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RESEARCH ARTICLE

Typical ranks of $m \times n \times (m-1)n$ tensors with $3 \leq m \leq n$ over the real number field

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Let $3 \leq m \leq n$. We study typical ranks of $m \times n \times (m-1)n$ tensors over the real number field. Let ρ be the Hurwitz-Radon function defined as $\rho(n) = 2^b + 8c$ for nonnegative integers a, b, c such that $n = (2a+1)2^{b+4c}$ and $0 \leq b < 4$. If $m \leq \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has two typical ranks $(m-1)n, (m-1)n+1$. In this paper, we show that the converse is also true: if $m > \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has only one typical rank $(m-1)n$.

Keywords: tensor; 3-way array; typical rank; Hurwitz-Radon number; perfect

AMS Subject Classification: 15A69, 14P10, 14Q20

1. Introduction

For positive integers m, n, p , an $m \times n \times p$ tensor is an element of tensor product of vector spaces V_1, V_2, V_3 of dimension m, n, p , respectively. If we choose bases of their vector spaces, we can identify a tensor with a 3-way array

$$(a_{ijk})_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p}.$$

Hitchcock [6] defined the rank of a tensor. An $m \times n \times p$ tensor of form

$$(x_i y_j z_k)_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p}$$

is called a rank one tensor. The rank of a tensor T , denoted by $\text{rank } T$, is defined as the minimal number of rank one tensors which describe T as a sum. The rank depends on the base field. For example there is a $2 \times 2 \times 2$ tensor over the real number field whose rank is 3 but is 2 as a tensor over the complex number field.

Throughout this paper, we assume that the base field is the real number field \mathbb{R} . Let $\mathbb{R}^{m \times n \times p}$ be the set of $m \times n \times p$ tensors with Euclidean topology. A number r is a typical rank of $m \times n \times p$ tensors if the set of tensors with rank r contains a nonempty open semi-algebraic set of $\mathbb{R}^{m \times n \times p}$ (see Theorem 2.1). We denote by $\text{typical_rank}_{\mathbb{R}}(m, n, p)$

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the set of typical ranks of $\mathbb{R}^{m \times n \times p}$. If s (resp. t) is the minimal (resp. maximal) number of $\text{typical_rank}_{\mathbb{R}}(m, n, p)$, then

$$\text{typical_rank}_{\mathbb{R}}(m, n, p) = [s, t] \cap \mathbb{Z},$$

the interval of all integers between s and t , including both, and s is equal to the generic rank of the set of $m \times n \times p$ tensors over the complex number field [5]. In the case where $m = 2$, the set of typical ranks of $2 \times n \times p$ tensor is well-known [15]:

$$\text{typical_rank}_{\mathbb{R}}(2, n, p) = \begin{cases} \{p\}, & n < p \leq 2n \\ \{2n\}, & 2n < p \\ \{p, p+1\}, & n = p \geq 2 \end{cases}$$

Suppose that $3 \leq m \leq n$. If $p > (m-1)n$ then the set of typical ranks of $m \times n \times p$ tensors is just $\{\min(p, mn)\}$. If $p = (m-1)n$ then the set of typical ranks of $m \times n \times p$ tensor is $\{p\}$ or $\{p, p+1\}$ [14]. Until our paper [13], only a few cases where $\text{typical_rank}_{\mathbb{R}}(m, n, (m-1)n) = \{(m-1)n, (m-1)n+1\}$ [3, 5] are known and we constructed infinitely many examples by using the concept of absolutely nonsingular tensors in [13]: If $m \leq \rho(n)$ then $\text{typical_rank}_{\mathbb{R}}(m, n, p) = \{p, p+1\}$, where $\rho(n)$ is the Hurwitz-Radon number given by $\rho(n) = 2^b + 8c$ for nonnegative integers a, b, c such that $n = (2a+1)2^{b+4c}$ and $0 \leq b < 4$.

n	1	2	3	4	5	6	7	8	9	10	11	12	16	32	64
$\rho(n)$	1	2	1	4	1	2	1	8	1	2	1	4	9	10	12

The purpose of this paper is to completely determine the set of typical ranks of $m \times n \times (m-1)n$ tensors:

Theorem 1.1: *Let $3 \leq m \leq n$ and $p = (m-1)n$. Then it holds*

$$\text{typical_rank}_{\mathbb{R}}(m, n, p) = \begin{cases} \{p\}, & m > \rho(n) \\ \{p, p+1\}, & m \leq \rho(n). \end{cases}$$

We denote an $m_1 \times m_2 \times m_3$ tensor (x_{ijk}) by $(\mathbf{X}_1; \dots; \mathbf{X}_{m_3})$, where $\mathbf{X}_t = (x_{ijt})$ is an $m_1 \times m_2$ matrix for each $1 \leq t \leq m_3$. Let $3 \leq m \leq n$ and $p = (m-1)n$. For an $n \times p \times m$ tensor $\mathbf{X} = (\mathbf{X}_1; \dots; \mathbf{X}_{m-1}; \mathbf{X}_m)$, let $\mathbf{H}(\mathbf{X})$ and $\hat{\mathbf{H}}(\mathbf{X})$ be a $p \times p$ matrix and an $mn \times p$ matrix respectively defined as follows.

$$\mathbf{H}(\mathbf{X}) = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_{m-1} \end{pmatrix}, \quad \hat{\mathbf{H}}(\mathbf{X}) = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{pmatrix}$$

Let

$$\mathcal{R} = \{\mathbf{X} \in \mathbb{R}^{n \times p \times m} \mid \mathbf{H}(\mathbf{X}) \text{ is nonsingular}\}.$$

This is a nonempty Zariski open set. For $\mathbf{X} = (\mathbf{X}_1; \dots; \mathbf{X}_{m-1}; \mathbf{X}_m) \in \mathcal{R}$, we see

$$\hat{\mathbf{H}}(\mathbf{X})\mathbf{H}(\mathbf{X})^{-1} = \begin{pmatrix} \mathbf{E}_n & & & \\ & \mathbf{E}_n & & \\ & & \ddots & \\ & & & \mathbf{E}_n \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_{m-1} \end{pmatrix},$$

where \mathbf{E}_n is the $n \times n$ identity matrix and $(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{m-1}) = \mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}$. Note that $\text{rank } \mathbf{X} \geq p$ for $\mathbf{X} \in \mathcal{R}$. Let h be an isomorphism from the set of $n \times p$ matrices to $\mathbb{R}^{n \times n \times (m-1)}$ given by

$$(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{m-1}) \mapsto (\mathbf{Y}_1; \mathbf{Y}_2; \dots; \mathbf{Y}_{m-1}).$$

Then $h(\mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}) \in \mathbb{R}^{n \times n \times (m-1)}$. We consider the following subsets of $\mathbb{R}^{n \times n \times (m-1)}$. For $\mathbf{Y} = (\mathbf{Y}_1; \mathbf{Y}_2; \dots; \mathbf{Y}_{m-1}) \in \mathbb{R}^{n \times n \times (m-1)}$ and $\mathbf{a} = (a_1, \dots, a_{m-1}, a_m)^T \in \mathbb{R}^m$, let

$$\mathbf{M}(\mathbf{a}, \mathbf{Y}) = \sum_{k=1}^{m-1} a_k \mathbf{Y}_k - a_m \mathbf{E}_n$$

and set

$$\mathcal{C} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times (m-1)} \mid |\mathbf{M}(\mathbf{a}, \mathbf{Y})| < 0 \text{ for some } \mathbf{a} \in \mathbb{R}^m\}$$

and

$$\mathcal{A} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times (m-1)} \mid |\mathbf{M}(\mathbf{a}, \mathbf{Y})| > 0 \text{ for all } \mathbf{a} \neq \mathbf{0}\}.$$

Here, $|\mathbf{M}(\mathbf{a}, \mathbf{Y})|$ is the determinant of the matrix $\mathbf{M}(\mathbf{a}, \mathbf{Y})$. The subsets \mathcal{C} and \mathcal{A} are open sets in Euclidean topology and $\mathcal{C} \cup \mathcal{A} = \mathbb{R}^{n \times n \times (m-1)}$. In [13], we show that \mathcal{A} is not empty if and only if $m \leq \rho(n)$ and that $\text{rank } \mathbf{X} > p$ for any $\mathbf{X} \in \mathcal{R}$ with $h(\mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}) \in \mathcal{A}$. In this paper, we show that there exists an open subset \mathcal{F} of \mathcal{C} such that $\overline{\mathcal{F}} = \mathcal{C}$ and $\text{rank } \mathbf{X} = p$ for any $\mathbf{X} \in \mathcal{R}$ with $h(\mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}) \in \mathcal{F}$.

2. Typical rank

Due to [11, 14] and others, a number r is a typical rank of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if the subset of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ of rank r has nonzero volume. De Silva and Lim [4] studied the space of real tensors and tensor rank with Euclidean topology and real semi-algebraic sets. They [4, Theorem 6.1] showed that for a positive integer r , the set of tensors with rank r is a semi-algebraic set by using the Tarski-Seidenberg principle (cf. [1]). In the sence, a number r is a typical rank if the set of tensors with rank r contains an open semi-algebraic set.

For $\mathbf{x} = (x_1, \dots, x_{m_1})^T \in \mathbb{C}^{m_1}$, $\mathbf{y} = (y_1, \dots, y_{m_2})^T \in \mathbb{C}^{m_2}$, and $\mathbf{z} = (z_1, \dots, z_{m_3})^T \in \mathbb{C}^{m_3}$, we denote $(x_i y_j z_k) \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ by $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$. Let $f_t: (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^t \rightarrow \mathbb{C}^{m_1 \times m_2 \times m_3}$ be a map given by

$$f_t(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}, \dots, \mathbf{x}_{t,1}, \mathbf{x}_{t,2}, \mathbf{x}_{t,3}) = \sum_{\ell=1}^t \mathbf{x}_{\ell,1} \otimes \mathbf{x}_{\ell,2} \otimes \mathbf{x}_{\ell,3}.$$

Let S be a subset of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. S is called semi-algebraic if it is a finite Boolean combination (that is, a finite composition of disjunctions, conjunctions and negations) of sets of the form

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid f(a_{111}, \dots, a_{m_1, m_2, m_3}) > 0\} \quad (1)$$

and

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid g(a_{111}, \dots, a_{m_1, m_2, m_3}) = 0\},$$

where f and g are polynomials in $m_1 m_2 m_3$ indeterminates $x_{111}, \dots, x_{m_1, m_2, m_3}$ over \mathbb{R} . Then S is an open semi-algebraic set if and only if it is expressed as a finite Boolean combinations of sets of the form (1), and it is a dense open semi-algebraic set if and only if it is a Zariski open set, that is, expressed as

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid g(a_{111}, \dots, a_{m_1, m_2, m_3}) \neq 0\}.$$

Theorem 2.1 ([5, Theorem 7.1]): *The space $\mathbb{R}^{m_1 \times m_2 \times m_3}$, $m_1, m_2, m_3 \in \mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $\mathcal{O}_1, \dots, \mathcal{O}_M$ satisfying the following properties.*

- (1) $\mathbb{R}^{m_1 \times m_2 \times m_3} \setminus \bigcup_{i=1}^M \mathcal{O}_i$ is a closed semi-algebraic set $\mathbb{R}^{m_1 \times m_2 \times m_3}$ of dimension less than $m_1 m_2 m_3$.
- (2) Each $\mathbf{T} \in \mathcal{O}_i$ has rank r_i for $i = 1, \dots, M$.
- (3) The number $\min(r_1, \dots, r_M)$ is equal to the generic rank $\text{grank}(m_1, m_2, m_3)$ of $\mathbb{C}^{m_1 \times m_2 \times m_3}$, that is, the minimal $t \in \mathbb{N}$ such that the closure of the image of f_t is equal to $\mathbb{C}^{m_1 \times m_2 \times m_3}$.
- (4) $\text{mtrank}(m_1, m_2, m_3) := \max(r_1, \dots, r_M)$ is the minimal $t \in \mathbb{N}$ such that the closure of $f_t((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^t)$ is equal to $\mathbb{R}^{m_1 \times m_2 \times m_3}$.
- (5) For each integer $r \in [\text{grank}(m_1, m_2, m_3), \text{mtrank}(m_1, m_2, m_3)]$, there exists $r_i = r$ for some integer $i \in [1, M]$.

Definition 2.2: A positive integer r is called a typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if

$$r \in [\text{grank}(m_1, m_2, m_3), \text{mtrank}(m_1, m_2, m_3)].$$

Put

$$\text{typical_rank}_{\mathbb{R}}(m_1, m_2, m_3) = [\text{grank}(m_1, m_2, m_3), \text{mtrank}(m_1, m_2, m_3)] \cap \mathbb{Z}.$$

This is the set of all typical ranks of $\mathbb{R}^{m_1 \times m_2 \times m_3}$.

We state basic facts. Some of them are seen in [4].

Proposition 2.3: *Let r be a positive integer and \mathcal{L} a nonempty open set of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. If every tensor of \mathcal{L} has rank r , then r is a typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$.*

Proof: Let $\mathcal{O}_1, \dots, \mathcal{O}_M$ be open connected disjoint semi-algebraic sets as in Theorem 2.1. Since $\dim(\mathbb{R}^{m_1 \times m_2 \times m_3} \setminus \bigcup_{i=1}^M \mathcal{O}_i) < m_1 m_2 m_3$, there exists $i \in [1, M]$ such that $\mathcal{L} \cap \mathcal{O}_i$ is not empty. \square

Proposition 2.4: *Let $m_1, m_2, m_3, m_4 \in \mathbb{N}$ with $m_3 < m_4$. Then*

$$\text{grank}(m_1, m_2, m_3) \leq \text{grank}(m_1, m_2, m_4)$$

and

$$\text{mtrank}(m_1, m_2, m_3) \leq \text{mtrank}(m_1, m_2, m_4).$$

Proof: Let \mathcal{L} be the nonempty Zariski open subset of $\mathbb{C}^{m_1 \times m_2 \times m_4}$ consisting of all tensors of rank $\text{grank}(m_1, m_2, m_4)$ and put

$$\mathcal{H} = \{(\mathbf{Y}_1; \mathbf{Y}_2; \dots; \mathbf{Y}_{m_3}) \in \mathbb{C}^{m_1 \times m_2 \times m_3} \mid (\mathbf{Y}_1; \mathbf{Y}_2; \dots; \mathbf{Y}_{m_4}) \in \mathcal{L}\}.$$

Then \mathcal{H} is a nonempty Zariski open set of $\mathbb{C}^{m_1 \times m_2 \times m_3}$. For the subset \mathcal{L}' of $\mathbb{C}^{m_1 \times m_2 \times m_3}$ consisting of all tensors of rank $\text{grank}(m_1, m_2, m_3)$, the intersection $\mathcal{H} \cap \mathcal{L}'$ is a nonempty Zariski open set. Since $\text{rank } \mathbf{Y} \leq \text{rank}(\mathbf{Y}; \mathbf{X})$ for $\mathbf{Y} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ and $(\mathbf{Y}; \mathbf{X}) \in \mathbb{C}^{m_1 \times m_2 \times m_4}$, we see

$$\text{grank}(m_1, m_2, m_3) \leq \text{grank}(m_1, m_2, m_4).$$

Next, take an open semi-algebraic set \mathcal{H} of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ consisting of tensors of rank $\text{mtrank}(m_1, m_2, m_3)$. Then there are $s \in \text{typical_rank}_{\mathbb{R}}(m_1, m_2, m_4)$ and an open semi-algebraic set \mathcal{O} of $\mathbb{R}^{m_1 \times m_2 \times m_4}$ consisting of tensors of rank s such that $\{(\mathbf{A}; \mathbf{B}) \mid \mathbf{A} \in \mathcal{H}, \mathbf{B} \in \mathbb{R}^{m_1 \times m_2 \times (m_4 - m_3)}\} \cap \mathcal{O} \neq \emptyset$. Thus

$$\text{mtrank}(m_1, m_2, m_3) \leq s \leq \text{mtrank}(m_1, m_2, m_4).$$

□

The action of $\text{GL}(m) \times \text{GL}(n) \times \text{GL}(p)$ on $\mathbb{R}^{m \times n \times p}$ is given as follows. Let $\mathbf{P} = (p_{ij}) \in \text{GL}(n)$, $\mathbf{Q} = (q_{ij}) \in \text{GL}(m)$, and $\mathbf{R} = (r_{ij}) \in \text{GL}(p)$. The tensor $(b_{ijk}) = (\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot (a_{ijk})$ is defined as

$$b_{ijk} = \sum_{s=1}^m \sum_{t=1}^n \sum_{u=1}^p p_{is} q_{jt} r_{ku} a_{stu}.$$

Therefore,

$$(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot (\mathbf{A}_1; \dots; \mathbf{A}_p) = \left(\sum_{u=1}^p r_{1u} \mathbf{P} \mathbf{A}_u \mathbf{Q}^T; \dots; \sum_{u=1}^p r_{pu} \mathbf{P} \mathbf{A}_u \mathbf{Q}^T \right).$$

Definition 2.5: Two tensors \mathbf{A} and \mathbf{B} is called *equivalent* if there exists $g \in \text{GL}(m) \times \text{GL}(n) \times \text{GL}(p)$ such that $\mathbf{B} = g \cdot \mathbf{A}$.

Proposition 2.6 (cf. [4, Lemma 2.3]): *If two tensors are equivalent, then they have the same rank.*

A $1 \times m_2 \times m_3$ tensor \mathbf{T} is an $m_2 \times m_3$ matrix and $\text{rank } \mathbf{T}$ is equal to the matrix rank. The following three propositions are well-known. The former two propositions easily follow from the definition.

Proposition 2.7: *Let $m_1, m_2, m_3 \in \mathbb{N}$ with $2 \leq m_1 \leq m_2 \leq m_3$. If $m_1 m_2 \leq m_3$, then typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ is only one integer $m_1 m_2$.*

Proposition 2.8: *Let $\mathbf{X} = (x_{ijk})$ be an $m_1 \times m_2 \times m_3$ tensor. For an $m_2 \times m_1 \times m_3$ tensor $\mathbf{Y} = (y_{jik})$ and an $m_1 \times m_3 \times m_2$ tensor $\mathbf{Z} = (z_{ikj})$, it holds that*

$$\text{rank } \mathbf{X} = \text{rank } \mathbf{Y} = \text{rank } \mathbf{Z}.$$

A rank decomposition implies the following proposition.

Proposition 2.9 (cf. [2, 14]): *An $m_1 \times m_2 \times m_3$ tensor $(\mathbf{Y}_1; \dots; \mathbf{Y}_{m_3})$ has rank less than or equal to r if and only if there are an $m_1 \times r$ matrix \mathbf{P} , an $r \times m_2$ matrix \mathbf{Q} , and $r \times r$ diagonal matrices $\mathbf{D}_1, \dots, \mathbf{D}_{m_3}$ such that $\mathbf{Y}_k = \mathbf{P}\mathbf{D}_k\mathbf{Q}$ for $1 \leq k \leq m_3$.*

For an integer $2 \leq m < n < 2m$, the number n is an only typical rank of $\mathbb{R}^{m \times n \times 2}$. Indeed, it is known that

Theorem 2.10 ([9]): *Let $2 \leq m < n$. There is an open dense semi-algebraic set \mathcal{O} of $\mathbb{R}^{m \times n \times 2}$ of which any tensor is equivalent to $((\mathbf{E}_m, \mathbf{O}); (\mathbf{O}, \mathbf{E}_m))$ which has rank $\min(n, 2m)$.*

Furthermore, by Proposition 2.4, $\text{typical_rank}_{\mathbb{R}}(m, m, 2)$ is equal to either $\{m\}$ or $\{m, m+1\}$. Let \mathcal{L} be an open subset of $\mathbb{R}^{m \times m \times 2}$ consisting of $(\mathbf{A}; \mathbf{B})$ such that \mathbf{A} is an $m \times m$ nonsingular matrix and all eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are distinct and contain non-real numbers. For $m \geq 2$, the set \mathcal{L} is not empty and any tensor of \mathcal{L} has rank $m+1$ (cf. [12, Theorem 4.6]) and therefore $\text{typical_rank}_{\mathbb{R}}(m, m, 2) = \{m, m+1\}$ by Proposition 2.3.

Theorem 2.11 ([14, Result 2]): *Let $m, n, \ell \in \mathbb{N}$ with $3 \leq m \leq n \leq u$. If $(m-1)n < u < mn$, typical rank of $\mathbb{R}^{m \times n \times u}$ is only one integer u .*

In fact we can construct a nonempty Zariski open subset consisting of tensors with rank $\leq u$. We omit the construction.

Corollary 2.12: *Let $3 \leq m \leq n$. Then the set of typical ranks of $m \times n \times (m-1)n$ tensors is either $\{(m-1)n\}$ or $\{(m-1)n, (m-1)n+1\}$.*

Proof: The typical rank of $\mathbb{R}^{m \times n \times ((m-1)n+1)}$ is only $(m-1)n+1$ by Theorem 2.11 and the minimal typical rank of $\mathbb{R}^{m \times n \times (m-1)n}$ is equal to $(m-1)n$, since it is equal to the generic rank of $\mathbb{C}^{m \times n \times (m-1)n}$. Thus the assertion follows from Proposition 2.4. \square

3. Characterization

From now on, let $3 \leq m \leq n$, $\ell = m-1$ and $p = (m-1)n$. For an $n \times n \times \ell$ tensor $(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell)$, consider an $n \times p \times m$ tensor $\mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = (\mathbf{X}_1; \dots; \mathbf{X}_m)$ given by

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_m \end{pmatrix} = \begin{pmatrix} \mathbf{E}_n & & \\ & \mathbf{E}_n & \\ & & \ddots \\ & & & \mathbf{E}_n \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_\ell \end{pmatrix}. \quad (2)$$

Note that $\text{rank } \mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) \geq p$, since $\text{rank } \mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell)$ is greater than or equal to the rank of the $mn \times p$ matrix (2). Generically, an $m \times n \times p$ tensor is equivalent to a tensor of type as $\mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell)$.

We denote by \mathcal{M} the set of tensors $\mathbf{Y} = (\mathbf{Y}_1; \dots; \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times \ell}$ such that there exist an $m \times p$ matrix (x_{ij}) and an $n \times p$ matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_p)$ such that

$$(x_{1j}\mathbf{Y}_1 + \cdots + x_{m-1,j}\mathbf{Y}_{m-1} - x_{mj}\mathbf{E}_n)\mathbf{a}_j = \mathbf{0} \quad (3)$$

for $1 \leq j \leq p$ and

$$\mathbf{B} := \begin{pmatrix} \mathbf{A}\mathbf{D}_1 \\ \vdots \\ \mathbf{A}\mathbf{D}_\ell \end{pmatrix} \quad (4)$$

is nonsingular, where $\mathbf{D}_k = \mathbf{Diag}(x_{k1}, \dots, x_{kp})$ for $1 \leq k \leq \ell$.

Lemma 3.1: $\text{rank } \mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = p$ if and only if $(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell) \in \mathcal{M}$.

Proof: Suppose that $\text{rank } \mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = p$. There are an $n \times p$ matrix \mathbf{A} , a $p \times p$ matrix \mathbf{Q} and $p \times p$ diagonal matrices \mathbf{D}_i such that $\mathbf{X}_k = \mathbf{A}\mathbf{D}_k\mathbf{Q}$ for $k = 1, \dots, m$. Since

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_\ell \end{pmatrix} = \mathbf{E}_p = \begin{pmatrix} \mathbf{A}\mathbf{D}_1 \\ \vdots \\ \mathbf{A}\mathbf{D}_\ell \end{pmatrix} \mathbf{Q},$$

\mathbf{B} is nonsingular. Then $(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell)\mathbf{B} = \mathbf{A}\mathbf{D}_m$ implies that $\sum_{k=1}^{\ell} \mathbf{Y}_k \mathbf{A} \mathbf{D}_k = \mathbf{A} \mathbf{D}_m$. Therefore, the j -th column vector \mathbf{a}_j of \mathbf{A} satisfies (3). Therefore $(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) \in \mathcal{M}$. It is easy to see that the converse is also true. \square

For an $n \times n \times \ell$ tensor $\mathbf{Y} = (\mathbf{Y}_1; \dots; \mathbf{Y}_\ell)$, we put

$$V(\mathbf{Y}) = \{\mathbf{a} \in \mathbb{R}^n \mid \sum_{k=1}^{\ell} x_k \mathbf{Y}_k \mathbf{a} = x_m \mathbf{a} \text{ for some } (x_1, \dots, x_m)^T \neq \mathbf{0}\}.$$

The set $V(\mathbf{Y})$ is not a vector subspace of \mathbb{R}^n . Let $\hat{V}(\mathbf{Y})$ be the smallest vector subspace of \mathbb{R}^n including $V(\mathbf{Y})$. Let

$$\mathcal{S} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid \dim \hat{V}(\mathbf{Y}) = n\}.$$

Proposition 3.2: $\mathcal{M} \subset \mathcal{S}$ holds.

Proof: Let $\mathbf{Y} \in \mathcal{M}$. Consider the matrix \mathbf{B} in (4) for any $m \times p$ matrix (x_{ij}) and any $n \times p$ matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_p)$ satisfying the equation (3). By column operations, \mathbf{B} is transformed to a $p \times p$ matrix having a form

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{O}_{n, p - \dim \hat{V}(\mathbf{Y})} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

where \mathbf{P}_{11} is an $n \times \dim \hat{V}(\mathbf{Y})$ submatrix of \mathbf{A} . Since \mathbf{B} is nonsingular, \mathbf{P}_{11} is also nonsingular, which implies that $\dim \hat{V}(\mathbf{Y}) = n$. \square

By Corollary 2.12, Lemma 3.1 and Proposition 3.2, we have the following

Proposition 3.3: If $\text{rank } \mathbf{X}(\mathbf{Y}) = p$ then $\mathbf{Y} \in \mathcal{S}$. In particular, $\overline{\mathcal{S}} \neq \mathbb{R}^{n \times n \times \ell}$ implies that $\text{typical_rank}_{\mathbb{R}}(m, n, p) = \{p, p+1\}$.

Theorem 3.4 ([13]): If $(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell; \mathbf{E}_n)$ is an absolutely nonsingular tensor, then it holds that $\text{rank } \mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) > p$.

Here $(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell; \mathbf{Y}_m)$ is called an absolutely nonsingular tensor if $|\sum_{k=1}^m x_k \mathbf{Y}_k| = 0$ implies $(x_1, \dots, x_m)^T = \mathbf{0}$. Therefore,

Proposition 3.5: $\dim \hat{V}(\mathbf{Y}) = 0$ if and only if $(\mathbf{Y}; \mathbf{E}_n)$ is an $n \times n \times m$ absolutely nonsingular tensor.

Note that there exists an $n \times n \times m$ absolutely nonsingular tensor if and only if m is less than or equal to the Hurwitz-Radon number $\rho(n)$ [13].

Proposition 3.6: *Let \mathbf{Y} and \mathbf{Z} be $n \times n \times m$ tensors. Suppose $(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot \mathbf{Y} = \mathbf{Z}$ for $(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \in \text{GL}(n) \times \text{GL}(n) \times \text{GL}(m)$. Then $V(\mathbf{Y}) = \mathbf{Q}^T V(\mathbf{Z}) = \{\mathbf{Q}^T \mathbf{y} \mid \mathbf{y} \in V(\mathbf{Z})\}$. In particular, $\dim \hat{V}(\mathbf{Z}) = \dim \hat{V}(\mathbf{Y})$.*

Proof: Suppose that $\sum_{k=1}^m x_k \mathbf{Z}_k \mathbf{y} = \mathbf{0}$. Then from the definition of the action, it follows that

$$\sum_{k=1}^m d_k \sum_{u=1}^m r_{ku} \mathbf{P} \mathbf{Y}_u \mathbf{Q}^T \mathbf{y} = \mathbf{P} \left(\sum_{u=1}^m \left(\sum_{k=1}^m d_k r_{ku} \mathbf{Y}_u \right) \right) \mathbf{Q}^T \mathbf{y} = \mathbf{0}.$$

Thus $\mathbf{Q}^T \mathbf{y} \in V(\mathbf{Y})$. □

Corollary 3.7: *\mathcal{S} is closed under the equivalence relation.*

The closure of the set of all $n \times p \times m$ tensors equivalent to $\mathbf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell)$ for some $\mathbf{Y}_1, \dots, \mathbf{Y}_\ell$ is $\mathbb{R}^{n \times p \times m}$. Furthermore, the following claim holds. Let \mathcal{H} be the set of $n \times p \times m$ tensors $(\mathbf{X}_1; \dots; \mathbf{X}_m)$ such that $\mathbf{A} = (\mathbf{X}_1^T, \dots, \mathbf{X}_m^T)$ is a nonsingular $p \times p$ matrix and $(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell)$ given by $(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = \mathbf{A}^{-1} \mathbf{X}_m$ lies in \mathcal{M} . Any tensor of \mathcal{H} has rank p . If \mathcal{M} is dense in $\mathbb{R}^{n \times n \times \ell}$ then \mathcal{H} is dense in $\mathbb{R}^{n \times p \times m}$.

4. Classes of $n \times n \times \ell$ tensors

We separate $\mathbb{R}^{n \times n \times \ell}$ into three classes \mathcal{A} , \mathcal{C} , and \mathcal{B} as follows. Let \mathcal{A} be the set of tensors \mathbf{Y} such that $(\mathbf{Y}; \mathbf{E}_n)$ is absolutely nonsingular. By Proposition 3.5, we have the following

Proposition 4.1: $\mathcal{A} \cap \mathcal{S} = \emptyset$.

From now on, we use symbols x_1, \dots, x_ℓ, x_m as indeterminates over \mathbb{R} . For $\mathbf{Y} = (\mathbf{Y}_1; \dots; \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times \ell}$, we define the $n \times n$ matrix with entries in $\mathbb{R}[x_1, \dots, x_\ell, x_m]$ as follows.

$$\mathbf{M}(\mathbf{x}, \mathbf{Y}) = \sum_{k=1}^{\ell} x_k \mathbf{Y}_k - x_m \mathbf{E}_n$$

Note that fixing a_1, \dots, a_ℓ , the determinant $|\mathbf{M}(\mathbf{a}, \mathbf{Y})|$ is positive for $a_m \ll 0$, where $\mathbf{a} = (a_1, \dots, a_\ell, a_m)^T$. Set

$$\mathcal{C} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\mathbf{a}, \mathbf{Y})| < 0 \text{ for some } \mathbf{a} \in \mathbb{R}^m\}.$$

Note that \mathcal{C} is not empty, and if n is not congruent to 0 modulo 4 then $\rho(n) \leq 2$ and thus \mathcal{A} is empty since $m \geq 3$. Set $\mathcal{B} = \mathbb{R}^{n \times n \times \ell} \setminus (\mathcal{A} \cup \mathcal{C})$. In particular, if n is odd, then $\mathcal{C} = \mathbb{R}^{n \times n \times \ell}$, and otherwise \mathcal{B} contains the zero tensor.

Proposition 4.2: *\mathcal{A} and \mathcal{C} are open subsets of $\mathbb{R}^{n \times n \times \ell}$.*

Recall that

$$\mathcal{A} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\mathbf{a}, \mathbf{Y})| > 0 \text{ for all } \mathbf{a} \neq \mathbf{0}\}.$$

Thus it holds

$$\mathcal{B} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid \begin{array}{l} |\mathbf{M}(\mathbf{b}, \mathbf{Y})| = 0 \text{ for some } \mathbf{b} \neq \mathbf{0} \text{ and} \\ |\mathbf{M}(\mathbf{a}, \mathbf{Y})| \geq 0 \text{ for all } \mathbf{a} \end{array}\}.$$

If n is even then \mathcal{B} contains a nonzero tensor in general. We give an example.

Example 4.3 Let $m = 3$ and $n = 6$. Note that \mathcal{A} is empty, since $\rho(6) = 2$. Let $\mathbf{Y} = (\mathbf{Y}_1; \mathbf{Y}_2)$ be a $6 \times 6 \times 2$ tensor given by

$$x_1 \mathbf{Y}_1 + x_2 \mathbf{Y}_2 = \begin{pmatrix} 0 & -x_2 & 0 & 0 & 0 & -x_1 \\ x_1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & x_1 & 0 & x_2 \\ -x_2 & 0 & 0 & 0 & x_1 & 0 \end{pmatrix}.$$

Then $|\mathbf{M}((a_1, a_2, a_3)^T, \mathbf{Y})| = a_3^2(a_1a_2 - a_3^2)^2 + (a_1^3 + a_2^3)^2 \geq 0$. The equality holds if $a_3 = 0$

and $a_1 = -a_2$. Thus $\mathbf{Y} \in \mathcal{B}$ and $\mathbf{Y} \notin \mathcal{S}$ since $\dim \hat{V}(\mathbf{Y}) = 1$. Let $\mathbf{X} = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ be a

6×6 matrix and let $\varepsilon \in \mathbb{R}$. If $x_1 = -1$ and $x_3 = \varepsilon c$, then

$$|\mathbf{M}((x_1, x_2, x_3)^T, (\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon \mathbf{X}))| = -\varepsilon^2(1 + \varepsilon^2 c^2)c((1 - \varepsilon c)^2 - c(1 + \varepsilon^2 c^2)).$$

Thus, for any $\varepsilon \neq 0$, there is $c_0 > 0$ such that $(1 - \varepsilon c_0)^2 - c_0(1 + \varepsilon^2 c_0^2) > 0$ and then $|\mathbf{M}((-1, 1, \varepsilon c_0)^T, (\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon \mathbf{X}))| < 0$, which implies that $(\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon \mathbf{X}) \in \mathcal{C}$.

Proposition 4.4: \mathcal{B} is a boundary of \mathcal{C} . In particular, $\mathbb{R}^{n \times n \times \ell}$ is a disjoint sum of \mathcal{A} and the closure $\overline{\mathcal{C}}$ of \mathcal{C} .

Proof: It suffices to show that $\mathcal{B} \subset \overline{\mathcal{C}}$ by Proposition 4.2. Let $\mathbf{Y} = (\mathbf{Y}_1; \dots; \mathbf{Y}_\ell) \in \mathcal{B}$. There are a nonzero vector $\mathbf{b} = (b_1, \dots, b_\ell, b_m)^T \in \mathbb{R}^n$ with $|\mathbf{M}(\mathbf{b}, \mathbf{Y})| = 0$ and an element $g \in \text{GL}(\ell)$ such that $g \cdot \mathbf{Y} = (\mathbf{Z}_1; \mathbf{Z}_2; \dots; \mathbf{Z}_\ell)$ and $\mathbf{Z}_1 = \sum_{k=1}^\ell b_k \mathbf{Y}_k$. Then $|\mathbf{Z}_1 - b_m \mathbf{E}_n| = 0$. Take a sequence $\{\mathbf{Z}_1^{(u)}\}_{u \geq 1}$ such that $|\mathbf{Z}_1^{(u)} - b_m \mathbf{E}_n| < 0$ and $\lim_{u \rightarrow \infty} \mathbf{Z}_1^{(u)} = \mathbf{Z}_1$. Thus, $(\mathbf{Z}_1^{(u)}; \mathbf{Z}_2; \dots; \mathbf{Z}_\ell) \in \mathcal{C}$ and then $g^{-1} \cdot (\mathbf{Z}_1^{(u)}; \mathbf{Z}_2; \dots; \mathbf{Z}_\ell) \in \mathcal{C}$. Therefore, $\mathbf{Y} \in \overline{\mathcal{C}}$. \square

Corollary 4.5: If \mathcal{A} is not empty then \mathcal{B} is a boundary of \mathcal{A} .

Proposition 4.6: If $m \leq \rho(n-1)$ then $\mathcal{C} \not\subset \mathcal{S}$, where $\rho(n-1)$ is a Hurwitz-Radon number.

Proof: Let $(\mathbf{A}_1; \dots; \mathbf{A}_\ell; \mathbf{E}_{n-1})$ be an $(n-1) \times (n-1) \times m$ absolutely nonsingular tensor. Put $\mathbf{B}_k = \text{Diag}(a_k, \mathbf{A}_k)$ for $1 \leq k \leq \ell$ and $\mathbf{B}_m = \text{Diag}(1, \mathbf{E}_{n-1}) = \mathbf{E}_n$, and $\mathbf{B} = (\mathbf{B}_1; \dots; \mathbf{B}_\ell)$. Then it is easy to see that $\mathbf{B} \in \mathcal{C}$ and $|\sum_{k=1}^\ell x_k \mathbf{B}_k - x_m \mathbf{B}_m| = 0$ implies $x_m = \sum_{k=1}^\ell a_k x_k$. Therefore $V(\mathbf{B}) = \{a(1, 0, \dots, 0)^T \in \mathbb{R}^n \mid a \in \mathbb{R}\}$. In particular $\mathbf{B} \notin \mathcal{S}$. \square

5. Proof of Theorem 1.1

In this section we show Theorem 1.1.

In the space of homogeneous polynomials in m variables, there exists a proper Zariski closed subset S such that if a polynomial does not belong to S then it is irreducible [7, Theorem 7], since $m \geq 3$. Let $P(m, n)$ be the set of homogeneous polynomials in m variables x_1, \dots, x_m with real coefficients of degree n such that the coefficient of x_m^n is one. Its dimension is $\binom{m+n-1}{m-1} - 1$. Let I_ℓ be a nonempty Zariski open subset of $P(m, n)$ such that any polynomial of I_ℓ is irreducible. Note that $|\mathbf{M}(\mathbf{x}, \mathbf{Y})| \in P(m, n)$.

A polynomial $f \in P(m, n)$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^m$ if both $f(\mathbf{e}) \neq 0$ and for all vectors $\mathbf{v} \in \mathbb{R}^m$, the univariate polynomial $t \mapsto f(\mathbf{v} - t\mathbf{e})$ has all real roots.

Theorem 5.1 ([8]): *A homogeneous polynomial f of $P(3, n)$ is hyperbolic with respect to the vector $e = (0, 0, 1)^T$ and satisfies $f(e) = 1$ if and only if there exist $n \times n$ symmetric matrices \mathbf{A}, \mathbf{B} such that f is given by*

$$f(x, y, z) = |x\mathbf{A} + y\mathbf{B} + z\mathbf{E}_n|.$$

Proposition 5.2: *The set*

$$\{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |-\mathbf{M}(\mathbf{x}, \mathbf{Y})| \in I_\ell\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n \times n \times \ell}$.

Proof: Let $f_\ell: \mathbb{R}^{n \times n \times \ell} \rightarrow P(m, n)$ be a map which sends \mathbf{Y} to $|-\mathbf{M}(\mathbf{x}, \mathbf{Y})|$ and put

$$\mathcal{J}_\ell := \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid f_\ell(\mathbf{Y}) \in I_\ell\}.$$

Since I_ℓ is a Zariski open set, \mathcal{J}_ℓ is a Zariski open subset of $\mathbb{R}^{n \times n \times \ell}$. Then it suffices to show that \mathcal{J}_ℓ is not empty. For $(\mathbf{Z}_1; \mathbf{Z}_2) \in \mathbb{R}^{n \times n \times 2}$, if the polynomial

$$|-\mathbf{M}((x_1, x_2, x_3)^T, (\mathbf{Z}_1; \mathbf{Z}_2))|$$

is irreducible, then for any $(\mathbf{Y}_3; \dots; \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times (\ell-2)}$, the polynomial

$$|-\mathbf{M}(\mathbf{x}, (\mathbf{Z}_1; \mathbf{Z}_2; \mathbf{Y}_3; \dots; \mathbf{Y}_\ell))|$$

is also irreducible. Therefore, it suffices to show the assertion only for $m = 3$.

The set of hyperbolic homogeneous polynomials of degree n contains an open subset [10]. Then there is an irreducible homogenous polynomial f of $P(3, n)$ which is hyperbolic with respect to $(e_1, e_2, e_3)^T$. Since $f(e_1, e_2, e_3) \neq 0$, it holds that $(e_1, e_2, e_3)^T \neq \mathbf{0}$. Thus there is an irreducible homogeneous polynomial $g \in P(3, n)$ such that g is hyperbolic with respect to $(0, 0, 1)^T$ and $g(0, 0, 1) = 1$. By Theorem 5.1, $g(x_1, x_2, x_3)$ is described as

$$|x_1\mathbf{Z}_1 + x_2\mathbf{Z}_2 + x_3\mathbf{E}_n| = |-\mathbf{M}((x_1, x_2, x_3)^T, (-\mathbf{Z}_1; -\mathbf{Z}_2))|$$

for some $n \times n$ matrices $\mathbf{Z}_1, \mathbf{Z}_2$. Thus, \mathcal{J}_2 is nonempty. This completes the proof. \square

Let $\tilde{\mathbf{x}} = (x_1, \dots, x_\ell)^T$ for $\mathbf{x} = (x_1, \dots, x_\ell, x_m)^T$, and put

$$\psi(\mathbf{x}, \mathbf{Y}) := \begin{pmatrix} (-1)^{n+1} |\mathbf{M}(\mathbf{x}, \mathbf{Y})_{n,1}| \\ (-1)^{n+2} |\mathbf{M}(\mathbf{x}, \mathbf{Y})_{n,2}| \\ \vdots \\ (-1)^{n+n} |\mathbf{M}(\mathbf{x}, \mathbf{Y})_{n,n}| \end{pmatrix} \in \mathbb{R}^n, \quad \tilde{\mathbf{x}} \otimes \psi(\mathbf{x}, \mathbf{Y}) := \begin{pmatrix} x_1 \psi(\mathbf{x}, \mathbf{Y}) \\ x_2 \psi(\mathbf{x}, \mathbf{Y}) \\ \vdots \\ x_\ell \psi(\mathbf{x}, \mathbf{Y}) \end{pmatrix} \in \mathbb{R}^p,$$

and

$$U(\mathbf{Y}) := \langle \tilde{\mathbf{u}} \otimes \psi(\mathbf{u}, \mathbf{Y}) \mid |\mathbf{M}(\mathbf{u}, \mathbf{Y})| = 0, \mathbf{u} \in \mathbb{R}^m \rangle.$$

Lemma 5.3: *If $\dim U(\mathbf{Y}) = p$, then $\mathbf{Y} \in \mathcal{M}$.*

Proof: Let $\dim U(\mathbf{Y}) = p$. Then there are $\mathbf{u}_j = (u_{1j}, \dots, u_{mj})^T \in U(\mathbf{Y})$ for $1 \leq j \leq p$ such that

$$\mathbf{B} = (\tilde{\mathbf{u}}_1 \otimes \psi(\mathbf{u}_1, \mathbf{Y}), \dots, \tilde{\mathbf{u}}_p \otimes \psi(\mathbf{u}_p, \mathbf{Y}))$$

is nonsingular. Note that $\mathbf{M}(\mathbf{u}_j, \mathbf{Y})\psi(\mathbf{u}_j, \mathbf{Y}) = \mathbf{0}$ for $1 \leq j \leq p$ and

$$\mathbf{B} = \begin{pmatrix} \mathbf{A}\mathbf{D}_1 \\ \vdots \\ \mathbf{A}\mathbf{D}_\ell \end{pmatrix},$$

where $\mathbf{A} = (\psi(\mathbf{u}_1, \mathbf{Y}), \dots, \psi(\mathbf{u}_p, \mathbf{Y}))$ and $\mathbf{D}_k = \mathbf{Diag}(u_{k1}, \dots, u_{kp})$ for $1 \leq k \leq \ell$. Thus $\mathbf{Y} \in \mathcal{M}$. \square

For an $n \times \ell$ matrix $\mathbf{C} = (c_1, \dots, c_\ell)$, we put

$$g(\mathbf{x}, \mathbf{Y}, \mathbf{C}) := \left| \begin{matrix} \mathbf{M}(\mathbf{x}, \mathbf{Y})^{<n} \\ \sum_{k=1}^{\ell} x_k \mathbf{c}_k^T \end{matrix} \right|,$$

where $\mathbf{M}(\mathbf{x}, \mathbf{Y})^{<n}$ is the $(n-1) \times n$ matrix obtained from $\mathbf{M}(\mathbf{x}, \mathbf{Y})$ by removing the n -th row.

Lemma 5.4: *Let $\mathbf{C} = (c_1, \dots, c_\ell)$ be an $n \times \ell$ matrix. The following claims are equivalent.*

- (1) $\dim U(\mathbf{Y}) = p$.
- (2) $g(\mathbf{a}, \mathbf{Y}, \mathbf{C}) = 0$ for any $\mathbf{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\mathbf{a}, \mathbf{Y})| = 0$ implies $\mathbf{C} = \mathbf{0}$.

Proof: Let $\mathbf{C} = (c_1, \dots, c_\ell)$ be an $n \times \ell$ matrix. Put $\mathbf{d} = (c_1^T, \dots, c_\ell^T)^T \in \mathbb{R}^p$. The inner product of this vector \mathbf{d} with $\tilde{\mathbf{a}} \otimes \psi(\mathbf{a}, \mathbf{Y})$ is equal to $g(\mathbf{a}, \mathbf{Y}, \mathbf{C})$. Therefore \mathbf{d} belongs to the orthogonal complement of $U(\mathbf{Y})$ if and only if $g(\mathbf{a}, \mathbf{Y}, \mathbf{C}) = 0$ for any $\mathbf{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\mathbf{a}, \mathbf{Y})| = 0$. Thus the assertion holds. \square

For any i and k with $1 \leq i \leq n-1$ and $1 \leq k \leq n$, let $s_i^{(k)}$ be an elementary symmetric polynomial of degree i with variables $\alpha_1, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n$. Put

$$\mathbf{S}_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ s_1^{(1)} & s_1^{(2)} & \dots & s_1^{(n)} \\ s_2^{(1)} & s_2^{(2)} & \dots & s_2^{(n)} \\ \vdots & \vdots & & \vdots \\ s_{n-1}^{(1)} & s_{n-1}^{(2)} & \dots & s_{n-1}^{(n)} \end{pmatrix}.$$

Lemma 5.5: *The determinant $|\mathbf{S}_n|$ of the $n \times n$ matrix \mathbf{S}_n is equal to*

$$\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j).$$

In particular, if $\alpha_1, \dots, \alpha_n$ are distinct from each other, then \mathbf{S}_n is nonsingular.

Proof: For any i and k with $1 \leq i \leq n-1$ and $2 \leq k \leq n-1$, let $t_i^{(k-1)}$ be an elementary symmetric polynomial of degree i with variables $\alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n$. For $1 \leq i \leq n-1$ and $1 \leq k \leq n$, we have $s_i^{(k)} - s_i^{(1)} = (\alpha_1 - \alpha_k)t_{i-1}^{(k-1)}$. Then

$$|\mathbf{S}_n| = \prod_{2 \leq k \leq n} (\alpha_1 - \alpha_k) \begin{vmatrix} 1 & 1 & \dots & 1 \\ t_1^{(1)} & t_1^{(2)} & \dots & t_1^{(n-1)} \\ \vdots & \vdots & & \vdots \\ t_{n-2}^{(1)} & t_{n-2}^{(2)} & \dots & t_{n-2}^{(n-1)} \end{vmatrix}.$$

Therefore we have the assertion by induction on n . □

The following lemma is obtained straightforwardly.

Lemma 5.6:

$$\begin{vmatrix} \alpha_1 + z & & & a_1 \\ & \alpha_2 + z & & a_2 \\ & & \ddots & \vdots \\ & & & \alpha_n + z & a_n \\ b_1 & b_2 & \dots & b_n & 0 \end{vmatrix} = -(z^{n-1}, z^{n-2}, \dots, 1) \mathbf{S}_n \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_n b_n \end{pmatrix}.$$

Proof: We see the left hand of the equation is equal to

$$\begin{aligned} & - \sum_{k=1}^n a_k b_k \frac{\prod_{1 \leq i \leq n} (\alpha_i + z)}{\alpha_k + z} \\ &= - \sum_{k=1}^n a_k b_k \left(\sum_{i=1}^n s_{i-1}^{(k)} \right) z^{n-i} \\ &= - \sum_{i=1}^n \left(\sum_{k=1}^n a_k b_k s_{i-1}^{(k)} \right) z^{n-i} \\ &= -(z^{n-1}, z^{n-2}, \dots, 1) \begin{pmatrix} \sum_{k=1}^n a_k b_k \\ \sum_{k=1}^n a_k b_k s_1^{(k)} \\ \vdots \\ \sum_{k=1}^n a_k b_k s_{n-1}^{(k)} \end{pmatrix}. \end{aligned}$$

□

Corollary 5.7: Let $\alpha_1, \dots, \alpha_{n-1}$ be distinct complex numbers, a_1, \dots, a_{n-1} nonzero complex numbers, and b_1, \dots, b_{n-1} complex numbers. If

$$\begin{vmatrix} \text{Diag}(\alpha_1, \dots, \alpha_{n-1}) + z \mathbf{E}_{n-1} & \mathbf{a} \\ \mathbf{b}^T & 0 \end{vmatrix} = 0$$

for any $z \in \mathbb{R}$, then $\mathbf{b} = \mathbf{0}$, where $\mathbf{a} = (a_1, \dots, a_{n-1})^T$ and $\mathbf{b} = (b_1, \dots, b_{n-1})^T$.

Proof: Since $\mathbf{S}_n \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_n b_n \end{pmatrix} = \mathbf{0}$ and \mathbf{S}_n is nonsingular, we have $(a_1 b_1, \dots, a_n b_n) = \mathbf{0}^T$. □

The set

$$\mathcal{U}_1 = \{Y \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\mathbf{x}, \mathbf{Y})| \text{ is irreducible}\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n \times n \times \ell}$ (see Proposition 5.2). Let W be the subset of $\mathbb{R}^{n \times n}$ consisting of matrices $\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^T & e \end{pmatrix}$ such that all eigenvalues of \mathbf{A} are distinct over the complex number field and every element of the vector $\mathbf{P}^{-1} \mathbf{b}$ is nonzero complex number where $\mathbf{A} \in \mathbb{R}^{(n-1) \times (n-1)}$, $\mathbf{P} \in \mathbb{C}^{(n-1) \times (n-1)}$ such that $\mathbf{P}^{-1} \mathbf{A} \mathbf{P}$ is a diagonal matrix. Note that the validity of the condition that every element of the vector $\mathbf{P}^{-1} \mathbf{b}$ is nonzero is

independent of the choice of \mathbf{P} . We put

$$\mathcal{U}_2 := \{(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times \ell} \mid \mathbf{Y}_k \in W, 1 \leq k \leq \ell\}.$$

The set \mathcal{U}_2 is a nonempty Zariski open subset of $\mathbb{R}^{n \times n \times \ell}$ and $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$ is also.

Lemma 5.8: *Let $\mathbf{Y} \in \mathcal{U}_2$ and $\mathbf{d}_1, \dots, \mathbf{d}_\ell \in \mathbb{R}^{n-1}$. If*

$$\left| \begin{array}{c} \mathbf{M}(\mathbf{a}, \mathbf{Y})^{<n} \\ \sum_{k=1}^{\ell} a_k \mathbf{d}_k^T \mathbf{0} \end{array} \right| = 0$$

for any $\mathbf{a} = (a_1, \dots, a_m)^T \in \mathbb{R}^m$, then $\mathbf{d}_1 = \dots = \mathbf{d}_\ell = \mathbf{0}$.

Proof: Let $1 \leq k \leq \ell$. Take $a_k = 1$ and $a_j = 0$ for $1 \leq j \leq \ell, j \neq k$ and put $\mathbf{Y}_k = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^T & e \end{pmatrix}$, where \mathbf{A} is an $(n-1) \times (n-1)$ matrix. Since $\mathbf{Y}_k \in W$, there are a matrix $\mathbf{P} \in \mathbb{C}^{(n-1) \times (n-1)}$ and distinct complex numbers $\alpha_1, \dots, \alpha_{n-1}$ such that

$$\text{Diag}(\mathbf{P}, 1)^{-1} \begin{pmatrix} (\mathbf{Y}_k - a_m \mathbf{E}_n)^{<n} \\ \mathbf{d}_k^T \mathbf{0} \end{pmatrix} \text{Diag}(\mathbf{P}, 1) = \begin{pmatrix} \text{Diag}(\alpha_1, \dots, \alpha_{n-1}) - a_m \mathbf{E}_{n-1} & \mathbf{P}^{-1} \mathbf{b} \\ \mathbf{d}_k^T \mathbf{P} & 0 \end{pmatrix}$$

and every element of $\mathbf{P}^{-1} \mathbf{b}$ is nonzero. Then we have $\mathbf{d}_k^T \mathbf{P} = \mathbf{0}^T$ by Corollary 5.7 and thus $\mathbf{d}_k = \mathbf{0}$. \square

The following lemma is essential for the proof of Theorem 1.1.

Lemma 5.9: $\mathcal{U} \cap \mathcal{C} \subset \mathcal{M}$. In particular, $\overline{\mathcal{C}} \subset \overline{\mathcal{M}}$ holds.

Proof: Let $\mathbf{Y} \in \mathcal{U} \cap \mathcal{C}$ and fix it. There exists $\mathbf{a} = (a_1, \dots, a_\ell, a_m)^T$ such that $|\mathbf{M}(\mathbf{a}, \mathbf{Y})| < 0$. Then there is an open neighborhood U of $(a_1, \dots, a_\ell)^T$ and a mapping $\mu: U \rightarrow \mathbb{R}$ such that

$$|\mathbf{M}\left(\begin{pmatrix} \mathbf{y} \\ \mu(\mathbf{y}) \end{pmatrix}, \mathbf{Y}\right)| = 0$$

for any $\mathbf{y} \in U$. Thus $|\mathbf{M}(\mathbf{x}, \mathbf{Y})| = 0$ determines an $(m-1)$ -dimensional algebraic set. Let \mathbf{C} be an $n \times \ell$ matrix. Now suppose that $g(\mathbf{a}, \mathbf{Y}, \mathbf{C}) = 0$ holds for any $\mathbf{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\mathbf{a}, \mathbf{Y})| = 0$. We show that $g(\mathbf{x}, \mathbf{Y}, \mathbf{C})$ is zero as a polynomial over elements of \mathbf{x} . As a contrary, assume that $g(\mathbf{x}, \mathbf{Y}, \mathbf{C})$ is not zero. The degree of $g(\mathbf{x}, \mathbf{Y}, \mathbf{C})$ corresponding to the m -th element of \mathbf{x} is less than n which is that of $|\mathbf{M}(\mathbf{x}, \mathbf{Y})|$. Furthermore, since $|\mathbf{M}(\mathbf{x}, \mathbf{Y})|$ is irreducible, $|\mathbf{M}(\mathbf{x}, \mathbf{Y})|$ and $g(\mathbf{x}, \mathbf{Y}, \mathbf{C})$ are coprime. Then there are polynomials $f_1(\mathbf{x}), f_2(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_\ell, x_m]$ and a nonzero polynomial $h(\tilde{\mathbf{x}}) \in \mathbb{R}[x_1, \dots, x_\ell]$ such that

$$f_1(\mathbf{x})|\mathbf{M}(\mathbf{x}, \mathbf{Y})| + f_2(\mathbf{x})g(\mathbf{x}, \mathbf{Y}, \mathbf{C}) = h(\tilde{\mathbf{x}})$$

as a polynomial over elements of \mathbf{x} , by Euclidean algorithm. However, we can take $\mathbf{b} \in U$ so that $h(\mathbf{b}) \neq 0$. Then the above equation does not hold at $\mathbf{x} = \begin{pmatrix} \mathbf{b} \\ \mu(\mathbf{b}) \end{pmatrix}$. Hence $g(\mathbf{x}, \mathbf{Y}, \mathbf{C})$

must be the zero polynomial over elements of \mathbf{x} . Let $\mathbf{c}_k^T = (c_{1k}, \dots, c_{nk})$. By seeing the coefficient of $x_m^{n-1} x_k$, we get $c_{nk} = 0$ for $1 \leq k \leq \ell$. Therefore $\mathbf{C} = \mathbf{O}$ by Lemma 5.8. By Lemmas 5.4 and 5.3 we get $\mathbf{Y} \in \mathcal{M}$. Therefore $\mathcal{U} \cap \mathcal{C}$ is a subset of \mathcal{M} . Since \mathcal{C} is open and \mathcal{U} is dense, we have $\overline{\mathcal{C}} = \overline{\mathcal{C} \cap \mathcal{U}}$ and then $\overline{\mathcal{C}}$ is a subset of $\overline{\mathcal{M}}$. \square

Theorem 5.10: $\overline{\mathcal{P}} = \overline{\mathcal{M}} = \overline{\mathcal{C}}$ holds.

Proof: We have $\overline{\mathcal{M}} \subset \overline{\mathcal{S}}$ by Proposition 3.2. By Propositions 4.1 and 4.4, the set \mathcal{S} is a subset of \mathcal{C} and then $\overline{\mathcal{S}} \subset \overline{\mathcal{C}}$. Therefore $\overline{\mathcal{S}} = \overline{\mathcal{M}} = \overline{\mathcal{C}}$ by Lemma 5.9. \square

Proof of Theorem 1.1: For almost all $\mathbf{Y} \in \mathcal{A}$, $\text{rank} X(\mathbf{Y}) = p + 1$ by Theorem 3.4. Since \mathcal{A} is an open set, if \mathcal{A} is not an empty set, then $\text{typical_rank}_{\mathbb{R}}(m, n, p) = \{p, p + 1\}$ ([13, Theorem 3.4]). Suppose that \mathcal{A} is empty. Then $\overline{\mathcal{M}} = \mathbb{R}^{n \times n \times \ell}$ and the closure of the set consisting of all $n \times p \times m$ tensors equivalent to $X(\mathbf{Y})$ for some $\mathbf{Y} \in \mathcal{M}$ is $\mathbb{R}^{n \times p \times m}$. Recall that any tensor $X(\mathbf{Y})$ for $\mathbf{Y} \in \mathcal{M}$ has rank p . By Theorem 2.1, p is the maximal typical rank of $\mathbb{R}^{n \times p \times m}$. Therefore,

$$\text{typical_rank}_{\mathbb{R}}(m, n, p) = \text{typical_rank}_{\mathbb{R}}(n, p, m) = \{p\}$$

holds.

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