respect to $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$ for each j.*

In other words, we have (2.16) and (2.17) for each R_λ and $\tau_{a,\theta}^{(j)}$. Let B, $D_\theta^{(j)}$, and $\Gamma_{a,\theta}^{(j)}$ be the generator of R_λ, the generator of $\mathcal{T}_\theta^{(j)}$, and the derivative of $\tau_{a,\theta}^{(j)}$ with respect to θ de ned by (2.10), respectively. For a pair of linear operators L_1, L_2 its commutator is de ned by $[L_1, L_2] = L_1L_2 - L_2L_1$. The next assumption represents the relation between the scaling $\mathcal{R}$ and translations $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$.

(T1) *For all $a, \theta \in \mathbb{R}$ and $j = 1, \cdots, n$ the inclusion*

$$\tau_{a,\theta}^{(j)}\big(\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D_\theta^{(j)}) \cap \mathrm{Dom}(\Gamma_{a,\theta}^{(j)})\big) \subset \mathrm{Dom}(B)$$

holds, and there is a $\mu_j > 0$ such that

$$[B, \tau_{a,\theta}^{(j)}]f + \theta\Gamma_{a,\theta}^{(j)}f = -a\mu_j D_\theta^{(j)}\tau_{a,\theta}^{(j)}f \tag{2.21}$$

holds for $f \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(\Gamma_{a,\theta}^{(j)}) \cap \mathrm{Dom}(D_\theta^{(j)})$.

(T2) *For any nontrivial f belonging to $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \cap_{j=1}^n \mathrm{Dom}(D_1^{(j)})$ the functions $\{Bf, D_1^{(1)}f, \cdots, D_1^{(n)}f\}$ are linearly independent.*

If $X = L^2(\mathbb{R})$ and if $(R_\lambda f)(x) = \lambda^{\frac{1}{2}}f(\lambda^{\frac{1}{2}}x)$ and $(\tau_a f)(x) = f(x+a)$, then **(T1)** formally holds with $\mu = \frac{1}{2}$, i.e., we have $[B, \tau_a] = -\frac{a}{2}\partial_x\tau_a$. Note that $\Gamma_{a,\theta} = 0$ in this case. The values μ_j in **(T1)** are related with the eigenvalues of A and they play important roles in our arguments. We set

$$\mu^* = \max\{\mu_1, \cdots, \mu_n, 1\}, \qquad \mu_* = \min\{\mu_1, \cdots, \mu_n, 1\}. \tag{2.22}$$

2.2.2. *Assumptions on A.* We recall that A is the generator of the strongly continuous semigroup $\Theta(t) = R_{e^t}e^{(e^t-1)\mathcal{A}}$. In our arguments it is more convenient to put assumptions on A than on $\mathcal{A}$. So the additional assumptions are given for A instead of $\mathcal{A}$ as follows. Let $\sigma(A)$ be the spectrum of A and let $r_{\mathrm{ess}}(e^{tA})$ be the radius of the essential spectrum of e^{tA}; see [6, Chapter IV] for de nitions.

(A1) *There is a positive number ϱ such that $\sigma(A) \subset \{0\} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq -\varrho\}$. Moreover, 0 is a simple eigenvalue of A in X.*

(A2) *There is a positive number ζ such that $\zeta > \max\{\varrho, \mu^*\}$ and $r_{\mathrm{ess}}(e^{tA}) \leq e^{-\zeta t}$.*

Let w_0 be the eigenfunction to the eigenvalue 0 of A normalized to be 1 in X. We will see that **(T1)** and **(T2)** are sufficient conditions for $D_1^{(j)}w_0$ to be an eigenfunction to the eigenvalue $-\mu_j$ of A if w_0 possesses a suitable regularity. Thus in this case $\varrho \leq \mu_*$ holds by **(A1)**.

Let us introduce the eigenprojections $\mathbf{P}_{0,0}$ and $\mathbf{Q}_{0,0}$, which are de ned by

$$\mathbf{P}_{0,0}f = < f, w_0^* > w_0, \qquad \mathbf{Q}_{0,0}f = f - \mathbf{P}_{0,0}f \tag{2.23}$$

where $<,>$ is a dual coupling of X and its dual space X^*, and w_0^* is the eigenfunction to the eigenvalue 0 of the adjoint operator A^* in X^* with $< w_0, w_0^* >= 1$. From **(A2)** the set $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) > -\zeta\}$ consists of nite numbers of eigenvalues with nite algebraic multiplicities; see [6, Corollary IV-2-11].

From the spectral mapping theorem we have

Proposition 2.3. *Assume that* **(A1)** *and* **(A2)** *hold. Then we have*

$$\|e^{tA}\mathbf{Q}_{0,0}\|_{\mathcal{B}(X)} \le C_\epsilon e^{-(\varrho-\epsilon)t}, \qquad t>0, \tag{2.24}$$

for any $\epsilon > 0$.

2.2.3. *Assumptions on $\mathcal{N}$.* Finally we give the assumptions on the nonlinear operator $\mathcal{N}$. For a linear operator T we denote by $\|\cdot\|_{\mathrm{Dom}(T)}$ the graph norm of T, i.e., $\|f\|_{\mathrm{Dom}(T)} = \|f\|_X + \|Tf\|_X$.

(N1) *$\mathcal{N}$ maps $\mathrm{Dom}(A)$ into $\mathbf{Q}_{0,0}X$ and there is $q > 0$ such that $\|\mathcal{N}(f)\|_X \le C\|f\|^{1+q}_{\mathrm{Dom}(A)}$ holds for any $\|f\|_{\mathrm{Dom}(A)} \le 1$.*

(N2) *There are $\alpha \in (0,1]$, $\beta \in [0,1)$, and $\epsilon_0 \in [0,\varrho)$ such that for each $t > 0$ the operator*

$$N(t,\cdot) = e^{tA}\mathcal{N}(\cdot) \tag{2.25}$$

is a $C^{1+\alpha}$ map from $\mathrm{Dom}(A)$ into $\mathbf{Q}_{0,0}X$ satisfying the estimate

$$\|N'(t,f)h - N'(t,g)h\|_X \le C\Big(\frac{1+t}{t}\Big)^{\beta} e^{-(\varrho-\epsilon_0)t}\|f-g\|_X^{\alpha}\|h\|_X, \tag{2.26}$$

for all $f,\ g,\ h \in \mathrm{Dom}(A)$ with a constant $C > 0$ depending only on α, β, ϱ, ϵ_0, and $M > 0$ when $\|f\|_X + \|g\|_X + \|h\|_X \le M$. Here $N'(t,f)$ is a Fréchet derivative of $N(t,\cdot)$ at f.

(N3) *There is a dense set $\mathcal{W}$ in X such that $\lambda R_\lambda \mathcal{N} = \mathcal{N} R_\lambda$ and $\tau^{(j)}_{a,\theta}\mathcal{N} = \mathcal{N}\tau^{(j)}_{a,\theta}$ hold in $\mathcal{W}$ for any $\lambda \in \mathbb{R}^\times$, $a \in \mathbb{R}^+$, $\theta \in \mathbb{R}$, and j.*

Note that from **(N1)** and **(N2)**, $N(t,0) = 0$ and $N'(t,0) = 0$ follows. Moreover, for each $t > 0$, $N(t,\cdot)$ is extended as a $C^{1+\alpha}$ map from X into $\mathbf{Q}_{0,0}X$.

Let A_{-1} be the generator of the semigroup e^{tA} in the negative order space X_{-1}:

$$X_{-1} = \mathrm{Dom}((-A+I)^{-1}), \qquad \|\cdot\|_{X_{-1}} = \|(-A+I)^{-1}\cdot\|_X. \tag{2.27}$$

Note that the domain of A_{-1} is just X; see [6, II-5]. Then the Laplace formula

$$(2.28)\qquad (-A+I)^{-1+\theta} = \frac{1}{\Gamma(1-\theta)}\int_0^\infty t^{-\theta}e^{-t}e^{-tA}dt, \qquad \theta \in [0,1),$$

with the Euler Γ function $\Gamma(1-\theta)$ and the estimates in **(N2)** lead to the following lemma.

Lemma 2.2. *The nonlinear operator $\mathcal{N}(\cdot)$ can be extended as a $C^{1+\alpha}$ map from X into X_{-1} and its Fréchet derivative $\mathcal{N}'$ is given by*

$$(2.29)\qquad \mathcal{N}'(f) = \frac{1}{\Gamma(1-\gamma)}(-A_{-1}+I)^{1-\gamma}\int_0^\infty t^{-\gamma}e^{-t}N'(t,f)dt,$$

for any $\gamma \in [0, 1-\beta)$. Moreover, the equality

$$(2.30)\qquad e^{tA_{-1}}\mathcal{N}'(f) = N'(t,f), \qquad t>0,$$

holds for any $f \in X$. Especially, $e^{tA_{-1}}\mathcal{N}'(f)$ is extended as a bounded linear operator from X into $\mathbf{Q}_{0,0}X$ and satisfies the estimates

$$(2.31)\quad \|e^{tA_{-1}}\mathcal{N}'(f)h - e^{tA_{-1}}\mathcal{N}'(g)h\|_X \le C_\epsilon(\frac{1+t}{t})^\beta e^{-(\varrho-\epsilon)t}\|f-g\|_X^\alpha\|h\|_X,$$

for any $\epsilon>0$. Here the constant C_ϵ is independent of $\|f\|_X$, $\|g\|_X$, $\|h\|_X$, and $t>0$.

Proof. We rst note that if $f \in X_{-1}$ then $(-A_{-1}+I)^\theta f = (-A+I)^\theta f$, and hence $(-A_{-1}+I)^{-\theta}f = (-A+I)^{-\theta}f$ for any $f\in X$. Moreover, if $f \in X$ then $e^{tA_{-1}}f = e^{tA}f$. From $N'(t,f)\in X$ and (**N2**) the right hand side of (2.29) makes sense for any $f\in X$. By the density argument, it suffices to prove (2.29) for any $f \in \mathrm{Dom}(A)$. Set $C_\gamma = \frac{1}{\Gamma(1-\gamma)}$. For any $f, h \in \mathrm{Dom}(A)$ we have

$$\begin{aligned}
&\|\mathcal{N}(f+h) - \mathcal{N}(f) - C_\gamma(-A_{-1}+I)^{1-\gamma}\int_0^\infty t^{-\gamma}e^{-t}N'(t,f)hdt\|_{X_{-1}}\\
=\;& \|(-A+I)^{-1}\{\mathcal{N}(f+h)-\mathcal{N}(f)\} - C_\gamma(-A_{-1}+I)^{-\gamma}\int_0^\infty t^{-\gamma}e^{-t}N'(t,f)hdt\|_X\\
=\;& \|C_\gamma(-A+I)^{-\gamma}\int_0^\infty t^{-\gamma}e^{-t}\{e^{tA}(\mathcal{N}(f+h)-\mathcal{N}(f)) - N'(t,f)h\}dt\|_X\\
=\;& \|C_\gamma(-A+I)^{-\gamma}\int_0^\infty t^{-\gamma}e^{-t}\{N(t,f+h)-N(t,f)-N'(t,f)h\}dt\|_X\\
\le\;& C\int_0^\infty t^{-\gamma}e^{-t}(\frac{1+t}{t})^\beta dt\|h\|_X^{1+\alpha} \le C\|h\|_X^{1+\alpha}.
\end{aligned}$$

So the Frechet derivative $\mathcal{N}'(f) : X \to X_{-1}$ is given by the right hand side of (2.29). To prove (2.30) we note that

$$e^{tA_{-1}} N'(s,f) = e^{tA} N'(s,f) = N'(t+s,f) = e^{sA} N'(t,f),$$

for $t, s > 0$ which can be seen from the density argument and the semigroup property of e^{tA}. Then we observe from (2.2) that

$$\begin{aligned} e^{tA_{-1}} \mathcal{N}'(f) &= C_\gamma(-A_{-1}+I)^{1-\gamma} \int_0^\infty s^{-\gamma} e^{-s} e^{tA_{-1}} N'(s,f)ds \\ &= C_\gamma(-A+I)^{1-\gamma} \int_0^\infty s^{-\gamma} e^{-s} e^{sA} N'(t,f)ds \\ &= N'(t,f). \end{aligned}$$

Hence (2.30) holds. The estimate (2.31) follows from the semigroup property $e^{(t+t_0)A_{-1}}\mathcal{N}'(f) = e^{tA_{-1}} e^{t_0 A_{-1}} \mathcal{N}'(f)$, Proposition 2.3, (2.30), and (**N2**). We omit the details here. This completes the proof of the lemma.

2.3. **Main results.** Let us state the main results in this paper. Due to the nonlinearity, we only deal with sufficiently small initial data and solutions. The rst result gives the existence of self-similar solutions to (E).

Theorem 2.1. *Assume that* (**E1**)*,* (**A1**)*,* (**N1**)*,* (**N2**)*, and* (**N3**) *hold. Let* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *be the scaling induced by* $\mathcal{R}$ *in* (**E1**)*, and* q, α *be the numbers in* (**N1**)*,* (**N2**)*. Then there is a number* $\delta_0 > 0$ *such that the following statement holds. There is a family of self-similar solutions* $\{R_{\frac{1}{t}} U_\delta\}_{|\delta|\le\delta_0}$ *to (E) with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *such that* U_δ *is* $C^{1+\alpha}$ *in* X *with respect to* δ *and written in the form* $U_\delta = \delta w_0 + v_\delta$ *for some* $v_\delta \in \mathbf{Q}_{0,0}X$ *with* $\|v_\delta\|_{\mathrm{Dom}(A)} \le C|\delta|^{1+q}$*.*

The second result is on the existence of time global solutions to (E) and their self-similar asymptotics at time in nity.

Theorem 2.2. *Assume that* (**E1**)*,* (**A1**)*,* (**A2**)*,* (**N1**)*,* (**N2**)*, and* (**N3**) *hold. Let* ϱ *be the number in* (**A1**)*. If* $\|\Omega_0\|_X$ *is sufficiently small, then there is a unique mild solution* $\Omega(t) \in C([0,\infty); X)$ *to (E) with initial data* Ω_0 *such that*

$$\|R_{1+t}\Omega(t) - U_\delta\|_X \le C(1+t)^{-\frac{\varrho}{2}} \|\Omega_0 - U_\delta\|_X, \qquad t > 0. \tag{2.32}$$

Here $\delta =< \Omega_0, w_0^* >$ *and* U_δ *is the function in Theorem 2.1.*

Remark 2.2. If there is a Banach space Y such that X is continuously embedded in Y and if $\|R_\lambda f\|_Y = K(\lambda)\|f\|_Y$ holds with a constant $K(\lambda)$

satisfying $\sup_{\lambda\le 1} K(\lambda) < \infty$, then from (2.32) we have

$$\begin{aligned}\|\Omega(t) - R_{\frac{1}{1+t}}U_\delta\|_Y &= K(\frac{1}{1+t})\|R_{1+t}\Omega(t) - U_\delta\|_Y \\ &\le CK(\frac{1}{1+t})\|R_{1+t}\Omega(t) - U_\delta\|_X \\ &\le CK(\frac{1}{1+t})(1+t)^{-\frac{\varrho}{2}}\|\Omega_0 - U_\delta\|_X. \qquad (2.33)\end{aligned}$$

Since $\|R_{\frac{1}{1+t}}U_\delta\|_Y = K(\frac{1}{1+t})\|U_\delta\|_Y$, this estimate shows that $R_{\frac{1}{1+t}}U_\delta$ gives an asymptotic pro le of $\Omega(t)$ at large t.

The estimate (2.32) in Theorem 2.2 implies that solutions are approximated by the self-similar solution in large time with accuracy up to $O(t^{-\frac{\varrho}{2}})$. In view of Proposition 2.3, the rate $O(t^{-\frac{\varrho}{2}})$ could be improved but in general at most up to $O(t^{-\varrho+\epsilon})$ for any $\epsilon > 0$. Our aim is to present an abstract method to capture more precise asymptotic pro les of solutions by making use of symmetries of equations, translation and scaling invariances. Especially, in many applications our method gives a suitable shift of the self-similar solution as an asymptotic approximation with accuracy beyond $O(t^{-\varrho})$.

For $y = (y_1, \cdots, y_{n+1})^\top \in \mathbb{R}^{n+1}$ we de ne the shift operator

$$S(y; f) = \tau^{(1)}_{y_1,1+y_{n+1}} \cdots \tau^{(n)}_{y_n,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} f. \qquad (2.34)$$

Note that if $\mathcal{O} \subset \mathbb{R}^{n+1}$ is a sufficiently small open ball centered at the origin, then $S(y; f)$ is a continuous map from $\mathcal{O}$ to X. The following lemma represents the relations between symmetries of (E_0) and the operator A.

Lemma 2.3. *Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, *and* **(A1)** *hold. Let* w_0 *be the eigenfunction for the eigenvalue* 0 *of* A *in* **(A1)** *with* $\|w_0\|_X = 1$. *Suppose that* $S(\cdot\ ; w_0) : \mathcal{O} \to X$ *is* C^1. *Then* Bw_0 *and* $D^{(j)}_1 w_0$ *are eigenfunctions of* A *for the eigenvalues* -1 *and* $-\mu_j$, *respectively. Moreover,* w_0, Bw_0 *and* $D^{(j)}_1 w_0$, $j = 1, \cdots, n$, *are linearly independent.*

If in addition **(A2)** holds, then the set $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge -\mu^*\}$ with $\mu^* = \max\{1, \mu_1, \cdots, \mu_n\}$ consists of nite numbers of eigenvalues with nite algebraic multiplicities (note that the relation $\mu^* \ge \varrho$ holds by **(A1)** and Lemma 2.3). Let $\mathbf{E}_0$ be the total eigenprojection to the eigenvalues $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge -\mu^*\}$, that is,

$$\mathbf{E}_0 = \frac{1}{2\pi i}\int_{\tilde{\gamma}} (\lambda - A)^{-1} d\lambda, \qquad (2.35)$$

where $\tilde{\gamma}$ is a suitable curve around $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge -\mu^*\}$.

Set

(2.36)
$$e_{0,0} = w_0, \qquad e_{0,n+1} = c_{0,n+1} B w_0, \qquad e_{0,j} = c_{0,j} D_1^{(j)} w_0, \qquad j = 1, \cdots, n.$$

Here $c_{0,j}$ is taken as $\|e_{0,j}\|_X = 1$. Then $\{e_{0,j}\}_{j=0}^{n+1}$ forms a part of the basis of the generalized eigenspace $\mathbf{E}_0 X = \{\mathbf{E}_0 f \mid f \in X\}$. So there are $\{e^*_{0,j}\}_{j=0}^{n+1} \subset X^*$ which forms a part of the basis of the generalized eigenspace associated with the eigenvalues $\{\mu \in \sigma(A^*) \mid \mathrm{Re}(\mu) \geq -\mu^*\}$ to the adjoint operator A^* and satis es the relations

$$< e_{0,j}, e^*_{0,k} >= \delta_{jk}, \tag{2.37}$$

where $<,>$ is a dual coupling of X and its dual space X^*, and δ_{jk} is kronecker's delta. By (**A1**) at least $e^*_{0,0} (= w^*_0)$ is the eigenfunction for the simple eigenvalue 0 of A^*. We set the projections as

$$\mathbf{P}_{0,j} f =< f, e^*_{0,j} > e_{0,j}, \qquad \mathbf{Q}_{0,j} f = f - \mathbf{P}_{0,j} f, \qquad 0 \leq j \leq n+1, \tag{2.38}$$

$$\mathbf{P}_0 f = \sum_{j=0}^{n+1} \mathbf{P}_{0,j} f, \qquad \mathbf{Q}_0 f = f - \mathbf{P}_0 f. \tag{2.39}$$

Note that $\mathbf{P}_0 X$ is a subset of $\mathbf{E}_0 X$, and, in general, $\mathbf{P}_0 X$ does not coincide with $\mathbf{E}_0 X$.

Let $-\nu_0$ be the the growth bound of $e^{t\mathbf{Q}_0 A \mathbf{Q}_0}$, that is,

(2.40)
$$-\nu_0 = \inf\{\mu \in \mathbb{R} \mid \exists C_\mu > 0 \text{ s.t. } \|e^{t\mathbf{Q}_0 A \mathbf{Q}_0} f\|_X \leq C_\mu e^{\mu t} \|f\|_X, \ \forall f \in \mathbf{Q}_0 X\}.$$

Note that we always have

$$\varrho \leq \nu_0 \leq \zeta, \tag{2.41}$$

where ϱ and ζ are the numbers in **(A1)** and **(A2)**.

Next we set

$$H(y_0, y; U_\delta) = S(y; U_{\delta + y_0}). \tag{2.42}$$

Then $H(y_0, y; U_\delta)$ is continuous from $(-\delta_0 + \delta, \delta_0 - \delta) \times \mathcal{O} \subset \mathbb{R}^{n+2}$ to X for each $\delta \in (-\delta_0, \delta_0)$.

Set $\mu_0 = 0$ and $\mu_{n+1} = 1$. We will show that each $-\mu_j$ is an eigenvalue of the linearized operator $A - \mathcal{N}'(U_\delta)$; see Section 6 for the precise realization of $A - \mathcal{N}'(U_\delta)$. Especially, $0 (= \mu_0)$ is shown to be a simple eigenvalue by the general perturbation theory. If $\nu_0 \geq \mu^*$ and each $-\mu_j$ is also a semisimple eigenvalue then we can derive the analogous estimate with (1.2) in the abstract settings. The main contribution of this paper is as follows.

Theorem 2.3. *Set* $\mu_{n+1} = 1$. *Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, **(A2)**, *and* **(N1)** - **(N3)** *hold. Suppose that* $S(y; w_0)$ *is* C^1 *near* $y = 0$ *and* $H(y_0, y; U_\delta)$ *is* $C^{1+\gamma}$ *near* $(y_0, y) = (0, 0)$ *for some* $\gamma > 0$. *Let* $\Omega(t)$ *be* *the mild solution in Theorem 2.2 with* $\delta \neq 0$ *and let* ν_0 *be the number in (2.40). Assume that* $\nu_0 \geq \mu^*$ *and* $\{-\mu_j\}_{j=1}^{n+1}$ *are semisimple eigenvalues of* $A - \mathcal{N}'(U_\delta)$*. Then there exist* $\eta(\delta) \in \mathbb{R}$ *and* $y^* \in \mathbb{R}^{n+1}$ *such that*

(2.43)
$$\|R_{1+t}\Omega(t) - S\big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}, \frac{y_{n+1}^*}{1+t}; U_\delta\big)\|_X \leq C_\epsilon (1+t)^{-\nu_0+\eta(\delta)+\epsilon},$$

holds for all $\epsilon > 0$ *and* $t \gg 1$*. Here* $\eta(\delta)$ *satisfies* $\lim_{\delta\to 0}\eta(\delta) = 0$ *and* C_ϵ *is independent of* $t \gg 1$*. Especially, if* $\nu_0 > \mu^*$ *and* $|\delta|$ *is sufficiently small then* $\{-\mu_j\}_{j=1}^{n+1}$ *are semisimple eigenvalues of* $A - \mathcal{N}'(U_\delta)$*, and thus (2.43) holds in this case.*

Remark 2.3. The distribution of the spectrum of A assumed in Theorem 2.3 is visualized in Fig.1 below. When $\nu_0 \geq \mu^*$ we can take $\varrho = \mu_* := \min_{1\leq j\leq n+1}\{\mu_j\}$ by **(A1)** and the de nitions of ν_0. So $\nu_0 \geq \mu^* \geq \mu_* = \varrho > 0$ holds, and the asymptotic pro le is improved by (2.43) if $\nu_0 > \mu_*$. We will give some examples in Section 7 such that $\nu_0 > \mu_*$ holds.

Remark 2.4. The value of $\eta(\delta)$ in Theorem 2.3 is determined by the spectrum of $A - \mathcal{N}'(U_\delta)$. Indeed, we will show that there exists an $\eta(\delta) \in \mathbb{R}$ with $\lim_{\delta\to 0}\eta(\delta) = 0$ such that

$$\sigma(A - \mathcal{N}'(U_\delta)) \subset \{-\mu_j\}_{j=0}^{n+1} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq -\nu_0 + \eta(\delta)\}. \tag{2.44}$$

The number $\eta(\delta)$ in Theorem 2.3 is nothing but $\eta(\delta)$ in (2.44). Moreover, in Theorem 2.3 if in addition $\zeta > \nu_0$ and $\{\mu \in \sigma(A - \mathcal{N}'(U_\delta)) \mid \mathrm{Re}(\mu) \geq -\nu_0 + \eta(\delta)\}$ consists of semisimple eigenvalues, then we can take $\epsilon = 0$ in (2.43); see Remark 6.4 for details.

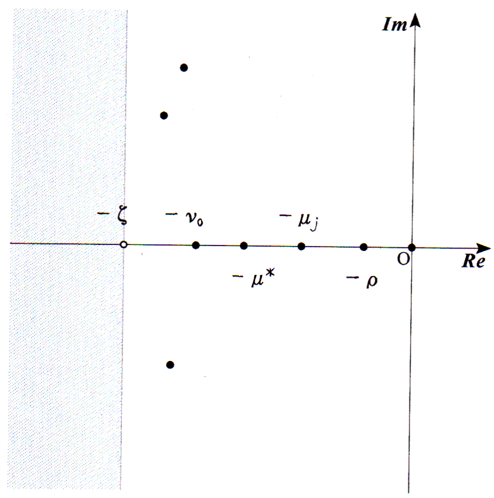

Fig. 1: The distribution of the spectrum $\sigma(A)$ assumed in Theorem 2.3. By **(A2)** the continuous spectrum is included in the gray region, and there are only isolated eigenvalues

with finite multiplicity in $\{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) > -\zeta\}$ (black dots in the figure). Each $-\mu_j$ is shown to be an eigenvalue of A by Lemma 2.3. The definition of ν_0 and the assumption $\nu_0 \geq \mu^*$ imply $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) > -\nu_0\} = \{-\mu_j \mid \mu_j < \nu_0,\ 0 \leq j \leq n+1\}$.

As a simple example of Theorem 2.3, let us consider the one dimensional viscous Burgers equations (1-B). In this case the operator $\mathcal{A}$ is the one dimensional Laplacian ∂_x^2, and a scaling $\mathcal{R} = \{R_\lambda\}_{\lambda \in \mathbb{R}^\times}$ and a translation $\mathcal{T} = \{\tau_a\}_{a \in \mathbb{R}^+}$ are taken as $R_\lambda f(x) = \lambda^{\frac{1}{2}} f(\lambda^{\frac{1}{2}} x)$ and $(\tau_a f)(x) = f(x+a)$, respectively. So the map $S(y;f)$, $y \in \mathbb{R}^2$, is de ned by

$$S(y;f) = \tau_{y_1} R_{\frac{1}{1+y_2}} f(x) = (1+y_2)^{-\frac{1}{2}} f(\frac{x+y_1}{\sqrt{1+y_2}}).$$

The space X is chosen as a polynomial weighted L^2 space, $L^2_m = L^2((1+x^2)^m dx)$ with $m > \frac{5}{2}$, which is continuously embedded in $L^1(\mathbb{R})$. In fact, we can see $\varrho = -\mu_1 = \frac{1}{2}$ and $\nu_0 = \min\{\frac{3}{2}, \frac{m}{2} - \frac{1}{4}\}$ in this case from the spectral property of $A = \partial_x^2 + \frac{x}{2}\partial_x + \frac{1}{2}$ in L^2_m. Hence by Theorem 2.3 we get

$$\begin{aligned} & \|R_{1+t}\Omega(t) - S((1+t)^{-\frac{1}{2}} y_1^*, (1+t)^{-1} y_2^*; U_\delta)\|_{L^1(\mathbb{R})} \\ \leq\ & C\|R_{1+t}\Omega(t) - S((1+t)^{-\frac{1}{2}} y_1^*, (1+t)^{-1} y_2^*; U_\delta)\|_{L^2_m} \\ \leq\ & C_\epsilon (1+t)^{-\min\{\frac{3}{2}, \frac{m}{2} - \frac{1}{4}\} + \eta(\delta) + \epsilon}, \end{aligned}$$

in other words,

$$\begin{aligned} & \|\Omega(t) - (1+t+y_2^*)^{-\frac{1}{2}} U_\delta\big(\frac{x+y_1^*}{\sqrt{1+t+y_2^*}}\big)\|_{L^1(\mathbb{R})} \\ \leq\ & C_\epsilon (1+t)^{-\min\{\frac{3}{2}, \frac{m}{2} - \frac{1}{4}\} + \eta(\delta) + \epsilon}. \end{aligned}$$

Thus we recover the estimate such as (1.2) for $p = 1$ and $m \geq \frac{7}{2}$. The details are given in Section 7.1.

In Theorem 2.3 we consider the shifts of U_δ with respect to both translations and scaling. In fact, we can also consider the shifts of U_δ with respect to only translations under weaker assumptions on A. Set

$$\tilde{\mu}^* = \max\{\mu_1, \cdots, \mu_n\}, \tag{2.45}$$

and

$$\tilde{\mathbf{P}}_0 f = \sum_{j=0}^{n} \mathbf{P}_{0,j} f, \qquad \tilde{\mathbf{Q}}_0 f = f - \tilde{\mathbf{P}}_0 f. \tag{2.46}$$

Let $-\tilde{\nu}_0$ be the the growth bound of $e^{t\tilde{\mathbf{Q}}_0 A \tilde{\mathbf{Q}}_0}$, that is,

(2.47)

$$-\tilde{\nu}_0 = \inf\{\mu \in \mathbb{R} \mid \exists C_\mu > 0 \text{ s.t. } \|e^{t\tilde{\mathbf{Q}}_0 A \tilde{\mathbf{Q}}_0} f\|_X \leq C_\mu e^{\mu t} \|f\|_X,\ \forall f \in \tilde{\mathbf{Q}}_0 X\}.$$

Instead of **(A2)** we consider the case

(A2)' *There is* $\zeta > \max\{\varrho, \tilde{\mu}^*\}$ *such that* $r_{\rm ess}(e^{tA}) \le e^{-\zeta t}$.

Since $\mu^* \ge \tilde{\mu}^*$, **(A2)'** is weaker than **(A2)** in general. For $\tilde{y} = (y_1, \cdots, y_n)^\top \in \mathbb{R}^n$ let us de ne the shift operator $\tilde{S}(\tilde{y}; f)$ by

$$\tilde{S}(\tilde{y}; f) = S(\tilde{y}, 0; f) = \tau^{(1)}_{y_1,1} \cdots \tau^{(n)}_{y_n,1} f. \tag{2.48}$$

Then we have

Theorem 2.4. *Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, **(A2)'**, *and* **(N1)** - **(N3)** *hold. Suppose that* $S(y; w_0)$ *is* C^1 *near* $y = 0$ *and* $H(y_0, y; U_\delta)$ *is* $C^{1+\gamma}$ *near* $(y_0, y) = (0, 0)$ *for some* $\gamma > 0$. *Let* $\Omega(t)$ *be the mild solution in Theorem 2.2 with* $\delta \ne 0$ *and let* $\tilde{\nu}_0$ *be the number in (2.47). Assume that* $\tilde{\nu}_0 \ge \tilde{\mu}^*$ *and* $\{-\mu_j\}_{j=1}^n$ *are semisimple eigenvalues of* $A - \mathcal{N}'(U_\delta)$. *Then there exist* $\tilde{\eta}(\delta) \in \mathbb{R}$ *and* $\tilde{y} \in \mathbb{R}^n$ *such that*

$$\|R_{1+t}\Omega(t) - \tilde{S}\big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}; U_\delta\big)\|_X \le C_\epsilon (1+t)^{-\tilde{\nu}_0 + \tilde{\eta}(\delta) + \epsilon}, \tag{2.49}$$

holds for all $\epsilon > 0$ *and* $t \gg 1$. *Here* $\tilde{\eta}(\delta)$ *satisfies* $\lim_{\delta\to 0} \tilde{\eta}(\delta) = 0$, C_ϵ *is independent of* $t \gg 1$. *Especially, if* $\tilde{\nu}_0 > \tilde{\mu}^*$ *and* $|\delta|$ *is sufficiently small then* $\{-\mu_j\}_{j=1}^n$ *are semisimple eigenvalues of* $A - \mathcal{N}'(U_\delta)$, *and thus (2.49) holds in this case.*

Remark 2.5. From (2.40), (2.47), and Lemma 2.3 we have $\nu_0 \ge \tilde{\nu}_0$ and $1 \ge \tilde{\nu}_0$. Hence, in general, the shift of self-similar solutions with respect to both translations and scaling, (2.43), can give more precise asymptotic pro le than the shift with respect to only translations, (2.49), if **(A2)** is satis ed. A typical example such that $\nu_0 > \tilde{\nu}_0$ follows is the viscous Burgers equations (1-B); see Section 7.1.1. Even when $\nu_0 = \tilde{\nu}_0$, the rate in (2.43) is better than the one in (2.49) if $\eta(\delta)$ is negative and $y^*_{n+1} \ne 0$. Indeed, we rst note that $\nu_0 = \tilde{\nu}_0 = 1$ in this case, for $\nu_0 \ge \mu^*$ is assumed in Theorem 2.3 and $1 \ge \tilde{\nu}_0$. Then if $\eta(\delta) < 0$ we have from (2.43),

$$\begin{aligned}
&\|R_{1+t}\Omega(t) - \tilde{S}\big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}; U_\delta\big)\|_X \\
\ge\ & \|S\big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}, 0; U_\delta\big) - S\big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}, \frac{y^*_{n+1}}{1+t}; U_\delta\big)\|_X \\
& -\|R_{1+t}\Omega(t) - S\big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}, \frac{y^*_{n+1}}{1+t}; U_\delta\big)\|_X \\
\ge\ & \frac{|y^*_{n+1}|}{1+t}\|BU_\delta\|_X - C(1+t)^{-1-c}
\end{aligned}$$

for some $c > 0$. Here we used the $C^{1+\gamma}$ regularity of $S(y; U_\delta)$ near $y = 0$ and the fact that $\partial_{y_{n+1}} S(y; U_\delta)|_{y=0} = BU_\delta$. Hence $\tilde{\eta}(\delta) + \epsilon = 0$ must hold in (2.49), which proves the above assertion. The examples such that $\nu_0 = \tilde{\nu}_0 = 1$ and $\eta(\delta) < 0$ hold are given in Section 7.1.2 and Section 7.1.3.

The proof of Theorem 2.4 is just a simple modi cation of the one of Theorem 2.3. So the details will be omitted in this paper; see Section 6.4. Applications of our main results to concrete nonlinear PDEs will be given in Section 7.

3. Invariant property and reduction by similarity transform

In this section we prove that (E) is invariant with respect to the scaling $\mathcal{R}$ and the translations $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $j = 1, \cdots, n$ under the assumptions stated in the previous section. We also derive an equation by the "similarity transform" which enables us to analyze large time behaviors of mild solutions to (E) in terms of a stability problem of stationary solutions to the new equation.

We rst consider the relation between $e^{t\mathcal{A}}\mathcal{N}$ and $e^{tA}\mathcal{N}$. Since $e^{sA} = R_{e^s} e^{(e^s-1)\mathcal{A}}$ we have for $f \in \mathrm{Dom}(A)$,

$$\mathcal{N}(t, f) := e^{t\mathcal{A}}\mathcal{N}(f) = R_{\frac{1}{1+t}} e^{\log(1+t)A}\mathcal{N}(f) = R_{\frac{1}{1+t}} N(\log(1+t), f). \tag{3.1}$$

Since there is a sequence $\{f_n\} \subset X$ such that $f_n \in \mathrm{Dom}(A)$ and $f_n \to f$ in X, $\mathcal{N}(t, \cdot)$ can be extended as a $C^{1+\alpha}$ map on X and the above equality holds for any $f \in X$. Moreover, by density arguments we have

Lemma 3.1. *Assume that* **(E1)**, **(E2)**, **(A1)**, **(N2)**, *and* **(N3)** *hold. Then it follows that*

$$\lambda R_\lambda \mathcal{N}(\lambda t, f) = \mathcal{N}(t, R_\lambda f), \qquad t > 0, \tag{3.2}$$

$$\tau_{a,t+\theta}^{(j)} \mathcal{N}(t, f) = \mathcal{N}(t, \tau_{a,\theta}^{(j)} f), \qquad t > 0, \tag{3.3}$$

for any $f \in X$, $\lambda \in \mathbb{R}^\times$, $a \in \mathbb{R}^+$, $\theta \in \mathbb{R}$, and j.

Proof. Here we give a proof for the rst equality only, since the second one is shown in the same way. From (2.16) and **(N3)** it follows that

$$\begin{aligned}\lambda R_\lambda \mathcal{N}(\lambda t, f) = \lim_{n\to\infty} \lambda R_\lambda e^{\lambda t\mathcal{A}}\mathcal{N}(f_n) &= \lim_{n\to\infty} e^{t\mathcal{A}}\lambda R_\lambda \mathcal{N}(f_n) \\ &= \lim_{n\to\infty} e^{t\mathcal{A}}\mathcal{N}(R_\lambda f_n) = \mathcal{N}(t, R_\lambda f).\end{aligned}$$

This completes the proof.

From **(N2)** and (3.1) we observe that the term $\int_s^t e^{(t-\tau)\mathcal{A}}\mathcal{N}(\Omega(\tau))d\tau$ in (2.13) makes sense for all $\Omega(t) \in C((0,\infty); X)$ by rewriting it as $\int_s^t \mathcal{N}(t-\tau, \Omega(\tau))d\tau$. Moreover, if $\Omega(t) \in C([0,\infty); X)$ then we can show $\lim_{t\to 0}\int_0^t \|\mathcal{N}(t-s, \Omega(s))\|_X ds = 0$. Hence as a corollary of Lemma 3.1 we have

Corollary 3.1. *Assume that* **(E1)**, **(E2)**, **(A1)**, **(N2)**, *and* **(N3)** *hold. Then (E) is invariant with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *and* $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $j = 1, \cdots, n$.

Proof. Let $\Omega(t) \in C([0,\infty); X)$ be a mild solution to (E) with initial data Ω_0. Then $\Theta_\lambda(\Omega)(t) \in C([0,\infty); X)$ and we have

$$\begin{aligned}
\Theta_\lambda(\Omega)(t) &= R_\lambda e^{\lambda t\mathcal{A}}\Omega_0 - R_\lambda \int_0^{\lambda t} \mathcal{N}(\lambda t - s, \Omega(s))ds \\
&= e^{t\mathcal{A}} R_\lambda \Omega_0 - \int_0^t \lambda R_\lambda \mathcal{N}(\lambda(t-s'), \Omega(\lambda s'))ds' \\
&= e^{t\mathcal{A}} R_\lambda \Omega_0 - \int_0^t \mathcal{N}(t-s', R_\lambda\Omega(\lambda s'))ds' \\
&= e^{t\mathcal{A}} R_\lambda \Omega_0 - \int_0^t \mathcal{N}(t-s', \Theta_\lambda(\Omega)(s'))ds'.
\end{aligned}$$

This implies that (E) is invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$. The invariance for $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$ is proved in a similar manner, so we omit the details. This completes the proof.

Next we consider the following integral equation.

$$\begin{aligned}
u(t) &= e^{tA}u_0 - \int_0^t N(t-s, u(s))ds \\
&= e^{tA}u_0 - \int_0^t e^{(t-s)A}\mathcal{N}(u(s))ds, \text{ if } u(t) \in C([0,\infty); \mathrm{Dom}(A)).
\end{aligned} \tag{3.4}$$

The main result in this section is as follows.

Lemma 3.2. *Assume that* **(E1)**, **(A1)**, **(N2)**, *and* **(N3)** *hold. If* $\Omega(t) \in C([0,\infty); X)$ *is a mild solution to (E) with initial data* Ω_0*, then* $u(t) = R_{e^t}\Omega(e^t - 1)$ *is a mild solution to (3.4) with initial data* Ω_0*. Conversely, if* $u(t) \in C([0,\infty); X)$ *is a mild solution to (3.4) with initial data* u_0*, then* $\Omega(t) = R_{\frac{1}{1+t}}u(\log(1+t))$ *is a mild solution to (E) with initial data* u_0*. Moreover,* $u \in X$ *is a stationary solution to (3.4) if and only if* $R_{\frac{1}{t}}u$ *is a self-similar solution to (E).*

Proof. Here we give a proof for the last assertion only. For any $t > s > 0$ and $u \in X$ we set

$$F(t,s;u) = R_{\frac{1}{t}} e^{\log\frac{t}{s}A}u - R_{\frac{1}{t}} \int_{\log s}^{\log t} N(\log t - \tau, u)d\tau.$$

From the de nition of e^{tA} and (3.1) we have

$$e^{\log\frac{t}{s}A} = R_{\frac{t}{s}} e^{(\frac{t}{s}-1)\mathcal{A}}, \qquad N(t,f) = R_{e^t}\mathcal{N}(e^t - 1, f).$$

Hence $F(t,s;u)$ is written as

$$\begin{aligned} F(t,s;u) &= R_{\frac{1}{s}} e^{\frac{1}{s}(t-s)\mathcal{A}} u - R_{\frac{1}{t}} \int_s^t \frac{1}{r} N(\log\frac{t}{r}, u) dr \\ &= e^{(t-s)\mathcal{A}} R_{\frac{1}{s}} u - R_{\frac{1}{t}} \int_s^t \frac{1}{r} R_{\frac{t}{r}} \mathcal{N}(\frac{t}{r}-1, u) dr \\ &= e^{(t-s)\mathcal{A}} R_{\frac{1}{s}} u - \int_s^t \frac{1}{r} R_{\frac{1}{r}} \mathcal{N}(\frac{t}{r}-1, u) dr \\ &= e^{(t-s)\mathcal{A}} R_{\frac{1}{s}} u - \int_s^t \mathcal{N}(t-r, R_{\frac{1}{r}} u) dr. \end{aligned}$$

In the last line we used Lemma 3.1. Note that $u \in X$ is a stationary solution to (3.4) if and only if u satis es

$$u = e^{\log\frac{t}{s}A} u - \int_{\log s}^{\log t} N(\log t - \tau, u) d\tau = R_t F(t,s;u),$$

for any $t > s > 0$. From the above calculation this is equivalent with

$$R_{\frac{1}{t}} u = F(t,s;u) = e^{(t-s)\mathcal{A}} R_{\frac{1}{s}} u - \int_s^t \mathcal{N}(t-r, R_{\frac{1}{r}} u) dr,$$

i.e., $R_{\frac{1}{t}} u$ is a self-similar solution to (E). This completes the proof.

4. Existence of self-similar solutions

In this section we prove the existence of self-similar solutions to (E), which is equivalent with the existence of stationary solutions to (3.4) by Lemma 3.2.

For this purpose we look for a stationary solution U_δ of the form $U_\delta = \delta w_0 + v$ where v belongs to $\mathbf{Q}_{0,0}X$. Then the equation for v is

$$-Av + \mathcal{N}(\delta w_0 + v) = 0. \tag{4.1}$$

Here we used the fact that $A\delta w_0 = 0$. Noting that $\mathcal{N}(\cdot) \in \mathbf{Q}_{0,0}X$, (4.1) in $\mathbf{Q}_{0,0}X$ can be solved by the usual contraction mapping theorem for the map

$$\Phi(\delta, f) = -(-A)^{-1}\mathcal{N}(\delta w_0 + f). \tag{4.2}$$

The main result of this section is as follows.

Theorem 4.1. *Assume that* **(E1)***,* **(A1)***,* **(N1)***,* **(N2)***, and* **(N3)** *hold. Then there is a positive number δ_0 such that if $|\delta| \le \delta_0$ then there is a unique solution v_δ to (4.1) in $\mathbf{Q}_{0,0}X$ satisfying the estimate*

$$\|v_\delta\|_{\mathrm{Dom}(A)} \le C|\delta|^{1+q}. \tag{4.3}$$

Moreover, v_δ is $C^{1+\alpha}$ in X with respect to δ, $|\delta| < \delta_0$. Thus, we have a self-similar solution to (E) which takes the form $R_{\frac{1}{t}}U_\delta$ with $U_\delta = \delta w_0 + v_\delta$.

Proof. Let δ_0 be a sufficiently small positive number. We show that $\Phi(\delta, \cdot)$ is a contraction mapping on the closed ball

$$B_{\delta_0} = \{f \in \mathbf{Q}_{0,0}X \mid \|f\|_{\mathrm{Dom}(A)} \le \delta_0 \},$$

if $|\delta| \le \delta_0 \ll 1$. Indeed, from **(N1)** we easily see that

$$\|\Phi(\delta, f)\|_{\mathrm{Dom}(A)} \le C\|\mathcal{N}(\delta w_0 + f)\|_X \le C\delta_0^{1+q}, \quad \text{if } f \in B_{\delta_0}.$$

Moreover, since the Laplace formula yields the representation

$$\begin{aligned}\Phi(\delta, f) - \Phi(\delta, g) &= \int_0^\infty e^{tA}(\mathcal{N}(\delta w_0 + f) - \mathcal{N}(\delta w_0 + g))dt \\ &= \int_0^\infty \int_0^1 N'(t, \delta w_0 + \theta f + (1-\theta)g)(f-g)d\theta dt,\end{aligned}$$

we have from **(N2)** that

$$\begin{aligned}&\|\Phi(\delta, f) - \Phi(\delta, g)\|_X \\ \le\ & C\int_0^\infty (\frac{1+t}{t})^\beta e^{-(\varrho-\epsilon_0)t}(|\delta| + \|f\|_X + \|g\|_X)^\alpha \|f-g\|_X dt \\ \le\ & C\delta_0^\alpha \|f-g\|_X.\end{aligned}$$

Combining these above, we observe that $\Phi(\delta, \cdot)$ is a contraction mapping on B_{δ_0} if $|\delta| \le \delta_0$ and δ_0 is small enough. Thus there is a unique xed point of $\Phi(\delta, \cdot)$ in B_{δ_0}. Let v_δ be the xed point. Then

$$\begin{aligned}\|v_\delta\|_{\mathrm{Dom}(A)} &\le \|\mathcal{N}(\delta w_0 + v_\delta)\|_X^{1+q} \\ &\le C|\delta|^{1+q} + C|\delta|^q\|v_\delta\|_X.\end{aligned}$$

Hence if $|\delta|$ is sufficiently small, then $\|v_\delta\|_{\mathrm{Dom}(A)} \le 2|\delta|^{1+q}$.

Finally, since $\Phi(\delta, \cdot)$ is $C^{1+\alpha}$ from X into $\mathbf{Q}_{0,0}X$, by the uniform contraction mapping principle we see that v_δ is $C^{1+\alpha}$ with respect to δ in X.

5. Global solvability of the evolution equations

In this section we prove the global existence of mild solutions to (3.4) for sufficiently small initial data:

$$u(t) = e^{tA}u_0 - \int_0^t N(t-s, u(s))ds.$$

Again by Lemma 3.2 this means the existence of time global solutions to (E) for small initial data.

The next result shows that the self-similar solution U_δ plays important roles for the large time behavior of solutions to (3.4).

Theorem 5.1. *Assume that* **(E1)**, **(A1)**, **(A2)**, **(N1)**, **(N2)**, *and* **(N3)** *hold. If* $\|u_0\|_X$ *is sufficiently small, then there is a unique mild solution* $u(t) \in C([0,\infty); X)$ *to (3.4) such that*

$$\sup_{t>0} e^{\frac{\varrho}{2}t}\|u(t) - U_\delta\|_X \leq C\|u_0 - U_\delta\|_X. \tag{5.1}$$

Here $\delta =< u_0, w_0^* >$.

Proof. We rst note that the smallness of $\|u_0\|_X$ leads to the smallness of $|\delta|$, which guarantees the existence of the stationary solution U_δ by Theorem 4.1. Let us consider the equation for $\omega(t) = u(t) - U_\delta \in \mathbf{Q}_{0,0}X$:

$$\partial_t \omega - A\omega = -\mathcal{N}(U_\delta + \omega) + \mathcal{N}(U_\delta), \quad t > 0. \tag{5.2}$$

The associated integral equation in X is

$$\begin{aligned} \omega(t) &= e^{tA}\omega_0 - \int_0^t N(t-s, U_\delta + \omega(s)) - N(t-s, U_\delta)ds \\ &=: \Upsilon(\omega), \end{aligned} \tag{5.3}$$

where $\omega_0 = u_0 - U_\delta$. Clearly this is equivalent with (3.4).

For $\theta \in [0,1]$ and the Banach space $\mathcal{B}$ let us introduce the function space $C_\theta([0,\infty); \mathcal{B})$ as follows.

(5.4)

$$C_\theta([0,\infty); \mathcal{B}) = \{f \in C([0,\infty); \mathcal{B}) \mid \|f\|_{C_\theta(\mathcal{B})} := \sup_{t>0} e^{\theta t}\|f(t)\|_{\mathcal{B}} < \infty\}.$$

This integral equation can be solved by the usual contraction mapping theorem on the ball

$$B_R = \{f \in C_{\frac{\varrho}{2}}([0,\infty); \mathbf{Q}_{0,0}X) \mid \|f\|_{C_{\frac{\varrho}{2}}(X)} \leq R\},$$

when $R = 2\|\Upsilon(0)\|_{C_{\frac{\varrho}{2}}(\mathbf{Q}_0 X)}$. Note that $\Upsilon(0)(t) = e^{tA}\omega_0$.

Since

$$\begin{aligned} &N(t-s, U_\delta + \omega_1(s)) - N(t-s, U_\delta + \omega_2(s)) \\ = &\int_0^1 N'(t-s, U_\delta + \sigma\omega_1 + (1-\sigma)\omega_2)d\sigma(\omega_1(s) - \omega_2(s)), \end{aligned}$$

from Lemma 2.2 we have the estimates for Υ such as

$$\begin{aligned}
&\|\Upsilon(\omega_1)-\Upsilon(\omega_2)\|_X(t)\\
\le\ & C\int_0^t e^{-\frac{3\varrho}{4}(t-s)}(\frac{1+t-s}{t-s})^\beta\\
&(\|U_\delta\|_X+\|\omega_1(s)\|_X+\|\omega_2(s)\|_X)^\alpha\|\omega_1(s)-\omega_2(s)\|_X ds\\
\le\ & C(|\delta|^\alpha+R^\alpha)\int_0^t e^{-\frac{3\varrho}{4}(t-s)}(\frac{1+t-s}{t-s})^\beta e^{-\frac{\varrho}{2}s}ds\|\omega_1-\omega_2\|_{C_{\frac{\varrho}{2}}(X)}\\
\le\ & C(|\delta|^\alpha+R^\alpha)e^{-\frac{\varrho}{2}t}\|\omega_1-\omega_2\|_{C_{\frac{\varrho}{2}}(X)}.
\end{aligned}$$

Since $\omega_0\in\mathbf{Q}_{0,0}X$, the function $\Upsilon(0)(t)$ is estimated as $\|\Upsilon(0)\|_{C_{\frac{\varrho}{2}}(X)}\le C\|u_0-U_\delta\|_X$ by Proposition 2.3. Hence Υ is a contraction mapping on B_R for sufficiently small $|\delta|$ and R. Note that $|\delta|$ and R can be taken small enough if $\|u_0\|_X$ is sufficiently small. Thus there is a unique solution to (5.3) in B_R if the initial data is sufficiently small. The proof is complete.

As a corollary of Theorem 5.1 we have from Lemma 3.2,

Corollary 5.1. *Assume that* **(E1)**, **(A1)**, **(A2)**, **(N1)**, **(N2)**, *and* **(N3)** *hold. If* $\|\Omega_0\|_X$ *is sufficiently small, then there is a unique mild solution* $\Omega(t)\in C([0,\infty);X)$ *to (E) such that*

$$\|R_{1+t}\Omega(t)-U_\delta\|_X\le C(1+t)^{-\frac{\varrho}{2}}\|\Omega_0-U_\delta\|_X,\qquad t>0. \tag{5.5}$$

Here $\delta=<\Omega_0,w_0^*>$.

This corollary immediately leads to Theorem 2.2.

6. More precise asymptotic profile by shift

In this section we study the asymptotic behavior of solutions obtained by Theorem 5.1 in more details by assuming that (E) possesses an invariance with respect to translations. The key step in our arguments is to introduce a linear transform with $n+2$ parameters $\mathbf{y}=(y_0,y),\ y=(y_1,\cdots,y_{n+1})$:

$$H(\mathbf{y};U_\delta)=S(y;U_{\delta+y_0}), \tag{6.1}$$

where

$$S(y;f)=\tau^{(1)}_{y_1,1+y_{n+1}}\cdots\tau^{(n)}_{y_n,1+y_{n+1}}R_{\frac{1}{1+y_{n+1}}}f. \tag{6.2}$$

We start from the next proposition.

Proposition 6.1. *Let* $f\in\mathrm{Dom}(A)$. *Assume that there is an open ball* $\mathcal{O}\subset\mathbb{R}^{n+1}$ *centered at the origin such that* $S(\cdot;f):\mathcal{O}\to X$ *is* C^1. *Let* $j=0,\cdots,n$. *Then for each* $y=(y_1,\cdots,y_{n+1})\in\mathcal{O}$ *we have* $S(0,\cdots,0,y_{j+1},\cdots,y_{n+1};f)\in$

$\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D^{(k)}_{1+y_{n+1}}) \cap \mathrm{Dom}(\Gamma^{(l)}_{y_l,1+y_{n+1}})$ *for all* $k = 1, \cdots, n$ *and* $l = 1, \cdots, j$.

Proof. For each $j = 0, \cdots, n$ we set

$$S_j(y_{j+1}, \cdots, y_{n+1}; f) = S(y; f)|_{y_1=\cdots=y_j=0}.$$

We will show by the backward induction that each S_j belongs to $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D^{(k)}_{1+y_{n+1}}) \cap \mathrm{Dom}(\Gamma^{(l)}_{y_l,1+y_{n+1}})$ for all $k = 1, \cdots, n$ and $l = 1, \cdots, j$ when $y \in \mathcal{O}$. Since $S(y; f)$ is assumed to be C^1 in $\mathcal{O}$, each $S_j(y_{j+1}, \cdots, y_{n+1}; f)$ is di erentiable if $y \in \mathcal{O}$. Especially, we have $S_j(y_{j+1}, \cdots, y_{n+1}; f) \in \mathrm{Dom}(D^{(k)}_{1+y_{n+1}})$ for all $k = 1, \cdots, n$ by the independent property of $\tau^{(j)}_{a_j,\theta}\tau^{(l)}_{a_l,\theta} = \tau^{(l)}_{a_l,\theta}\tau^{(j)}_{a_j,\theta}$ for $1 \le j, l \le n$.

Let us consider $S_n(y_{n+1}; f) = R_{\frac{1}{1+y_{n+1}}} f$. From the C^1 regularity of S_n we have $f \in \mathrm{Dom}(B)$ and so is true for $S_n(y_{n+1}; f)$. Then from the relation of $e^{(e^t-1)\mathcal{A}} f = R_{e^{-t}} e^{tA} f$ and the assumption of $f \in \mathrm{Dom}(A)$, we have $f \in \mathrm{Dom}(\mathcal{A})$. So from the invariant property of $R_\lambda e^{\lambda t\mathcal{A}} = e^{t\mathcal{A}} R_\lambda$, we also have $S_n(y_{n+1}; f) = R_{\frac{1}{1+y_{n+1}}} f \in \mathrm{Dom}(\mathcal{A})$. Since both $S(y; f)|_{y_i=0, i\neq l, n+1} = \tau^{(l)}_{y_l,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} f$ and $R_{\frac{1}{1+y_{n+1}}} f$ are C^1 with respect to y_{n+1} for each $l = 1, \cdots, n$, we can check that $R_{\frac{1}{1+y_{n+1}}} f \in \mathrm{Dom}(\Gamma^{(l)}_{y_l,1+y_{n+1}})$. This gives the desired regularity for S_n. Suppose that $S_{j+1}(y_{j+2}, \cdots, y_{n+1}; f)$ belongs to $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D^{(k)}_{1+y_{n+1}}) \cap \mathrm{Dom}(\Gamma^{(l)}_{y_l,1+y_{n+1}})$ for all $k = 1, \cdots, n$ and $l = 1, \cdots, j+1$. Then we have from **(T1)** that

$$S_j(y_{j+1}, \cdots, y_{n+1}; f) = \tau^{(j)}_{y_j,1+y_{n+1}} S_{j+1}(y_{j+2}, \cdots, y_{n+1}; f) \in \mathrm{Dom}(B).$$

Furthermore, from the invariant property of (2.17) we see

$$e^{t\mathcal{A}} \tau^{(j)}_{y_j,1+y_{n+1}} S_{j+1}(y_{j+2}, \cdots, y_{n+1}; f) = \tau^{(j)}_{y_j,t+1+y_{n+1}} e^{t\mathcal{A}} S_{j+1}(y_{j+2}, \cdots, y_{n+1}; f).$$

Hence, $\tau^{(j)}_{y_j,1+y_{n+1}} S_{j+1}(y_{j+2}, \cdots, y_{n+1}; f) \in \mathrm{Dom}(\mathcal{A})$ from the assumption of $S_{j+1}(y_{j+2}, \cdots, y_{n+1}; f) \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(\Gamma^{(j)}_{y_j,1+y_{n+1}})$. Now it remains to show $S_j(y_{j+1}, \cdots, y_{n+1}; f) \in \mathrm{Dom}(\Gamma^{(l)}_{y_l,1+y_{n+1}})$ for all $l = 1, \cdots, j$. But this can be veri ed from the fact that $\tau^{(l)}_{y_l,1+y_{n+1}} S_j(y_{j+1}, \cdots, y_{n+1}; f) = S(y; f)|_{y_i=0, i\neq l, j+1, \cdots, n}$ is C^1 with respect to y_{n+1} for $l = 1, \cdots, j$. This completes the proof.

Let $H(\cdot;U_\delta):(-\delta_0+\delta,\delta_0-\delta)\times\mathcal{O}\to X$ be C^1, where $\mathcal{O}\subset\mathbb{R}^{n+1}$ is an open ball centered at the origin. Then we observe that

$$\begin{aligned} \partial_{y_0}H(0;U_\delta) &= \partial_\delta U_\delta, && (6.3)\\ \partial_{y_l}H(0;U_\delta) &= D_1^{(l)}U_\delta, \quad 1\le l\le n, && (6.4)\\ \partial_{y_{n+1}}H(0;U_\delta) &= BU_\delta. && (6.5) \end{aligned}$$

Our aim is to determine the parameters $\mathbf{y}(t)=(y_0(t),y(t))$ so that $\|u(t)-H(\mathbf{y}(t);U_\delta)\|_X$ decays faster than $\|u(t)-U_\delta\|_X$. Let us formulate our problem precisely. Recall that A_{-1} is the generator of e^{tA} in the negative order space X_{-1} with the domain X. We consider the equation for $v(t)=u(t)-H(\mathbf{y}(t);U_\delta)$ in X_{-1} where $\mathbf{y}(t)\in\mathbb{R}^{n+2}$ is determined later. For simplicity of notations, we write $V_\delta(t)$ for $H(\mathbf{y}(t);U_\delta)$. Then we obtain the equation in X_{-1} such as

$$\partial_t v-(A_{-1}-\mathcal{N}'(U_\delta))v=T_\delta v+F_\delta(v)+J(V_\delta), \tag{6.6}$$

where

$$\begin{aligned} &V_\delta(t)=H(\mathbf{y}(t);U_\delta),\\ &T_\delta v(t)=\{\mathcal{N}'(U_\delta)-\mathcal{N}'(V_\delta(t))\}v(t),\\ &F_\delta(v)(t)=-\int_0^1\big\{\mathcal{N}'(V_\delta(t)+\tau v(t))-\mathcal{N}'(V_\delta(t))\big\}v(t)d\tau,\\ &J(V_\delta)(t)=-\partial_t V_\delta(t)+A_{-1}V_\delta(t)-\mathcal{N}(V_\delta(t)), \end{aligned}$$

and

$$\mathcal{N}'(f)=(-A_{-1}+I)\int_0^\infty e^{-t}N'(t,f)dt.$$

Let us consider the linearized operator $A_{-1}-\mathcal{N}'(U_\delta)$ in X_{-1} with the domain

$$\{f\in X_{-1}\mid (A_{-1}-\mathcal{N}'(U_\delta))f\in X_{-1}\}.$$

It is not difficult to see that $\mathrm{Dom}(A_{-1}-\mathcal{N}'(U_\delta))=X$. Indeed, by Lemma 2.2 we can write

$$\mathcal{N}'(U_\delta)=C_\theta(-A_{-1}+I)^{1-\theta}\int_0^\infty t^{-\theta}e^{-t}N'(t,U_\delta)dt \tag{6.7}$$

with $\theta\in[0,1-\beta)$. Thus the interpolation inequality yields that $\mathcal{N}'(U_\delta)$ is relatively A_{-1}-bounded in X_{-1} with the bound 0, and hence, $\mathrm{Dom}(A_{-1}-\mathcal{N}'(U_\delta))=\mathrm{Dom}(A_{-1})=X$. Let L_δ be the part of $A_{-1}-\mathcal{N}'(U_\delta)$ in X, that is,

$$\begin{aligned} \mathrm{Dom}(L_\delta) &= \{f\in X\mid (A_{-1}-\mathcal{N}'(U_\delta))f\in X\}, && (6.8)\\ L_\delta f &:= (A_{-1}-\mathcal{N}'(U_\delta))f. \end{aligned}$$

Then by using Lemma 2.2 we can apply the perturbation theory of Desch-Schappacher to L_δ; see [6, III-3-3].

Lemma 6.1. *The operator L_δ above generates the strongly continuous semigroup e^{tL_δ} in X, and satisfies the equation*

$$e^{tL_\delta} = e^{tA} - \int_0^t e^{(t-s)A_{-1}} \mathcal{N}'(U_\delta) e^{sL_\delta} ds. \tag{6.9}$$

Moreover, the estimate $\|e^{tL_\delta} f\|_X \le C(t)\|f\|_X$ follows with a constant $C(t) \ge 1$ which is independent of δ such as $|\delta| \le \delta_0$.

Proof. From Lemma 2.2 we have for any $f(s) \in C([0,t_0);X)$,

$$\begin{aligned} \|\int_0^{t_0} e^{(t_0-s)A_{-1}} N'(U_\delta) f(s) ds\|_X &\le C|\delta|^\alpha \int_0^{t_0} (\frac{1+t_0-s}{t_0-s})^\beta e^{-\frac{\varrho}{2}(t_0-s)} \|f(s)\|_X ds \\ &\le C(t_0)|\delta_0|^\alpha \sup_{0<s<t_0} \|f(s)\|_X, \end{aligned}$$

where $0 < C(t_0) < 1$ if t_0 is sufficiently small. Hence from [6, Corollary III-3-3] the operator L_δ generates the strongly continuous semigroup in X which satis es (6.9). Note that the above t_0 can be chosen independent of δ with $|\delta| \le \delta_0$. Thus from the above inequality it is easy to get the estimate

$$\sup_{0<t<t_0} \|e^{tL_\delta} f\|_X \le C \sup_{0<t<t_0} \|e^{tA} f\|_X \le C\|f\|_X,$$

where C and t_0 are independent of δ with $|\delta| \le \delta_0$. By the semigroup property we have $\|e^{tL_\delta} f\|_X \le C(t)\|f\|_X$ with a constant $C(t)$ which is independent of δ with $|\delta| \le \delta_0$. This completes the proof.

6.1. **Spectral property of the linearized operator.** We rst investigate the spectral property of L_δ in X which is directly related with the time decay of solutions to (6.9). Note that the value $|\delta|$ is always assumed to be sufficiently small and U_δ satis es the estimate in Theorem 4.1. Let $\sigma(L_\delta)$ be the set of the spectrum of L_δ in X and we denote by $\sharp U$ the number of the elements of U.

Lemma 6.2. *Set $\mu_0 = 0$, $\mu_{n+1} = 1$. Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, **(A2)**, *and* **(N1)** - **(N3)** *hold. Suppose that $S(y;w_0)$ and $H(y_0,y;U_\delta)$ are C^1 near $y=0$ and $(y_0,y)=(0,0)$, respectively. If $|\delta|$ is sufficiently small then there is an $\eta(\delta)$ such that $\lim_{\delta\to 0}\eta(\delta) = 0$ and*

$$\sigma(L_\delta) \subset \{-\mu_j\}_{j=0}^{n+1} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\nu_0 + \eta(\delta)\}. \tag{6.10}$$

Here ν_0 is the number given by (2.40). When $\delta \ne 0$ the functions $\partial_\delta U_\delta$, BU_δ, and $D_1^{(j)} U_\delta$ with $j = 1, \cdots, n$ are linearly independent eigenfunctions of L_δ for the eigenvalues μ_0, $-\mu_{n+1}$, and $-\mu_j$, respectively. Moreover, if $\nu_0 > \mu_j$ and $|\delta|$ is sufficiently small, then $-\mu_j$ is a semisimple eigenvalue of L_δ with *multiplicity* $\sharp\{\mu_k|\ \mu_k = \mu_j\}$. *Especially,* $\mu_0(=0)$ *is a simple eigenvalue of* L_δ.

Remark 6.1. The assumption **(A2)** is not essential to prove BU_δ and $D_1^{(j)}U_\delta$ are eigenfunctions. Especially, Lemma 6.2 and its proof yield Lemma 2.3 by taking $\mathcal{N} = 0$.

Before proving Lemma 6.2 let us start from the next proposition by which we have the bound for the essential spectrum of L_δ in X.

Proposition 6.2. *It follows that*

$$\lim_{\delta\to 0} \mathrm{dist}(\sigma(e^A), \sigma(e^{L_\delta})) = 0. \tag{6.11}$$

Especially, there is $\eta'(\delta) \in \mathbb{R}$ *such that* $\lim_{\delta\to 0}\eta'(\delta) = 0$ *and*

$$r_{\mathrm{ess}}(e^{tL_\delta}) \le e^{-(\zeta-\eta'(\delta))t} \tag{6.12}$$

hold.

Proof. From Lemma 6.1 e^{tL_δ} is strongly continuous semigroup in X and

$$\|e^{tL_\delta} f\|_X \le C(t)\|f\|_X, \qquad t > 0,$$

holds where $C(t)$ is a constant independent of δ with $|\delta| \le \delta_0$. Set

$$\Lambda(t)f = e^{tL_\delta} f - e^{tA} f = -\int_0^t e^{(t-s)A_{-1}}\mathcal{N}'(U_\delta)e^{sL_\delta} f ds.$$

Then by Lemma 2.2 we have

$$\begin{aligned} \|\Lambda(t)f\|_X &\le C|\delta|^\alpha \int_0^t (t-s)^{-\beta}\|e^{sL_\delta} f\|_X ds \\ &\le C|\delta|^\alpha \int_0^t (t-s)^{-\beta} ds \|f\|_X \\ &\le C|\delta|^\alpha\|f\|_X. \end{aligned}$$

Here the above constant C does not depend on $\|f\|_X$ and δ with $|\delta| \le \delta_0$. Hence e^{L_δ} converges to e^A in $\mathcal{B}(X)$. So (6.11) follows from the general perturbation theory of linear operators; see [17, Remark IV-3-3]. Thus (6.12) holds by (6.11), **(A2)**, and the equality $\log r_{\mathrm{ess}}(e^{L_\delta}) = \frac{1}{t}\log r_{\mathrm{ess}}(e^{tL_\delta})$; see [6, Proposition IV-2-10]. This completes the proof.

Corollary 6.1. *For each* $\epsilon > 0$ *the set*

$$\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}\ (\mu) > -\zeta + \eta'(\delta) + \epsilon\}$$

consists of finite numbers of eigenvalues with finite algebraic multiplicities.

Proof. The assertion follows from [6, Corollary IV-2-11]. We omit the details here.

Remark 6.2. If $\Lambda(t) = e^{tL_\delta} - e^{tA}$ is compact, then the essential spectrum of e^{tL_δ} is the same as the one of e^{tA}. Especially, we have $\eta'(\delta) = 0$ in this case.

Proof of Lemma 6.2. Let us prove that $-\mu_j$ is an eigenvalue of L_δ. Since $H(y_0, y; U_\delta)$ is assumed to be C^1 with respect to (y_0, y), from Proposition 6.1, $R_{\frac{1}{1+y_{n+1}}} U_{\delta+y_0} \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D^{(k)}_{1+y_{n+1}}) \cap \mathrm{Dom}(\Gamma^{(l)}_{y_l,1})$ and $\tau^{(l)}_{y_l,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} U_{\delta+y_0} \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D^{(k)}_{1+y_{n+1}})$ for each $1 \leq k, l \leq n$ when $\sum_{j=0}^{n+1} |y_j| \ll 1$. We start from the equality

$$-AU_\delta + \mathcal{N}(U_\delta) = 0, \tag{6.13}$$

and by acting $\tau^{(j)}_{a,1}$ with $|a| \ll 1$ we have from $A = \mathcal{A} + B$ in $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B)$ and $[\mathcal{A}, \tau^{(j)}_{a,1+y_{n+1}}] = \Gamma^{(j)}_{a,1+y_{n+1}}$,

$$-\mathcal{A}\tau^{(j)}_{a,1} U_\delta + \Gamma^{(j)}_{a,1} U_\delta + [B, \tau^{(j)}_{a,1}] U_\delta - B\tau^{(j)}_{a,1} U_\delta + \tau^{(j)}_{a,1} \mathcal{N}(U_\delta) = 0.$$

Hence by **(T1)** and **(N3)** it follows that

$$A\tau^{(j)}_{a,1} U_\delta - \mathcal{N}(\tau^{(j)}_{a,1} U_\delta) = -a\mu_j D^{(j)}_1 \tau^{(j)}_{a,1} U_\delta,$$

that is,

$$A\frac{\tau^{(j)}_{a,1} U_\delta - U_\delta}{a} - \frac{\mathcal{N}(\tau^{(j)}_{a,1} U_\delta) - \mathcal{N}(U_\delta)}{a} = -\mu_j D^{(j)}_1 \tau^{(j)}_{a,1} U_\delta.$$

Since $U_\delta \in \mathrm{Dom}(D^{(j)}_1)$ we can take the limit $a \to 0$ in X_{-1} and obtain

$$\big(A_{-1} - \mathcal{N}'(U_\delta)\big) D^{(j)}_1 U_\delta = -\mu_j D^{(j)}_1 U_\delta. \tag{6.14}$$

This implies that $D^{(j)}_1 U_\delta \in \mathrm{Dom}(L_\delta)$. Since U_δ is not a trivial function, $D^{(j)}_1 U_\delta$ is not trivial either from **(T2)**. Hence $D^{(j)}_1 U_\delta$ is an eigenfunction for the eigenvalue $-\mu_j$ of L_δ.

Next we show BU_δ is an eigenfunction for the eigenvalue -1 of L_δ. Let $|\lambda - 1| \ll 1$. We act λR_λ on both sides of (6.13). Then from $\lambda R_\lambda \mathcal{A} = \mathcal{A} R_\lambda$ and **(N3)** we have

$$\mathcal{A} R_\lambda U_\delta + \lambda R_\lambda B U_\delta - \mathcal{N}(R_\lambda U_\delta) = 0,$$

i.e,

$$A\frac{R_\lambda U_\delta - U_\delta}{\lambda - 1} - \frac{\mathcal{N}(R_\lambda U_\delta) - \mathcal{N}(U_\delta)}{\lambda - 1} = -R_\lambda B U_\delta.$$

By taking the limit $\lambda \to 1$ in X_{-1}, we observe that $BU_\delta \in \mathrm{Dom}(L_\delta)$ and from **(T2)** it is an eigenfunction for the eigenvalue -1 of L_δ. Similarly we can easily see from (6.13) that $\partial_\delta U_\delta$ is an eigenvalue of the eigenvalue 0 of L_δ. From **(T2)** it is clear that $\partial_\delta U_\delta$, BU_δ, and $D^{(j)}_1 U_\delta$ with $j = 1, \cdots, n$ are linearly independent.

Since $\sigma(A) \subset \{-\mu_j\}_{j=0}^{n+1} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\nu_0\}$ by the de nition of ν_0 in (2.40), the continuity of $\sigma(e^{L_\delta})$ as in (6.11) yields (6.10) for some $\eta(\delta)$ satisfying

$$\lim_{\delta\to 0}\eta(\delta) = 0, \qquad -\zeta + \eta'(\delta) \le -\nu_0 + \eta(\delta), \tag{6.15}$$

where $\eta'(\delta)$ is the number in (6.12). This completes the proofs except for the last statement in the lemma.

Let us prove that if $\nu_0 > \mu_j$ then $-\mu_j$ is a semisimple eigenvalue of L_δ with multiplicity $\sharp\{\mu_k \mid \mu_k = \mu_j\}$. For this purpose we rst observe that $-\mu_j$ is a semisimple eigenvalue of A, and hence, of A_{-1}, with multiplicity $\sharp\{\mu_k \mid \mu_k = \mu_j\}$ when $\nu_0 > \mu_j$. Indeed, since $\nu_0 > \mu_j$ the space $\mathrm{Ker}(A+\mu_j I)$ is spanned by $\{e_{0,k} \mid \mu_k = \mu_j\}$, otherwise there is an eigenfunction in $\mathbf{Q}_0 X$ (see (2.39) for the de nition of $\mathbf{Q}_0$) of the eigenvalue $-\mu_j$ to $\mathbf{Q}_0 A\mathbf{Q}_0$, which contradicts with $\nu_0 > \mu_j$ by (2.40) and the spectral mapping theorem. Hence the geometric multiplicity of the eigenvalue $-\mu_j$ to A is $\sharp\{\mu_k \mid \mu_k = \mu_j\}$. Assume that there is a nontrivial function $f \in \mathrm{Ker}(A + \mu_j I)^2$ such that $f \notin \mathrm{Ker}(A + \mu_j I)$. Then since $(A + \mu_j I)f \in \mathrm{Ker}(A + \mu_j I)$ we have

$$(A + \mu_j I)f = \sum_{\mu_k=\mu_j} a_k e_{0,k},$$

for some $a_k \in \mathbb{C}$, which yields

$$(\mathbf{Q}_0 A\mathbf{Q}_0 + \mu_j I)\mathbf{Q}_0 f = 0.$$

Since $f \notin \mathrm{Ker}(A+\mu_j I)$ we have $\mathbf{Q}_0 f \ne 0$. Hence $-\mu_j$ must be an eigenvalue of $\mathbf{Q}_0 A\mathbf{Q}_0$, which again contradicts with $\nu_0 > \mu_j$ by (2.40). Thus we have $\mathrm{Ker}(A + \mu_j I)^2 = \mathrm{Ker}(A + \mu_j I)$. On the other hand, since the rank of the eigenprojection around the eigenvalue $-\mu_j$ of A is nite by (**A2**) and [6, Corollary IV-2-11], $-\mu_j$ must be a pole of the resolvent of A; see [6, Section IV-1-17]. Thus by [19, Remark A.2.4], $-\mu_j$ is a semisimple eigenvalue of A.

Finally we prove that the eigenvalue $-\mu_j$ of L_δ is semisimple if $\nu_0 > \mu_j$ and $|\delta|$ is sufficiently small. From Corollary 6.1 $-\mu_j$ is an isolated eigenvalue with nite algebraic multiplicity. So it suffices to show $\mathrm{Ker}(L_\delta + \mu_j I)^2 = \mathrm{Ker}(L_\delta + \mu_j I)$ as above. We note that, since L_δ is a part of $A_{-1} - \mathcal{N}'(U_\delta)$ in X, the eigenvalues of L_δ are also eigenvalues of $A_{-1} - \mathcal{N}'(U_\delta)$ to which the general perturbation theory for linear operators is easily applied. For the operator $A_{-1} - \mathcal{N}'(U_\delta)$ the rank of the eigenprojection around $-\mu_j$ is invariant if $|\delta|$ is sufficiently small, so it is $\sharp\{\mu_k \mid \mu_k = \mu_j\} < \infty$ and $-\mu_j$ is a pole of the resolvent of $A_{-1} - \mathcal{N}'(U_\delta)$ by [6, Section IV-1-17]. On the other hand, the geometric multiplicity of the eigenvalue $-\mu_j$ to $A_{-1} - \mathcal{N}'(U_\delta)$ is large than or equal to $\sharp\{\mu_k \mid \mu_k = \mu_j\}$, for we have found cooresponding eigenfunctions. Hence the rank of the eigenprojection around $-\mu_j$ coincides with the geometric multiplicity of $-\mu_j$. This implies $\mathrm{Ker}(L_\delta + \mu_j I)^2 =$

$\mathrm{Ker}(L_\delta + \mu_j I)$ since L_δ is a part of $A_{-1} - \mathcal{N}'(U_\delta)$ in X. This completes the proof of the lemma.

Proposition 6.3. *The map* $N_\delta(t,\cdot) = e^{tL_\delta}\mathcal{N} : \mathrm{Dom}(A) \to X$ *can be extended as a* $C^{1+\alpha}$ *map on* X *and satisfies the estimate*

$$\|N_\delta'(t,f)h - N_\delta'(t,g)h\|_X \le C(\frac{1+t}{t})^\beta e^{-\frac{\varrho}{2}t}\|f-g\|_X^\alpha \|h\|_X, \tag{6.16}$$

for any $f, g, h \in X$. *Here* $N_\delta'(t,f)$ *is the Fréchet derivative of* $N_\delta(t,\cdot)$ *at* f. *Especially,* $\mathcal{N}$ *can be extended as a* $C^{1+\alpha}$ *map from* X *into the negative order space* $\mathrm{Dom}(L_{\delta,-1})$ *with* $\|\cdot\|_{\mathrm{Dom}(L_{\delta,-1})} = \|(-L_\delta + I)^{-1}\cdot\|_X$, *and the relation*

$$e^{tL_{\delta,-1}}\mathcal{N}'(f) = N_\delta'(t,f), \tag{6.17}$$

holds for any $f \in X$.

*Proof.*The assertion that $\mathcal{N}$ is extended as a $C^{1+\alpha}$ map from X into $\mathrm{Dom}(D_{\delta,-1})$ is shown as in Lemma 2.2. Let $f, g \in \mathrm{Dom}(A)$. Then from Lemma 6.1 we have

$$\begin{aligned} & e^{tL_\delta}\mathcal{N}'(f)h - e^{tL_\delta}\mathcal{N}'(g)h \\ = \ & e^{tA}\mathcal{N}'(f)h - e^{tA}\mathcal{N}'(g)h - \int_0^t e^{(t-s)A_{-1}}\mathcal{N}'(U_\delta)\big(e^{sL_\delta}\mathcal{N}'(f)h - e^{sL_\delta}\mathcal{N}'(g)h\big)ds. \end{aligned}$$

Then using (2.31), we get

$$\begin{aligned} & \|e^{tL_\delta}\mathcal{N}'(f)h - e^{tL_\delta}\mathcal{N}'(g)h\|_X \\ \le \ & \|e^{tA}\mathcal{N}'(f)h - e^{tA}\mathcal{N}'(g)h\|_X \\ & + C\int_0^t (\frac{1+t-s}{t-s})^\beta e^{-\frac{3\varrho}{4}(t-s)}\|U_\delta\|_X^\alpha \|e^{sL_\delta}\mathcal{N}'(f)h - e^{sL_\delta}\mathcal{N}'(g)h\|_X ds \\ \le \ & \|e^{tA}\mathcal{N}'(f)h - e^{tA}\mathcal{N}'(g)h\|_X \\ & + C|\delta|^\alpha \int_0^t (\frac{(1+t-s)(1+s)}{(t-s)s})^\beta e^{-\frac{3\varrho}{4}(t-s)}e^{-\frac{\varrho}{2}s}ds \\ & \qquad \times \sup_{t>0}(\frac{t}{1+t})^\beta e^{\frac{\varrho}{2}t}\|e^{tL_\delta}\mathcal{N}'(f)h - e^{tL_\delta}\mathcal{N}'(g)h\|_X, \end{aligned}$$

which implies from (2.31) that

$$\begin{aligned} & \sup_{t>0}(\frac{t}{1+t})^\beta e^{\frac{\varrho}{2}t}\|e^{tL_\delta}\mathcal{N}'(f)h - e^{tL_\delta}\mathcal{N}'(g)h\|_X \\ \le \ & C\sup_{t>0}(\frac{t}{1+t})^\beta e^{\frac{\varrho}{2}t}\|e^{tA}\mathcal{N}'(f)h - e^{tA}\mathcal{N}'(g)h\|_X \le C\|f-g\|_X^\alpha\|h\|_X. \end{aligned}$$

The relation (6.17) follows from the argument just as in the proof of (2.30) in Lemma 2.2. This completes the proof.

From Lemma 6.1 and Proposition 6.3 we convert (6.6) to the integral equation

$$v(t) = e^{tL_\delta} v_0 + \int_0^t e^{(t-s)L_\delta} \big\{ T_\delta v(s) + F_\delta(v)(s) + J(V_\delta(s)) \big\} ds. \tag{6.18}$$

Here $e^{(t-s)L_\delta} T_\delta v(s)$ and $e^{(t-s)L_\delta} F_\delta(v)(s)$ are interpreted as

$$\begin{aligned} e^{(t-s)L_\delta} T_\delta v(s) &= \big\{ N'_\delta(t-s, U_\delta) - N'_\delta(t-s, V_\delta(s)) \big\} v(s), \\ e^{(t-s)L_\delta} F_\delta(v)(s) &= -\int_0^1 \big\{ N'_\delta(t-s, V_\delta(s) + \tau v(s)) - N'_\delta(t-s, V_\delta(s)) \big\} v(s) d\tau. \end{aligned}$$

6.2. **Determination of $\mathbf{y}(t)$.** In this section we nd suitable parameters $\mathbf{y}(t) = (y_0(t), \cdots, y_{n+1}(t))^\top$ such that $\|u(t) - H(\mathbf{y}(t); U_\delta)\|_X$ decays faster than $\|u(t) - U_\delta\|_X$. For this purpose the next lemma for $J(V_\delta)(t) = -\partial_t V_\delta(t) + A_{-1} V_\delta(t) - \mathcal{N}(V_\delta(t))$ with $V_\delta(t) = H(\mathbf{y}(t); U_\delta)$ is important.

Lemma 6.3. *Assume that $H(\mathbf{y}; U_\delta)$ is C^1 near $\mathbf{y} = 0$. Let $\mathbf{y}(t) \in C^1((0,\infty); \mathbb{R}^{n+2})$ with $\sup_{t>0} |\mathbf{y}(t)| \ll 1$ be given. Set $\mu_0 = 0$ and $\mu_{n+1} = 1$, and let μ_j be the number in Lemma 6.2 for $j = 1, \cdots, n$. Then for any stationary solution U_δ of (3.4) with $|\delta| \ll 1$, we have*

$$J(V_\delta)(t) = -\sum_{j=0}^{n+1} \partial_{y_j} H(\mathbf{y}(t); U_\delta) \cdot \big(y'_j(t) + \mu_j y_j(t) \big). \tag{6.19}$$

Proof. From Proposition 6.1 we have $H(\mathbf{y}; U_\delta) \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B)$ and $\tau^{(j+1)}_{y_{j+1}, 1+y_{n+1}} \cdots \tau^{(n)}_{y_n, 1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} U_{\delta+y_0} \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(\Gamma^{(j)}_{y_j, 1+y_{n+1}})$ for each $j = 1, \cdots, n-1$. We start from the equality

$$\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) = \mathcal{A} H(\mathbf{y}; U_\delta) - \mathcal{N}(H(\mathbf{y}; U_\delta)), \tag{6.20}$$

which follows by regarding y_{n+1} as a time variable. Indeed, since $U_{\delta+y_0}$ is a stationary solution to (3.4), $R_{\frac{1}{t}} U_{\delta+y_0}$ is a self-similar solution to (E) from Lemma 3.2. Especially, $R_{\frac{1}{\frac{1}{2}+t}} U_{\delta+y_0}$ is a mild solution to (E) with initial data $R_2 U_{\delta+y_0}$. Then from Corollary 3.1 we see $\tau^{(n)}_{y_n, \frac{1}{2}+t} R_{\frac{1}{\frac{1}{2}+t}} U_{\delta+y_0}$ is a mild solution to (E) with initial data $\tau^{(n)}_{y_n, \frac{1}{2}} R_2 U_{\delta+y_0}$. Thus again by from Corollary 3.1 we have $\tau^{(n-1)}_{y_{n-1}, \frac{1}{2}+t} \tau^{(n)}_{y_n, \frac{1}{2}+t} R_{\frac{1}{\frac{1}{2}+t}} U_{\delta+y_0}$ is a mild solution to (E) with initial data $\tau^{(n-1)}_{y_{n-1}, \frac{1}{2}} \tau^{(n)}_{y_n, \frac{1}{2}} R_2 U_{\delta+y_0}$. Repeating this, we observe that $\tau^{(1)}_{y_1, \frac{1}{2}+t} \cdots \tau^{(n)}_{y_n, \frac{1}{2}+t} R_{\frac{1}{\frac{1}{2}+t}} U_{\delta+y_0}$ is a mild solution to (E) with initial data $\tau^{(1)}_{y_1, \frac{1}{2}} \cdots \tau^{(n)}_{y_n, \frac{1}{2}} R_2 U_{\delta+y_0}$. Hence, by setting $t = y_{n+1} + \frac{1}{2}$ we get (6.20). On the other hand, the direct calculation yields that

(6.21)

$$\begin{aligned}
&\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) \\
= &\sum_{j=1}^{n} \tau^{(1)}_{y_1,1+y_{n+1}} \cdots \tau^{(j-1)}_{y_{j-1},1+y_{n+1}} \Gamma^{(j)}_{y_j,1+y_{n+1}} \tau^{(j+1)}_{y_{j+1},1+y_{n+1}} \cdots \tau^{(n)}_{y_n,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} U_\delta \\
&-\frac{1}{1+y_{n+1}} \tau^{(1)}_{y_1,1+y_{n+1}} \cdots \tau^{(n)}_{y_n,1+y_{n+1}} B R_{\frac{1}{1+y_{n+1}}} U_\delta.
\end{aligned}$$

Hence

$$\begin{aligned}
&\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) + \frac{1}{1+y_{n+1}} B H(\mathbf{y}; U_\delta) \\
= &\sum_{j=1}^{n} \tau^{(1)}_{y_1,1+y_{n+1}} \cdots \tau^{(j-1)}_{y_{j-1},1+y_{n+1}} \Gamma^{(j)}_{y_j,1+y_{n+1}} \tau^{(j+1)}_{y_{j+1},1+y_{n+1}} \cdots \tau^{(n)}_{y_n,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} U_\delta \\
+ &\frac{1}{1+y_{n+1}} \sum_{j=1}^{n} \tau^{(1)}_{y_1,1+y_{n+1}} \cdots \tau^{(j-1)}_{y_{j-1},1+y_{n+1}} [B, \tau^{(j)}_{y_j,1+y_{n+1}}] \tau^{(j+1)}_{y_{j+1},1+y_{n+1}} \cdots \\
&\cdots \tau^{(n)}_{y_n,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} U_\delta.
\end{aligned}$$

By **(T1)** we get

$$\begin{aligned}
&(1+y_{n+1})\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) + B H(\mathbf{y}; U_\delta) \\
= &-\sum_{j=1}^{n} \tau^{(1)}_{y_1,1+y_{n+1}} \cdots \tau^{(j-1)}_{y_{j-1},1+y_{n+1}} y_j \mu_j D^{(j)}_{y_j,1+y_{n+1}} \tau^{(j)}_{y_j,1+y_{n+1}} \cdots \tau^{(n)}_{y_n,1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} U_\delta \\
= &-\sum_{j=1}^{n} \mu_j y_j D^{(j)}_{y_j,1+y_{n+1}} H(\mathbf{y}; U_\delta) = -\sum_{j=1}^{n} \mu_j y_j \partial_{y_j} H(\mathbf{y}; U_\delta).
\end{aligned}$$

Here we used the independent property of $\{\mathcal{T}^{(j)}_\theta\}$ assumed in **(E2)**. Combining this with (6.20), we have

$$\begin{aligned}
&A H(\mathbf{y}; U_\delta) - \mathcal{N}(H(\mathbf{y}; U_\delta)) \\
= &\mathcal{A} H(\mathbf{y}; U_\delta) - \mathcal{N}(H(\mathbf{y}; U_\delta)) + B H(\mathbf{y}; U_\delta) \\
= &\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) - (1+y_{n+1})\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) - \sum_{j=1}^{n} \mu_j y_j \partial_{y_j} H(\mathbf{y}; U_\delta) \\
= &-y_{n+1}\partial_{y_{n+1}} H(\mathbf{y}; U_\delta) - \sum_{j=1}^{n} \mu_j y_j \partial_{y_j} H(\mathbf{y}; U_\delta).
\end{aligned}$$

Now the desired relation easily follows. This completes the proof.

Now let us derive the ODEs which determines $\mathbf{y}(t)$. Below we assume that $\delta \neq 0$. As in Lemma 6.3 we put $\mu_0 = 0$ and $\mu_{n+1} = 1$. Recalling Lemma 6.2, we set

$$e_{\delta,0} = c_{\delta,0}\partial_\delta U_\delta, \tag{6.22}$$

$$e_{\delta,j} = c_{\delta,j} D_1^{(j)} U_\delta, \quad 1 \leq j \leq n, \tag{6.23}$$

$$e_{\delta,n+1} = c_{\delta,n+1} B U_\delta. \tag{6.24}$$

Here each $c_{\delta,j}$ is taken so that $\|e_{\delta,j}\|_X = 1$. Let us introduce the eigenprojections $\mathbf{P}_{\delta,j},\ j = 0, \cdots, n+1$ by

$$\mathbf{P}_{\delta,j} f =< f, e^*_{\delta,j} > e_{\delta,j}, \tag{6.25}$$

Here the functions $\{e^*_{\delta,j}\}_{j=0}^{n+1}$ satisfy the relation

$$< e_{\delta,l}, e^*_{\delta,j} >= \delta_{jl}, \tag{6.26}$$

and $e^*_{\delta,0}$ is the eigenfunction of the adjoint operator L^*_δ for the simple eigenvalue 0, $e^*_{\delta,j}$ is one of functions which form the generalized eigenspace for the eigenvalue $-\mu_j$ of L^*_δ when $j \geq 1$. Note that if $-\mu_j$ is a semisimple eigenvalue of L_δ then $e^*_{\delta,j}$ is also an eigenfunction of L^*_δ for the eigenvalue $-\mu_j$.

Then we introduce the projection

$$\mathbf{P}_\delta = \sum_{j=0}^{n+1} \mathbf{P}_{\delta,j}, \qquad \mathbf{Q}_\delta = I - \mathbf{P}_\delta. \tag{6.27}$$

Then from Lemma 6.2 and Proposition 6.3 we have

Lemma 6.4. *Let ν_0 and $\eta(\delta)$ be the numbers in Lemma 6.2. Let $N_\delta(t,f) = e^{tL_\delta}\mathcal{N}(f)$ be the nonlinear operator in Proposition 6.3. For any $\epsilon > 0$ set $\nu_{\epsilon,\delta} = \nu_0 - \eta(\delta) - \epsilon$. Then there is a positive constant C_ϵ such that*

$$\|\mathbf{Q}_\delta e^{tL_\delta} f\|_X \leq C_\epsilon e^{-\nu_{\epsilon,\delta} t} \|\mathbf{Q}_\delta f\|_X, \tag{6.28}$$

$$\|\mathbf{Q}_\delta \{N'_\delta(t,f) - N'_\delta(t,g)\} h\|_X \leq C_\epsilon \left(\frac{1+t}{t}\right)^\beta e^{-\nu_{\epsilon,\delta} t} \|f-g\|_X^\alpha \|h\|_X, \tag{6.29}$$

hold for any f, h and $g \in X$.

Proof. We note that, although $\mathbf{Q}_\delta e^{tL_\delta} = \mathbf{Q}_\delta e^{tL_\delta}\mathbf{Q}_\delta$ holds, $\mathbf{Q}_\delta X$ is not invariant under the action of e^{tL_δ} in general, for each $-\mu_j$ is not assumed to be a semisimple eigenvalue of L_δ in this lemma.

From the semigroup property $\mathbf{Q}_\delta e^{(t+s)L_\delta} = \mathbf{Q}_\delta e^{tL_\delta}\mathbf{Q}_\delta e^{sL_\delta}$ the growth bound of $\mathbf{Q}_\delta e^{tL_\delta}$ is given by the spectral radious of $\mathbf{Q}_\delta e^{tL_\delta}$, which is denoted by $r(\mathbf{Q}_\delta e^{tL_\delta})$. Hence it suffices to show that $r(\mathbf{Q}_\delta e^{tL_\delta})$ is less than or equal to $e^{-t\nu_{0,\delta}}$. To prove this we recall that if $\nu_0 > \mu_j$ then $-\mu_j$ is a semisimple eigenvalue of L_δ with multiplicity $\sharp\{\mu_k \mid \mu_k = \mu_j\}$ by Lemma 6.2. Especially, for such μ_j the corresponding eigenspace is spanned by $\{e_{\delta,k} \mid \mu_k = \mu_j\}$ and it must be included in $\mathbf{P}_\delta X$. By considering the specral projection $\mathbf{P}'_\delta$ on the eigenspace for the eigenvalues $\{-\mu_j \mid \nu_0 > \mu_j\}$, we may consider the problem in $\mathbf{Q}'_\delta X$ where $\mathbf{Q}'_\delta = I - \mathbf{P}'_\delta$ since $\mathbf{Q}'_\delta X$ is invariant under the action of e^{tL_δ} and $\mathbf{Q}_\delta X \subset \mathbf{Q}'_\delta X$. Then from Lemma 6.2 and the spectral mapping theorem for eigenvalues, we have

$$\begin{aligned} r(\mathbf{Q}_\delta e^{tL_\delta}) &\leq r(\mathbf{Q}'_\delta e^{tL_\delta}) \\ &\leq \max\{r_{ess}(\mathbf{Q}'_\delta e^{tL_\delta}), e^{-\min\{\mu_j \mid \nu_0 \leq \mu_j\}t}, e^{-\nu_{0,\delta}t}\} \\ &= \max\{r_{ess}(e^{tL_\delta}), e^{-\min\{\mu_j \mid \nu_0 \leq \mu_j\}t}, e^{-\nu_{0,\delta}t}\}. \end{aligned} \tag{6.30}$$

Assume that there is no μ_j satisfying $\nu_0 \leq \mu_j$. By (6.15) we have

$$r_{ess}(e^{tL_\delta}) \leq e^{-(\zeta - \eta'(\delta))t} \leq e^{-(\nu_0 - \eta(\delta))t}.$$

Hence $r(\mathbf{Q}_\delta e^{tL_\delta}) \leq e^{-\nu_{0,\delta}t}$ in this case by (6.30), which gives (6.28).

Next we assume that there is a μ_j such that $\nu_0 \leq \mu_j$. Then $\zeta > \mu^* \geq \nu_0$ by **(A2)**, and so we may assume that $|\eta'(\delta)| + |\eta(\delta)| < \zeta - \mu^*$. Hence we have

$$r_{ess}(e^{tL_\delta}) \leq e^{-(\zeta - \eta'(\delta))t} < e^{-(\nu_0 - \eta(\delta))t}, \tag{6.31}$$

uniformly in $|\delta| \ll 1$ in this case. This implies that $\eta(\delta)$ is determined by the behavior of $\sigma(L_\delta)$ around $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) = -\nu_0\}$ which consists of finite number of eigenvalues with finite multiplicity. If there is no μ_j satisfying $\mu_j = \nu_0$ then we have $r(\mathbf{Q}_\delta e^{tL_\delta}) \leq e^{-\nu_{0,\delta}t}$ from (6.30) and (6.31), and thus (6.28) follows. Assume that there is a μ_j such that $\mu_j = \nu_0$. In this case we need to consider how $\eta(\delta)$ is determined. Since the eigenvalues of L_δ are continuously depending on δ by (6.11), $\eta(\delta)$ is taken as the maximum of the following two quantities l_1, l_2 when $|\delta| \ll 1$:

$$l_1 = \begin{cases} \max\{\mathrm{Re}(\mu(\delta)) - \nu_0 \mid \mu(\delta) \in \sigma(L_\delta)\backslash\{-\nu_0\} \text{ is a bifurcation from} \\ \qquad \text{some eigenvalue } \mu(0) \text{ of } A \text{ with } \mathrm{Re}(\mu(0)) = -\nu_0\}, \\ -\infty \quad \text{if the set } \{\mu(\delta) \in \sigma(L_\delta)\backslash\{-\nu_0\} \mid \mu(\delta) \text{ is a bifurcation from} \\ \qquad \text{some eigenvalue } \mu(0) \text{ of } A \text{ with } \mathrm{Re}(\mu(0)) = -\nu_0\} \text{ is empty,} \end{cases}$$

$$l_2 = \begin{cases} 0 \quad \text{if the rank of the total eigenprojection for the eigenvalue } -\nu_0 \\ \qquad \text{of } L_\delta \text{ is strictly larger than } \sharp\{\mu_k \mid \mu_k = \nu_0\}, \\ -\infty \text{ otherwise.} \end{cases}$$

We note that the value of $\max\{l_1, l_2\}$ can not be $-\infty$; otherwise we must have $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) = -\nu_0\} = \{-\nu_0\}$ from $l_1 = -\infty$ and its algebraic multiplicity is $\sharp\{\mu_k \mid \mu_k = \nu_0\}$ from $l_2 = -\infty$. Thus the eigenspace of the eigenvalues $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) = -\nu_0\}$ is just spanned by $\{e_{0,k} \mid \mu_k = \nu_0\}$, which contradicts with the definition of ν_0 in (2.40).

From the de nitions of l_1 and l_2 the inequality $r(\mathbf{Q}_\delta e^{tL_\delta}) \le e^{-\nu_{0,\delta} t}$ holds also in this case. Indeed, if $\eta(\delta)$ is negative then $l_2 = -\infty$ and thus $-\nu_0$ is a semisimple eigenvalue of L_δ whose eigenspace is spanned by $\{e_{\delta,k} \mid \mu_k = \nu_0\}$. So from (6.31) we have instead of (6.30),

$$\begin{aligned} r(\mathbf{Q}_\delta e^{tL_\delta}) &\le \max\{r_{ess}(e^{tL_\delta}), e^{-\min\{\mu_j \mid \nu_0<\mu_j\}}, e^{-\nu_{0,\delta} t}\} \\ &= e^{-\nu_{0,\delta} t}, \end{aligned}$$

for $|\delta| \ll 1$. When $\eta(\delta) \ge 0$ the desired inclusion clearly holds from (6.30) and (6.31). Hence we get (6.28).

The estimate (6.29) for $t \in (0,1]$ directly follows from (6.16). For $t > 1$ we rst note the relation

$$N'_\delta(t+t_0, f) = e^{(t+t_0)L_{\delta,-1}} \mathcal{N}(f) = e^{tL_{\delta,-1}} N'_\delta(t_0, f) = e^{tL_\delta} N'_\delta(t_0, f),$$

for $0 < t_0 < \frac{1}{2}$, where we used (6.17). Then we get (6.29) from (6.28) and (6.16). This completes the proof.

Let $\eta' > 0$ be a sufficiently small number satisfying $\eta' > \eta'(\delta)$, where $\eta'(\delta)$ is the number in Corollary 6.1. Let $\mathbf{E}_\delta$ be the total eigenprojection for the eigenvalues $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) > -\zeta + \eta'\}$. Then from (6.18) we have (6.32)

$$\mathbf{E}_\delta v(t) = e^{(t-T)L_\delta} \mathbf{E}_\delta v(T) + \int_T^t e^{(t-s)L_\delta} \mathbf{E}_\delta \big\{ T_\delta v(s) + F_\delta(v)(s) + J(V_\delta(s)) \big\} ds.$$

Since $\mathbf{E}_\delta \big\{ T_\delta v(t) + F_\delta(v)(t) + J(V_\delta(t)) \big\} \in C([0,\infty); X)$, $\mathbf{E}_\delta v(t)$ is di erentiable in X with respect to t and we obtain

$$\frac{d}{dt} \mathbf{E}_\delta v(t) - L_\delta \mathbf{E}_\delta v(t) = \mathbf{E}_\delta \big\{ T_\delta v(t) + F_\delta(v)(t) + J(V_\delta(t)) \big\}. \tag{6.33}$$

Our aim is to construct $(n+2)$ parameters $\mathbf{y}(t)$ so that $v(t)$ belongs to $\mathbf{Q}_\delta X$ for all $t \ge T$ with large T. From now on we assume that $\{-\mu_j\}_{j=1}^{n+1}$ are semisimple eigenvalues of $A - \mathcal{N}'(U_\delta)$. Then we have $\mathbf{P}_\delta L_\delta \mathbf{Q}_\delta = 0$. By the fact that $\mathbf{P}_\delta \mathbf{E}_\delta = \mathbf{E}_\delta \mathbf{P}_\delta = \mathbf{P}_\delta$, from (6.33) the requirement $\mathbf{P}_\delta v(t) = 0$ leads to the equation

$$-\mathbf{P}_\delta J(V_\delta) = \mathbf{P}_\delta T_\delta v + \mathbf{P}_\delta F_\delta(v), \qquad t > T, \tag{6.34}$$

and the initial condition at the initial time $T \ge 0$,

$$\mathbf{P}_\delta u(T) = \mathbf{P}_\delta H(\mathbf{y}(T); U_\delta). \tag{6.35}$$

Eq.(6.34) and (6.35) are equivalent with the ODE system as follows. For $0 \le j,\ l \le n+1$, let $k_{j,l}(t)$, $\mathcal{T}_{\delta,j}(t)$ and $\mathcal{F}_{\delta,j}(t)$ be functions de ned by

$$\begin{aligned} k_{j,l}(t) &=< \partial_{y_l} H(\mathbf{y}(t); U_\delta) - \partial_{y_l} H(0; U_\delta), e^*_{\delta,j} >, \\ \mathcal{T}_{\delta,j}(t) &=< T_\delta v, e^*_{\delta,j} >, \\ \mathcal{F}_{\delta,j}(t) &=< F_\delta(v), e^*_{\delta,j} > . \end{aligned}$$

As in Lemma 6.3 we set $\mu_0 = 0$ and $\mu_{n+1} = 1$. Let $\tilde{\Pi} = (\tilde{d}_{ij})_{1\le i,j\le n}$ be an $n \times n$ matrix whose component $\tilde{d}_{ij}$ is given by

$$\tilde{d}_{ij} = \mu_j \delta_{ij}, \qquad 1 \le i, j \le n. \tag{6.36}$$

Set $k_j(t) = (k_{j,1}(t), \cdots, k_{j,n}(t))^\top$. From (6.34), (6.3)-(6.5), and Lemma 6.3, we get the equation for $y_0(t)$ such as

$$\begin{aligned} (1 + k_{0,0}(t))y_0' &= -k_0(t) \cdot (\tilde{y}' + \tilde{\Pi}_\mu \tilde{y}) - k_{0,n+1}(t)(y_{n+1}' + y_{n+1}) \\ &\quad + \mathcal{T}_{\delta,0}(t) + \mathcal{F}_{\delta,0}(t). \end{aligned} \tag{6.37}$$

Similarly we have for $\tilde{y}(t)$,

$$\begin{aligned} (I + \tilde{\mathcal{K}}(t))(\tilde{y}' + \tilde{\Pi}\tilde{y}) &= -\tilde{k}^{(1)}(t)y_0' - \tilde{k}^{(n)}(t)(y_{n+1}' + y_{n+1}) \\ &\quad + \tilde{\mathcal{T}}_\delta(t) + \tilde{\mathcal{F}}_\delta(t), \end{aligned} \tag{6.38}$$

where $\tilde{\mathcal{T}}_\delta(t) = (\mathcal{T}_{\delta,1}(t), \cdots, \mathcal{T}_{\delta,n}(t))^\top$, $\tilde{\mathcal{F}}_\delta(t) = (\mathcal{F}_{\delta,1}(t), \cdots, \mathcal{F}_{\delta,n}(t))^\top$, and $\tilde{\mathcal{K}}(t) = (k_1(t), \cdots, k_n(t))^\top$ is an $n \times n$ matrix with the vector $k_j(t)$. The vectors $\tilde{k}^{(1)}(t)$ and $\tilde{k}^{(n)}(t)$ are 1-st and n-th columns of the transposed matrix of $\tilde{\mathcal{K}}$.

Finally we have for $y_{n+1}(t)$,

$$\begin{aligned} (1 + k_{n+1,n+1}(t))(y_{n+1}' + y_{n+1}) &= -k_{n+1,0}(t)y_0' - k_{n+1}(t) \cdot (\tilde{y}' + \tilde{\Pi}_\mu \tilde{y}) \\ &\quad + \mathcal{T}_{\delta,n+1}(t) + \mathcal{F}_{\delta,n+1}(t). \end{aligned} \tag{6.39}$$

We consider the ODE system for $\mathbf{y}(t)$ with the initial time $T \ge 0$ under the initial condition

$$< u(T), e^*_{\delta,j} >=< H(\mathbf{y}(T); U_\delta), e^*_{\delta,j} >, \tag{6.40}$$

for each $j = 0, \cdots, n+1$. We set

$$\begin{aligned} \mathcal{T}_\delta(t) &= (\mathcal{T}_{\delta,0}(t), \tilde{\mathcal{T}}(t), \mathcal{T}_{\delta,n+1}(t))^\top, \\ \mathcal{F}_\delta(t) &= (\mathcal{F}_{\delta,0}(t), \tilde{\mathcal{F}}_\delta(t), \mathcal{F}_{\delta,n+1}(t))^\top. \end{aligned}$$

Then the system (6.37)-(6.39) can be written in the form

$$(I + K(t))(\mathbf{y}'(t) + \Pi\mathbf{y}(t)) = \mathcal{T}_\delta(t) + \mathcal{F}_\delta(t),$$

where $K(t)$ is an $(n+2)\times(n+2)$ matrix whose components are given by linear combinations of $k_{j,l}(t)$, and $\Pi = (d_{ij})_{0\le i,j\le n+1}$ is an $(n+2)\times(n+2)$ matrix whose component d_{ij} is given by

$$d_{ij} = \mu_j \delta_{ij}, \qquad 0 \le i, j \le n+1. \tag{6.41}$$

By the de nitions of $\{k_{j,l}(t)\}_{0\le j,l\le n+1}$, it is not difficult to prove

Proposition 6.4. *Assume that $H(\mathbf{y}; U_\delta)$ is $C^{1+\gamma}$ in X near $\mathbf{y} = 0$. Let $k_{j,l}(t)$ be given in (6.37). Then*

$$\sum_{j=0}^{n+1}\sum_{l=0}^{n+1} |k_{j,l}(t)| \ \le\ C|\mathbf{y}(t)|^\gamma, \tag{6.42}$$

where C does not depend on $|\delta| \ll 1$ and $t > 0$. Especially, we have

$$\|K(t)\|_{\mathbb{R}^{n+2}\to\mathbb{R}^{n+2}} \le C_0|\mathbf{y}(t)|^\gamma, \tag{6.43}$$

for $|\mathbf{y}(t)| \le 1$, where C_0 is independent of t and δ.

We expect that $|\mathbf{y}(t)|$ is sufficiently small. If this is true, the inverse of $I + K(t)$ exists. Then we have the equation

$$\mathbf{y}'(t) + \Pi\mathbf{y}(t) = (I + K(t))^{-1}(\mathcal{T}_\delta(t) + \mathcal{F}_\delta(t)) =: \mathbf{W}(t, \mathbf{y}(t)). \tag{6.44}$$

Let us write

$$\mathbf{W}(t, \mathbf{y}(t)) = (W_0(t, \mathbf{y}(t)), W(t, \mathbf{y}(t)))^\top,$$

where

$$W(t, \mathbf{y}(t)) = (W_1(t, \mathbf{y}(t)), \cdots, W_{n+1}(t, \mathbf{y}(t)))^\top.$$

The representation (6.44) is useful since the right-hand side of (6.44) does not depend on the time derivative of $\mathbf{y}(t)$. In order to solve (6.44) we derive some estimates of $\mathbf{W}(t, \mathbf{y}(t))$.

Proposition 6.5. *Let T_δ and F_δ be given by (6.6). Assume that $H(\mathbf{y}; U_\delta)$ is C^1 in X near $\mathbf{y} = 0$. Then for any $f \in X$ we have*

$$\|(-L_\delta + I)^{-1}T_\delta f\|_X \le C|\mathbf{y}(t)|^\alpha \|f\|_X, \tag{6.45}$$

$$\|(-L_\delta + I)^{-1}F_\delta(f)\|_X \le C\|f\|_X^{1+\alpha}. \tag{6.46}$$

Here C is independent of $|\delta| \ll 1$ and $t > 0$. As a consequence, we have

$$|\mathcal{T}_\delta(t)| \le C|\mathbf{y}(t)|^\alpha \|v(t)\|_X, \tag{6.47}$$

$$|\mathcal{F}_\delta(t)| \le C\|v(t)\|_X^{1+\alpha}. \tag{6.48}$$

Especially, if $H(\mathbf{y}; U_\delta)$ is $C^{1+\gamma}$ in X near $\mathbf{y} = 0$ then we have

$$|\mathbf{W}(t, \mathbf{y}(t))| \le C(|\mathbf{y}(t)|^\alpha + \|v(t)\|_X^\alpha)\|v(t)\|_X. \tag{6.49}$$

Proof. We give the proof of (6.45) only. By the Laplace formula we have

$$\begin{aligned}(-L_\delta + I)^{-1} T_\delta f &= \int_0^\infty e^{-s} e^{sL_\delta}\{\mathcal{N}'(U_\delta) - \mathcal{N}'(V_\delta(t))\} f ds \\ &= \int_0^\infty e^{-s}\{N_\delta'(s, U_\delta) - N_\delta'(s, V_\delta(t))\} f ds.\end{aligned}$$

Hence from (6.16) and the C^1 regularity of $H(\mathbf{y}; f)$ with respect to $\mathbf{y}$, we get

$$\begin{aligned}\|(-L_\delta + I)^{-1} T_\delta f\|_X &\leq C\int_0^\infty (\frac{1+s}{s})^\beta e^{-s} ds \|U_\delta - V_\delta(t)\|_X^\alpha \|f\|_X \\ &\leq C|\mathbf{y}(t)|^\alpha \|f\|_X.\end{aligned}$$

This completes the proof.

We look for a solution $\mathbf{y}(t)$ which decays at time in nity. However, the equation for $y_0(t)$ does not lead to the time decay apparently. To overcome this difficulty we consider the following integral equation

$$y_0(t) = -\int_t^\infty W_0(s, \mathbf{y}(s)) ds, \tag{6.50}$$

$$y_i(t) = e^{-\mu_i(t-T)} y_i(T) + \int_T^t e^{-\mu_i(t-s)} W_i(s, \mathbf{y}(s)) ds, \qquad 1 \leq i \leq n, \tag{6.51}$$

$$y_{n+1}(t) = e^{-t+T} y_{n+1}(T) + \int_T^t e^{-(t-s)} W_{n+1}(s, \mathbf{y}(s)) ds, \tag{6.52}$$

which is equivalent with (6.37)-(6.39) if $\lim_{t\to\infty} |y_0(t)| = 0$. The problem here is that solutions to (6.50)-(6.52) might not satisfy the initial condition (6.40) for $j = 0$. In fact, we can show the initial condition for $j = 0$ is automatically satis ed if $\mathbf{y}(t)$ is a solution to (6.50)-(6.52) decaying at time in nity. The main result in this section is as follows.

Proposition 6.6. *Let $u(t)$ be the solution in Theorem 5.1 with the initial data $u_0 \in X$ and $\delta \neq 0$. Then for sufficiently large $T > 0$ there is a solution $\mathbf{y}(t) \in C^1([T, \infty); \mathbb{R}^{n+2})$ to (6.50)-(6.52) satisfying the initial condition (6.40) and the estimate*

$$|\mathbf{y}'(t)| + |\mathbf{y}(t)| \leq c' e^{-\frac{\varrho}{3}(1+\alpha)(t-T)}, \qquad t > T, \tag{6.53}$$

where $c' > 0$ is a sufficiently small constant depending only on $\|u(T) - U_\delta\|_X$ and $T > 0$.

Proof. From Theorem 5.1 we have $\|u(t) - U_\delta\|_X \leq e^{-\frac{\varrho}{2}t}\|u_0 - U_\delta\|_X$. Especially, by taking T large enough we may assume that $\|u(T) - U_\delta\|_X \leq \kappa$ where κ is so small as we want if it is needed. Let C_0 be the constant in

(6.43). Let $r_1, r_2 \in (0, (2C_0)^{-\frac{1}{\gamma}})$ be small numbers with $r_1 \le r_2$ and consider the closed convex set

(6.54)
$$B_{r_1,r_2} = \{\mathbf{y}(t) \in C([T,\infty);\mathbb{R}^{n+2}) \mid |\mathbf{y}(T)| \le r_1, \quad \sup_{t>T} e^{\frac{\varrho}{3}(t-T)}|\mathbf{y}(t)| \le r_2\}.$$

For $\mathbf{y} \in B_{r_1,r_2}$ we consider the map $\mathbf{Y}(t;\mathbf{y}) = (Y_0(t;\mathbf{y}), \cdots, Y_{n+1}(t;\mathbf{y}))^\top$ de ned by

$$Y_0(t;y) = -\int_t^\infty W_0(s,\mathbf{y}(s))ds,$$
$$Y_j(t;\mathbf{y}) = e^{-\mu_j(t-T)}Y_j(T;\mathbf{y}) + \int_T^t e^{-\mu_j(t-s)}W_j(s,\mathbf{y}(s))ds, \quad j \ge 1$$

where the initial data $Y(T;\mathbf{y}) = (Y_1(T;\mathbf{y}), \cdots, Y_{n+1}(T;\mathbf{y}))^\top$ is determined by the relation

$$< u(T), e^*_{\delta,j} >=< H(Y_0(T;\mathbf{y}), Y(T;\mathbf{y}); U_\delta), e^*_{\delta,j} >, \tag{6.55}$$

for $j = 1, \cdots, n+1$. The existence of such $Y(T;\mathbf{y})$ will be proved later.

Our aim is to nd a xed point of the map $\mathbf{Y}(t;\mathbf{y})$ on B_{r_1,r_2} by the Schauder xed point theorem. For this purpose, let us rst estimate $W(t,\mathbf{y}(t))$ for $\mathbf{y} \in B_{r_1,r_2}$. By the estimate of U_δ and the de nition of $H(\mathbf{y};\cdot)$, we see that

$$\|U_\delta - H(\mathbf{y}(t);U_\delta)\|_X \le C|\mathbf{y}(t)|. \tag{6.56}$$

Since we already have $\|u(t) - U_\delta\|_X \le \kappa e^{-\frac{\varrho}{2}(t-T)}$ for any $t \ge T$ with sufficiently small $\kappa > 0$, it follows that

$$\begin{aligned} \|v(t)\|_X = \|u(t) - H(\mathbf{y}(t);U_\delta)\|_X &\le \|u(t) - U_\delta\|_X + \|U_\delta - H(\mathbf{y}(t);U_\delta)\|_X \\ &\le \kappa e^{-\frac{\varrho}{2}t} + C|\mathbf{y}(t)|. \end{aligned}$$

Hence we have

$$(|\mathbf{y}(t)|^\alpha + \|v(t)\|_X^\alpha)\|v(t)\|_X \le C(r_2^{1+\alpha} + \kappa^{1+\alpha})e^{-\frac{\varrho}{3}(1+\alpha)(t-T)}. \tag{6.57}$$

Hence from Proposition 6.5 we have

$$|\mathbf{W}(t,\mathbf{y}(t))| \le C(r_2^{1+\alpha} + \kappa^{1+\alpha})e^{-\frac{\varrho}{3}(1+\alpha)(t-T)}. \tag{6.58}$$

This yields that

$$|Y_0(t;\mathbf{y})| \le C(r_2^{1+\alpha} + \kappa^{1+\alpha})e^{-\frac{\varrho}{3}(1+\alpha)(t-T)}, \tag{6.59}$$
$$|Y_j(t;\mathbf{y})| \le e^{-\mu_j(t-T)}|Y_j(T;\mathbf{y})| + C(r_2^{1+\alpha} + \kappa^{1+\alpha})e^{-\frac{\varrho}{3}(1+\alpha)(t-T)}. \tag{6.60}$$

Here we used the fact $\varrho \le \min\{\mu_1, \cdots, \mu_{n+1}\}$ and thus $\mu_j > \frac{\varrho}{3}(1+\alpha)$ for each $j = 1, \cdots, n+1$.

Next we prove the existence and the uniqueness of

$$Y(T;\mathbf{y}) = (Y_1(T;\mathbf{y}), \cdots, Y_{n+1}(T;\mathbf{y}))^\top$$

by the implicit function theorem. Note that $u(T)$ is written in the form

$$u(T) = U_\delta + \xi_0 \bar{\omega}(T)$$

where $\bar{\omega}(T) \in \mathbf{Q}_{0,0}X$ with $\|\bar{\omega}(T)\|_X = 1$, and sufficiently small $\xi_0 \geq 0$. For $\phi = (\phi_1, \cdots, \phi_{n+1}) \in \mathbb{R}^{n+1}$ and $\xi_1 \in \mathbb{R}$ we consider the vector function

$$R(\xi_0, \xi_1, \phi) = (R_1(\xi_0, \xi_1, \phi), \cdots, R_{n+1}(\xi_0, \xi_1, \phi))^\top$$

where

$$R_j(\xi_0, \xi_1, \phi) \quad = \quad < U_\delta + \xi_0 \bar{\omega}(T), e^*_{\delta,j} > - < H(\xi_1, \phi; U_\delta), e^*_{\delta,j} > .$$

Clearly $R(\xi_0, \xi_1, \phi)$ is C^1 near the origin. Since $H(0; U_\delta) = U_\delta$, we see that $R_j(0,0,0) = 0$ for each j. Moreover, since

$$< \partial_{y_l} H(0; U_\delta), e^*_{\delta,j} >= c^{-1}_{\delta,l} \delta_{jl},$$

we have $|\det(\nabla_\phi R(0,0,0))| = |\Pi_{j=1}^{n+1} c^{-1}_{\delta,j}| > 0$. Hence by the implicit function theorem there is a C^1 function $\phi(\xi_0, \xi_1)$ such that $R(\xi_0, \xi_1, \phi(\xi_0, \xi_1)) = 0$ if $|\xi_0|$ and $|\xi_1|$ are sufficiently small. Thus if $\|u(T) - U_\delta\|_X$ and $|Y_0(T;\mathbf{y})|$ are sufficiently small, then there is a unique $Y(T, \mathbf{y}) \in \mathbb{R}^{n+1}$ satisfying (6.55). Moreover, from the equality

$$\partial_{\xi_i} R_j + \sum_{l=1}^{n+1} \partial_{\phi_l} R_j \partial_{\xi_i} \phi_l = 0, \qquad i = 0, 1, \quad j = 1, \cdots, n,$$

and

$$\phi_l(\xi_0, \xi_1) = \sum_{i=0}^{1} \int_0^1 (\partial_{\xi_i} \phi_l)(\theta \xi_0, \theta \xi_1) \xi_i d\theta,$$

we have

$$|\phi_l(\xi_0, \xi_1)| \leq C(|\delta|)(|\xi_0| + |\xi_1|).$$

Noting that $|\xi_0| = \|u(T) - U_\delta\|_X \leq \kappa$ and $|\xi_1| = |Y_0(T;\mathbf{y})| \leq C(r_2^{1+\alpha} + \kappa^{1+\alpha})$ from (6.59), we get

$$|Y(T;\mathbf{y})| \leq C(\kappa + r_2^{1+\alpha}). \tag{6.61}$$

Hence by combining this with (6.60) it is easy to see that $\mathbf{Y}(t;\mathbf{y})$ is a completely continuous mapping from B_{r_1,r_2} into itself by the Ascoli-Arzela theorem for sufficiently small κ and $r_1 \leq r_2$. Then by the Schauder xed point theorem we have $\mathbf{y} \in B_{r_1,r_2}$ such that $\mathbf{y}(t) = \mathbf{Y}(t;\mathbf{y})$ for any $t \geq T$. It is clear that the xed point $\mathbf{y}(t)$ satis es the estimate (6.53). It remains to prove this xed point $\mathbf{y}(t)$ satis es the initial condition (6.40). From the de nition of $\mathbf{Y}(t;\mathbf{y})$, (6.40) holds for $j = 1, \cdots, n+1$. So it suffices to check (6.40) for $j = 0$. Note that we already know $\mathbf{E}_\delta v(t) = \mathbf{E}_\delta\big(u(t) -$

$H(\mathbf{y}(t); U_\delta)\big)$ vanishes at time in nity and satis es the equation (6.33). By the construction of $\mathbf{y}(t)$ and (6.34), the right-hand side of (6.33) belongs to $(I - \mathbf{P}_{\delta,0})X$. Thus $\mathbf{E}_\delta v(t)$ satis es the integral equation of the form

$$\mathbf{E}_\delta v(t) = e^{(t-T)L_\delta}\mathbf{E}_\delta v(T) + \int_T^t e^{(t-s)L_\delta}\mathbf{E}_\delta (I - \mathbf{P}_{\delta,0})S(s)ds \tag{6.62}$$

where

$$S(t) \quad = \quad T_\delta v(t) + F_\delta(v(t)) + J(V_\delta)(t).$$

By Proposition 6.5, Lemma 6.3, and (6.44) it follows that

$$\|\mathbf{E}_\delta(I - \mathbf{P}_{\delta,0})S(t)\|_X \le C\|v(t)\|_X(|\mathbf{y}(t)|^\alpha + \|v(t)\|_X^\alpha) + |\mathbf{W}(t, \mathbf{y}(t))| \tag{6.63}$$

for a constant C independent of $t > T$. In particular, we have $\|\mathbf{E}_\delta(I - \mathbf{P}_{\delta,0})S(t)\|_X \le Ce^{-\frac{\varrho}{3}(1+\alpha)t}$. Hence $\| \int_0^t e^{(t-s)L_\delta}\mathbf{E}_\delta(I - \mathbf{P}_{\delta,0})S(s)ds\|_X$ converges to zero at time in nity. Since $\|\mathbf{E}_\delta v(t)\|_X$ also converges to zero at time in nity, we have $\lim_{t\to\infty} \|e^{(t-T)L_\delta}\mathbf{E}_\delta v(T)\|_X = 0$. But this implies that

$$< u(T), e^*_{\delta,0} >=< H(\mathbf{y}(T); U_\delta), e^*_{\delta,0} >,$$

otherwise $\|e^{(t-T)L_\delta}\mathbf{E}_\delta v(T)\|_X$ can not vanish at time in nity. The proof of Proposition 6.6 is now complete.

6.3. **Estimate of** $u(t) - H(\mathbf{y}(t); U_\delta)$**.** In this section we calculate $\|v(t)\|_X$ more precisely by using the integral equation (6.18) for the initial data $v(T) = u(T) - H(\mathbf{y}(T); U_\delta) \in \mathbf{Q}_\delta X$, which leads to Theorem 2.3. The main result in this section is as follows.

Theorem 6.1. *Let $u(t)$ be the mild solution to (E) obtained in Theorem 5.1 with the initial data $u_0 \in X$ and $\delta \neq 0$. Let $\mathbf{y}(t) \in C^1([T, \infty); \mathbb{R}^{n+2})$ be the parameters in Proposition 6.6. Then for any $\epsilon > 0$, $v(t) = u(t) - H(\mathbf{y}(t); U_\delta)$ satisfies*

$$\|v(t)\|_X \le Ce^{-(\nu_0 - \eta(\delta) - \epsilon)(t-T)}, \qquad \forall t > T. \tag{6.64}$$

Here C depends only on ϵ, δ, and $\|u(T) - U_\delta\|_X$.

Remark 6.3. *From the relation $v(t) = u(t) - H(\mathbf{y}(t); U_\delta)$ we see that*

$$\|u(t) - H(\mathbf{y}(t); U_\delta)\|_X \le Ce^{-(\nu_0 - \eta(\delta) - \epsilon)t}, \ t > T. \tag{6.65}$$

Thus, if the spectrum of L_δ in $\mathbf{Q}_\delta X$ is included in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < -\nu_0\}$ (this means $\eta(\delta)$ is negative), then the above convergence rate becomes better than $e^{-\nu_0 t}$.

Proof. From the construction of $\mathbf{y}(t)$ we have $\mathbf{Q}_\delta v(t) = v(t)$ for $t \geq T$. Hence $v(t)$ satis es

(6.66)

$$v(t) = \mathbf{Q}_\delta e^{(t-T)L_\delta} v(T) + \int_T^t \mathbf{Q}_\delta e^{(t-s)L_\delta} \big(T_\delta v(s) + F_\delta(v(s)) + J(V_\delta)(s)\big) ds.$$

From Lemma 6.4 we have for any $\epsilon > 0$,

$$\begin{aligned} \|\mathbf{Q}_\delta e^{(t-s)L_\delta} T_\delta v(s)\|_X &\leq C_\epsilon (\frac{1+t-s}{t-s})^\beta e^{-\nu_{\epsilon,\delta}(t-s)} |\mathbf{y}(s)|^\alpha \|v(s)\|_X, \\ \|\mathbf{Q}_\delta e^{(t-s)L_\delta} F_\delta(v(s))\|_X &\leq C_\epsilon (\frac{1+t-s}{t-s})^\beta e^{-\nu_{\epsilon,\delta}(t-s)} \|v(s)\|_X^{1+\alpha}. \end{aligned}$$

Moreover, from Lemma 6.3 and (6.44) we have

$$\begin{aligned} \mathbf{Q}_\delta J(V_\delta)(t) &= -\mathbf{Q}_\delta \nabla_{\mathbf{y}} H(\mathbf{y}(t); U_\delta) \cdot (\mathbf{y}' + \Pi \mathbf{y}) \\ &= -\mathbf{Q}_\delta \nabla_{\mathbf{y}} H(\mathbf{y}(t); U_\delta) \cdot \mathbf{W}(t, \mathbf{y}(t)) \\ &= -\mathbf{Q}_\delta \big(\nabla_{\mathbf{y}} H(\mathbf{y}(t); U_\delta) - \nabla_{\mathbf{y}} H(0; U_\delta)\big) \cdot \mathbf{W}(t, \mathbf{y}(t)). \end{aligned}$$

This gives

$$\begin{aligned} & \|\mathbf{Q}_\delta e^{(t-s)L_\delta} J(V_\delta)(s)\|_X \\ \leq\ & C_\epsilon e^{-\nu_{\epsilon,\delta}(t-s)} |\mathbf{y}(s)|^\gamma |\mathbf{W}(s, \mathbf{y}(s))| \\ \leq\ & C_\epsilon e^{-\nu_{\epsilon,\delta}(t-s)} |\mathbf{y}(s)|^\gamma (|\mathbf{y}(s)|^\alpha + \|v(s)\|_X^\alpha) \|v(s)\|_X. \end{aligned}$$

Then, combining these above, we get

$$\begin{aligned} & \|\mathbf{Q}_\delta e^{(t-s)L_\delta} \big(T_\delta v(s) + F_\delta(v(s)) + J(V_\delta)(s)\big)\|_X \\ \leq\ & C_\epsilon (\frac{1+t-s}{t-s})^\beta e^{-\nu_{\epsilon,\delta}(t-s)} \big(|\mathbf{y}(s)|^\alpha + \|v(s)\|_X^\alpha\big) \|v(s)\|_X \\ \leq\ & C_\epsilon (\frac{1+t-s}{t-s})^\beta e^{-\nu_{\epsilon,\delta}(t-s)} e^{-\frac{\varrho}{3}\alpha(s-T)} \|v(s)\|_X. \end{aligned}$$

This yields for any $t \geq T' \geq T$,

$$\begin{aligned} & \|v(t)\|_X \\ \leq\ & C_\epsilon e^{-\nu_{\epsilon,\delta}(t-T')} \|v(T')\|_X \\ & + C_\epsilon \int_{T'}^t (\frac{1+t-s}{t-s})^\beta e^{-\nu_{\epsilon,\delta}(t-s)} e^{-\frac{\varrho}{3}\alpha(s-T)} \|v(s)\|_X ds \\ \leq\ & C_\epsilon e^{-\nu_{\epsilon,\delta}(t-T')} \|v(T')\|_X \\ & + C_\epsilon e^{-\frac{\varrho}{6}\alpha(T'-T)} \int_{T'}^t (\frac{1+t-s}{t-s})^\beta e^{-\nu_{\epsilon,\delta}(t-s)} e^{-\frac{\varrho}{6}\alpha(s-T)} \|v(s)\|_X ds. \end{aligned}$$

By taking T' sufficiently large, we can take $C_\epsilon \|v(T')\|_X$ and $C_\epsilon e^{-\frac{\varrho}{6}\alpha(T'-T)}$ sufficiently small. Then it is not difficult to get the estimate (6.64). We omit the details here. This completes the proof.

Using Theorem 6.1, we can improve the decay estimate of $\mathbf{W}(t,\mathbf{y}(t))$. Let us recall that $\mu_{n+1}=1$ and μ^*, μ_* are de ned by

$$\mu^* \;=\; \max\{\mu_1,\cdots,\mu_{n+1}\}, \qquad \mu_*=\min\{\mu_1,\cdots,\mu_{n+1}\}.$$

Proposition 6.7. *Let $\mathbf{y}(t)\in C^1([T,\infty);\mathbb{R}^{n+2})$ be the parameters constructed in Proposition 6.6. Then for sufficiently small $\epsilon>0$ we have*

$$|\mathbf{W}(t,\mathbf{y}(t))| \;\le\; Ce^{-(\nu_{\epsilon,\delta}+\alpha\min\{\mu_*,\ \nu_{\epsilon,\delta}\})(t-T)}. \tag{6.67}$$

Proof. From Proposition 6.6 and Theorem 6.1 we have

$$(|\mathbf{y}(t)|^\alpha+\|v(t)\|_X^\alpha)\|v(t)\|_X\le C(e^{-\frac{9}{3}\alpha(1+\alpha)(t-T)}+e^{-\alpha\nu_{\epsilon,\delta}(t-T)})e^{-\nu_{\epsilon,\delta}(t-T)}.$$

Hence we rst get the estimate $|\mathbf{W}(t,\mathbf{y}(t))|\le Ce^{-\nu_{\epsilon,\delta}(t-T)}$ from (6.49). Then from (6.50)-(6.52) we get the estimates for $\mathbf{y}(t)$ such as

$$|\mathbf{y}(t)|\le Ce^{-\min\{\mu_*,\ \nu_{\epsilon,\delta}\}(t-T)},$$

if $\epsilon>0$ is sufficiently small. Then again by (6.49) we have the improved estimate (6.67). This completes the proof.

From Proposition 6.7 we obtain the precise decay estimates for $\mathbf{y}(t)$.

Corollary 6.2. *Let $\mathbf{y}(t)\in C^1([T,\infty);\mathbb{R}^{n+2})$ be the parameters constructed in Proposition 6.6. Assume that $\nu_0\ge\mu_j$. Then the limit $y_j^*=\lim_{t\to\infty}e^{\mu_j t}y_j(t)$ exists and*

$$|e^{\mu_j t}y_j(t)-y_j^*|\le Ce^{-(\nu_{\epsilon,\delta}+\alpha\min\{\mu_*,\ \nu_{\epsilon,\delta}\}-\mu_j)(t-T)}, \tag{6.68}$$

holds.

Proof. From (6.67) we observe that $e^{\mu_j t}W_j(t,\mathbf{y}(t))$ is integrable over (T,∞) since the value $\nu_{\epsilon,\delta}+\alpha\min\{\mu_*,\nu_{\epsilon,\delta}\}-\mu_j$ is strictly positive if $\nu_0\ge\mu_j$ and ϵ, $|\delta|$ are sufficiently small by the de nition $\nu_{\epsilon,\delta}=\nu_0-\eta(\delta)-\epsilon$. Hence we have

$$\int_T^t e^{-\mu_j(t-s)}W_j(s,\mathbf{y}(s))ds=e^{-\mu_j t}\Big(\int_T^\infty e^{\mu_j s}W_j(s,\mathbf{y}(s))ds-\int_t^\infty e^{\mu_j s}W_j(s,\mathbf{y}(s))ds\Big).$$

Especially, for $y_j^*=e^{\mu_j T}y_j(T)+\int_T^\infty e^{\mu_j s}W_j(s,\mathbf{y}(s))ds$ we have the estimate (6.68). This completes the proof.

Proof of Theorem 2.3. We are now in position to show Theorem 2.3. Let $\Omega(t)\in C([0,\infty);X)$ be the solution to (E) with initial data Ω_0 with $\|\Omega_0\|_X\ll 1$. Then by Theorem 2.2 and Lemma 3.2, $u(\tau)=R_{e^\tau}\Omega(e^\tau-1)$ is the unique solution to (3.4) with initial data Ω_0. Let $\mathbf{y}(\tau)=(y_0(\tau),y(\tau))\in C^1([T,\infty);\mathbb{R}^{n+2})$ be the parameters constructed in Proposition 6.6. We note that $S(y(\tau);U_\delta) = H(0,y(\tau);U_\delta)$. Then from Theorem 6.1 and Corollary 6.2 with $j=0$ we have

$$\begin{aligned}\|u(\tau) - S(y(\tau);U_\delta)\|_X &\le \|u(\tau) - H(\mathbf{y}(\tau);U_\delta)\|_X + \|H(\mathbf{y}(\tau);U_\delta) - H(0,y(\tau);U_\delta)\|_X \\ &\le Ce^{-(\nu_0-\eta(\delta)-\epsilon)(\tau-T)} + C|y_0(\tau)| \\ &\le Ce^{-(\nu_0-\eta(\delta)-\epsilon)(\tau-T)}, \qquad \tau > T \gg 1.\end{aligned}$$

From $u(\tau) = R_{e^\tau}\Omega(e^\tau - 1)$ we have

$$\|R_{e^\tau}\Omega(e^\tau-1) - S(y(\tau);U_\delta)\|_X \le Ce^{-(\nu_0-\eta(\delta)-\epsilon)(\tau-T)}, \qquad \tau > T \gg 1,$$

that is,

$$\|R_{1+t}\Omega(t) - S(\xi(t);U_\delta)\|_X \le C(1+t)^{-\nu_0+\eta(\delta)+\epsilon}, \qquad t \gg 1, \tag{6.69}$$

where $\xi(t) = y(\log(1+t))$. Assume now that $\nu_0 \ge \mu^*$. Then by Corollary 6.2 there are $y^* = (y_1^*, \cdots, y_{n+1}^*)$ such that (6.68) holds. Then from (6.69) we conclude that

(6.70)

$$\|R_{1+t}\Omega(t) - S(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}, \frac{y_{n+1}^*}{1+t};U_\delta)\|_X \le C(1+t)^{-\nu_0+\eta(\delta)+\epsilon},$$

for $t \gg 1$. If $\nu_0 > \mu^*$ then by Lemma 6.2 $\{-\mu_j\}_{j=1}^{n+1}$ must be semisimple eigenvalues of L_δ if $|\delta|$ is sufficiently small. Hence (6.70) holds in this case. This completes the proof of Theorem 2.3.

Remark 6.4. Let $\eta'(\delta)$ be the number in (6.12). By Corollary 6.1 the set $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) > -\zeta + \eta'(\delta)\}$ consists of isolated eigenvalues with nite multiplicity. Hence, if $\zeta > \nu_0$ and $|\delta|$ is sufficiently small then the set $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) \ge -\nu_0 + \eta(\delta) = -\nu_{0,\delta}\}$ consists of nite number of eigenvalues with nite multiplicity. In this case if all eigenvalues in $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) \ge -\nu_{0,\delta}\}$ are semisimple then we can take $\epsilon = 0$ in (6.64) and (6.67), and hence, in (6.70). Indeed, by considering the spectral decomposition for the semisimple eiegnvalues $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) \ge -\nu_{0,\delta}\}$ it is not difficult to verify

$$\|\mathbf{Q}_\delta e^{tL_\delta} f\|_X \le Ce^{-\nu_{0,\delta}t}\|\mathbf{Q}_\delta f\|_X,$$

in Lemma 6.4. Then the calculations of $\|v(t)\|_X$ above imply

$$\|v(t)\|_X \le Ce^{-\nu_{0,\delta}(t-T)},$$

and thus we rst obtain the estimate $|\mathbf{W}(t,\mathbf{y}(t))| \le Ce^{-(\alpha\min\{\mu_*,\nu_{\epsilon,\delta}\}+\nu_{0,\delta})(t-T)}$ by (6.67). This yields $|\mathbf{y}(t)| \le Ce^{-\min\{\mu_*,\nu_{0,\delta}\}(t-T)}$ from (6.50)-(6.52). Then again by (6.67) we have the improved estimate

$$|\mathbf{W}(t,\mathbf{y}(t))| \le Ce^{-(\nu_{0,\delta}+\alpha\min\{\mu_*,\ \nu_{0,\delta}\})(t-T)}.$$

This proves the assertion.

6.4. **Comments on Theorem 2.4.** Theorem 2.4 is proved in the same way as in Theorem 2.3. Indeed, it suffices to determine $y(t) = (y_0(t), \tilde{y}(t)) \in C^1([T, \infty); \mathbb{R}^{n+1})$ so that $\tilde{v}(t) = u(t) - H(y_0(t), \tilde{y}(t), 0; U_\delta)$ belongs to $\tilde{\mathbf{Q}}_0 X$ for all $t \geq T$, which leads to the ODEs for $(y_0(t), \tilde{y}(t))$ as in the case of Theorem 2.3. We can solve this ODEs by using the equality in Lemma 6.3:

$$J(\tilde{V}_\delta)(t) = -\sum_{j=0}^{n} \partial_{y_j} H(y_0(t), \tilde{y}(t), 0; U_\delta) \cdot \big(y_j'(t) + \mu_j y_j(t)\big), \tag{6.71}$$

where

$$\begin{aligned} &\tilde{V}_\delta(t) = H(y_0(t), \tilde{y}(t), 0; U_\delta), \\ &J(\tilde{V}_\delta)(t) = -\partial_t J(\tilde{V}_\delta)(t) + A_{-1}\tilde{V}_\delta(t) - \mathcal{N}(\tilde{V}_\delta(t)). \end{aligned}$$

Under the assumptions of Theorem 2.4 we can show, instead of (6.10) in Lemma 6.2,

$$\sigma(L_\delta) \subset \{-\mu_j\}_{j=0}^{n+1} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq -\tilde{\nu}_0 + \tilde{\eta}(\delta)\} \tag{6.72}$$

for some $\tilde{\eta}(\delta) \in \mathbb{R}$ satisfying $\lim_{\delta \to 0} \tilde{\eta}(\delta) = 0$. Then if we set $\tilde{\mu}^* = \max\{\mu_1, \cdots, \mu_n\}$, $\tilde{\mu}_* = \min\{\mu_1, \cdots, \mu_n\}$, and $\tilde{\nu}_{\delta,\epsilon} = \tilde{\nu}_0 - \tilde{\eta}(\delta) - \epsilon$ for $\epsilon > 0$, as in Theorem 6.1 and Corollary 6.2, we obtain the estimates

$$\|\tilde{v}(t)\|_X \leq C e^{-\tilde{\nu}_{\delta,\epsilon}(t-T)}, \tag{6.73}$$

$$|y(t)| \leq C e^{-\min\{\tilde{\mu}_*, \tilde{\nu}_{\delta,\epsilon}\}(t-T)}. \tag{6.74}$$

Then by the relation $R_{1+t}\Omega(t) = u(\log(1+t))$ we have

$$\|R_{1+t}\Omega(t) - \tilde{S}(\tilde{\xi}(t); U_\delta)\|_X \leq C(1+t)^{-\nu_{\delta,\epsilon}}, \qquad t \gg 1, \tag{6.75}$$

where $\tilde{\xi}(t) = \tilde{y}(\log(1+t))$. If $\tilde{\nu}_0 \geq \tilde{\mu}^*$ then the limit $y_j^* = \lim_{t\to\infty} e^{\mu_j t} y_j(t)$ exists and

$$|e^{\mu_j t} y_j(t) - y_j^*| \leq C e^{-(\tilde{\nu}_{\epsilon,\delta} + \alpha \min\{\tilde{\mu}_*,\ \tilde{\nu}_{\epsilon,\delta}\} - \mu_j)(t-T)}, \tag{6.76}$$

holds. Hence we have

$$\|R_{1+t}\Omega(t) - \tilde{S}((1+t)^{-\mu_1} y_1^*, \cdots, (1+t)^{-\mu_n} y_n^*; U_\delta)\|_X \leq C(1+t)^{-\tilde{\nu}_{\delta,\epsilon}}, \qquad t \gg 1. \tag{6.77}$$

The details are omitted here. Especially, if $\tilde{\nu}_0 > \tilde{\mu}^*$ and $|\delta|$ is sufficiently small, then $\{-\mu_j\}_{j=1}^n$ must be semisimple eigenvalues of L_δ as in the proof of Theorem 2.3. Hence (6.77) holds in this case. This complets the proof of Theorem 2.4.

7. Applications

In this section we give several applications of our arguments.

7.1. **Nonlinear heat-convection equations.** A typical example of $\mathcal{A}$ is the Laplacian in the Eucledian space $\mathbb{R}^n$, $n \geq 1$. In this case (E_0) is the heat equation (H), i.e.,

$$\partial_t \Omega - \Delta \Omega = 0, \qquad t > 0, \quad x \in \mathbb{R}^n. \tag{H}$$

For $m \geq 0$ let L^2_m be a weighted L^2 space de ned by

$$L^2_m = \{ f \in L^2(\mathbb{R}^n) \mid \|f\|^2_{L^2_m} = \int_{\mathbb{R}^n} (1 + |x|^2)^m |f(x)|^2 dx < \infty \}. \tag{7.1}$$

Then Δ generates a strongly continuous semigroup $e^{t\Delta}$ in L^2_m and is given by

$$(e^{t\Delta} f)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy. \tag{7.2}$$

For each $\lambda > 0$, $a \in \mathbb{R}$, and $j = 1, \cdots, n$, we set

$$(R_\lambda f)(x) = \lambda^{\frac{n}{2}} f(\lambda^{\frac{1}{2}} x), \tag{7.3}$$

$$(\tau_a^{(j)} f)(x) = f(x_1, \cdots, x_{j-1}, x_j + a, x_{j+1}, \cdots, x_n). \tag{7.4}$$

Then by the density arguments $\mathcal{R} = \{R_\lambda\}_{\lambda \in \mathbb{R}^\times}$ and $\mathcal{T}^{(j)} = \{\tau_a^{(j)}\}_{a \in \mathbb{R}}$ are shown to be a scaling and a translation in L^2_m, respectively. Moreover, by setting $\mathcal{T}_\theta^{(j)} = \mathcal{T}^{(j)}$ for each $\theta \in \mathbb{R}$, we have independent one parameter families of translations $\{\mathcal{T}_\theta^{(j)}\}_{\theta \in \mathbb{R}}$, $j = 1, \cdots, n$; see De nition 2.1.

The generators of $\mathcal{R}$ and $\mathcal{T}_\theta^{(j)}$ are respectively given by

$$B = \frac{x}{2} \cdot \nabla + \frac{n}{2}, \qquad \mathrm{Dom}(B) = \{ f \in L^2_m \mid x \cdot \nabla f \in L^2_m \}, \tag{7.5}$$

$$D_\theta^{(j)} = D^{(j)} = \partial_{x_j}, \quad \mathrm{Dom}(D_\theta^{(j)}) = \{ f \in L^2_m \mid \partial_{x_j} f \in L^2_m \}. \tag{7.6}$$

Clearly $\Gamma_{a,\theta}^{(j)}(f) = \lim_{h \to 0} \frac{\tau_{a,\theta+h}^{(j)}(f) - \tau_{a,\theta}^{(j)}(f)}{h} = \lim_{h \to 0} \frac{\tau_a^{(j)}(f) - \tau_a^{(j)}(f)}{h} = 0$ for each a and θ in this case.

Then we have

Proposition 7.1. *Let $\frac{m}{2} > \frac{n}{4} + 1$ and $X = L^2_m$. Then under the setting of (H), (7.3), and (7.4), the conditions* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, *and* **(A2)** *are satisfied with $\mu_j = \frac{1}{2}$, $\varrho = \frac{1}{2}$, and $\zeta = \frac{m}{2} - \frac{n}{4}$. The eigenprojection $\mathbf{P_{0,0}}$ for the eigenvalue 0 of A (see (2.23)) is given by*

$$\mathbf{P_{0,0}} f = (\int_{\mathbb{R}^n} f(y) dy) G, \tag{7.7}$$

and the number ν_0 defined by (2.40) is 1 when $n \geq 2$ and $\min\{2, \frac{m}{2} - \frac{1}{4}\}$ when $n = 1$. Moreover, if $\frac{m}{2} > \frac{n}{4} + \frac{1}{2}$ and $X = L^2_m$ then the conditions **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, *and* **(A2)'** *are satisfied with $\mu_j = \frac{1}{2}$, $\varrho = \frac{1}{2}$, and $\zeta = \frac{m}{2} - \frac{n}{4}$. the number $\tilde{\nu}_0$ defined by (2.47) is $\min\{1, \frac{m}{2} - \frac{n}{4}\}$ in this case.*

Proof. It is easy to check $R_\lambda e^{\lambda t\Delta} = e^{t\Delta}R_\lambda$ and $\tau^{(j)}_{a,t+\theta}e^{t\Delta} = e^{t\Delta}\tau^{(j)}_{a,\theta}$, which gives **(E1)** and **(E2)**. The condition **(T1)** follows with $\mu_j = \frac{1}{2}$ from (7.5) and (7.6). To check **(T2)** we rst note that $|x|f(x) \in L^2(\mathbb{R}^n)$ if $f \in L^2_m$ with $\frac{m}{2} > \frac{n}{4} + 1$. Let $f \in \mathrm{Dom}(B) \cap \cap_{j=1}^n \mathrm{Dom}(D_1^{(j)})$ and suppose that

$$a_0 Bf + \sum_{j=1}^n a_j D_1^{(j)} f = a_0(\frac{x}{2}\cdot\nabla f + \frac{n}{2}f) + \sum_{j=1}^n a_j \partial_{x_j} f = 0, \tag{7.8}$$

where $a_j \in \mathbb{R}$ for each $j = 0, \cdots, n$. If $a_0 \neq 0$ then multiplying both sides above by f and integrating over $\mathbb{R}^n$, we get from the integration by parts, $\frac{a_0 n}{4}\|f\|^2_{L^2} = 0$, i.e., $f = 0$. If $a_0 = 0$ and there is an a_j with $j \geq 1$, then multiplying both sides of (7.8) by $x_j f$ and integrating over $\mathbb{R}^n$, we have $a_j\|f\|^2_{L^2} = 0$. That is, $f = 0$. Hence **(T2)** holds.

We note that when $\mathcal{A} = \Delta$ the semigroup $e^{tA} = e^{(1-e^{-t})\mathcal{A}}R_{e^t}$ is explicitly given by

$$(e^{tA}f)(x) = \frac{e^{\frac{nt}{2}}}{(4\pi a(t))^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4a(t)}} f(ye^t)dy, \qquad a(t) = 1 - e^{-t}. \tag{7.9}$$

and

$$A = \Delta + \frac{x}{2}\cdot\nabla + \frac{n}{2}, \qquad \mathrm{Dom}(A) = \{f \in L^2_m \mid (\Delta + \frac{x}{2}\cdot\nabla)f \in L^2_m\}.$$

In [9] Gallay and Wayne proved that the bound of the essential spectrum of e^{tA} and the spectrum of A in L^2_m are given by

$$r_{ess}(e^{tA}) = e^{-(\frac{m}{2}-\frac{n}{4})t}, \tag{7.10}$$

$$\sigma(A) = \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq -\frac{m}{2} + \frac{n}{4}\} \cup \{-\frac{k}{2} \mid k = 0, 1, 2, \cdots\}. \tag{7.11}$$

Moreover, if $k \in \mathbb{N} \cup \{0\}$ satis es $-\frac{k}{2} > -\frac{m}{2} + \frac{n}{4}$ then $-\frac{k}{2}$ is a semisimple eigenvalue with multiplicity $\frac{(n+k-1)!}{k!(n-1)!}$ and the associated eigenspace is spanned by the Hermite functions $\{\partial_x^\beta G\}_{|\beta|=k}$ where G is the n dimensional Gaussian

$$G(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}. \tag{7.12}$$

This gives **(A1)** and **(A2)** when $\frac{m}{2} > \frac{n}{4} + 1$ and **(A2)'** when $\frac{m}{2} > \frac{n}{4} + \frac{1}{2}$. The eigenprojection $\mathbf{P_{0,0}}$ is easily calculated and given by (7.7). The proof is completed.

For later use we consider the relations between the domains $\mathrm{Dom}(\mathcal{A})$, $\mathrm{Dom}(A)$, and $\mathrm{Dom}(B)$ in the case of $\mathcal{A} = \Delta$ and $B = \frac{x}{2}\cdot\nabla + \frac{n}{2}$ in L^2_m. Set

$$\rho(x) = (1 + |x|^2)^{\frac{1}{2}}.$$

Then $f \in L^2_m$ if and only if $\rho^m f \in L^2(\mathbb{R}^n)$. For each $s \in \mathbb{N}$ we introduce the Sobolev space H^s_m by

$$H^s_m = \{f \in L^2_m \mid \partial_x^\beta f \in L^2_m,\ |\beta| \le s\}.$$

Proposition 7.2. *Let* $m \ge 0$. *Then* $\mathrm{Dom}(\Delta) = H^2_m$. *In particular,* $\rho^m \partial_{x_j} f \in W^{1,2}(\mathbb{R}^n)$ *for each* j.

Proof. For $R > 0$ let ϕ_R be the function on $[0, \infty)$ such that $0 \le \phi_R \le 1$, $\phi'_R \le 0$, $\phi_R(r) = 1$ if $r \ge R$ and $\phi_R = 0$ if $r \ge 2R$. Set $\chi_R(x) = \phi_R(|x|^2)$. For $f \in \mathrm{Dom}(\Delta)$ in L^2_m we have by the integration by parts,

(7.13)

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_R \rho^{2m} |\nabla f|^2 dx \;&=\; -\int_{\mathbb{R}^n} \chi_R \rho^{2m} \Delta f f dx + 2m \int_{\mathbb{R}^n} \chi_R |f|^2 \nabla \cdot (x\rho^{2m-2}) dx \\ &\quad -4m \int_{\mathbb{R}^n} \rho^{2m-2} |f|^2 x \cdot \nabla \chi_R dx + \int_{\mathbb{R}^n} \rho^{2m} |f|^2 \Delta \chi_R dx.\end{aligned}$$

Then it is not difficult to see if $f,\ \rho^m \Delta f \in L^2(\mathbb{R}^n)$ then $\rho^m \partial_{x_j} f \in L^2(\mathbb{R}^n)$ for each j by taking the limit $R \to \infty$ in (7.13). Then the assertion follows from the equality

$$\Delta(\rho^m f) = \rho^m \Delta f + 4m\rho^{m-1} \nabla f \cdot x + f \Delta \rho^m.$$

Indeed, since the right-hand side of this equality belongs to $L^2(\mathbb{R}^n)$, each second order derivatives of $\rho^m f$ belongs to $L^2(\mathbb{R}^n)$ by using the Calderon-Zygmund inequality. Then the interpolation argument yields $\rho^m f \in W^{2,2}(\mathbb{R}^n)$. Now it is easy to see that $f \in H^2_m$. This completes the proof.

As stated in (2.20), the inclusion $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \subset \mathrm{Dom}(A)$ holds in general. When $\mathcal{A} = \Delta$ in L^2_m we have the equality as follows.

Proposition 7.3. *Let* $m \ge 0$ *and* $\mathcal{A} = \Delta$ *in* L^2_m. *Then* $\mathrm{Dom}(A) = \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B)$ *with equivalent graph norms.*

Proof. We use the cut-o function χ_R in Proposition 7.2. The direct calculation yields the equality

$$\begin{aligned}(7.14)\ &\int_{\mathbb{R}^n} \chi_R \rho^{2m} |\Delta f + \frac{x}{2} \cdot \nabla f|^2 dx \\ =\ &\int_{\mathbb{R}^n} \chi_R \rho^{2m} |\Delta f|^2 dx + \int_{\mathbb{R}^n} \chi_R \rho^{2m} |\frac{x}{2} \cdot \nabla f|^2 (1 - \frac{4m}{1+|x|^2}) dx \\ &- \int_{\mathbb{R}^n} \rho^{2m} |x \cdot \nabla f|^2 \phi'_R dx + \int_{\mathbb{R}^n} \chi_R \rho^{2m} |\nabla f|^2 (\frac{n}{2} - 1 + m - \frac{m}{1+|x|^2}) dx.\end{aligned}$$

Combining this with (7.13) and $\phi_R' \le 0$, we obtain the inequality

$$\begin{aligned}
&\int_{\mathbb{R}^n} \chi_R \rho^{2m} |\Delta f|^2 dx + \int_{\mathbb{R}^n} \chi_R \rho^{2m} |\frac{x}{2}\cdot \nabla f|^2 (1 - \frac{4m}{1+|x|^2}) dx \\
\le\ & 2\int_{\mathbb{R}^n} \chi_R \rho^{2m} |\Delta f + \frac{x}{2}\cdot\nabla f|^2 dx + C\int_{\mathbb{R}^n} \rho^{2m}|f|^2 dx.
\end{aligned} \tag{7.15}$$

Since it is easy to see $\mathrm{Dom}(A) \subset W^{1,2}_{loc}(\mathbb{R}^n)$, by taking the limit $R \to \infty$ in (7.15) we have the estimate

$$\|f\|_{\mathrm{Dom}(\mathcal{A})} + \|f\|_{\mathrm{Dom}(B)} \le C\|f\|_{\mathrm{Dom}(A)}.$$

The inverse inequality is proved similarly. We omit the details here. This completes the proof.

Let $\mathcal{O} \subset \mathbb{R}^n$ be a small open ball centered at the origin. For $y = (\tilde{y}, y_{n+1}) \in \mathcal{O}$ with $\tilde{y} = (y_1, \cdots, y_n)$, we set

$$\begin{aligned}
S(y; f) &= \tau^{(1)}_{y_1, 1+y_{n+1}} \cdots \tau^{(n)}_{y_n, 1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} f \\
&= \tau^{(1)}_{y_1} \cdots \tau^{(n)}_{y_n} R_{\frac{1}{1+y_{n+1}}} f \\
&= (1+y_{n+1})^{-\frac{n}{2}} f(\frac{\cdot + \tilde{y}}{(1+y_{n+1})^{\frac{1}{2}}}).
\end{aligned} \tag{7.16}$$

Then, since G is rapidly decreasing and smooth we have

Proposition 7.4. *The map $S(\cdot\ ; G) : \mathcal{O} \to L^2_m$ is C^∞.*

Especially, the facts $\partial_{x_j} G$ and $(x\cdot\nabla + n)G$ are the eigenfunctions of the eigenvalues $-\frac{1}{2}$ and -1 are interpreted as the symmetry of the heat equation from Lemma 2.3.

7.1.1. *Convection-diffusion equations (n-B).* In this section we apply our results to convection-di usion equations (n-B) with $p = \frac{1}{n}$, i.e.,

$$\partial_t \Omega - \Delta\Omega + a\cdot\nabla(|\Omega|^{\frac{1}{n}}\Omega) = 0, \quad t > 0, \quad x \in \mathbb{R}^n. \tag{7.17}$$

Let $X = L^2_m$. Then the nonlinear term $\mathcal{N}(f) = a\cdot\nabla(|f|^{\frac{1}{n}} f) = (1 + \frac{1}{n})|f|^{\frac{1}{n}} a\cdot\nabla f$ makes sense for $f \in \mathrm{Dom}(\Delta)$. Indeed, we have from the Hölder inequality,

$$\|\mathcal{N}(f)\|_{L^2_m} \le (1+\frac{1}{n})|a| \|f\|^{\frac{1}{n}}_{L^{1+\frac{1}{n}}} \|\rho^m \nabla f\|_{L^{1+\frac{1}{n}}}, \tag{7.18}$$

which is bounded if $f \in \mathrm{Dom}(\Delta)$ by Proposition 7.2 and the Gagliardo-Nirenberg inequality. Now we prove

Proposition 7.5. *Let* $\frac{m}{2} > \frac{n}{4} + \frac{1}{2}$ *and* $X = L^2_m$. *Then* $\mathcal{N}(f) = a \cdot \nabla(|f|^{\frac{1}{n}} f)$ *satisfies the conditions* **(N1)**, **(N2)**, *and* **(N3)** *with* $q = \alpha = \frac{1}{n}$, $\beta = \frac{3}{4}$, *and* $\epsilon_0 = 0$.

Proof. It is easy to check **(N3)**. We will show **(N1)** and **(N2)**. From (7.18), Proposition 7.2, Proposition 7.3, and the Gagliardo-Nirenberg inequality, we have $\|\mathcal{N}(f)\|_{L^2_m} \le C\|f\|^{1+\frac{1}{n}}_{\mathrm{Dom}(A)}$. Moreover, $\mathcal{N}$ maps $\mathrm{Dom}(A)$ into $\mathbf{Q}_{0,0}X$ by (7.7). Hence **(N1)** follows. To prove **(N2)** we recall the estimates for e^{tA} (see (7.9)) obtained in [9]:

$$\|\nabla e^{tA} f\|_{L^2_m} \le \frac{C}{a(t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{2})+\frac{1}{2}}}\|\rho^m f\|_{L^q}, \qquad 1 \le q \le 2, \tag{7.19}$$

where $a(t) = 1 - e^{-t}$. Then by the relation $\partial_{x_j} e^{tA} = e^{\frac{t}{2}} e^{tA} \partial_{x_j}$ we have

$$\|e^{tA}\partial_{x_j} f\|_{L^2_m} \le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{2})+\frac{1}{2}}}\|\rho^m f\|_{L^q}, \qquad 1 \le q \le 2. \tag{7.20}$$

Thus $N(t,f) = e^{tA}\mathcal{N}(f)$ is estimated as

$$\begin{aligned}
\|N(t,f) - N(t,g)\|_{L^2_{m+\frac{m}{n}}} &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|\rho^{m+\frac{m}{n}}(|f|^{\frac{1}{n}}f - |g|^{\frac{1}{n}}g)\|_{L^{\frac{2n}{n+1}}} \\
&\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|\rho^{\frac{m}{n}}(|f|+|g|)^{\frac{1}{n}}\rho^m(f-g)\|_{L^{\frac{2n}{n+1}}} \\
&\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|\rho^{\frac{m}{n}}(|f|+|g|)^{\frac{1}{n}}\|_{L^{2n}}\|\rho^m(f-g)\|_{L^2} \\
&\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}(\|f\|_{L^2_m} + \|g\|_{L^2_m})^{\frac{1}{n}}\|f-g\|_{L^2_m}. \qquad (7.21)
\end{aligned}$$

Note that we have the additional decay for $N(t,f)$ if $m > 0$. Similarly, by the relation $N'(t,f)h = (1+\frac{1}{n})e^{tA}a \cdot \nabla(|f|^{\frac{1}{n}}h)$ we have

$$\|N'(t,f)h - N'(t,g)h\|_{L^2_{m+\frac{m}{n}}} \le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|f-g\|^{\frac{1}{n}}_{L^2_m}\|h\|_{L^2_m}. \tag{7.22}$$

We omit the details here. This completes the proof.

Let $\frac{m}{2} > \frac{n}{4} + \frac{1}{2}$. Then by Theorem 2.1 there is a $\delta_0 > 0$ such that for each δ with $|\delta| \le \delta_0$ there is a $U_\delta \in L^2_m$ which gives a self-similar solution $R_{\frac{1}{1+t}} U_\delta$ to (7.17) and is $C^{1+\frac{1}{n}}$ in L^2_m with respect to δ. Note that U_δ is a solution to

$$-\Delta U - \frac{x}{2} \cdot \nabla U - \frac{n}{2}U = a \cdot \nabla(|U|^{\frac{1}{n}}U), \qquad x \in \mathbb{R}^n, \tag{7.23}$$

with $\int_{\mathbb{R}^n} U(x)dx = \delta$ by the de nition of the projection $\mathbf{P}_{0,0}$. On the other hand, in [1] it is proved that there is a unique solution $\tilde{U}_\delta$ to (7.23) with $\int_{\mathbb{R}^n} \tilde{U}_\delta(x)dx = \delta$ for all $\delta \in \mathbb{R}$ which belongs to $H^2_G \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$, where H^2_G is the Gaussian weighted L^2 space de ned by

$$L^2_G = \{f \in L^2(\mathbb{R}^n) \mid \|f\|^2_{L^2_G} = \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{G(x)} < \infty\}, \tag{7.24}$$

$$H^s_G = \{f \in L^2_G \mid \partial^\beta_x f \in L^2_G,\ |\beta| \leq s\}. \tag{7.25}$$

Moreover, the estimate of solutions such as

$$\|\tilde{U}_\delta\|_{L^\infty} + \|\tilde{U}_\delta\|_{H^1_G} \leq C|\delta|, \tag{7.26}$$

can be veri ed by the pointwise estimates of $\tilde{U}_\delta$ obtain by [18]. We will show $U_\delta = \tilde{U}_\delta$. Indeed, in [8] it is proved that the self-similar solution $R_{\frac{1}{1+t}}\tilde{U}_\delta$ attracts any solution $\Omega(t) \in C([1,\infty); L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$ to (7.17) with $\int_{\mathbb{R}^n} \Omega(t,x)dx = \delta$ in the sense of (1.1). This implies $U_\delta = \tilde{U}_\delta$. Hence we have $U_\delta \in H^2_G \cap W^{2,p}(\mathbb{R}^n)$ with (7.26), and U_δ is $C^{1+\frac{1}{n}}$ in L^2_m for all $m \geq 0$ at least for sufficiently small $|\delta|$. Note that U_δ has the form $U_\delta = \delta G + v_\delta$ with $\int_{\mathbb{R}^n} v_\delta(x)dx = 0$ by the construction of Theorem 4.1.

Let $|\delta| < \delta_0$. For $(y_0, y) \in (-\delta_0 + \delta, \delta_0 - \delta) \times \mathcal{O} \subset \mathbb{R}^{n+2}$ we set

$$H(y_0, y; U_\delta) = S(y; U_{\delta+y_0}) = (1 + y_{n+1})^{-\frac{n}{2}} U_{\delta+y_0}\Big(\frac{\cdot + \tilde{y}}{(1+y_{n+1})^{\frac{1}{2}}}\Big). \tag{7.27}$$

Then the regularity of U_δ leads to

Proposition 7.6. *Let $\frac{m}{2} > \frac{n}{4} + \frac{1}{2}$. Then $H(y_0, y; U_\delta)$ is $C^{1+\frac{1}{n}}$ as a mapping from $(-\delta_0 + \delta, \delta_0 - \delta) \times \mathcal{O}$ into L^2_m.*

Proof. Since H^1_G is included in $\mathrm{Dom}(B)$ in L^2_m, the fact $U_\delta \in H^2_G$ implies that $H(y_0, y; U_\delta)$ is in fact C^2 in L^2_m with respect to y for each xed y_0. Since we already know that U_δ is $C^{1+\frac{1}{n}}$ with respect to δ in L^2_m, $H(y_0, y; U_\delta)$ is also $C^{1+\frac{1}{n}}$ with respect to y_0 in L^2_m. Hence it suffices to show, for example, U_δ is di erentiable in H^1_G with respect to δ. To prove this, let us recall that the operator $A = \Delta + \frac{x}{2} \cdot \nabla + \frac{n}{2}$ is realized as a self-adjoint operator in L^2_G, denoted by A_∞ in order to avoid confusions, and $H^1_G = \mathrm{Dom}((-A_\infty + I)^{\frac{1}{2}})$ with equivalent norms; see [7]. Moreover, the duality arguments as in [11, Proposition 2.1] shows that $(-A_\infty)^{-\frac{1}{2}}\partial_{x_j}$ is extended as a bounded operator in L^2_G. Now let us consider the operator $(-A_\infty)^{-1}\partial_{x_j}$. By decomposing $(-A_\infty)^{-1}\partial_{x_j} = (-A_\infty)^{-\frac{1}{2}}(-A_\infty)^{-\frac{1}{2}}\partial_{x_j}$, we have the estimate

$$\|(-A_\infty)^{-1}\partial_{x_j} f\|_{H^1_G} \leq C\|f\|_{L^2_G}, \tag{7.28}$$

for all $f \in L^2_G$. Since $U_\delta = \delta G + v_\delta$ solves the equation in L^2_G:

$$v_\delta = (-A_\infty)^{-1}\mathcal{N}(U_\delta),$$

we have for sufficiently small $|h| \le |\delta|$,

$$v_{\delta+h} - v_\delta = (-A_\infty)^{-1}\int_0^1 \mathcal{N}'(\tau U_{\delta+h} + (1-\tau)U_\delta)(v_{\delta+h} - v_\delta + hG)d\tau,$$

where

$$\mathcal{N}'(f)g = (1+\frac{1}{n})a\cdot\nabla(|f|^{\frac{1}{n}}g).$$

Then from (7.26) and (7.28) we have

$$\begin{aligned}\|v_{\delta+h} - v_\delta\|_{H^1_G} &\le C\int_0^1 \||\tau U_{\delta+h} + (1-\tau)U_\delta|^{\frac{1}{n}}(v_{\delta+h} - v_\delta + hG)\|_{L^2_G}d\tau \\ &\le C|\delta|^{\frac{1}{n}}(\|v_{\delta+h} - v_\delta\|_{L^2_G} + |h|),\end{aligned}$$

and hence,

$$\|\frac{v_{\delta+h} - v_\delta}{h}\|_{H^1_G} \le C|\delta|^{\frac{1}{n}}, \tag{7.29}$$

where C does not depend on h.

Set $\omega_h = \frac{v_{\delta+h}-v_\delta}{h}$. Then we have

$$\begin{aligned}\omega_h - \omega_{h'} = (-A_\infty)^{-1}\int_0^1 &\{\mathcal{N}'(\tau U_{\delta+h} + (1-\tau)U_\delta)(\omega_h + G) \\ &-\mathcal{N}'(\tau U_{\delta+h'} + (1-\tau)U_\delta)(\omega_{h'} + G)\}d\tau\end{aligned}$$

Similar calculations as in (7.29) yields

$$\begin{aligned}\|\omega_h - \omega_{h'}\|_{H^1_G} &\le C\||U_{\delta+h} - U_{\delta+h'}|^{\frac{1}{n}}(\omega_{h'} + G)\|_{L^2_G} \\ &\le C\|U_{\delta+h} - U_{\delta+h'}\|^{\frac{1}{n}}_{L^{1+\frac{1}{n}}}(\|\omega_{h'}\|_{L^{1+\frac{1}{n}}} + C) \\ &\le C\|U_{\delta+h} - U_{\delta+h'}\|^{\frac{1}{n}}_{H^1_G}(\|\omega_{h'}\|_{H^1_G} + C) \le C|h-h'|^{\frac{1}{n}}.\end{aligned}$$

Thus ω_h converges to $\partial_\delta v_\delta$ is H^1_G as $h \to 0$, which completes the proof.

Now we can apply Theorem 2.4 to (7.17) and obtain the following theorem.

Theorem 7.1. *Let $\frac{m}{2} > \frac{n}{4} + \frac{1}{2}$. Assume that $\Omega(t) \in C([0,\infty); L^2_m)$ be the solution to (7.17) satisfying $\|\Omega(0)\|_{L^2_m} \ll 1$ and $\int_{\mathbb{R}^n}\Omega(0,x)dx = \delta \ne 0$. Then there are $\tilde{\eta}(\delta) \in \mathbb{R}$ and $\tilde{y}^* \in \mathbb{R}^n$ such that $\lim_{\delta\to 0}\tilde{\eta}(\delta) = 0$ and*

(7.30)

$$\|\Omega(t) - (1+t)^{-\frac{n}{2}}U_\delta(\frac{\cdot + \tilde{y}^*}{\sqrt{1+t}})\|_{L^p} \le C_\epsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\min\{1,\frac{m}{2}-\frac{n}{4}\}+\tilde{\eta}(\delta)+\epsilon},$$

holds for all $t \gg 1$, $\epsilon > 0$, *and* $1 \le p \le 2$.

When $n = 1$ if the initial data is more localized then we can consider the shifts of U_δ with respect to both translations and scaling, which recovers the results of (1.2).

Theorem 7.2. *Let* $m > \frac{5}{2}$. *Assume that* $\Omega(t) \in C([0,\infty); L^2_m)$ *be the solution to (7.17) satisfying* $\|\Omega(0)\|_{L^2_m} \ll 1$ *and* $\int_{\mathbb{R}} \Omega(0,x)dx = \delta \neq 0$. *Then there are* $\eta(\delta) \in \mathbb{R}$ *and* $y^* \in \mathbb{R}^2$ *such that* $\lim_{\delta\to 0} \eta(\delta) = 0$ *and*
(7.31)

$$\|\Omega(t) - \frac{1}{\sqrt{1+t+y_2^*}} U_\delta(\frac{\cdot + y_1^*}{\sqrt{1+t+y_2^*}})\|_{L^p} \le C_\epsilon (1+t)^{-\min\{\frac{3}{2}, \frac{m}{2}-\frac{1}{4}\} - \frac{1}{2}(1-\frac{1}{p}) + \eta(\delta)},$$

holds for all $t \gg 1$, $\epsilon > 0$, *and* $1 \le p \le 2$.

Remark 7.1. If we can show -1 is a semisimple eigenvalue of the linearized operator $L_\delta = \Delta + \frac{x}{2}\cdot\nabla + \frac{n}{2} - \mathcal{N}'(U_\delta)$, then we have the analogous estimate with (7.31) also for $n \ge 2$. However, the authors do not know if -1 is a semisimple eigenvalue of L_δ or not when $n \ge 2$. The sign of $\eta(\delta)$ is not determined either.

Proof of Theorem 7.1 and Theorem 7.2. We rst note that $-\frac{1}{2}$ is a semisimple eigenvalue of L_δ. Indeed, for (7.17) we have $\tilde{\nu}_0 = \min\{1, \frac{m}{2} - \frac{n}{4}\}$ and $\tilde{\mu}^* = \tilde{\mu}_* = \mu_j = \frac{1}{2}$ with $j = 1, \cdots, n$, by Proposition 7.1. Then by Lemma 6.2, $-\frac{1}{2}$ is a semisimple eigenvalue of L_δ if $|\delta|$ is sufficiently small. Hence from Theorem 2.3 there is $\tilde{y}^* = (y_1^*, \cdots, y_n^*) \in \mathbb{R}^n$ such that

$$\|R_{1+t}\Omega(t) - \tilde{S}(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \cdots, \frac{y_n^*}{(1+t)^{\frac{1}{2}}}; U_\delta)\|_{L^2_m} \le C(1+t)^{-\min\{1, \frac{m}{2}-\frac{n}{4}\} + \eta(\delta) + \epsilon},$$

Thus from $\|R_\lambda f\|_{L^p} = \lambda^{\frac{n}{2}(1-\frac{1}{p})}\|f\|_{L^p}$ we have

$$\|\Omega(t) - R_{\frac{1}{1+t}}\tilde{S}\big(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \cdots, \frac{y_n^*}{(1+t)^{\frac{1}{2}}}; U_\delta\big)\|_{L^p} \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{p}) - \min\{1, \frac{m}{2}-\frac{n}{4}\} + \eta(\delta) + \epsilon},$$

for $1 \le p \le 2$. It is not difficult to see

$$R_{\frac{1}{1+t}}\tilde{S}\big(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \cdots, \frac{y_n^*}{(1+t)^{\frac{1}{2}}}; U_\delta\big) = (1+t)^{-\frac{n}{2}} U_\delta(\frac{x+\tilde{y}^*}{\sqrt{1+t}}),$$

which gives Theorem 7.1.

When $n = 1$ we have from (7.10) and (7.11) that $\nu_0 = \min\{\frac{3}{2}, \frac{m}{2} - \frac{1}{4}\} > 1 = \mu^*$. Hence by Theorem 2.3 there is $y^* = (y_1^*, y_2^*) \in \mathbb{R}^2$ such that

$$\|\Omega(t) - R_{\frac{1}{1+t}}S\big(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \frac{y_2^*}{1+t}; U_\delta\big)\|_{L^p} \le C(1+t)^{-\min\{\frac{3}{2}, \frac{m}{2}-\frac{1}{4}\} - \frac{1}{2}(1-\frac{1}{p}) + \eta(\delta) + \epsilon},$$

which gives the desired estimate. This completes the proof of Theorem 7.2.

7.1.2. *Two dimensional vorticity equations.* In this section we consider the two dimensional vorticity equations for viscous incompressible flows:

$$\partial_t\Omega - \Delta\Omega + \nabla\cdot(\Omega\nabla^{\perp}(-\Delta)^{-1}\Omega) = 0, \quad t>0, \quad x\in\mathbb{R}^2, \tag{2-V}$$

where $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})^{\top}$ and $\nabla^{\perp}(-\Delta)^{-1}f$ is explicitly given as

$$\nabla^{\perp}(-\Delta)^{-1}f = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}}{|x-y|^2}f(y)dy,$$

where $x^{\perp} = (-x_2, x_1)^{\top}$. In this case $\mathcal{A} = \Delta$ and $\mathcal{N}(f) = \nabla\cdot(f\nabla^{\perp}(-\Delta)^{-1}f)$.

Proposition 7.7. *Let $m>3$ and $X = L^2_m$. Then $\mathcal{N}(f) = \nabla\cdot(f\nabla^{\perp}(-\Delta)^{-1}f)$ satisfies the conditions* **(N1)**, **(N2)**, *and* **(N3)** *with $q=\alpha=1$, $\beta=\frac{3}{4}$, and $\epsilon_0 = 0$.*

Proof. Since **(N3)** is easy to see, we will check only **(N1)** and **(N2)**. We note that $\mathcal{N}(f) = (\nabla^{\perp}(-\Delta)^{-1}f, \nabla)f$. From the Hardy-Littlewood-Sobolev inequality we have $\|\nabla^{\perp}(-\Delta)^{-1}f\|_{L^4} \le C\|f\|_{L^{\frac{4}{3}}} \le C\|f\|_{L^2_m}$ if $m>1$. Hence by Proposition 7.2 and Proposition 7.3 we have

$$\begin{aligned}\|\mathcal{N}(f)\|_{L^2_m} = \|\rho^m\mathcal{N}(f)\|_{L^2} &\le C\|\nabla^{\perp}(-\Delta)^{-1}f\|_{L^4}\|\rho^m\nabla f\|_{L^4}\\ &\le C\|f\|_{L^2_m}\|f\|_{\mathrm{Dom}(\mathcal{A})} \le \|f\|^2_{\mathrm{Dom}(A)}.\end{aligned}$$

Together with (7.7), this shows **(N1)**.

Next we consider **(N2)**. Since $\mathcal{N}$ is a bilinear form, it suffices to give the estimate for $e^{tA}\mathcal{M}(f,g) := e^{tA}\nabla\cdot(g\nabla^{\perp}(-\Delta)^{-1}f)$. Then from (7.20) we have

$$\begin{aligned}\|e^{tA}\mathcal{M}(f,g)\|_{L^2_m} &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|\rho^m g\nabla^{\perp}(-\Delta)^{-1}f\|_{L^{\frac{4}{3}}}\\ &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|\rho^m g\|_{L^2}\|\nabla^{\perp}(-\Delta)^{-1}f\|_{L^4}\\ &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|g\|_{L^2_m}\|f\|_{L^2_m}.\end{aligned}$$

This completes the proof.

For (2-V) the equation for U_δ is

$$-\Delta U - \frac{x}{2}\cdot\nabla U - \frac{n}{2}U + \nabla\cdot(U\nabla^{\perp}(-\Delta)^{-1}U) = 0, \qquad x\in\mathbb{R}^2, \tag{7.32}$$

with $\int_{\mathbb{R}^2}U(x)dx = \delta$. In [10] Gallay and Wayne proved that $U_\delta = \delta G$ is the unique solution to (7.32) in $L^1(\mathbb{R}^2)$ by proving the global stability of δG; see also [13, 12, 20]. Especially, the function $H(y_0, y; U_\delta)$ defined by (7.27) is C^∞ as a mapping from $\mathbb{R}\times\mathcal{O}\subset\mathbb{R}^{n+2}$ to L^2_m.

When $\Omega_\delta(t) \in C([0,\infty); L^1(\mathbb{R}^n)) \cap C((0,\infty); L^\infty(\mathbb{R}^2))$ is a solution to (2-V) with $\int_{\mathbb{R}^2} \Omega(0,x)dx = \delta$, the large time behavior of $\Omega_\delta(t)$ is described as

$$(7.33) \quad \|\Omega_\delta(t) - \delta t^{-1} G(\frac{\cdot}{\sqrt{t}})\|_{L^p} = o(t^{-1+\frac{1}{p}}), \qquad t \to \infty, \quad 1 \le p \le \infty.$$

This was obtained by [13, 3, 12] for sufficiently small $|\delta|$, and the smallness condition for δ was removed by [10]. Theorem 2.3 leads to

Theorem 7.3. *Let $m > 3$. Then for any $\delta \in \mathbb{R}$ with $0 < |\delta| \ll 1$ there exists a negative number $\eta(\delta)$ such that $\lim_{\delta\to 0} \eta(\delta) = 0$ and the following statements hold.*

Assume that $\Omega(t) \in C([0,\infty); L^2_m)$ be the solution to (2-V) satisfying $\int_{\mathbb{R}^2} \Omega(0,x)dx = \delta$. Then there is a $y^* = (\tilde{y}^*, y_3^*)^\top \in \mathbb{R}^2 \times \mathbb{R}$ such that $\Omega(t)$ satisfies

$$(7.34) \quad \|\Omega(t) - \delta(1+t+y_3^*)^{-1} G(\frac{\cdot + \tilde{y}^*}{\sqrt{1+t+y_3^*}})\|_{L^p} \le C_\epsilon (1+t)^{-2+\frac{1}{p}+\eta(\delta)+\epsilon},$$

for all $t \gg 1$, $\epsilon > 0$, and $1 \le p \le 2$.

Remark 7.2. In [10] the second and third asymptotic expansions are established when $m > 3$ without any restriction of δ by obtaining the estimate like (7.34). So the results of Theorem 7.3 are not essentially new.

Proof of Theorem 7.3. Since $\nu_0 = 1 > \frac{1}{2} = \mu_j$, $j = 1,2$, $-\frac{1}{2}$ is a semisimple eigenvalue of L_δ by Lemma 6.2. Furthermore, from [10, Remark 4.9] we observe that -1 is a simple eigenvalue of L_δ and $\eta(\delta)$ is strictly negative if δ is not zero. Hence from Theorem 2.3 there are $y_3^* \in \mathbb{R}^3$ such that

$$\|R_{1+t}\Omega(t) - S((1+t)^{-\frac{1}{2}} y_1^*, (1+t)^{-\frac{1}{2}} y_2^*, (1+t)^{-1} y_3^*; U_\delta)\|_{L^2_m} \le C(1+t)^{-1+\eta(\delta)+\epsilon},$$

where $U_\delta = \delta G$. Now we can get (7.34) as in the proof of Theorem 7.2. This completes the proof.

7.1.3. *Keller-Segel systems.* In this section we consider the two dimensional parabolic systems modelling chemotaxis:

$$\text{(KS)} \quad \begin{cases} \partial_t \Omega^{(1)} - \Delta \Omega^{(1)} + \nabla \cdot (\Omega^{(1)} \nabla \Omega^{(2)}) = 0, & t > 0, \ x \in \mathbb{R}^2, \\ \partial_t \Omega^{(2)} - \Delta \Omega^{(2)} - \Omega^{(1)} = 0, & t > 0, \ x \in \mathbb{R}^2. \end{cases}$$

For (KS) the existence of self-similar solutions is proved in [2] and the stability estimate (7.35) below is obtained by [23] when the initial data $(\Omega^{(1)}(0), \Omega^{(2)}(0))$ satis es $(1+|x|^2)\Omega^{(1)}(0) \in L^1(\mathbb{R}^2)$, $\partial_{x_j}\Omega^{(2)}(0) \in L^1(\mathbb{R}^2)$ for each j, and $\|\Omega^{(1)}(0)\|_{L^1}$ and $\|\nabla\Omega^{(2)}(0)\|_{L^2}$ are sufficiently small:

$$(7.35) \quad \|\Omega^{(1)}(t,\cdot) - t^{-1} U_\delta^{(1)}(\frac{\cdot}{\sqrt{t}})\|_{L^p(\mathbb{R}^2)} = O(t^{-1+\frac{1}{p}-\sigma}), \quad \tau \to \infty, \quad \frac{3}{4} \le p \le 2.$$

Here $(t^{-1}U_\delta^{(1)}(\frac{x}{\sqrt{t}}), U_\delta^{(2)}(\frac{x}{\sqrt{t}}))$ is the self-similar solution to (KS) with $\int_{\mathbb{R}^2} U_\delta^{(1)}(x)dx = \int_{\mathbb{R}^2} \Omega^{(1)}(x,0)dx =: \delta$, and σ is a constant in $(0,\frac{1}{2})$. The value of σ is not explicitly determined in [23]. The results in Section 2 can be applied also for (KS), but the arguments to check the conditions stated in Section 2.2 become more complicated. Moreover, for (KS) the detailed analysis of the value $\eta(\delta)$ in Lemma 6.2 is possible by applying the perturbation theory of linear operators. The detailed discussion will be given in [15] and we just state a result in [15] here.

Theorem 7.4. *Let $m > 2$. Assume that $\|(\Omega^{(1)}(0), \Omega^{(2)}(0))\|_{L^2_m \times H^1_{m-2}} \ll 1$ and $\int_{\mathbb{R}^2} \Omega^{(1)}(0,x)dx = \delta \neq 0$. Then there exists a unique solution $(\Omega^{(1)}(t), \Omega^{(2)}(t)) \in C([0,\infty); L^2_m \times H^1_{m-2})$ to (KS) such that the following statements hold.*

There are $\tilde{\eta}(\delta) \in \mathbb{R}$ and $\tilde{y}^* \in \mathbb{R}^2$ such that $\lim_{\delta\to 0} \tilde{\eta}(\delta) = 0$ and

(7.36)
$$\|\Omega^{(1)}(t) - (1+t)^{-1}U_\delta^{(1)}(\frac{\cdot + \tilde{y}^*}{\sqrt{1+t}})\|_{L^p} \leq C_\epsilon (1+t)^{-1+\frac{1}{p}-\min\{1,\frac{m-1}{2}\}+\tilde{\eta}(\delta)+\epsilon},$$

holds for all $t \gg 1$, $\epsilon > 0$, and $1 \leq p \leq 2$. Furthermore, if $m > 3$ then there are $\eta(\delta) \in \mathbb{R}$ and $(\tilde{y}^*, y_3^*) \in \mathbb{R}^2 \times \mathbb{R}$ such that $\lim_{\delta\to 0} \eta(\delta) = 0$ and

$$\|\Omega^{(1)}(t) - (1+t+y_3^*)^{-1}U_\delta^{(1)}(\frac{\cdot + \tilde{y}^*}{\sqrt{1+t+y_3^*}})\|_{L^p} \leq C_\epsilon (1+t)^{-2+\frac{1}{p}+\eta(\delta)}, \tag{7.37}$$

holds for all $t \gg 1$ and $1 \leq p \leq 2$. Moreover, $\eta(\delta)$ is positive (negative) if δ is positive (negative).

Remark 7.3. Let $m > 3$. Est. (7.37) implies that $\tilde{\eta}(\delta) + \epsilon = 0$ if $\delta < 0$ and $\tilde{\eta}(\delta) + \epsilon = \eta(\delta)$ if $\delta > 0$ in (7.36). Therefore, (7.37) gives more precise asymptotic pro le than (7.36) if $m > 3$ and $\delta < 0$; see Remark 2.5.

Remark 7.4. Instead of (KS) we can also apply our abstract results to the Keller-Segel system of a parabolic-elliptic type:

$$\text{(KS')} \quad \begin{cases} \partial_t \Omega^{(1)} - \Delta\Omega^{(1)} + \nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)}) = 0, & t > 0, \quad x \in \mathbb{R}^2, \\ -\Delta\Omega^{(2)} = \Omega^{(1)}, & t > 0, \quad x \in \mathbb{R}^2. \end{cases}$$

We can show the estimate like (7.36) for solutions to (KS') when $\|\Omega^{(1)}(0)\|_{L^2_m}$ is sufficiently small for some $m > 2$, but the details are omitted here.

7.2. **One dimensional Vlasov-Poisson-Fokker-Plank equations.** In this section we consider the one dimensional Vlasov-Poisson-Fokker-Plank equations without friction:

$$\partial_t\Omega + u\partial_x\Omega + E_\pm(\Omega)\partial_u\Omega - \partial_u^2\Omega = 0, \qquad t > 0, \quad (x,u) \in \mathbb{R}\times\mathbb{R}. \tag{7.38}$$

Here $\Omega = \Omega(t, x, u)$, and

$$E_{\pm}(\Omega) = \pm\frac{1}{2}\mathrm{sgn}(x) *_x \int_{\mathbb{R}} \Omega(t, x, u)du. \tag{7.39}$$

For simplicity we consider the case $E(\Omega) = E_+(\Omega)$ here. In [4] the existence of global and classical solutions to (7.38) is proved for some class of initial data. We consider large time behavior of solutions to (7.38) under the condition that the mass of initial data is zero. For known results on multi-dimensional case, see the results and references in [14].

By using the Fourier transform we can see that the semigroup associated with $\partial_u^2 - u\partial_x$ is given by

$$e^{t\mathcal{A}}f = \frac{\sqrt{3}}{2\pi t^2}\int_{\mathbb{R}^2} e^{-\frac{3}{t^3}\{x-y-\frac{t}{2}(u+v)\}^2 - \frac{(u-v)^2}{4t}} f(y, v)dydv. \tag{7.40}$$

It is not difficult to see that for any compactly supported function f, the function $e^{t\mathcal{A}}f$ is smooth as a function of (t, x, u) if $t > 0$, and satis es

$$\partial_t\Omega + u\partial_x\Omega - \partial_u^2\Omega = 0, \qquad t > 0, \quad (x, u) \in \mathbb{R}\times\mathbb{R}, \tag{7.41}$$

pointwisely. We note that in [25] fundamental solutions to linear Vlasov-Poisson-Fokker-Plank equations are studied in details for any dimensions.

In this section the spaces L^2_m and H^s_m are given by

$$L^2_m = \{f \in L^2(\mathbb{R}^2) \mid \|f\|^2_{L^2_m} = \int_{\mathbb{R}^2}(1 + x^2 + u^2)^m|f(x, u)|^2dxdu < \infty\},$$
$$H^s_m = \{f \in L^2_m \mid \partial_x^{\beta_1}\partial_u^{\beta_2} f \in L^2_m,\ 0 \le \beta_1 + \beta_2 \le s\}.$$

By setting $e^{0\mathcal{A}} = I$, we can check that $\{e^{t\mathcal{A}}\}_{t\ge 0}$ is a strongly continuous semigroup in L^2_m for each $m \ge 0$. The associated generator is again denoted by $\mathcal{A}$. The following estimates for $e^{t\mathcal{A}}$ are useful.

Proposition 7.8. *Assume that $m \ge 0$. Let s be a nonnegative integer and $j_1,\ j_2 \in \{0, 1\}$. Then for any $\epsilon > 0$ we have*

$$\|e^{t\mathcal{A}}f\|_{H^s_m} \le C_\epsilon e^{\epsilon t}\|f\|_{H^s_m}, \qquad t > 0, \tag{7.42}$$

$$\|\partial_x^{j_1}\partial_u^{j_2} e^{t\mathcal{A}}f\|_{H^s_m} \le \frac{C_\epsilon e^{\epsilon t}}{t^{\frac{3j_1+j_2}{2}}}\|f\|_{H^s_m}, \qquad t > 0, \tag{7.43}$$

$$\|e^{t\mathcal{A}}\partial_u f\|_{H^s_m} \le \frac{C_\epsilon e^{\epsilon t}}{t^{\frac{1}{2}}}\|f\|_{H^s_m}, \qquad t > 0. \tag{7.44}$$

If $m = 0$ and $s = 0$ then we can take $\epsilon = 0$ in the above estimates. As a consequence, we have

$$\|\partial_u(I - \mathcal{A})^{-1}f\|_{L^2_m} \le C\|f\|_{L^2_m}. \tag{7.45}$$

Proof. We rewrite (7.40) as

$$e^{t\mathcal{A}}f = \frac{\sqrt{3}}{2\pi t^2}\int_{\mathbb{R}^2} e^{-\frac{3y^2}{t^3}-\frac{v^2}{4t}} f(x-y-\frac{t}{2}(2u-v), u-v)\,dydv. \tag{7.46}$$

Set

$$\tilde{f}(x,u;y,v,t) = f(x-y-\frac{t}{2}(2u-v), u-v).$$

Then we can check the equality

$$\partial_x^{\beta_1}\partial_u^{\beta_2}\tilde{f}(x,u;y,v,t) = \sum_{0\le l\le\beta_2} c_l t^l (\partial_x^{\beta_1+l}\partial_u^{\beta_2-l}f)(x-y-\frac{t}{2}(2u-v), u-v),$$

where each c_l is a suitable constant and $c_0 = 1$.

Then by the inequality

$$\rho^m = (1+x^2+u^2)^{\frac{m}{2}} \le C(1+|x-y-\frac{t}{2}(2u-v)|^m + |y|^m + (1+t^m)|u-v|^m + (1+t^m)|v|^m),$$

and the Minkovski inequality, we have

$$\|\rho^m\partial_x^{\beta_1}\partial_u^{\beta_2}e^{t\mathcal{A}}f\|_{L^2} \le C(1+t^{m'})\|f\|_{H_m^{\beta_1+\beta_2}}, \qquad t>0, \tag{7.47}$$

for some $m' \ge m$, which gives (7.42). Est. (7.43) is proved similarly from the equality

$$\partial_x^{j_1}\partial_u^{j_2}e^{t\mathcal{A}}f = \frac{\sqrt{3}}{2\pi t^2}\int_{\mathbb{R}^2}\big((\partial_v^{j_2}-\frac{t}{2}\partial_y^{j_1})\partial_y^{j_1}e^{-\frac{3y^2}{t^3}-\frac{v^2}{4t}}\big)f(x-y-\frac{t}{2}(2u-v), u-v)\,dydv.$$

To prove (7.44) we note that

$$\begin{aligned}
&(\partial_x^{\beta_1}\partial_u^{\beta_2+1}f)(x-y-\frac{t}{2}(2u-v), u-v)\\
= \;&\partial_x^{\beta_1}\partial_u^{\beta_2+1}\tilde{f}(x,u;y,v,t) - \sum_{1\le l\le\beta_2+1} c_l t^l(\partial_x^{\beta_1+l}\partial_u^{\beta_2+1-l}f)(x-y-\frac{t}{2}(2u-v), u-v).
\end{aligned}$$

Thus we have from (7.43) and $\partial_x e^{t\mathcal{A}}f = e^{t\mathcal{A}}\partial_x f$,

$$\begin{aligned}
\|\partial_x^{\beta_1}\partial_u^{\beta_2}e^{t\mathcal{A}}\partial_u f\|_{L^2_m} &\le \|\partial_x^{\beta_1}\partial_u^{\beta_2+1}e^{t\mathcal{A}}f\|_{L^2_m} + \sum_{1\le l\le\beta_2+1}|c_l|t^l\|e^{t\mathcal{A}}\partial_x^{\beta_1+l}\partial_u^{\beta_2+1-l}f\|_{L^2_m}\\
&\le \|\partial_x^{\beta_1}\partial_u^{\beta_2+1}e^{t\mathcal{A}}f\|_{L^2_m} + C\sum_{1\le l\le\beta_2+1}t^l\|\partial_x e^{t\mathcal{A}}\partial_x^{\beta_1+l-1}\partial_u^{\beta_2+1-l}f\|_{L^2_m}\\
&\le C(1+t^{m'})t^{-\frac{1}{2}}\|f\|_{H_m^{\beta_1+\beta_2}} + C\sum_{1\le l\le\beta_2+1}t^l(1+t^{m'})t^{-\frac{3}{2}}\|f\|_{H_m^{\beta_1+\beta_2}}\\
&\le C(1+t^{m''})t^{-\frac{1}{2}}\|f\|_{H_m^{\beta_1+\beta_2}},
\end{aligned}$$

for some $m'' \geq m'$. Est. (7.45) is obtained from (7.43) and the Laplace formula

$$\partial_u (I-\mathcal{A})^{-1} f = \int_0^\infty e^{-t}\partial_u e^{t\mathcal{A}} f dt.$$

This completes the proof.

For each $\lambda > 0$ and $a, \theta \in \mathbb{R}$, set

$$(R_\lambda f)(x) = \lambda^{\frac{5}{2}} f(\lambda^{\frac{3}{2}} x, \lambda^{\frac{1}{2}} u), \tag{7.48}$$

$$(\tau_{a,\theta} f)(x) = f(x+\theta a, u+a). \tag{7.49}$$

Then $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ and $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ with $\mathcal{T}_\theta = \{\tau_{a,\theta}\}_{a\in\mathbb{R}^+}$ are a scaling and a strongly continuous one parameter family of translations in L^2_m, respectively. The generators of $\mathcal{R}$ and $\mathcal{T}_\theta$ are given by

$$B = \frac{3x}{2}\partial_x + \frac{u}{2}\partial_u + \frac{5}{2},\ \mathrm{Dom}(B) = \{f \in L^2_m \mid (\frac{3x}{2}\partial_x + \frac{u}{2}\partial_u) f \in L^2_m\},$$

$$D_\theta = \theta\partial_x + \partial_u, \qquad \mathrm{Dom}(D_\theta) = \{f \in L^2_m \mid (\theta\partial_x + \partial_u) f \in L^2_m\}.$$

Furthermore, $\Gamma_{a,\theta}$ is given by

$$\Gamma_{a,\theta} = a\partial_x \tau_{a,\theta}, \qquad \mathrm{Dom}(\Gamma_{a,\theta}) = \{f \in L^2_m \mid a\partial_x \tau_{a,\theta} f \in L^2_m\}.$$

For $j,\ k \in \mathbb{N}\cup\{0\}$ we introduce a function $H_{j,k}$ by

$$H_{j,k} = c_{j,k}\partial_x^j(\partial_x+\partial_u)^k e^{-3(x-\frac{u}{2})^2-\frac{1}{4}u^2}, \qquad c_{j,k} = (-\frac{1}{3})^{|j|}\frac{\sqrt{3}}{2\pi j!k!}. \tag{7.50}$$

Let $m > 1$ and let $L^2_{m,0}$ be a subspace of L^2_m de ned by

$$L^2_{m,0} = \{f \in L^2_m \mid \int_{\mathbb{R}^2} f(x,u) dx du = 0\}. \tag{7.51}$$

Then we have

Proposition 7.9. *Let $q \in \mathbb{N}$ with $q \geq 3$. Let $m \geq \frac{q}{2} + \frac{7}{4}$ and $X = L^2_{m,0}$. Then under the setting of (7.41), (7.48), and (7.49), the conditions* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, *and* **(A2)** *hold with $n = 1$, $\mu_1 = \frac{1}{2}$, $\varrho = \frac{1}{2}$, and $\zeta = \frac{q}{2}$. The eigenprojection $\mathbf{P_{0,0}}$ for the eigenvalue 0 of A (see (2.23)) is given by*

$$\mathbf{P_{0,0}} f = (\int_{\mathbb{R}^2} u f(x,u) dx du) H_{0,1}, \tag{7.52}$$

and the number ν_0 defined by (2.40) is 1. Moreover, if $q \in \mathbb{N}$ with $q \geq 2$ and $m > \frac{q}{2} + \frac{7}{4}$ and $X = L^2_{m,0}$ then the conditions **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, *and* **(A2)'** *are satisfied with $\mu_j = \frac{1}{2}$, $\varrho = \frac{1}{2}$, and $\zeta = \frac{q}{2}$. the number $\tilde{\nu}_0$ defined by (2.47) is 1 also in this case.*

Proof. It is easy to see from (7.40), (7.48), and (7.49) that $R_\lambda e^{\lambda t\mathcal{A}} = e^{t\mathcal{A}}R_\lambda$ and $\tau_{a,\theta+t}e^{t\mathcal{A}} = e^{t\mathcal{A}}\tau_{a,\theta}$ for each $t > 0$, which implies **(E1)** and **(E2)**. Let $f \in \mathrm{Dom}(B) \cap \mathrm{Dom}(D_\theta) \cap \mathrm{Dom}(\Gamma_{a,\theta})$. We will show $\tau_{a,\theta}f \in \mathrm{Dom}(B)$. Indeed, if $a = 0$ then $\tau_{0,\theta} = I$ and thus $f \in \mathrm{Dom}(B)$. If $a \neq 0$ then $\partial_x f \in L^2_m$ by (7.2). Thus $\partial_u f$ also belongs to L^2_m since $D_\theta f = (\theta\partial_x + \partial_u)f \in L^2_m$. The assertion $\tau_{a,\theta}f \in \mathrm{Dom}(B)$ follows from the equality

$$(\frac{3x}{2}\partial_x + \frac{u}{2}\partial_u)\tau_{a,\theta}f = (\frac{3(x+\theta a)}{2}\partial_x + \frac{u+a}{2}\partial_u)\tau_{a,\theta}f - (\frac{3\theta a}{2}\partial_x + \frac{a}{2}\partial_u)\tau_{a,\theta}f.$$

The condition **(T1)** is now veri ed from the above equality and (7.2). Suppose that $f \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D_1)$ satis es

$$a_1Bf + a_2D_1f = 0. \tag{7.53}$$

Then by (7.45) we have $\partial_u f \in L^2_m$, and hence, $\partial_x f \in L^2_m$ by the de nition of D_1. If $a_1 \neq 0$ then we multiply both sides of (7.53) by f and integrate over $\mathbb{R}^2$, which yields by integration by parts that $\frac{3a_1}{2}\|f\|^2_{L^2} = 0$, i.e., $f = 0$. If $a_1 = 0$ then multiplying both sides of (7.53) by xf and integrating over $\mathbb{R}^2$, we have $\frac{a_2}{2}\|f\|^2_{L^2} = 0$. This implies **(T2)**.

We note that $e^{tA} = e^{(1-e^{-t})\mathcal{A}}R_{e^t}$ is expressed as

$$e^{tA}f = \frac{\sqrt{3}e^{\frac{5t}{2}}}{2\pi a(t)^2}\int_{\mathbb{R}^2} e^{-\frac{3}{a(t)^3}\{x-y-\frac{a(t)}{2}(u+v)\}^2-\frac{(u-v)^2}{4a(t)}} f(ye^{\frac{3t}{2}}, ve^{\frac{t}{2}})dydv, \tag{7.54}$$

where $a(t) = 1-e^{-t}$. Let $m \geq \frac{q}{2}+\frac{7}{4}$. In [14] it is proved that $r_{ess}(e^{tA}) \leq e^{-\frac{qt}{2}}$ in L^2_m and the spectrum of A in L^2_m is estimated as

$$\sigma(A) \subset \{\mathrm{Re}(\mu) \leq -\frac{q}{2}\} \cup \{\frac{1}{2} - \frac{l}{2} \mid l = 1, 2, \cdots\}. \tag{7.55}$$

Moreover, if $q+1 > l$ and $l \in \mathbb{N}\cup\{0\}$ then $\frac{1}{2} - \frac{l}{2}$ is a semisimple eigenvalue and its eigenspace is spanned by $\{H_{j,k}\}_{3j+k=l}$; the associated eigenprojection is given by

$$\mathcal{P}_l f = \sum_{3j+k=l} < f, H^*_{j,k} > H_{j,k}, \tag{7.56}$$

where

$$< f, g >= \int_{\mathbb{R}^2} f(x,u)g(x,u)e^{3(x-\frac{u}{2})^2+\frac{1}{4}u^2}dxdu,$$

and $H^*_{j,k} = (\partial_x + 3\partial_u)^j(\partial_x + 2\partial_u)^k e^{-3(x-\frac{u}{2})^2-\frac{1}{4}u^2}$. Especially, the direct calculations show that

$$\mathcal{P}_0 f = (\int_{\mathbb{R}^2} f(x,u)dxdu)H_{0,0}, \qquad \mathcal{P}_1 f = (\int_{\mathbb{R}^2} uf(x,u)dxdu)H_{0,1}.$$

Hence, the rst and the second eigenvalues of A in $L^2_{m,0}$ are simple and given by 0 and $-\frac{1}{2}$, respectively, and the eigenprojection for the eigenvalue 0 is

$$\mathbf{P_{0,0}}f = \mathcal{P}_1 f = (\int_{\mathbb{R}^2} uf(x,u)dxdu)H_{0,1}. \tag{7.57}$$

Since the third eigenvalue of A in $L^2_{m,0}$ is -1, we observe that ν_0 de ned by (2.40) is equal to 1. This completes the proof.

Let $\mathcal{O} \subset \mathbb{R}^2$ be a small open ball centered at the origin. Let $f \in L^2_{m,0}$. For (7.38) the map $S(\cdot; f) : \mathcal{O} \to L^2_{m,0}$ is de ned by

$$\begin{aligned} S(y;f)(x,u) &= (\tau_{y_1,1+y_2} R_{\frac{1}{1+y_2}} f)(x,u) \\ &= (1+y_2)^{-\frac{5}{2}} f\big(\frac{x+y_1(1+y_2)}{(1+y_2)^{\frac{3}{2}}}, \frac{u+y_1}{(1+y_2)^{\frac{1}{2}}}\big). \end{aligned} \tag{7.58}$$

From the de nition of $H_{0,1}$ we have

Proposition 7.10. *Let $m \geq 0$. Then the map $S(\cdot; H_{0,1}) : \mathcal{O} \to L^2_{m,0}$ is C^∞.*

Let us give the estimates for derivatives of $e^{tA}f$, which are essentially obtained in [14].

Proposition 7.11. *Let $m \geq \frac{9}{4}$. Then for any nonnegative integer s and $j_1, j_2 \in \{0,1\}$ we have*

$$\|\partial_x^{j_1}\partial_u^{j_2} e^{tA} f\|_{H^s_m} \leq C(1+t^{-\frac{3j_1+j_2}{2}})\|f\|_{H^s_m}, \qquad t>0, \tag{7.59}$$

for any $f \in L^2_{m,0} \cap H^s_m$. We also have

$$\|e^{tA}\partial_u f\|_{H^s_m} \leq C(1+t^{-\frac{1}{2}})\|f\|_{H^s_m}, \qquad t>0, \tag{7.60}$$

for any $f \in L^2_m \cap H^s_m$.

Remark 7.5. If $m \geq \frac{11}{4}$ and if $f \in \mathbf{Q}_{0,0}L^2_{m,0} \cap H^s_m$, then (7.59) is replaced by

$$\|\partial_x^{j_1}\partial_u^{j_2} e^{tA} f\|_{H^s_m} \leq C(1+t^{-\frac{3j_1+j_2}{2}})e^{-\frac{t}{2}}\|f\|_{H^s_m}, \qquad t>0. \tag{7.61}$$

This is proved from the estimate $\|e^{tA}f\|_{L^2_m} \leq Ce^{-\frac{t}{2}}\|f\|_{L^2_m}$ for $f \in \mathbf{Q}_{0,0}L^2_{m,0}$, (7.59), and the semigroup property of e^{tA}. We omit the details here.

Proof of Proposition 7.11. Est. (7.59) is already observed in [14], but we give the proof for convenience to the reader. Let $0 < t \leq 2$. Recalling the relation $e^{tA} = e^{(1-e^{-t})\mathcal{A}}R_{e^t}$, we have from (7.43) that

$$\|\partial_x^{j_1}\partial_u^{j_2} e^{tA} f\|_{H^s_m} \leq Ct^{-\frac{3j_1+j_2}{2}}\|R_{e^t}f\|_{H^s_m} \leq Ct^{-\frac{3j_1+j_2}{2}}\|f\|_{H^s_m}, \quad 0<t\leq 2.$$

For the case $t \geq 2$ we use the semigroup property and obtain $\|\partial_x^{j_1}\partial_u^{j_2} e^{tA} f\|_{H^s_m} \leq C\|e^{(t-1)A} f\|_{L^2_m} \leq C\|f\|_{L^2_m}$. Let us prove (7.60). It suffices to consider the case $t \leq 2$ by the semigroup property. Then from (7.44) we get

$$\|e^{tA}\partial_u f\|_{H^s_m} = \|R_{e^t} e^{(e^t-1)\mathcal{A}}\partial_u f\|_{H^s_m} \leq C\|e^{(e^t-1)\mathcal{A}}\partial_u f\|_{H^s_m} \leq Ct^{-\frac{1}{2}}\|f\|_{H^s_m}.$$

This completes the proof.

Next we consider the nonlinear term $\mathcal{N}(f) = E(f)\partial_u f$. From (7.39) it is easy to see that for $f \in L^2_m$ with $m > 1$,

(7.62)

$$\|\mathcal{N}(f)\|_{L^2_m} \leq \|E(f)\|_{L^\infty}\|\partial_u f\|_{L^2_m} \leq \|f\|_{L^1}\|\partial_u f\|_{L^2_m} \leq C\|f\|_{L^2_m}\|\partial_u f\|_{L^2_m}.$$

Then we have

Proposition 7.12. *Let $m \geq \frac{11}{4}$ and $X = L^2_{m,0}$. Then $\mathcal{N}(f) = E(f)\partial_u f$ satisfies the conditions* **(N1)**, **(N2)**, *and* **(N3)** *with $q = \alpha = 2$, $\beta = \frac{1}{2}$, and $\epsilon_0 = 0$.*

Proof. From (7.59) with $l = 0$ and $k = 1$ we have $\partial_u(I-A)^{-1}$ is a bounded operator in $L^2_{m,0}$ as in the proof of (7.45). So (7.62) yields $\|\mathcal{N}(f)\|_{L^2_m} \leq C\|f\|_{L^2_m}\|f\|_{\mathrm{Dom}(A)}$.

Let us show that $\mathcal{N}$ maps $\mathrm{Dom}(A) \cap L^2_{m,0}$ into

$$\mathbf{Q}_{0,0}L^2_{m,0} = \{f \in L^2_{m,0} \mid \int_{\mathbb{R}^2} uf(x,u)dxdu = 0\}.$$

Indeed, since $\int_{\mathbb{R}^2} f(x,u)dxdu = 0$ we have

$$E(f) = \int_{-\infty}^x \int_{\mathbb{R}} f(y,v)dvfy =: F(x).$$

Then it follows that by integration by parts,

$$\begin{aligned}\int_{\mathbb{R}^2} uE(f)\partial_u f dxdu &= -\int_{\mathbb{R}^2} E(f)fdxdu \\ &= -\int_{\mathbb{R}} F(x)(\int_{\mathbb{R}} f(x,u)du)dx \\ &= -\int_{\mathbb{R}} F(x)F'(x)dx = 0.\end{aligned}$$

This proves the claim, and **(N1)** holds. The condition **(N3)** is easily checked, so we omit the proof of it. Finally we consider **(N2)**. Let $N(t,f) =$

$e^{tA}\mathcal{N}(f)$ with $f \in L^2_{m,0}$. If $t \le 1$ then we have from (7.60),

$$\begin{aligned} \|N(t,f) - N(t,g)\|_{L^2_m} &= \|e^{tA}\partial_u\big(E(f-g)f + E(g)(f-g)\big)\|_{L^2_m} \\ &\le Ct^{-\frac{1}{2}}\|E(f-g)f + E(g)(f-g)\|_{L^2_m} \\ &\le Ct^{-\frac{1}{2}}\|f-g\|_{L^2_m}(\|f\|_{L^2_m} + \|g\|_{L^2_m}). \end{aligned}$$

When $t \ge 1$ we rst note that $\partial_u\big(E(f-g)f + E(g)(f-g)\big) \in \mathbf{Q}_{0,0}L^2_{m,0}$ if $f, g \in L^2_{m,0} \cap \mathrm{Dom}(A)$. Thus in this case, by using the estimate

$$\|e^{tA}f\|_{L^2_m} \le Ce^{-\frac{t}{2}}\|f\|_{L^2_m}$$

for $f \in \mathbf{Q}_{0,0}L^2_{m,0}$ and (7.60), we have

$$\begin{aligned} \|N(t,f) - N(t,g)\|_{L^2_m} &= \|e^{(t-1)A}e^{1A}\partial_u\big(E(f-g)f + E(g)(f-g)\big)\|_{L^2_m} \\ &\le Ce^{-\frac{t}{2}}\|e^{1A}\partial_u\big(E(f-g)f + E(g)(f-g)\big)\|_{L^2_m} \\ &\le Ce^{-\frac{t}{2}}\|f-g\|_{L^2_m}(\|f\|_{L^2_m} + \|g\|_{L^2_m}). \end{aligned}$$

From the density arguments the above inequality is valid for any $f, g \in L^2_{m,0}$. Since the Frechet derivative of $N(t,\cdot)$ is formally given by $N'(t,f)h = e^{tA}\partial_u\big(E(f)h + E(h)f\big)$, by the similar arguments as above, we have

$$\|N'(t,f)h - N'(t,g)h\|_{L^2_m} \le C(\frac{1+t}{t})^{\frac{1}{2}}e^{-\frac{t}{2}}\|f-g\|_{L^2_m}\|h\|_{L^2_m}.$$

Hence **(N2)** follows and the proof of Proposition 7.12 is completed.

Let $m \ge \frac{11}{4}$. Then by Theorem 2.1 there is a $\delta_0 > 0$ such that for each δ with $|\delta| \le \delta_0$ there is a $U_\delta \in L^2_m$ which gives a self-similar solution $R_{\frac{1}{1+t}}U_\delta$ to (7.38) and is C^2 in L^2_m with respect to δ. Note that U_δ is of the form $U_\delta = \delta H_{0,1} + v_\delta$ from Theorem 4.1, and $v_\delta \in \mathbf{Q}_{0,0}L^2_{m,0} \cap \mathrm{Dom}(A)$ solves

$$v_\delta = (-A)^{-1}\mathcal{N}(\delta H_{0,1} + v_\delta) = \int_0^\infty e^{tA}\mathcal{N}(\delta H_{0,1} + v_\delta)dt. \tag{7.63}$$

Since $v_\delta \in \mathbf{Q}_{0,0}L^2_{m,0} \cap \mathrm{Dom}(A)$ implies $\partial_u v_\delta \in L^2_m$, $\mathcal{N}(\delta H_{0,1} + v_\delta)$ belongs to $\mathbf{Q}_{0,0}L^2_{m,0}$ by (7.62). In order to apply Theorem 2.3 we need the regularity of $v_\delta \in H^2_{m+2}$. For this purpose we resolve (7.63) in H^2_{m+2} below.

Proposition 7.13. *Let $m \ge \frac{11}{4}$. If $|\delta|$ is sufficiently small, then there is a solution v_δ to (7.63) in H^2_{m+2} such that $\|v_\delta\|_{H^2_{m+2}} \le \bar{C}|\delta|^2$ for some constant $\bar{C} > 0$. Moreover, this solution is unique in the set $\{f \in L^2_m \mid \|f\|_{L^2_m} \le \bar{C}|\delta|\}$, and v_δ is C^2 in H^2_{m+2} with respect to δ.*

Proof. We rst prove the uniqueness. Let $\tilde{v}_1$ and $\tilde{v}_2$ be two solutions to (7.63) with $\|\tilde{v}_i\|_{L^2_m} \le \bar{C}|\delta|$. Recalling the de nition $N(t,f) = e^{tA}\mathcal{N}(f)$, we have

$$\tilde{v}_1 - \tilde{v}_2 = \int_0^\infty N'(t, \delta H_{0,1} + \tilde{v}_1)(\tilde{v}_1 - \tilde{v}_2)dt - \int_0^\infty N(t, \tilde{v}_1 - \tilde{v}_2)dt.$$

Then from **(N2)** we observe that

$$\|\tilde{v}_1 - \tilde{v}_2\|_{L^2_m} \le C\bar{C}|\delta| \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{t}{2}} \|\tilde{v}_1 - \tilde{v}_2\|_{L^2_m} dt \le C\bar{C}|\delta| \|\tilde{v}_1 - \tilde{v}_2\|_{L^2_m}.$$

Thus if $|\delta|$ is small enough, we have $\tilde{v}_1 = \tilde{v}_2$. Next we nd a solution to (7.63) in the ball $B_\delta = \{f \in H^2_{m+2} \mid \|f\|_{H^2_{m+2}} \le \bar{C}|\delta|^2\}$. The proof is just as same as in Theorem 4.1. For $v \in B_\delta$ we set

(7.64)
$$\Psi(v) = \int_0^\infty e^{tA}\mathcal{N}(\delta H_{0,1} + v)dt = \int_0^\infty e^{tA}\partial_u\big(E(\delta H_{0,1} + v)(\delta H_{0,1} + v)\big)dt.$$

From (7.60) and (7.61) we can see that

$$\|e^{tA}\partial_u E(f)f\|_{H^2_{m+2}} \le C(\frac{1+t}{t})^{\frac{1}{2}} e^{-\frac{t}{2}} \|E(f)f\|_{H^2_{m+2}},$$

holds. Combining with the estimate $\|E(f)f\|_{H^2_{m+2}} \le C\|f\|^2_{H^2_{m+2}}$, we get

$$\|\Psi(v)\|_{H^2_{m+2}} \le C\|\delta H_{0,1} + v\|^2_{H^2_{m+2}}.$$

Similarly we also have

$$\|\Psi(v_1) - \Psi(v_2)\|_{H^2_{m+2}} \le C(|\delta| + \|v_1\|_{H^2_{m+2}} + \|v_2\|_{H^2_{m+2}})\|v_1 - v_2\|_{H^2_{m+2}}.$$

These estimates are enough to conclude that Ψ is a contraction mapping in B_δ, and hence there is a unique xed point v_δ in B_δ. Since $\mathcal{N}$ is a bilinear form, it is also easy to see that v_δ is C^2 in H^2_{m+2} with respect to δ. This completes the proof.

By Theorem 2.2 and the spectral property of A it is not difficult to show that the solution $\Omega(t) \in C([0,\infty); L^2_{m,0})$ to (7.38) with $\|\Omega(0)\|_{L^2_m} \ll 1$ satis es

$$\|\Omega(t) - (1+t)^{-\frac{5}{2}} U_\delta\big(\frac{x}{(1+t)^{\frac{3}{2}}}, \frac{u}{(1+t)^{\frac{1}{2}}}\big)\|_{L^p} \le C_\epsilon (1+t)^{-3+\frac{2}{p}+\epsilon}, \tag{7.65}$$

for all $t \gg 1$, $\epsilon > 0$, and $1 \le p \le 2$, where $\delta = \int_{\mathbb{R}^2} u\Omega(0,x,u)dxdu$. We will improve the asymptotic pro le by considering a shift of the self-similar solution.

Let $|\delta| < \delta_0 \ll 1$. For $(y_0, y) \in (-\delta_0 + \delta, \delta_0 - \delta) \times \mathcal{O} \subset \mathbb{R}^3$ we set

(7.66)

$$H(y_0, y; U_\delta) = S(y; U_{\delta+y_0}) = (1+y_2)^{-\frac{5}{2}} U_{\delta+y_0}\Big(\frac{x + y_1(1+y_2)}{(1+y_2)^{\frac{3}{2}}}, \frac{u+y_1}{(1+y_2)^{\frac{1}{2}}}\Big).$$

Then Proposition 7.13 immediately leads to

Corollary 7.1. *Let $m \geq \frac{11}{4}$. Then $H(y_0, y; U_\delta)$ is C^2 as a mapping from $(-\delta_0 + \delta, \delta_0 - \delta) \times \mathcal{O}$ into L^2_m.*

Proof. From Proposition 7.13 we see $U_\delta = \delta H_{0,0} + v_\delta \in H^2_{m+2}$ and is C^2 in H^2_{m+2} with respect to δ. Then from the de nition of (7.66) we have the claim. The proof is completed.

Now we can apply Theorem 2.4 to (7.38) and obtain

Theorem 7.5. *Let $m > \frac{11}{4}$. Assume that $\Omega(t) \in C([0,\infty); L^2_{m,0})$ is the solution to (7.38) satisfying $\|\Omega(0)\|_{L^2_m} \ll 1$ and $\int_{\mathbb{R}^2} u\Omega(0, x, u)dxdu = \delta \neq 0$. Then there are $\tilde{\eta}(\delta) \in \mathbb{R}$ and $y_1^* \in \mathbb{R}$ such that $\lim_{\delta\to 0} \tilde{\eta}(\delta) = 0$ and*

(7.67)

$$\|\Omega(t) - (1+t)^{-\frac{5}{2}} U_\delta\Big(\frac{x + (1+t)y_1^*}{(1+t)^{\frac{3}{2}}}, \frac{u + y_1^*}{(1+t)^{\frac{1}{2}}}\Big)\|_{L^p} \leq C_\epsilon (1+t)^{-\frac{7}{2}+\frac{2}{p}+\tilde{\eta}(\delta)+\epsilon},$$

holds for all $t \gg 1$, $\epsilon > 0$, and $1 \leq p \leq 2$.

Remark 7.6. The assumption that the mass of initial data is zero is essential in our arguments, although it will be less physical. The large time behavior of solutions to (7.38) for general initial data seems to be difficult questions. We remark that, for multi-dimensional Vlasov-Poisson-Fokker-Planck equations, the higher order asymptotic expansions of small solutions at large time are already established in [14] by using the invariant manifolds theory.

Proof of Theorem 7.5. Since $\tilde{\nu}_0 = 1 > \frac{1}{2} = \mu_1$ by Proposition 7.9, we see from Theorem 2.4 that there are $\tilde{\eta}(\delta)$ and $y_1^* \in \mathbb{R}$ such that

$$\|R_{1+t}\Omega(t) - \tilde{S}((1+t)^{-\frac{1}{2}} y_1^*; U_\delta)\|_{L^2_m} \leq C(1+t)^{-1+\tilde{\eta}(\delta)+\epsilon},$$

if $t \gg 1$. Hence from $\|R_\lambda f\|_{L^p} = \lambda^{2(1-\frac{1}{p})+\frac{1}{2}} \|f\|_{L^p}$ we have

$$\|\Omega(t) - R_{\frac{1}{1+t}} \tilde{S}((1+t)^{-\frac{1}{2}} y_1^*; U_\delta)\|_{L^p} \leq C(1+t)^{-\frac{7}{2}+\frac{2}{p}+\tilde{\eta}(\delta)+\epsilon},$$

for $1 \leq p \leq 2$, which gives (7.67). The proof is completed.

Remark 7.7. If we could show -1 is a semisimple eigenvalue of L_δ then we would also apply Theorem 2.3 to obtain more precise asymptotic pro le as in the case of the Keller-Segel system (Theorem 7.4).

References

[1] J. Aguirre, M. Escobedo, E. Zuazua, Self-similar solutions of a convection-diffusion equation and related elliptic problems, Commun. P. D. E. 15 (2) (1990) 139-157.

[2] P. Biler, Local and global solvability of some parabolic systems modelling chemotaxis, Adv. Math. Sci. Appl. 8 (2) (1998) 715-743.

[3] A. Carpio, Asymptotic behavior for the vorticity equations in dimensions two and three, Commun. P. D. E. 19 (1994) 827-872.

[4] P. Degond, Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimensions, Ann. Sci. Ecole Norm. Sup. (4) 19 (4) (1986) 519-542.

[5] I.-L. Chern, T.-P. Liu, Convergence to diffusion waves of solutions for viscous conservation laws, Commun. Math. Phys. 110 (1987) 503-517.

[6] K.-J. Engel, R. Nagel, One-Parameter semigroups for linear evolution equations, Graduate Texts in Mathematics, New York, Springer-Verlag, 2000.

[7] M. Escobedo, O. Kavian, Variational problems related to self-similar solutions for the heat equation, Nonlinear Anal. TMA 11 (1987) 1103-1133.

[8] M. Escobedo, E. Zuazua, Large time behavior for convection-diffusion equations in $\mathbb{R}^n$, J. Funct. Anal. 100 (1991) 1190-161.

[9] Th. Gallay, C. E. Wayne, Invariant Manifolds and the Long-Time Asymptotics of the Navier-Stokes and Vorticity Equations on $\mathbb{R}^2$, Arch. Rational Mech. Anal. 163 (2002) 209-258.

[10] Th. Gallay, C. E. Wayne, Global Stability of vortex solutions of the two dimensional Navier-Stokes equation, Comm. Math. Phys. 255 (1) (2005) 97-129.

[11] Th. Gallay, C. E. Wayne, Existence and stability of asymmetric Burgers vortices, J. Math. Fluid Mech. 9 (2007) 243-261.

[12] Y. Giga, M.-H. Giga, Nonlinear Partial Differential Equation, Self-similar solutions and asymptotic behavior, Kyoritsu, 1999 (in Japanese), English version to be published by Birkhäuser.

[13] Y. Giga, T. Kambe, Large time behavior of the vorticity of two dimensional viscous flow and its application to vortex formation, Commun. Math. Phys. 117 (1988) 549-568.

[14] Y. Kagei, Invariant manifolds and long-time asymptotics for the Vlasov-Poisson-Fokker-Planck equation, SIAM J. Math. Anal. 33 (2) (2001) 489-507.

[15] Y. Kagei, Y. Maekawa, On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis.

[16] M. Kato, Sharp asymptotics for a parabolic system of chemotaxis in one space dimension, Osaka Univ. Research Report in Math. 07-03.

[17] T. Kato, Perturbation Theory for Linear Operators, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[18] S. Kawashima, Self-similar solutions of a convection-diffusion equation, Lecture Notes in Num. Appl. Anal. 12 (1993) 123-136.

[19] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 1995.

[20] Y. Maekawa, Spectral properties of the linearization at the Burgers vortex in the high rotation limit, to appear in J. Math. Fluid Mech.

[21] J. C. Miller, A. J. Bernoff, Rates of convergence to self-similar solutions of Burgers's equation, Stud. Appl. Math. 111 (1) (2003) 29-40.

[22] T. Nagai, T. Yamada, Large time behavior of bounded solutions to a parabolic system of chemotaxis in the whole space, J. Math. Anal. Appl. 336 (2007) 704-726.

[23] Y. Naito, Asymptotically self-similar solutions for the parabolic system modelling chemotaxis, Self-similar solutions of nonlinear PDE, Banach Center Publ., 74, Polish Acad. Sci. Warsaw (2006) 149-160.

[24] K. Nishihara, Asymptotic profile of solutions to a parabolic system of chemotaxis in one dimensional space, preprint.

[25] H. D. Victory, B. P. O 'Dwyer, On classical solutions of Vlasov-Poisson Fokker-Planck systems, Indiana Univ. Math. J. 39 (1990) 105-156.

[26] C. E. Wayne, Invariant manifolds for parabolic partial differential equations on unbounded domains, Arch. Rational Mech. Anal. 138 (3) (1997) 279-306.

[27] T. Yanagisawa, Asymptotic behavior of solutions to the viscous Burgers equation, Osaka J. Math. 44 (1) (2007) 99-119.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2 Eiji ONODERA
The intial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in nfinite extensions over a p-adic field

MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials

MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization

MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5 Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings

MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2-projection