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Typical ranks of $m \times n \times (m-1)n$ tensors with $3 \leq m \leq n$ over the real number field

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Let $3 \leq m \leq n$. We study typical ranks of $m \times n \times (m-1)n$ tensors over the real number field. Let $\rho$ be the Hurwitz-Radon function defined as $\rho(n) = 2^b + 8c$ for nonnegative integers $a, b, c$ such that $n = (2a + 1)^2 + b$ and $0 \leq b < 4$. If $m \leq \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has two typical ranks $(m-1)n, (m-1)n + 1$. In this paper, we show that the converse is also true: if $m > \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has only one typical rank $(m-1)n$.

Keywords: tensor; 3-way array; typical rank; Hurwitz-Radon number; perfect

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1. Introduction

For positive integers $m, n, p$, an $m \times n \times p$ tensor is an element of tensor product of vector spaces $V_1, V_2, V_3$ of dimension $m, n, p$, respectively. If we choose bases of their vector spaces, we can identify a tensor with a 3-way array

$$(a_{ijk})_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p}.$$

Hitchcock [6] defined the rank of a tensor. An $m \times n \times p$ tensor of form

$$(x_{ij}y_{jk})_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p}$$

is called a rank one tensor. The rank of a tensor $T$, denoted by rank$T$, is defined as the minimal number of rank one tensors which describe $T$ as a sum. The rank depends on the base field. For example there is a $2 \times 2 \times 2$ tensor over the real number field whose rank is 3 but is 2 as a tensor over the complex number field.

Throughout this paper, we assume that the base field is the real number field $\mathbb{R}$. Let $\mathbb{R}^{m \times n \times p}$ be the set of $m \times n \times p$ tensors with Euclidean topology. A number $r$ is a typical rank of $m \times n \times p$ tensors if the set of tensors with rank $r$ contains a nonempty open semi-algebraic set of $\mathbb{R}^{m \times n \times p}$ (see Theorem 2.1). We denote by typical rank$_{\mathbb{R}}(m, n, p)$

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the set of typical ranks of $\mathbb{R}^{m \times n \times p}$. If $s$ (resp. $t$) is the minimal (resp. maximal) number of typical rank $\text{rank}_R(m, n, p)$, then

$$\text{typical rank}_R(m, n, p) = [s, t] \cap \mathbb{Z},$$

the interval of all integers between $s$ and $t$, including both, and $s$ is equal to the generic rank of the set of $m \times n \times p$ tensors over the complex number field $[5]$. In the case where $m = 2$, the set of typical ranks of $2 \times n \times p$ tensor is well-known $[15]$:

$$\text{typical rank}_R(2, n, p) = \begin{cases} \{p\}, & n < p \leq 2n \\ \{2n\}, & 2n < p \\ \{p, p+1\}, & n = p \geq 2 \end{cases}$$

Suppose that $3 \leq m \leq n$. If $p > (m-1)n$ then the set of typical ranks of $m \times n \times p$ tensors is just $\{\min(p, mn)\}$. If $p = (m-1)n$ then the set of typical ranks of $m \times n \times p$ tensor is $\{p\}$ or $\{p, p+1\}$ $[14]$. Until our paper $[13]$, only a few cases where typical rank $\text{rank}_R(m, n, (m-1)n) = \{p, (m-1)n\}$ $[3, 5]$ are known and we constructed infinitely many examples by using the concept of absolutely nonsingular tensors in $[13]$: If $m \leq \rho(n)$ then typical rank $\text{rank}_R(m, n, p) = \{p, p+1\}$, where $\rho(n)$ is the Hurwitz-Radon number given by $\rho(n) = 2^b + 8c$ for nonnegative integers $a, b, c$ such that $n = (2a+1)2^{b+4c}$ and $0 \leq b < 4$.

$$\begin{array}{c|cccccccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 16 & 32 & 64 \\
 \hline
 \rho(n) & 1 & 2 & 1 & 4 & 1 & 4 & 1 & 8 & 1 & 2 & 1 & 4 & 9 & 10 & 12 \\
\end{array}$$

The purpose of this paper is to completely determine the set of typical ranks of $m \times n \times (m-1)n$ tensors:

**Theorem 1.1:** Let $3 \leq m \leq n$ and $p = (m-1)n$. Then it holds

$$\text{typical rank}_R(m, n, p) = \begin{cases} \{p\}, & m > \rho(n) \\ \{p, p+1\}, & m \leq \rho(n) \end{cases}$$

We denote an $m_1 \times m_2 \times m_3$ tensor $(x_{ijk})$ by $(X_1; \ldots; X_{m_3})$, where $X_t = (x_{ijk})$ is an $m_1 \times m_2$ matrix for each $1 \leq t \leq m_3$. Let $3 \leq m \leq n$ and $p = (m-1)n$. For an $n \times p \times m$ tensor $X = (X_1; \ldots; X_{m-1}; X_m)$, let $H(X)$ and $\hat{H}(X)$ be a $p \times p$ matrix and an $mn \times p$ matrix respectively defined as follows.

$$H(X) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix}, \quad \hat{H}(X) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix}$$

Let

$$\mathcal{R} = \{X \in \mathbb{R}^{n \times p \times m} \mid H(X) \text{ is nonsingular}\}.$$
This is a nonempty Zariski open set. For $X = (X_1; \ldots; X_m) \in \mathcal{R}$, we see

$$\hat{H}(X)H(X)^{-1} = \begin{pmatrix} E_n & \cdots & E_n \\ E_n & \cdots & E_n \\ \vdots & \ddots & \vdots \\ Y_1 & \cdots & Y_m \end{pmatrix},$$

where $E_n$ is the $n \times n$ identity matrix and $(Y_1, Y_2, \ldots, Y_m) = X_mH(X)^{-1}$. Note that $\text{rank } X \geq p$ for $X \in \mathcal{R}$. Let $h$ be an isomorphism from the set of $n \times p$ matrices to $\mathbb{R}^{n \times n \times (m-1)}$ given by

$$(Y_1, Y_2, \ldots, Y_m) \mapsto (Y_1; Y_2; \ldots; Y_m).$$

Then $h(X_mH(X)^{-1}) \in \mathbb{R}^{n \times n \times (m-1)}$. We consider the following subsets of $\mathbb{R}^{n \times n \times (m-1)}$. For $Y = (Y_1; Y_2; \ldots; Y_m) \in \mathbb{R}^{n \times n \times (m-1)}$ and $a = (a_1, \ldots, a_m, a_m) \in \mathbb{R}^m$, let

$$M(a, Y) = \sum_{k=1}^{m-1} a_k Y_k - a_m E_n$$

and set

$$\mathcal{C} = \{ Y \in \mathbb{R}^{n \times n \times (m-1)} \mid |M(a, Y)| < 0 \text{ for some } a \in \mathbb{R}^m \}$$

and

$$\mathcal{A} = \{ Y \in \mathbb{R}^{n \times n \times (m-1)} \mid |M(a, Y)| > 0 \text{ for all } a \neq 0 \}.$$

Here, $|M(a, Y)|$ is the determinant of the matrix $M(a, Y)$. The subsets $\mathcal{C}$ and $\mathcal{A}$ are open sets in Euclidean topology and $\mathcal{C} \cup \mathcal{A} = \mathbb{R}^{n \times n \times (m-1)}$. In [13], we show that $\mathcal{A}$ is not empty if and only if $m \leq p(n)$ and that $\text{rank } X > p$ for any $X \in \mathcal{R}$ with $h(X_mH(X)^{-1}) \in \mathcal{A}$. In this paper, we show that there exists an open subset $\mathcal{F}$ of $\mathcal{C}$ such that $\mathcal{F} = \mathcal{C}$ and $\text{rank } X = p$ for any $X \in \mathcal{R}$ with $h(X_mH(X)^{-1}) \in \mathcal{F}$.

2. Typical rank

Due to [11, 14] and others, a number $r$ is a typical rank of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if the subset of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ of rank $r$ has nonzero volume. De Silva and Lim [4] studied the space of real tensors and tensor rank with Euclidean topology and real semi-algebraic sets. They [4, Theorem 6.1] showed that for a positive integer $r$, the set of tensors with rank $r$ is a semi-algebraic set by using the Tarski-Seidenberg principle (cf. [1]). In the sense, a number $r$ is a typical rank if the set of tensors with rank $r$ contains an open semi-algebraic set.

For $x = (x_1, \ldots, x_{m_1})^T \in \mathbb{C}^{m_1}$, $y = (y_1, \ldots, y_{m_2})^T \in \mathbb{C}^{m_2}$, and $z = (z_1, \ldots, z_{m_3})^T \in \mathbb{C}^{m_3}$, we denote $(x_1 y_2 z_3) \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ by $x \otimes y \otimes z$. Let $f_t : (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^t \rightarrow \mathbb{C}^{m_1 \times m_2 \times m_3}$ be a map given by

$$f_t(x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{t,1}, x_{t,2}, x_{t,3}) = \sum_{t=1}^{t} x_{t,1} \otimes x_{t,2} \otimes x_{t,3}.$$
Let $S$ be a subset of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. $S$ is called semi-algebraic if it is a finite Boolean combination (that is, a finite composition of disjunctions, conjunctions and negations) of sets of the form

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid f(a_{111}, \ldots, a_{m_1, m_2, m_3}) > 0\}$$

and

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid g(a_{111}, \ldots, a_{m_1, m_2, m_3}) = 0\},$$

where $f$ and $g$ are polynomials in $m_1 m_2 m_3$ indeterminates $x_{111}, \ldots, x_{m_1, m_2, m_3}$ over $\mathbb{R}$. Then $S$ is an open semi-algebraic set if and only if it is expressed as a finite Boolean combinations of sets of the form (1), and it is a dense open semi-algebraic set if and only if it is a Zariski open set, that is, expressed as

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid g(a_{111}, \ldots, a_{m_1, m_2, m_3}) \neq 0\}.$$

**Theorem 2.1** ([5, Theorem 7.1]): The space $\mathbb{R}^{m_1 \times m_2 \times m_3}$, $m_1, m_2, m_3 \in \mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $\mathcal{O}_1, \ldots, \mathcal{O}_M$ satisfying the following properties.

1. $\mathbb{R}^{m_1 \times m_2 \times m_3} \setminus \bigcup_{i=1}^{M} \mathcal{O}_i$ is a closed semi-algebraic set $\mathbb{R}^{m_1 \times m_2 \times m_3}$ of dimension less than $m_1 m_2 m_3$.
2. Each $\mathcal{O}_i$ has rank $r_i$ for $i = 1, \ldots, M$.
3. The number $\min(r_1, \ldots, r_M)$ is equal to the generic rank $\text{grank}(m_1, m_2, m_3)$ of $\mathbb{C}^{m_1 \times m_2 \times m_3}$, that is, the minimal $t \in \mathbb{N}$ such that the closure of the image of $f_i$ is equal to $\mathbb{C}^{m_1 \times m_2 \times m_3}$.
4. $\text{mtrank}(m_1, m_2, m_3) := \max(r_1, \ldots, r_M)$ is the minimal $t \in \mathbb{N}$ such that the closure of $f_i((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^t)$ is equal to $\mathbb{R}^{m_1 \times m_2 \times m_3}$.
5. For each integer $r \in [\text{grank}(m_1, m_2, m_3), \text{mtrank}(m_1, m_2, m_3)]$, there exists $r_i = r$ for some integer $i \in [1, M]$.

**Definition 2.2:** A positive integer $r$ is called a typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if

$r \in [\text{grank}(m_1, m_2, m_3), \text{mtrank}(m_1, m_2, m_3)].$

Put

$$\text{typical\_rank}_\mathbb{R}(m_1, m_2, m_3) = [\text{grank}(m_1, m_2, m_3), \text{mtrank}(m_1, m_2, m_3)] \cap \mathbb{Z}.$$

This is the set of all typical ranks of $\mathbb{R}^{m_1 \times m_2 \times m_3}$.

We state basic facts. Some of them are seen in [4].

**Proposition 2.3:** Let $r$ be a positive integer and $\mathcal{L}$ a nonempty open set of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. If every tensor of $\mathcal{L}$ has rank $r$, then $r$ is a typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$.

**Proof:** Let $\mathcal{O}_1, \ldots, \mathcal{O}_M$ be open connected disjoint semi-algebraic sets as in Theorem 2.1. Since $\dim(\mathbb{R}^{m_1 \times m_2 \times m_3} \setminus \bigcup_{i=1}^{M} \mathcal{O}_i) < m_1 m_2 m_3$, there exists $i \in [1, M]$ such that $\mathcal{L} \cap \mathcal{O}_i$ is not empty. □

**Proposition 2.4:** Let $m_1, m_2, m_3, m_4 \in \mathbb{N}$ with $m_3 < m_4$. Then

$$\text{grank}(m_1, m_2, m_3) \leq \text{grank}(m_1, m_2, m_4)$$
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and

$$\text{mtrank} (m_1, m_2, m_3) \leq \text{mtrank} (m_1, m_2, m_4).$$

**Proof:** Let $\mathcal{L}$ be the nonempty Zariski open subset of $\mathbb{C}^{m_1 \times m_2 \times m_3}$ consisting of all tensors of rank $\text{rank} (m_1, m_2, m_4)$ and put

$$\mathcal{H} = \{(Y_1; Y_2; \ldots; Y_m) \in \mathbb{C}^{m_1 \times m_2 \times m_3} \mid (Y_1; Y_2; \ldots; Y_m) \in \mathcal{L}\}.$$  

Then $\mathcal{H}$ is a nonempty Zariski open set of $\mathbb{C}^{m_1 \times m_2 \times m_3}$. For the subset $\mathcal{L}'$ of $\mathbb{C}^{m_1 \times m_2 \times m_3}$ consisting of all tensors of rank $\text{rank} (m_1, m_2, m_3)$, the intersection $\mathcal{H} \cap \mathcal{L}'$ is a nonempty Zariski open set. Since $\text{rank} \ Y \leq \text{rank} (Y; X)$ for $Y \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ and $(Y; X) \in \mathbb{C}^{m_1 \times m_2 \times m_4}$, we see

$$\text{rank} (m_1, m_2, m_3) \leq \text{rank} (m_1, m_2, m_4).$$

Next, take an open semi-algebraic set $\mathcal{H}'$ of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ consisting of tensors of rank $\text{mtrank} (m_1, m_2, m_3)$. Then there are $s \in \text{typical rank}_R (m_1, m_2, m_4)$ and an open semi-algebraic set $\mathcal{O}'$ of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ consisting of tensors of rank $s$ such that $\{(A; B) \mid A \in \mathcal{H}', B \in \mathbb{R}^{m_1 \times m_2 \times (m_4 - m_3)} \} \cap \mathcal{O}' \neq \emptyset$. Thus

$$\text{mtrank} (m_1, m_2, m_3) \leq s \leq \text{mtrank} (m_1, m_2, m_4).$$

The action of $\text{GL}(m) \times \text{GL}(n) \times \text{GL}(p)$ on $\mathbb{R}^{m \times n \times p}$ is given as follows. Let $P = (p_{ij}) \in \text{GL}(n)$, $Q = (q_{ij}) \in \text{GL}(m)$, and $R = (r_{ij}) \in \text{GL}(p)$. The tensor $(b_{ijk}) = (P, Q, R) \cdot (a_{ijk})$ is defined as

$$b_{ijk} = \sum_{s=1}^{m} \sum_{r=1}^{n} \sum_{u=1}^{p} p_{ir} q_{jr} r_{ku} a_{su}.$$

Therefore,

$$(P, Q, R) \cdot (A_1; \ldots; A_p) = \left( \sum_{u=1}^{p} r_{1u} PA_u Q^T; \ldots; \sum_{u=1}^{p} r_{pu} PA_u Q^T \right).$$

**Definition 2.5:** Two tensors $A$ and $B$ is called equivalent if there exists $g \in \text{GL}(m) \times \text{GL}(n) \times \text{GL}(p)$ such that $B = g \cdot A$.

**Proposition 2.6 (cf. [4, Lemma 2.3]):** If two tensors are equivalent, then they have the same rank.

A $1 \times m_2 \times m_3$ tensor $T$ is an $m_2 \times m_3$ matrix and rank $T$ is equal to the matrix rank. The following three propositions are well-known. The former two propositions easily follow from the definition.

**Proposition 2.7:** Let $m_1, m_2, m_3 \in \mathbb{N}$ with $2 \leq m_1 \leq m_2 \leq m_3$. If $m_1m_2 \leq m_3$, then typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ is only one integer $m_1m_2$.

**Proposition 2.8:** Let $X = (x_{ijk})$ be an $m_1 \times m_2 \times m_3$ tensor. For an $m_2 \times m_1 \times m_3$ tensor $Y = (x_{ijk})$ and an $m_1 \times m_3 \times m_2$ tensor $Z = (x_{ijk})$, it holds that

$$\text{rank} X = \text{rank} Y = \text{rank} Z.$$
A rank decomposition implies the following proposition.

**Proposition 2.9 (cf. [2, 14]):** An $m_1 \times m_2 \times m_3$ tensor $\{Y_1;\ldots;Y_{m_3}\}$ has rank less than or equal to $r$ if and only if there are an $m_1 \times r$ matrix $F$, an $r \times m_2$ matrix $Q$, and an $r \times r$ diagonal matrices $D_1,\ldots,D_{m_3}$ such that $Y_k = FD_kQ$ for $1 \leq k \leq m_3$.

For an integer $2 \leq m < n < 2m$, the number $n$ is an only typical rank of $\mathbb{R}^{m \times n \times 2}$. Indeed, it is known that

**Theorem 2.10 ([9]):** Let $2 \leq m < n$. There is an open dense semi-algebraic set $\mathcal{O}$ of $\mathbb{R}^{m \times n \times 2}$ of which any tensor is equivalent to $((E_m,O),(O,E_m))$ which has rank $\min(n,2m)$.

Furthermore, by Proposition 2.4, typical rank $R(m,m,2)$ is equal to either $\{m\}$ or $\{m,m+1\}$. Let $\mathcal{L}$ be an open subset of $\mathbb{R}^{m \times m \times 2}$ consisting of $(A,B)$ such that $A$ is an $m \times m$ nonsingular matrix and all eigenvalues of $A^{-1}B$ are distinct and contain non-real numbers. For $m \geq 2$, the set $\mathcal{L}$ is not empty and any tensor of $\mathcal{L}$ has rank $m+1$ (cf. [12, Theorem 4.6]) and therefore typical rank $R(m,m,2) = \{m,m+1\}$ by Proposition 2.3.

**Theorem 2.11 ([14, Result 2]):** Let $m,n,\ell \in \mathbb{N}$ with $3 \leq m \leq n \leq u$. If $(m-1)n < u < mn$, typical rank of $\mathbb{R}^{m \times n \times u}$ is only one integer $u$.

In fact we can construct a nonempty Zariski open subset consisting of tensors with rank $u$. We omit the construction.

**Corollary 2.12:** Let $3 \leq m \leq n$. Then the set of typical ranks of $m \times n \times (m-1)n$ tensors is either $\{(m-1)n\}$ or $\{(m-1)n,(m-1)n+1\}$.

**Proof:** The typical rank of $\mathbb{R}^{m \times n \times (m-1)n+1}$ is only $(m-1)n+1$ by Theorem 2.11 and the minimal typical rank of $\mathbb{R}^{m \times n \times (m-1)n}$ is equal to $(m-1)n$, since it is equal to the generic rank of $\mathbb{C}^{m \times n \times (m-1)n}$. Thus the assertion follows from Proposition 2.4. \qed

### 3. Characterization

From now on, let $3 \leq m \leq n$, $\ell = m-1$ and $p = (m-1)n$. For an $n \times n \times \ell$ tensor $(Y_1;\ldots;Y_\ell)$, consider an $n \times p \times n$ tensor $X(Y_1,\ldots,Y_\ell) = (X_1;\ldots;X_m)$ given by

$$
\begin{pmatrix}
X_1 \\
\vdots \\
X_m
\end{pmatrix} = 
\begin{pmatrix}
E_n \\
E_n \\
\ddots \\
E_n
\end{pmatrix}
\begin{pmatrix}
Y_1 Y_2 \cdots Y_\ell
\end{pmatrix}.
$$

(2)

Note that rank $X(Y_1,\ldots,Y_\ell) \geq p$, since rank $X(Y_1,\ldots,Y_\ell)$ is greater than or equal to the rank of the $mn \times p$ matrix $(2)$. Generically, an $m \times n \times p$ tensor is equivalent to a tensor of type as $X(Y_1,\ldots,Y_\ell)$.

We denote by $\mathcal{M}$ the set of tensors $Y = (Y_1;\ldots;Y_\ell) \in \mathbb{R}^{n \times n \times \ell}$ such that there exist an $m \times p$ matrix $(x_{ij})$ and an $n \times p$ matrix $A = (a_1,\ldots,a_p)$ such that

$$(x_{1j}Y_1 + \cdots + x_{m-1,j}Y_{m-1} - x_{mj}E_n)a_j = 0
$$

(3)
Theorem 3.4  

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for $1 \leq j \leq p$ and

$$B := \begin{pmatrix} AD_1 \\ \vdots \\ AD_\ell \end{pmatrix}$$

is nonsingular, where $D_k = \text{Diag}(x_{k1}, \ldots, x_{kp})$ for $1 \leq k \leq \ell$.

**Lemma 3.1:** \( \text{rank} X(Y_1, \ldots, Y_\ell) = p \) if and only if \( (Y_1; \ldots; Y_\ell) \in \mathcal{M} \).

**Proof:** Suppose that \( \text{rank} X(Y_1, \ldots, Y_\ell) = p \). There are an \( n \times p \) matrix $A$, a \( p \times p \) matrix $Q$ and \( p \times p \) diagonal matrices $D_k$ such that $X_k = AD_kQ$ for $k = 1, \ldots, m$. Since

$$\begin{pmatrix} X_1 \\ \vdots \\ X_\ell \end{pmatrix} = E_p \begin{pmatrix} AD_1 \\ \vdots \\ AD_\ell \end{pmatrix} Q,$$

$B$ is nonsingular. Then \( (Y_1, \ldots, Y_\ell)B = AD_m \) implies that \( \sum_{k=1}^\ell Y_k AD_k = AD_m \). Therefore, the $j$-th column vector $a_j$ of $A$ satisfies (3). Therefore $(Y_1, \ldots, Y_\ell) \in \mathcal{M}$. It is easy to see that the converse is also true. \( \Box \)

For an \( n \times n \times \ell \) tensor $Y = (Y_1; \ldots; Y_\ell)$, we put

$$V(Y) = \{ a \in \mathbb{R}^n \mid \sum_{k=1}^\ell x_k Y_k a = x_m a \text{ for some } (x_1, \ldots, x_m)^T \neq 0 \}.$$ 

The set $V(Y)$ is not a vector subspace of $\mathbb{R}^n$. Let $\hat{V}(Y)$ be the smallest vector subspace of $\mathbb{R}^n$ including $V(Y)$. Let

$$\mathcal{S} = \{ Y \in \mathbb{R}^{n \times n \times \ell} \mid \dim \hat{V}(Y) = n \}.$$ 

**Proposition 3.2:** \( \mathcal{M} \subset \mathcal{S} \) holds.

**Proof:** Let $Y \in \mathcal{M}$. Consider the matrix $B$ in (4) for any $m \times p$ matrix $(x_{ij})$ and any $n \times p$ matrix $A = (a_1, \ldots, a_p)$ satisfying the equation (3). By column operations, $B$ is transformed to a $p \times p$ matrix having a form

$$\begin{pmatrix} P_{11} & O_{n,p-\dim \hat{V}(Y)} \\ P_{21} & P_{22} \end{pmatrix}$$

where $P_{11}$ is an $n \times \dim \hat{V}(Y)$ submatrix of $A$. Since $B$ is nonsingular, $P_{11}$ is also nonsingular, which implies that $\dim \hat{V}(Y) = n$. \( \Box \)

By Corollary 2.12, Lemma 3.1 and Proposition 3.2, we have the following

**Proposition 3.3:** If $\text{rank} X(Y) = p$ then $Y \in \mathcal{S}$. In particular, $\mathcal{S} \neq \mathbb{R}^{n \times n \times \ell}$ implies that typical $\text{rank}_S(m, n, p) = \{ p, p+1 \}$.

**Theorem 3.4** ([13]): If $Y_1; \ldots; Y_\ell; E_n$ is an absolutely nonsingular tensor, then it holds that $\text{rank} X(Y_1, \ldots, Y_\ell) > p$.

Here $(Y_1; \ldots; Y_\ell; Y_m)$ is called an absolutely nonsingular tensor if $|\sum_{k=1}^m x_k Y_k| = 0$ implies $(x_1, \ldots, x_m)^T = 0$. Therefore,

**Proposition 3.5:** $\dim \hat{V}(Y) = 0$ if and only if $(Y; E_n)$ is an $n \times n \times m$ absolutely nonsingular tensor.
Note that there exists an \( n \times n \times m \) absolutely nonsingular tensor if and only if \( m \) is less than or equal to the Hurwitz-Radon number \( \rho(n) \) [13].

**Proposition 3.6:** Let \( \mathbf{Y} \) and \( \mathbf{Z} \) be \( n \times n \times m \) tensors. Suppose \( (\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot \mathbf{Y} = \mathbf{Z} \) for \( (\mathbf{P}, \mathbf{Q}, \mathbf{R}) \in \text{GL}(n) \times \text{GL}(n) \times \text{GL}(m) \). Then \( \mathbf{V}(\mathbf{Y}) = \mathbf{Q}^T \mathbf{V}(\mathbf{Z}) = \{ \mathbf{Q}^T \mathbf{y} \mid \mathbf{y} \in \mathbf{V}(\mathbf{Z}) \} \). In particular, \( \dim \mathbf{V}(\mathbf{Z}) = \dim \mathbf{V}(\mathbf{Y}) \).

**Proof:** Suppose that \( \sum_{k=1}^{m} q_k \mathbf{Y}_k = 0 \). Then from the definition of the action, it follows that

\[
\sum_{k=1}^{m} d_k \sum_{u=1}^{p} r_{ku} \mathbf{P}_u \mathbf{Q}_u \mathbf{Q}^T \mathbf{y} = \mathbf{P}(\sum_{u=1}^{m} \sum_{k=1}^{p} d_k r_{ku} \mathbf{Y}_u) \mathbf{Q}^T \mathbf{y} = 0.
\]

Thus \( \mathbf{Q}^T \mathbf{y} \in \mathbf{V}(\mathbf{Y}) \).

**Corollary 3.7:** \( \mathcal{S} \) is closed under the equivalence relation.

The closure of the set of all \( n \times p \times m \) tensors equivalent to \( \mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) \) for some \( \mathbf{Y}_1, \ldots, \mathbf{Y}_\ell \) is \( \mathbb{R}^{n \times p \times m} \). Furthermore, the following claim holds. Let \( \mathcal{H} \) be the set of \( n \times p \times m \) tensors \( (\mathbf{X}_1, \ldots, \mathbf{X}_m) \) such that \( \mathbf{A} = (\mathbf{X}_1^T, \ldots, \mathbf{X}_m^T) \) is a nonsingular \( p \times p \) matrix and \( (\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) \) given by \( (\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = \mathbf{A}^{-1} \mathbf{X}_m \) lies in \( \mathcal{H} \). Any tensor of \( \mathcal{H} \) has rank \( p \). If \( \mathcal{H} \) is dense in \( \mathbb{R}^{n \times p \times m} \) then \( \mathcal{H} \) is dense in \( \mathbb{R}^{n \times p \times m} \).

4. **Classes of \( n \times n \times \ell \) tensors**

We separate \( \mathbb{R}^{n \times n \times \ell} \) into three classes \( \mathcal{A} \), \( \mathcal{C} \), and \( \mathcal{B} \) as follows. Let \( \mathcal{A} \) be the set of tensors \( \mathbf{Y} \) such that \( (\mathbf{Y}; \mathbf{E}_n) \) is absolutely nonsingular. By Proposition 3.5, we have the following

**Proposition 4.1:** \( \mathcal{A} \cap \mathcal{S} = \emptyset \).

From now on, we use symbols \( x_1, \ldots, x_\ell, x_m \) as indeterminates over \( \mathbb{R} \). For \( \mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times \ell} \), we define the \( n \times n \) matrix with entries in \( \mathbb{R}[x_1, \ldots, x_\ell, x_m] \) as follows.

\[
\mathbf{M}(\mathbf{x}, \mathbf{Y}) = \sum_{k=1}^{\ell} x_k \mathbf{Y}_k - x_m \mathbf{E}_n
\]

Note that fixing \( a_1, \ldots, a_\ell \), the determinant \( |\mathbf{M}(a, \mathbf{Y})| \) is positive for \( a_m \ll 0 \), where \( a = (a_1, \ldots, a_\ell, a_m)^T \). Set

\[
\mathcal{C} = \{ \mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(a, \mathbf{Y})| < 0 \text{ for some } a \in \mathbb{R}^m \}.
\]

Note that \( \mathcal{C} \) is not empty, and if \( n \) is not congruent to 0 modulo 4 then \( \rho(n) \leq 2 \) and thus \( \mathcal{A} \) is empty since \( m \geq 3 \). Set \( \mathcal{B} = \mathbb{R}^{n \times n \times \ell} \setminus (\mathcal{A} \cup \mathcal{C}) \). In particular, if \( n \) is odd, then \( \mathcal{C} = \mathbb{R}^{n \times n \times \ell} \), and otherwise \( \mathcal{B} \) contains the zero tensor.

**Proposition 4.2:** \( \mathcal{A} \) and \( \mathcal{C} \) are open subsets of \( \mathbb{R}^{n \times n \times \ell} \).

Recall that

\[
\mathcal{A} = \{ \mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(a, \mathbf{Y})| > 0 \text{ for all } a \neq 0 \}.
\]

Thus it holds

\[
\mathcal{B} = \{ \mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(b, \mathbf{Y})| = 0 \text{ for some } b \neq 0 \text{ and } |\mathbf{M}(a, \mathbf{Y})| \geq 0 \text{ for all } a \neq 0 \}.
\]
If \( n \) is even then \( \mathcal{B} \) contains a nonzero tensor in general. We give an example.

**Example 4.3** Let \( m = 3 \) and \( n = 6 \). Note that \( \mathcal{A} \) is empty, since \( \rho(6) = 2 \). Let \( Y = (Y_1; Y_2) \) be a \( 6 \times 6 \times 2 \) tensor given by
\[
\begin{pmatrix}
x_1 Y_1 + x_2 Y_2 =
\end{pmatrix}
= \begin{pmatrix}
0 & -x_2 & 0 & 0 & 0 & -x_1 \\
x_1 & 0 & x_2 & 0 & 0 \\
0 & x_1 & 0 & x_2 & 0 \\
0 & 0 & x_1 & 0 & -x_2 \\
0 & 0 & 0 & x_1 & 0 \\
-x_2 & 0 & 0 & 0 & x_1
\end{pmatrix}.
\]

Then \(|M((a_1, a_2, a_3)^T, Y)| = a_3^2(a_1 a_2 - a_3)^2 + (a_1^3 + a_2^3) \geq 0\). The equality holds if \( a_3 = 0 \) and \( a_1 = -a_2 \). Thus \( Y \in \mathcal{B} \) and \( Y \notin \mathcal{A} \) since \( \text{dim} \hat{V}(Y) = 1 \). Let \( X = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \) be a
\( 6 \times 6 \) matrix and let \( \varepsilon \in \mathbb{R} \). If \( x_1 = -1 \) and \( x_3 = \varepsilon c \), then
\[
|M((x_1, x_2, x_3)^T, (Y_1; Y_2 + \varepsilon X))| = -\varepsilon^2(1 + \varepsilon^2 c^2)c((1 - \varepsilon c)^2 - (1 + \varepsilon^2 c^2))
\]
Thus, for any \( \varepsilon \neq 0 \), there is \( c_0 > 0 \) such that \( (1 - \varepsilon c_0)^2 - c_0(1 + \varepsilon^2 c_0^2) > 0 \) and then \(|M((-1, 1, \varepsilon c_0)^T, (Y_1; Y_2 + \varepsilon X))| < 0\), which implies that \((Y_1; Y_2 + \varepsilon X) \in \mathcal{C}\).

**Proposition 4.4:** \( \mathcal{B} \) is a boundary of \( \mathcal{C} \). In particular, \( \mathbb{R}^{n \times n \times \ell} \) is a disjoint sum of \( \mathcal{A} \) and the closure \( \overline{\mathcal{C}} \) of \( \mathcal{C} \).

**Proof:** It suffices to show that \( \mathcal{B} \subset \overline{\mathcal{C}} \) by Proposition 4.2. Let \( Y = (Y_1; \ldots; Y_{\ell}) \in \mathcal{B} \).
There are a nonzero vector \( b = (b_1, \ldots, b_{\ell}, b_m)^T \in \mathbb{R}^n \) with \(|M(b, Y)| = 0 \) and an element \( g \in \text{GL}(\ell) \) such that \( g \cdot Y = (Z_1; Z_2; \ldots; Z_{\ell}) \) and \( Z_1 = \sum_{k=1}^{\ell} b_k Y_k \). Then \(|Z_1 - b_m E_m| = 0\).
Take a sequence \( \{Z_{1u}\}_{u \geq 1} \) such that \(|Z_{1u} - b_m E_m| < 0 \) and \( \lim_{u \to \infty} Z_{1u} = Z_1 \). Thus, \( Z_{1u}; Z_2; \ldots; Z_{\ell} \in \mathcal{C} \) and then \( g^{-1} \cdot (Z_{1u}; Z_2; \ldots; Z_{\ell}) \in \mathcal{C} \). Therefore, \((Y_1; \ldots; Y_{\ell}) \in \overline{\mathcal{C}} \). \( \square \)

**Corollary 4.5:** If \( \mathcal{A} \) is not empty then \( \mathcal{B} \) is a boundary of \( \mathcal{A} \).

**Proposition 4.6:** If \( m \leq \rho(n-1) \) then \( \mathcal{C} \notin \mathcal{A} \), where \( \rho(n-1) \) is a Hurwitz-Radon number.

**Proof:** Let \((A_1; \ldots; A_{\ell}; E_{n-1})\) be an \((n-1) \times (n-1) \times m \) absolutely nonsingular tensor. Put \( B_k = \text{Diag}(a_k, A_k) \) for \( 1 \leq k \leq \ell \) and \( B_m = \text{Diag}(1, E_{n-1}) = E_m \), and \( B = (B_1; \ldots; B_{\ell}) \).
Then it is easy to see that \( B \in \mathcal{C} \) and \(|\sum_{k=1}^{\ell} x_k B_k - x_m B_m| = 0 \) implies \( x_m = \sum_{k=1}^{\ell} a_k x_k \).
Therefore \( V(B) = \{a(1, 0, \ldots, 0)^T \in \mathbb{R}^n \mid a \in \mathbb{R}\} \). In particular \( B \notin \mathcal{A} \). \( \square \)

5. Proof of Theorem 1.1

In this section we show Theorem 1.1.

In the space of homogeneous polynomials in \( m \) variables, there exists a proper Zariski closed subset \( S \) such that if a polynomial does not belong to \( S \) then it is irreducible [7, Theorem 7], since \( m \geq 3 \). Let \( P(m,n) \) be the set of homogeneous polynomials in \( m \) variables \( x_1, \ldots, x_m \) with real coefficients of degree \( n \) such that the coefficient of \( x_m^n \) is one. Its dimension is \( \binom{m+n-1}{m-1} - 1 \). Let \( I_\ell \) be a nonempty Zariski open subset of \( P(m,n) \) such that any polynomial of \( I_\ell \) is irreducible. Note that \(|-M(x, Y)| \in P(m,n)\).

A polynomial \( f \in P(m,n) \) is hyperbolic with respect to a vector \( e \in \mathbb{R}^m \) if both \( f(e) \neq 0 \) and for all vectors \( v \in \mathbb{R}^m \), the univariate polynomial \( t \mapsto f(v - te) \) has all real roots.
**Theorem 5.1** ([8]): A homogeneous polynomial $f$ of $P(3,n)$ is hyperbolic with respect to the vector $e = (0,0,1)^T$ and satisfies $f(e) = 1$ if and only if there exist $n \times n$ symmetric matrices $A$, $B$ such that $f$ is given by

$$f(x,y,z) = |xA + yB + zE_n|.$$ 

**Proposition 5.2:** The set

$$\{Y \in \mathbb{R}^{n \times \ell} \mid | - M(x,Y)| \in I \}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n \times \ell}$.

**Proof:** Let $f_\ell : \mathbb{R}^{n \times \ell} \to P(m,n)$ be a map which sends $Y$ to $| - M(x,Y)|$ and put

$$\mathcal{S}_\ell := \{Y \in \mathbb{R}^{n \times \ell} \mid f_\ell(Y) \in I \}.$$ 

Since $I$ is a Zariski open set, $\mathcal{S}_\ell$ is a Zariski open subset of $\mathbb{R}^{n \times \ell}$. Then it suffices to show that $\mathcal{S}_\ell$ is not empty. For $(Z_1; Z_2) \in \mathbb{R}^{n \times 2}$, if the polynomial

$$| - M((x_1,x_2,x_3)^T,(Z_1; Z_2))|$$

is irreducible, then for any $(Y_1; \ldots; Y_\ell) \in \mathbb{R}^{n \times (\ell-2)}$, the polynomial

$$| - M((x_1Z_1 + x_2Z_2 + x_3E_n)| = | - M((x_1,x_2,x_3)^T,(-Z_1;-Z_2))|$$

is also irreducible. Therefore, it suffices to show the assertion only for $m = 3$.

The set of hyperbolic homogeneous polynomials of degree $n$ contains an open subset [10]. Then there is an irreducible homogenous polynomial $f$ of $P(3,n)$ which is hyperbolic with respect to $(e_1,e_2,e_3)^T$. Since $f(e_1,e_2,e_3) \neq 0$, it holds that $(e_1,e_2,e_3)^T \neq 0$. Thus there is an irreducible homogeneous polynomial $g \in P(3,n)$ such that $g$ is hyperbolic with respect to $(0,0,1)^T$ and $g(0,0,1) = 1$. By Theorem 5.1, $g(x_1,x_2,x_3)$ is described as

$$|x_1Z_1 + x_2Z_2 + x_3E_n| = | - M((x_1,x_2,x_3)^T,(-Z_1;-Z_2))|$$

for some $n \times n$ matrices $Z_1,Z_2$. Thus, $\mathcal{S}_2$ is nonempty. This completes the proof.  

Let $\tilde{x} = (x_1, \ldots, x_\ell)^T$ for $x = (x_1, \ldots, x_\ell,x_m)^T$, and put

$$\psi(x,Y) := \begin{pmatrix} (-1)^{n+1}|M(x,Y)_{n,1}| \\ (-1)^{n+2}|M(x,Y)_{n,2}| \\ \vdots \\ (-1)^{n+n}|M(x,Y)_{n,n}| \end{pmatrix} \in \mathbb{R}^n, \quad \tilde{x} \otimes \psi(x,Y) := \begin{pmatrix} x_1\psi(x,Y) \\ x_2\psi(x,Y) \\ \vdots \\ x_\ell\psi(x,Y) \end{pmatrix} \in \mathbb{R}^P,$$

and

$$U(Y) := \{\tilde{u} \otimes \psi(u,Y) \mid |M(u,Y)| = 0, \ u \in \mathbb{R}^m\}.$$ 

**Lemma 5.3:** If $\dim U(Y) = p$, then $Y \in \mathcal{M}$.

**Proof:** Let $\dim U(Y) = p$. Then there are $u_j = (u_{1,j}, \ldots, u_{m,j})^T \in U(Y)$ for $1 \leq j \leq p$ such that

$$B = (\tilde{u}_1 \otimes \psi(u_1,Y), \ldots, \tilde{u}_p \otimes \psi(u_p,Y))$$
is nonsingular. Note that $\mathbf{M}(u_j, \mathbf{Y}) \psi(u_j, \mathbf{Y}) = 0$ for $1 \leq j \leq p$ and

$$
\mathbf{B} = \begin{pmatrix}
\mathbf{A} \mathbf{D}_1 \\
\vdots \\
\mathbf{A} \mathbf{D}_n
\end{pmatrix},
$$

where $\mathbf{A} = (\psi(u_1, \mathbf{Y}), \ldots, \psi(u_p, \mathbf{Y}))$ and $\mathbf{D}_k = \text{Diag}(u_{k1}, \ldots, u_{kp})$ for $1 \leq k \leq \ell$. Thus $\mathbf{Y} \in \mathcal{M}$.

Proof: For any $\mathbf{Y}$, the orthogonal complement of $\mathbf{U}$ is $\mathbf{C}$.

In particular, if $\mathbf{Y}$ is a $(n - 1) \times n$ matrix obtained from $\mathbf{M}(\mathbf{x}, \mathbf{Y})$ by removing the $n$-th row.

**Lemma 5.4:** Let $\mathbf{C} = (c_1, \ldots, c_\ell)$ be an $n \times \ell$ matrix. The following claims are equivalent.

1. $\dim U(\mathbf{Y}) = p$.
2. $g(\mathbf{a}, \mathbf{Y}, \mathbf{C}) = 0$ for any $\mathbf{a} \in \mathbb{R}^n$ with $|\mathbf{M}(\mathbf{a}, \mathbf{Y})| = 0$ implies $\mathbf{C} = \mathbf{0}$.

**Proof:** Let $\mathbf{C} = (c_1, \ldots, c_\ell)$ be an $n \times \ell$ matrix. Put $\mathbf{d} = (c_1^T, \ldots, c_\ell^T)^T \in \mathbb{R}^p$. The inner product of this vector $\mathbf{d}$ with $\mathbf{a} \otimes \psi(\mathbf{a}, \mathbf{Y})$ is equal to $g(\mathbf{a}, \mathbf{Y}, \mathbf{C})$. Therefore $\mathbf{d}$ belongs to the orthogonal complement of $U(\mathbf{Y})$ if and only if $g(\mathbf{a}, \mathbf{Y}, \mathbf{C}) = 0$ for any $\mathbf{a} \in \mathbb{R}^n$ with $|\mathbf{M}(\mathbf{a}, \mathbf{Y})| = 0$. Thus the assertion holds.

For any $i$ and $k$ with $1 \leq i \leq n - 1$ and $1 \leq k \leq n$, let $s_i^{(k)}$ be an elementary symmetric polynomial of degree $i$ with variables $\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. Put

$$
\mathbf{S}_n = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
s_1^{(1)} & s_1^{(2)} & \ldots & s_1^{(n)} \\
s_2^{(1)} & s_2^{(2)} & \ldots & s_2^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
s_{n-1}^{(1)} & s_{n-1}^{(2)} & \ldots & s_{n-1}^{(n)}
\end{pmatrix}.
$$

**Lemma 5.5:** The determinant $|\mathbf{S}_n|$ of the $n \times n$ matrix $\mathbf{S}_n$ is equal to

$$
\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j).
$$

In particular, if $\alpha_1, \ldots, \alpha_n$ are distinct from each other, then $\mathbf{S}_n$ is nonsingular.

**Proof:** For any $i$ and $k$ with $1 \leq i \leq n - 1$ and $2 \leq k \leq n - 1$, let $t_i^{(k-1)}$ be an elementary symmetric polynomial of degree $i$ with variables $\alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. For $1 \leq i \leq n - 1$ and $1 \leq k \leq n$, we have $s_i^{(k)} - s_i^{(1)} = (\alpha_1 - \alpha_k)t_i^{(k-1)}$. Then

$$
|\mathbf{S}_n| = \prod_{2 \leq k \leq n} (\alpha_k - \alpha_n) \begin{vmatrix}
1 & 1 & \ldots & 1 \\
\frac{t_1^{(1)}}{t_1^{(1)}} & \frac{t_1^{(2)}}{t_1^{(1)}} & \ldots & \frac{t_1^{(n-1)}}{t_1^{(1)}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{t_{n-2}^{(1)}}{t_{n-2}^{(1)}} & \frac{t_{n-2}^{(2)}}{t_{n-2}^{(1)}} & \ldots & \frac{t_{n-2}^{(n-1)}}{t_{n-2}^{(1)}}
\end{vmatrix}.
$$
Therefore we have the assertion by induction on $n$. □

The following lemma is obtained straightforwardly.

**Lemma 5.6:**

\[
\begin{vmatrix}
\alpha_1 + z & a_1 \\
\alpha_2 + z & a_2 \\
\vdots & \vdots \\
\alpha_n + z & a_n \\
b_1 & b_2 & \ldots & b_n & 0
\end{vmatrix}
= -(\alpha_1^{n-1}, \alpha_2^{n-1}, \ldots, 1)S_n
\begin{pmatrix}
a_1b_1 \\
a_2b_2 \\
\vdots \\
a_nb_n
\end{pmatrix}.
\]

**Proof:** We see the left hand of the equation is equal to

\[-\sum_{k=1}^{n} a_kb_k \prod_{1 \leq i \leq n}(\alpha_i + z) \alpha_k + z = -\sum_{k=1}^{n} a_kb_k \left( \sum_{i=1}^{n} s_{i-1}^{(k)} \right) z^{n-i} = -\sum_{i=1}^{n} \left( \sum_{k=1}^{n} a_kb_k s_{i-1}^{(k)} \right) z^{n-i} = -(\alpha_1^{n-1}, \alpha_2^{n-1}, \ldots, 1) \begin{pmatrix}
\sum_{k=1}^{n} a_kb_k \\
\sum_{k=1}^{n} a_kb_k s_{1-1}^{(k)} \\
\vdots \\
\sum_{k=1}^{n} a_kb_k s_{n-1}^{(k)}
\end{pmatrix}.\]

□

**Corollary 5.7:** Let $\alpha_1, \ldots, \alpha_{n-1}$ be distinct complex numbers, $a_1, \ldots, a_{n-1}$ nonzero complex numbers, and $b_1, \ldots, b_{n-1}$ complex numbers. If

\[
\begin{vmatrix}
\text{Diag}(\alpha_1, \ldots, \alpha_{n-1}) + zE_{n-1} \\
b^T
\end{vmatrix} = 0
\]

for any $z \in \mathbb{R}$, then $b = 0$, where $a = (a_1, \ldots, a_{n-1})^T$ and $b = (b_1, \ldots, b_{n-1})^T$.

**Proof:** Since $S_n \begin{pmatrix}
a_1b_1 \\
a_2b_2 \\
\vdots \\
a_nb_n
\end{pmatrix} = 0$ and $S_n$ is nonsingular, we have $(a_1b_1, \ldots, a_nb_n) = 0^T$. □

The set

$\mathcal{Y}_1 = \{ Y \in \mathbb{R}^{n \times n \times \ell} \mid |M(x, Y)| \text{ is irreducible} \}$

is a nonempty Zariski open subset of $\mathbb{R}^{n \times n \times \ell}$ (see Proposition 5.2). Let $W$ be the subset of $\mathbb{R}^{n \times n}$ consisting of matrices $\begin{pmatrix}
A & b \\
e^T & e
\end{pmatrix}$ such that all eigenvalues of $A$ are distinct over the complex number field and every element of the vector $P^{-1}b$ is nonzero complex number where $A \in \mathbb{R}^{(n-1) \times (n-1)}$, $P \in \mathbb{C}^{(n-1) \times (n-1)}$ such that $P^{-1}AP$ is a diagonal matrix. Note that the validity of the condition that every element of the vector $P^{-1}b$ is nonzero is
Typical ranks of $m \times n \times (m-1)n$ tensors

independent of the choice of $P$. We put
\[ \mathcal{V}_2 := \{ (Y_1; \ldots; Y_\ell) \in \mathbb{R}^{n \times n \times \ell} \mid Y_k \in W, 1 \leq k \leq \ell \}. \]

The set $\mathcal{V}_2$ is a nonempty Zariski open subset of $\mathbb{R}^{n \times n \times \ell}$ and $\mathcal{V} := \mathcal{V}_1 \cap \mathcal{V}_2$ is also.

**Lemma 5.8:** Let $Y \in \mathcal{V}_2$ and $d_1, \ldots, d_\ell \in \mathbb{R}^{n-1}$. If
\[ \frac{|M(a, Y)|}{\sum_{k=1}^\ell a_k d_k^T} = 0 \]
for any $a = (a_1, \ldots, a_m)^T \in \mathbb{R}^m$, then $d_1 = \cdots = d_\ell = 0$.

**Proof:** Let $1 \leq k \leq \ell$. Take $a_k = 1$ and $a_j = 0$ for $1 \leq j \leq \ell, j \neq k$ and put $Y_k = \left( \begin{array}{cc} A & b \\ c^T & e \end{array} \right)$, where $A$ is an $(n-1) \times (n-1)$ matrix. Since $Y_k \in W$, there are a matrix $P \in \mathbb{C}^{(n-1) \times (n-1)}$ and distinct complex numbers $\alpha_1, \ldots, \alpha_{n-1}^*$ such that
\[ \text{Diag}(P, 1)^{-1} \begin{pmatrix} Y_k - a_m e_n \\ d_k^T \end{pmatrix} = 0 \]
and every element of $P^{-1}b$ is nonzero. Then we have $d_k^T P = 0^T$ by Corollary 5.7 and thus $d_k = 0$. \hfill $\square$

The following lemma is essential for the proof of Theorem 1.1.

**Lemma 5.9:** $\mathcal{V} \cap \mathcal{C} \subseteq \mathcal{M}$. In particular, $\mathcal{C} \subseteq \mathcal{M}$ holds.

**Proof:** Let $Y \in \mathcal{V} \cap \mathcal{C}$ and fix it. There exists $a = (a_1, \ldots, a_\ell, a_m)$ such that $|M(a, Y)| < 0$. Then there is an open neighborhood $U$ of $(a_1, \ldots, a_\ell)^T$ and a mapping $\mu : U \rightarrow \mathbb{R}$ such that
\[ |M\left( \begin{pmatrix} y \\ \mu(y) \end{pmatrix}, Y \right)| = 0 \]
for any $y \in U$. Thus $|M(x, Y)| = 0$ determines an $(m-1)$-dimensional algebraic set. Let $C$ be an $n \times \ell$ matrix. Now suppose that $g(a, Y, C) = 0$ holds for any $a \in \mathbb{R}^m$ with $|M(a, Y)| = 0$. We show that $g(x, Y, C)$ is zero as a polynomial over elements of $x$. As a contrary, assume that $g(x, Y, C)$ is not zero. The degree of $g(x, Y, C)$ corresponding to the $m$-th element of $x$ is less than $n$ which is that of $|M(x, Y)|$. Furthermore, since $|M(x, Y)|$ is irreducible, $|M(x, Y)|$ and $g(x, Y, C)$ are coprime. Then there are polynomials $f_1(x)$, $f_2(x) \in \mathbb{R}[x_1, \ldots, x_\ell, x_m]$ and a nonzero polynomial $h(\overline{x}) \in \mathbb{R}[x_1, \ldots, x_\ell]$ such that
\[ f_1(x)|M(x, Y)| + f_2(x)g(x, Y, C) = h(\overline{x}) \]
as a polynomial over elements of $x$, by Euclidean algorithm. However, we can take $b \in U$ so that $h(b) \neq 0$. Then the above equation does not hold at $x = \left( \begin{array}{c} b \\ \mu(b) \end{array} \right)$. Hence $g(x, Y, C)$ must be the zero polynomial over elements of $x$. Let $c_k = (c_{1k}, \ldots, c_{nk})$. By seeing the coefficient of $x_m^{n-1} x_k$, we get $c_{nk} = 0$ for $1 \leq k \leq \ell$. Therefore $C = 0$ by Lemma 5.8. By Lemmas 5.4 and 5.3 we get $Y \in \mathcal{M}$. Therefore $\mathcal{V} \cap \mathcal{C}$ is a subset of $\mathcal{M}$. Since $\mathcal{C}$ is open and $\mathcal{V}$ is dense, we have $\mathcal{C} = \mathcal{C} \cap \mathcal{M}$ and thus $\mathcal{C}$ is a subset of $\mathcal{M}$. \hfill $\square$

**Theorem 5.10:** $\mathcal{T} = \overline{\mathcal{M}} = \overline{\mathcal{C}}$ holds.
**Proof** : We have \( M \subset S \) by Proposition 3.2. By Propositions 4.1 and 4.4, the set \( S \) is a subset of \( C \) and then \( S \subset C \). Therefore \( S = M = C \) by Lemma 5.9. \( \square \)

**Proof of Theorem 1.1**: For almost all \( Y \in \mathcal{A} \), \( \text{rank} X(Y) = p + 1 \) by Theorem 3.4. Since \( \mathcal{A} \) is an open set, if \( \mathcal{A} \) is not an empty set, then typical\_rank\_R(m,n,p) = \{p, p + 1\} ([13, Theorem 3.4]). Suppose that \( \mathcal{A} \) is empty. Then \( \mathcal{M} = \mathbb{R}^{n \times n \times \ell} \) and the closure of the set consisting of all \( n \times p \times m \) tensors equivalent to \( X(Y) \) for some \( Y \in \mathcal{M} \) is \( \mathbb{R}^{n \times p \times m} \). Recall that any tensor \( X(Y) \) for \( Y \in \mathcal{M} \) has rank \( p \). By Theorem 2.1, \( p \) is the maximal typical rank of \( \mathbb{R}^{n \times p \times m} \). Therefore,

\[
\text{typical\_rank}\_R(m,n,p) = \text{typical\_rank}\_R(n,p,m) = \{p\}
\]

holds.

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**References**


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