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RESEARCH ARTICLE

Typical ranks of $m \times n \times (m-1)n$ tensors with $3 \le m \le n$ over the real number field

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Let $3 \le m \le n$. We study typical ranks of $m \times n \times (m-1)n$ tensors over the real number field. Let ρ be the Hurwitz-Radon function defined as $\rho(n) = 2^b + 8c$ for nonnegative integers a, b, c such that $n = (2a+1)2^{b+4c}$ and $0 \le b < 4$. If $m \le \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has two typical ranks $(m-1)n, (m-1)n+1$. In this paper, we show that the converse is also true: if $m > \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has only one typical rank $(m-1)n$.

Keywords: tensor; 3-way array; typical rank; Hurwitz-Radon number; perfect

AMS Subject Classification: 15A69, 14P10, 14Q20

1. Introduction

For positive integers m, n, p, an $m \times n \times p$ tensor is an element of tensor product of vector spaces V_1, V_2, V_3 of dimension m, n, p, respectively. If we choose bases of their vector spaces, we can identify a tensor with a 3-way array

$$(a_{ijk})_{1 \le i \le m,\ 1 \le j \le n,\ 1 \le k \le p}.$$

Hitchcock [6] defined the rank of a tensor. An $m \times n \times p$ tensor of form

$$(x_i y_j z_k)_{1 \le i \le m,\ 1 \le j \le n,\ 1 \le k \le p}$$

is called a rank one tensor. The rank of a tensor T, denoted by $\operatorname{rank} T$, is defined as the minimal number of rank one tensors which describe T as a sum. The rank depends on the base field. For example there is a $2 \times 2 \times 2$ tensor over the real number field whose rank is 3 but is 2 as a tensor over the complex number field.

Throughout this paper, we assume that the base field is the real number field $\mathbb{R}$. Let $\mathbb{R}^{m \times n \times p}$ be the set of $m \times n \times p$ tensors with Euclidean topology. A number r is a typical rank of $m \times n \times p$ tensors if the set of tensors with rank r contains a nonempty open semi-algebraic set of $\mathbb{R}^{m \times n \times p}$ (see Theorem 2.1). We denote by $\operatorname{typical\_rank}_{\mathbb{R}}(m, n, p)$

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the set of typical ranks of $\mathbb{R}^{m\times n\times p}$. If s (resp. t) is the minimal (resp. maximal) number of $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p)$, then

$$\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p) = [s,t]\cap\mathbb{Z},$$

the interval of all integers between s and t, including both, and s is equal to the generic rank of the set of $m\times n\times p$ tensors over the complex number field [5]. In the case where $m=2$, the set of typical ranks of $2\times n\times p$ tensor is well-known [15]:

$$\mathrm{typical\_rank}_{\mathbb{R}}(2,n,p) = \begin{cases} \{p\}, & n<p\le 2n \\ \{2n\}, & 2n<p \\ \{p,p+1\}, & n=p\ge 2 \end{cases}$$

Suppose that $3\le m\le n$. If $p>(m-1)n$ then the set of typical ranks of $m\times n\times p$ tensors is just $\{\min(p,mn)\}$. If $p=(m-1)n$ then the set of typical ranks of $m\times n\times p$ tensor is $\{p\}$ or $\{p,p+1\}$ [14]. Until our paper [13], only a few cases where $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,(m-1)n) = \{(m-1)n,(m-1)n+1\}$ [3, 5] are known and we constructed infinitely many examples by using the concept of absolutely nonsingular tensors in [13]: If $m\le\rho(n)$ then $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p)=\{p,p+1\}$, where $\rho(n)$ is the Hurwitz-Radon number given by $\rho(n)=2^b+8c$ for nonnegative integers a,b,c such that $n=(2a+1)2^{b+4c}$ and $0\le b<4$.

n	1	2	3	4	5	6	7	8	9	10	11	12	16	32	64
$\rho(n)$	1	2	1	4	1	2	1	8	1	2	1	4	9	10	12

The purpose of this paper is to completely determine the set of typical ranks of $m\times n\times(m-1)n$ tensors:

Theorem 1.1 : *Let $3\le m\le n$ and $p=(m-1)n$. Then it holds*

$$\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p) = \begin{cases} \{p\}, & m>\rho(n) \\ \{p,p+1\}, & m\le\rho(n). \end{cases}$$

We denote an $m_1\times m_2\times m_3$ tensor (x_{ijk}) by $(\mathbf{X}_1;\dots;\mathbf{X}_{m_3})$, where $\mathbf{X}_t=(x_{ijt})$ is an $m_1\times m_2$ matrix for each $1\le t\le m_3$. Let $3\le m\le n$ and $p=(m-1)n$. For an $n\times p\times m$ tensor $\mathsf{X}=(\mathbf{X}_1;\dots;\mathbf{X}_{m-1};\mathbf{X}_m)$, let $\mathbf{H}(\mathsf{X})$ and $\hat{\mathbf{H}}(\mathsf{X})$ be a $p\times p$ matrix and an $mn\times p$ matrix respectively defined as follows.

$$\mathbf{H}(\mathsf{X}) = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_{m-1} \end{pmatrix}, \quad \hat{\mathbf{H}}(\mathsf{X}) = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{pmatrix}$$

Let

$$\mathscr{R} = \{\mathsf{X}\in\mathbb{R}^{n\times p\times m} \mid \mathbf{H}(\mathsf{X}) \text{ is nonsingular}\}.$$

This is a nonempty Zariski open set. For $\mathsf{X} = (\mathbf{X}_1; \ldots; \mathbf{X}_{m-1}; \mathbf{X}_m) \in \mathscr{R}$, we see

$$\hat{\mathbf{H}}(\mathsf{X})\mathbf{H}(\mathsf{X})^{-1} = \begin{pmatrix} \mathbf{E}_n & & & \\ & \mathbf{E}_n & & \\ & & \ddots & \\ & & & \mathbf{E}_n \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_{m-1} \end{pmatrix},$$

where $\mathbf{E}_n$ is the $n \times n$ identity matrix and $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{m-1}) = \mathbf{X}_m \mathbf{H}(\mathsf{X})^{-1}$. Note that $\operatorname{rank} \mathsf{X} \geq p$ for $\mathsf{X} \in \mathscr{R}$. Let h be an isomorphism from the set of $n \times p$ matrices to $\mathbb{R}^{n \times n \times (m-1)}$ given by

$$(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{m-1}) \mapsto (\mathbf{Y}_1; \mathbf{Y}_2; \ldots; \mathbf{Y}_{m-1}).$$

Then $h(\mathbf{X}_m \mathbf{H}(\mathsf{X})^{-1}) \in \mathbb{R}^{n \times n \times (m-1)}$. We consider the following subsets of $\mathbb{R}^{n \times n \times (m-1)}$. For $\mathsf{Y} = (\mathbf{Y}_1; \mathbf{Y}_2; \ldots; \mathbf{Y}_{m-1}) \in \mathbb{R}^{n \times n \times (m-1)}$ and $\boldsymbol{a} = (a_1, \ldots, a_{m-1}, a_m)^T \in \mathbb{R}^m$, let

$$\mathbf{M}(\boldsymbol{a}, \mathsf{Y}) = \sum_{k=1}^{m-1} a_k \mathbf{Y}_k - a_m \mathbf{E}_n$$

and set

$$\mathscr{C} = \{\mathsf{Y} \in \mathbb{R}^{n \times n \times (m-1)} \mid |\mathbf{M}(\boldsymbol{a}, \mathsf{Y})| < 0 \text{ for some } \boldsymbol{a} \in \mathbb{R}^m\}$$

and

$$\mathscr{A} = \{\mathsf{Y} \in \mathbb{R}^{n \times n \times (m-1)} \mid |\mathbf{M}(\boldsymbol{a}, \mathsf{Y})| > 0 \text{ for all } \boldsymbol{a} \neq \mathbf{0}\}.$$

Here, $|\mathbf{M}(\boldsymbol{a}, \mathsf{Y})|$ is the determinant of the matrix $\mathbf{M}(\boldsymbol{a}, \mathsf{Y})$. The subsets $\mathscr{C}$ and $\mathscr{A}$ are open sets in Euclidean topology and $\overline{\mathscr{C}} \cup \overline{\mathscr{A}} = \mathbb{R}^{n \times n \times (m-1)}$. In [13], we show that $\mathscr{A}$ is not empty if and only if $m \leq \rho(n)$ and that $\operatorname{rank} \mathsf{X} > p$ for any $\mathsf{X} \in \mathscr{R}$ with $h(\mathbf{X}_m \mathbf{H}(\mathsf{X})^{-1}) \in \mathscr{A}$. In this paper, we show that there exists an open subset $\mathscr{F}$ of $\mathscr{C}$ such that $\overline{\mathscr{F}} = \overline{\mathscr{C}}$ and $\operatorname{rank} \mathsf{X} = p$ for any $\mathsf{X} \in \mathscr{R}$ with $h(\mathbf{X}_m \mathbf{H}(\mathsf{X})^{-1}) \in \mathscr{F}$.

2. Typical rank

Due to [11, 14] and others, a number r is a typical rank of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if the subset of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ of rank r has nonzero volume. De Silva and Lim [4] studied the space of real tensors and tensor rank with Euclidean topology and real semi-algebraic sets. They [4, Theorem 6.1] showed that for a positive integer r, the set of tensors with rank r is a semi-algebraic set by using the Tarski-Seidenberg principle (cf. [1]). In the sence, a number r is a typical rank if the set of tensors with rank r contains an open semi-algebraic set.

For $\boldsymbol{x} = (x_1, \ldots, x_{m_1})^T \in \mathbb{C}^{m_1}$, $\boldsymbol{y} = (y_1, \ldots, y_{m_2})^T \in \mathbb{C}^{m_2}$, and $\boldsymbol{z} = (z_1, \ldots, z_{m_3})^T \in \mathbb{C}^{m_3}$, we denote $(x_i y_j z_k) \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ by $\boldsymbol{x} \otimes \boldsymbol{y} \otimes \boldsymbol{z}$. Let $f_t \colon (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^t \to \mathbb{C}^{m_1 \times m_2 \times m_3}$ be a map given by

$$f_t(\boldsymbol{x}_{1,1}, \boldsymbol{x}_{1,2}, \boldsymbol{x}_{1,3}, \ldots, \boldsymbol{x}_{t,1}, \boldsymbol{x}_{t,2}, \boldsymbol{x}_{t,3}) = \sum_{\ell=1}^{t} \boldsymbol{x}_{\ell,1} \otimes \boldsymbol{x}_{\ell,2} \otimes \boldsymbol{x}_{\ell,3}.$$

Let S be a subset of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. S is called semi-algebraic if it is a finite Boolean combination (that is, a finite composition of disjunctions, conjunctions and negations) of sets of the form

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid f(a_{111}, \ldots, a_{m_1,m_2,m_3}) > 0\} \tag{1}$$

and

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid g(a_{111}, \ldots, a_{m_1,m_2,m_3}) = 0\},$$

where f and g are polynomials in $m_1m_2m_3$ indeterminates $x_{111}, \ldots, x_{m_1,m_2,m_3}$ over $\mathbb{R}$. Then S is an open semi-algebraic set if and only if it is expressed as a finite Boolean combinations of sets of the form (1), and it is a dense open semi-algebraic set if and only if it is a Zariski open set, that is, expressed as

$$\{(a_{ijk}) \in \mathbb{R}^{m_1 \times m_2 \times m_3} \mid g(a_{111}, \ldots, a_{m_1,m_2,m_3}) \neq 0\}.$$

Theorem 2.1 ([5, Theorem 7.1])**:** *The space $\mathbb{R}^{m_1 \times m_2 \times m_3}$, $m_1, m_2, m_3 \in \mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $\mathscr{O}_1, \ldots, \mathscr{O}_M$ satisfying the following properties.*

(1) $\mathbb{R}^{m_1 \times m_2 \times m_3} \smallsetminus \cup_{i=1}^{M} \mathscr{O}_i$ *is a closed semi-algebraic set* $\mathbb{R}^{m_1 \times m_2 \times m_3}$ *of dimension less than $m_1m_2m_3$.*
(2) Each $\mathbf{T} \in \mathscr{O}_i$ has rank r_i for $i = 1, \ldots, M$.
(3) The number $\min(r_1, \ldots, r_M)$ *is equal to the generic rank* $\operatorname{grank}(m_1, m_2, m_3)$ *of* $\mathbb{C}^{m_1 \times m_2 \times m_3}$*, that is, the minimal $t \in \mathbb{N}$ such that the closure of the image of f_t is equal to $\mathbb{C}^{m_1 \times m_2 \times m_3}$.*
(4) $\operatorname{mtrank}(m_1, m_2, m_3) := \max(r_1, \ldots, r_M)$ *is the minimal $t \in \mathbb{N}$ such that the closure of $f_t((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^t)$ is equal to $\mathbb{R}^{m_1 \times m_2 \times m_3}$.*
(5) For each integer $r \in [\operatorname{grank}(m_1, m_2, m_3), \operatorname{mtrank}(m_1, m_2, m_3)]$, there exists $r_i = r$ for some integer $i \in [1, M]$.

Definition 2.2: A positive integer r is called a typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if

$$r \in [\operatorname{grank}(m_1, m_2, m_3), \operatorname{mtrank}(m_1, m_2, m_3)].$$

Put

$$\operatorname{typical\_rank}_{\mathbb{R}}(m_1, m_2, m_3) = [\operatorname{grank}(m_1, m_2, m_3), \operatorname{mtrank}(m_1, m_2, m_3)] \cap \mathbb{Z}.$$

This is the set of all typical ranks of $\mathbb{R}^{m_1 \times m_2 \times m_3}$.

We state basic facts. Some of them are seen in [4].

Proposition 2.3: *Let r be a positive integer and $\mathscr{L}$ a nonempty open set of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. If every tensor of $\mathscr{L}$ has rank r, then r is a typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$.*

Proof: Let $\mathscr{O}_1, \ldots, \mathscr{O}_M$ be open connected disjoint semi-algebraic sets as in Theorem 2.1. Since $\dim(\mathbb{R}^{m_1 \times m_2 \times m_3} \smallsetminus \cup_{i=1}^{M} \mathscr{O}_i) < m_1m_2m_3$, there exists $i \in [1, M]$ such that $\mathscr{L} \cap \mathscr{O}_i$ is not empty. $\square$

Proposition 2.4: *Let $m_1, m_2, m_3, m_4 \in \mathbb{N}$ with $m_3 < m_4$. Then*

$$\operatorname{grank}(m_1, m_2, m_3) \leq \operatorname{grank}(m_1, m_2, m_4)$$

and

$$\operatorname{mtrank}(m_1,m_2,m_3)\le\operatorname{mtrank}(m_1,m_2,m_4).$$

Proof: Let $\mathscr{L}$ be the nonempty Zariski open subset of $\mathbb{C}^{m_1\times m_2\times m_4}$ consisting of all tensors of rank $\operatorname{grank}(m_1,m_2,m_4)$ and put

$$\mathscr{H}=\{(\mathbf{Y}_1;\mathbf{Y}_2;\dots;\mathbf{Y}_{m_3})\in\mathbb{C}^{m_1\times m_2\times m_3}\mid(\mathbf{Y}_1;\mathbf{Y}_2;\dots;\mathbf{Y}_{m_4})\in\mathscr{L}\}.$$

Then $\mathscr{H}$ is a nonempty Zariski open set of $\mathbb{C}^{m_1\times m_2\times m_3}$. For the subset $\mathscr{L}'$ of $\mathbb{C}^{m_1\times m_2\times m_3}$ consisting of all tensors of rank $\operatorname{grank}(m_1,m_2,m_3)$, the intersection $\mathscr{H}\cap\mathscr{L}'$ is a nonempty Zariski open set. Since $\operatorname{rank}\mathsf{Y}\le\operatorname{rank}(\mathsf{Y};\mathsf{X})$ for $\mathsf{Y}\in\mathbb{C}^{m_1\times m_2\times m_3}$ and $(\mathsf{Y};\mathsf{X})\in\mathbb{C}^{m_1\times m_2\times m_4}$, we see

$$\operatorname{grank}(m_1,m_2,m_3)\le\operatorname{grank}(m_1,m_2,m_4).$$

Next, take an open semi-algebraic set $\mathscr{H}$ of $\mathbb{R}^{m_1\times m_2\times m_3}$ consisting of tensors of rank $\operatorname{mtrank}(m_1,m_2,m_3)$. Then there are $s\in\operatorname{typical\_rank}_{\mathbb{R}}(m_1,m_2,m_4)$ and an open semi-algebraic set $\mathscr{O}$ of $\mathbb{R}^{m_1\times m_2\times m_4}$ consisting of tensors of rank s such that $\{(\mathsf{A};\mathsf{B})\mid\mathsf{A}\in\mathscr{H},\ \mathsf{B}\in\mathbb{R}^{m_1\times m_2\times(m_4-m_3)}\}\cap\mathscr{O}\ne\emptyset$. Thus

$$\operatorname{mtrank}(m_1,m_2,m_3)\le s\le\operatorname{mtrank}(m_1,m_2,m_4).$$

□

The action of $\mathrm{GL}(m)\times\mathrm{GL}(n)\times\mathrm{GL}(p)$ on $\mathbb{R}^{m\times n\times p}$ is given as follows. Let $\mathbf{P}=(p_{ij})\in\mathrm{GL}(n)$, $\mathbf{Q}=(q_{ij})\in\mathrm{GL}(m)$, and $\mathbf{R}=(r_{ij})\in\mathrm{GL}(p)$. The tensor $(b_{ijk})=(\mathbf{P},\mathbf{Q},\mathbf{R})\cdot(a_{ijk})$ is defined as

$$b_{ijk}=\sum_{s=1}^{m}\sum_{t=1}^{n}\sum_{u=1}^{p}p_{is}q_{jt}r_{ku}a_{stu}.$$

Therefore,

$$(\mathbf{P},\mathbf{Q},\mathbf{R})\cdot(\mathbf{A}_1;\dots;\mathbf{A}_p)=(\sum_{u=1}^{p}r_{1u}\mathbf{P}\mathbf{A}_u\mathbf{Q}^T;\dots;\sum_{u=1}^{p}r_{pu}\mathbf{P}\mathbf{A}_u\mathbf{Q}^T).$$

Definition 2.5: Two tensors A and B is called *equivalent* if there exists $g\in\mathrm{GL}(m)\times\mathrm{GL}(n)\times\mathrm{GL}(p)$ such that $\mathsf{B}=g\cdot\mathsf{A}$.

Proposition 2.6 (cf. [4, Lemma 2.3])**:** *If two tensors are equivalent, then they have the same rank.*

A $1\times m_2\times m_3$ tensor T is an $m_2\times m_3$ matrix and $\operatorname{rank}\mathsf{T}$ is equal to the matrix rank. The following three propositions are well-known. The former two propositions easily follow from the definition.

Proposition 2.7: *Let $m_1,m_2,m_3\in\mathbb{N}$ with $2\le m_1\le m_2\le m_3$. If $m_1m_2\le m_3$, then typical rank of $\mathbb{R}^{m_1\times m_2\times m_3}$ is only one integer m_1m_2.*

Proposition 2.8: *Let $\mathsf{X}=(x_{ijk})$ be an $m_1\times m_2\times m_3$ tensor. For an $m_2\times m_1\times m_3$ tensor $\mathsf{Y}=(x_{jik})$ and an $m_1\times m_3\times m_2$ tensor $\mathsf{Z}=(x_{ikj})$, it holds that*

$$\operatorname{rank}\mathsf{X}=\operatorname{rank}\mathsf{Y}=\operatorname{rank}\mathsf{Z}.$$

A rank decomposition implies the following proposition.

Proposition 2.9 (cf. [2, 14])**:** *An $m_1 \times m_2 \times m_3$ tensor $(\mathbf{Y}_1; \dots; \mathbf{Y}_{m_3})$ has rank less than or equal to r if and only if there are an $m_1 \times r$ matrix $\mathbf{P}$, an $r \times m_2$ matrix $\mathbf{Q}$, and $r \times r$ diagonal matrices $\mathbf{D}_1, \dots, \mathbf{D}_{m_3}$ such that $\mathbf{Y}_k = \mathbf{P}\mathbf{D}_k\mathbf{Q}$ for $1 \le k \le m_3$.*

For an integer $2 \le m < n < 2m$, the number n is an only typical rank of $\mathbb{R}^{m\times n\times 2}$. Indeed, it is known that

Theorem 2.10 ([9])**:** *Let $2 \le m < n$. There is an open dense semi-algebraic set $\mathscr{O}$ of $\mathbb{R}^{m\times n\times 2}$ of which any tensor is equivalent to $((\mathbf{E}_m, \mathbf{O}); (\mathbf{O}, \mathbf{E}_m))$ which has rank* $\min(n, 2m)$.

Furthermore, by Proposition 2.4, $\operatorname{typical\_rank}_{\mathbb{R}}(m,m,2)$ is equal to either $\{m\}$ or $\{m, m+1\}$. Let $\mathscr{L}$ be an open subset of $\mathbb{R}^{m\times m\times 2}$ consisting of $(\mathbf{A};\mathbf{B})$ such that $\mathbf{A}$ is an $m\times m$ nonsingular matrix and all eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are distinct and contain non-real numbers. For $m \ge 2$, the set $\mathscr{L}$ is not empty and any tensor of $\mathscr{L}$ has rank $m+1$ (cf. [12, Theorem 4.6]) and therefore $\operatorname{typical\_rank}_{\mathbb{R}}(m,m,2) = \{m, m+1\}$ by Proposition 2.3.

Theorem 2.11 ([14, Result 2])**:** *Let $m, n, \ell \in \mathbb{N}$ with $3 \le m \le n \le u$. If $(m-1)n < u < mn$, typical rank of $\mathbb{R}^{m\times n\times u}$ is only one integer u.*

In fact we can construct a nonempty Zariski open subset consisting of tensors with rank $\le u$. We omit the construction.

Corollary 2.12: *Let $3 \le m \le n$. Then the set of typical ranks of $m \times n \times (m-1)n$ tensors is either $\{(m-1)n\}$ or $\{(m-1)n, (m-1)n+1\}$.*

Proof: The typical rank of $\mathbb{R}^{m\times n\times((m-1)n+1)}$ is only $(m-1)n+1$ by Theorem 2.11 and the minimal typical rank of $\mathbb{R}^{m\times n\times(m-1)n}$ is equal to $(m-1)n$, since it is equal to the generic rank of $\mathbb{C}^{m\times n\times(m-1)n}$. Thus the assertion follows from Proposition 2.4. $\square$

3. Characterization

From now on, let $3 \le m \le n$, $\ell = m-1$ and $p = (m-1)n$. For an $n\times n\times \ell$ tensor $(\mathbf{Y}_1; \dots; \mathbf{Y}_\ell)$, consider an $n \times p \times m$ tensor $\mathsf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = (\mathbf{X}_1; \dots; \mathbf{X}_m)$ given by

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_m \end{pmatrix} = \begin{pmatrix} \mathbf{E}_n & & & \\ & \mathbf{E}_n & & \\ & & \ddots & \\ & & & \mathbf{E}_n \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_\ell \end{pmatrix}. \tag{2}$$

Note that $\operatorname{rank}\mathsf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) \ge p$, since $\operatorname{rank}\mathsf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell)$ is greater than or equal to the rank of the $mn \times p$ matrix (2). Generically, an $m\times n\times p$ tensor is equivalent to a tensor of type as $\mathsf{X}(\mathbf{Y}_1, \dots, \mathbf{Y}_\ell)$.

We denote by $\mathscr{M}$ the set of tensors $\mathsf{Y} = (\mathbf{Y}_1; \dots; \mathbf{Y}_\ell) \in \mathbb{R}^{n\times n\times \ell}$ such that there exist an $m \times p$ matrix (x_{ij}) and an $n\times p$ matrix $\mathbf{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_p)$ such that

$$(x_{1j}\mathbf{Y}_1 + \cdots + x_{m-1,j}\mathbf{Y}_{m-1} - x_{mj}\mathbf{E}_n)\boldsymbol{a}_j = \mathbf{0} \tag{3}$$

for $1 \leq j \leq p$ and

$$\mathbf{B} := \begin{pmatrix} \mathbf{AD}_1 \\ \vdots \\ \mathbf{AD}_\ell \end{pmatrix} \tag{4}$$

is nonsingular, where $\mathbf{D}_k = \mathbf{Diag}(x_{k1}, \cdots, x_{kp})$ for $1 \leq k \leq \ell$.

Lemma 3.1: $\operatorname{rank}\mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = p$ *if and only if* $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathscr{M}$.

Proof: Suppose that $\operatorname{rank}\mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = p$. There are an $n \times p$ matrix $\mathbf{A}$, a $p \times p$ matrix $\mathbf{Q}$ and $p \times p$ diagonal matrices $\mathbf{D}_i$ such that $\mathbf{X}_k = \mathbf{AD}_k\mathbf{Q}$ for $k = 1, \ldots, m$. Since

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_\ell \end{pmatrix} = \mathbf{E}_p = \begin{pmatrix} \mathbf{AD}_1 \\ \vdots \\ \mathbf{AD}_\ell \end{pmatrix} \mathbf{Q},$$

$\mathbf{B}$ is nonsingular. Then $(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell)\mathbf{B} = \mathbf{AD}_m$ implies that $\sum_{k=1}^{\ell} \mathbf{Y}_k\mathbf{AD}_k = \mathbf{AD}_m$. Therefore, the j-th column vector $\boldsymbol{a}_j$ of $\mathbf{A}$ satisfies (3). Therefore $(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) \in \mathscr{M}$. It is easy to see that the converse is also true. $\square$

For an $n \times n \times \ell$ tensor $\mathbf{Y} = (\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell)$, we put

$$V(\mathbf{Y}) = \{\boldsymbol{a} \in \mathbb{R}^n \mid \sum_{k=1}^{\ell} x_k \mathbf{Y}_k \boldsymbol{a} = x_m \boldsymbol{a} \text{ for some } (x_1, \ldots, x_m)^T \neq \mathbf{0}\}.$$

The set $V(\mathbf{Y})$ is not a vector subspace of $\mathbb{R}^n$. Let $\hat{V}(\mathbf{Y})$ be the smallest vector subspace of $\mathbb{R}^n$ including $V(\mathbf{Y})$. Let

$$\mathscr{S} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid \dim \hat{V}(\mathbf{Y}) = n\}.$$

Proposition 3.2: $\mathscr{M} \subset \mathscr{S}$ *holds.*

Proof: Let $\mathbf{Y} \in \mathscr{M}$. Consider the matrix $\mathbf{B}$ in (4) for any $m \times p$ matrix (x_{ij}) and any $n \times p$ matrix $\mathbf{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$ satisfying the equation (3). By column operations, $\mathbf{B}$ is transformed to a $p \times p$ matrix having a form

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{O}_{n,p-\dim \hat{V}(\mathbf{Y})} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

where $\mathbf{P}_{11}$ is an $n \times \dim \hat{V}(\mathbf{Y})$ submatrix of $\mathbf{A}$. Since $\mathbf{B}$ is nonsingular, $\mathbf{P}_{11}$ is also nonsingular, which implies that $\dim \hat{V}(\mathbf{Y}) = n$. $\square$

By Corollary 2.12, Lemma 3.1 and Proposition 3.2, we have the following

Proposition 3.3: *If* $\operatorname{rank}\mathbf{X}(\mathbf{Y}) = p$ *then* $Y \in \mathscr{S}$. *In particular,* $\overline{\mathscr{S}} \neq \mathbb{R}^{n \times n \times \ell}$ *implies that* $\operatorname{typical\_rank}_{\mathbb{R}}(m,n,p) = \{p, p+1\}$.

Theorem 3.4 ([13])**:** *If* $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell; \mathbf{E}_n)$ *is an absolutely nonsingular tensor, then it holds that* $\operatorname{rank}\mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) > p$.

Here $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell; \mathbf{Y}_m)$ is called an absolutely nonsingular tensor if $|\sum_{k=1}^{m} x_k \mathbf{Y}_k| = 0$ implies $(x_1, \ldots, x_m)^T = \mathbf{0}$. Therefore,

Proposition 3.5: $\dim \hat{V}(\mathbf{Y}) = 0$ *if and only if* $(\mathbf{Y}; \mathbf{E}_n)$ *is an* $n \times n \times m$ *absolutely nonsingular tensor.*

Note that there exists an $n \times n \times m$ absolutely nonsingular tensor if and only if m is less than or equal to the Hurwitz-Radon number $\rho(n)$ [13].

Proposition 3.6: *Let* Y *and* Z *be* $n \times n \times m$ *tensors. Suppose* $(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot \mathsf{Y} = \mathsf{Z}$ *for* $(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \in \mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(m)$. *Then* $V(\mathsf{Y}) = \mathbf{Q}^T V(\mathsf{Z}) = \{\mathbf{Q}^T \boldsymbol{y} \mid \boldsymbol{y} \in V(\mathsf{Z})\}$. *In particular,* $\dim \hat{V}(\mathsf{Z}) = \dim \hat{V}(\mathsf{Y})$.

Proof: Suppose that $\sum_{k=1}^m x_k \mathbf{Z}_k \boldsymbol{y} = \mathbf{0}$. Then from the definition of the action, it follows that

$$\sum_{k=1}^m d_k \sum_{u=1}^m r_{ku} \mathbf{P} \mathbf{Y}_u \mathbf{Q}^T \boldsymbol{y} = \mathbf{P}(\sum_{u=1}^m (\sum_{k=1}^m d_k r_{ku} \mathbf{Y}_u)) \mathbf{Q}^T \boldsymbol{y} = \mathbf{0}.$$

Thus $\mathbf{Q}^T \boldsymbol{y} \in V(\mathsf{Y})$. □

Corollary 3.7: $\mathscr{S}$ *is closed under the equivalence relation.*

The closure of the set of all $n \times p \times m$ tensors equivalent to $\mathsf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell)$ for some $\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell$ is $\mathbb{R}^{n \times p \times m}$. Furthermore, the following claim holds. Let $\mathscr{H}$ be the set of $n \times p \times m$ tensors $(\mathbf{X}_1; \ldots; \mathbf{X}_m)$ such that $\mathbf{A} = (\mathbf{X}_1^T, \ldots, \mathbf{X}_\ell^T)$ is a nonsingular $p \times p$ matrix and $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell)$ given by $(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = \mathbf{A}^{-1} \mathbf{X}_m$ lies in $\mathscr{M}$. Any tensor of $\mathscr{H}$ has rank p. If $\mathscr{M}$ is dense in $\mathbb{R}^{n \times n \times \ell}$ then $\mathscr{H}$ is dense in $\mathbb{R}^{n \times p \times m}$.

4. Classes of $n \times n \times \ell$ tensors

We separate $\mathbb{R}^{n \times n \times \ell}$ into three classes $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{B}$ as follows. Let $\mathscr{A}$ be the set of tensors Y such that $(\mathbf{Y}; \mathbf{E}_n)$ is absolutely nonsingular. By Proposition 3.5, we have the following

Proposition 4.1: $\mathscr{A} \cap \mathscr{S} = \varnothing$.

From now on, we use symbols $x_1, \ldots, x_\ell, x_m$ as indeterminates over $\mathbb{R}$. For $\mathsf{Y} = (\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times \ell}$, we define the $n \times n$ matrix with entries in $\mathbb{R}[x_1, \ldots, x_\ell, x_m]$ as follows.

$$\mathbf{M}(\boldsymbol{x}, \mathsf{Y}) = \sum_{k=1}^{\ell} x_k \mathbf{Y}_k - x_m \mathbf{E}_n$$

Note that fixing $a_1, \ldots, a_\ell$, the determinant $|\mathbf{M}(\boldsymbol{a}, \mathsf{Y})|$ is positive for $a_m \ll 0$, where $\boldsymbol{a} = (a_1, \ldots, a_\ell, a_m)^T$. Set

$$\mathscr{C} = \{\mathsf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\boldsymbol{a}, \mathsf{Y})| < 0 \text{ for some } \boldsymbol{a} \in \mathbb{R}^m\}.$$

Note that $\mathscr{C}$ is not empty, and if n is not congruent to 0 modulo 4 then $\rho(n) \leq 2$ and thus $\mathscr{A}$ is empty since $m \geq 3$. Set $\mathscr{B} = \mathbb{R}^{n \times n \times \ell} \smallsetminus (\mathscr{A} \cup \mathscr{C})$. In particular, if n is odd, then $\mathscr{C} = \mathbb{R}^{n \times n \times \ell}$, and otherwise $\mathscr{B}$ contains the zero tensor.

Proposition 4.2: $\mathscr{A}$ *and* $\mathscr{C}$ *are open subsets of* $\mathbb{R}^{n \times n \times \ell}$.

Recall that

$$\mathscr{A} = \{\mathsf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\boldsymbol{a}, \mathsf{Y})| > 0 \text{ for all } \boldsymbol{a} \neq \mathbf{0}\}.$$

Thus it holds

$$\mathscr{B} = \{\mathsf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid \begin{array}{l} |\mathbf{M}(\boldsymbol{b}, \mathsf{Y})| = 0 \text{ for some } \boldsymbol{b} \neq \mathbf{0} \text{ and} \\ |\mathbf{M}(\boldsymbol{a}, \mathsf{Y})| \geq 0 \text{ for all } \boldsymbol{a} \end{array}\}.$$

If n is even then $\mathscr{B}$ contains a nonzero tensor in general. We give an example.

Example 4.3 Let $m = 3$ and $n = 6$. Note that $\mathscr{A}$ is empty, since $\rho(6) = 2$. Let $\mathbf{Y} = (\mathbf{Y}_1; \mathbf{Y}_2)$ be a $6 \times 6 \times 2$ tensor given by

$$x_1\mathbf{Y}_1 + x_2\mathbf{Y}_2 = \begin{pmatrix} 0 & -x_2 & 0 & 0 & 0 & -x_1 \\ x_1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & x_1 & 0 & x_2 \\ -x_2 & 0 & 0 & 0 & x_1 & 0 \end{pmatrix}.$$

Then $|\mathbf{M}((a_1,a_2,a_3)^T, \mathbf{Y})| = a_3^2(a_1a_2 - a_3^2)^2 + (a_1^3 + a_2^3)^2 \geq 0$. The equality holds if $a_3 = 0$ and $a_1 = -a_2$. Thus $\mathbf{Y} \in \mathscr{B}$ and $\mathbf{Y} \notin \mathscr{S}$ since $\dim \hat{V}(\mathbf{Y}) = 1$. Let $\mathbf{X} = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ be a 6×6 matrix and let $\varepsilon \in \mathbb{R}$. If $x_1 = -1$ and $x_3 = \varepsilon c$, then

$$|\mathbf{M}((x_1,x_2,x_3)^T, (\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon\mathbf{X}))| = -\varepsilon^2(1+\varepsilon^2c^2)c((1-\varepsilon c)^2 - c(1+\varepsilon^2c^2)).$$

Thus, for any $\varepsilon \neq 0$, there is $c_0 > 0$ such that $(1-\varepsilon c_0)^2 - c_0(1+\varepsilon^2c_0^2)) > 0$ and then $|\mathbf{M}((-1,1,\varepsilon c_0)^T, (\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon\mathbf{X}))| < 0$, which implies that $(\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon\mathbf{X}) \in \mathscr{C}$.

Proposition 4.4: *$\mathscr{B}$ is a boundary of $\mathscr{C}$. In particular, $\mathbb{R}^{n\times n\times \ell}$ is a disjoint sum of $\mathscr{A}$ and the closure $\overline{\mathscr{C}}$ of $\mathscr{C}$.*

Proof: It suffices to show that $\mathscr{B} \subset \overline{\mathscr{C}}$ by Proposition 4.2. Let $\mathbf{Y} = (\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathscr{B}$. There are a nonzero vector $\boldsymbol{b} = (b_1, \ldots, b_\ell, b_m)^T \in \mathbb{R}^n$ with $|\mathbf{M}(\boldsymbol{b}, \mathbf{Y})| = 0$ and an element $g \in \mathrm{GL}(\ell)$ such that $g \cdot \mathbf{Y} = (\mathbf{Z}_1; \mathbf{Z}_2; \ldots; \mathbf{Z}_\ell)$ and $\mathbf{Z}_1 = \sum_{k=1}^{\ell} b_k\mathbf{Y}_k$. Then $|\mathbf{Z}_1 - b_m\mathbf{E}_n| = 0$. Take a sequence $\{\mathbf{Z}_1^{(u)}\}_{u\geq 1}$ such that $|\mathbf{Z}_1^{(u)} - b_m\mathbf{E}_n| < 0$ and $\lim_{u\to\infty} \mathbf{Z}_1^{(u)} = \mathbf{Z}_1$. Thus, $(\mathbf{Z}_1^{(u)}; \mathbf{Z}_2; \ldots; \mathbf{Z}_\ell) \in \mathscr{C}$ and then $g^{-1} \cdot (\mathbf{Z}_1^{(u)}; \mathbf{Z}_2; \ldots; \mathbf{Z}_\ell) \in \mathscr{C}$. Therefore, $\mathbf{Y} \in \overline{\mathscr{C}}$. $\square$

Corollary 4.5: *If $\mathscr{A}$ is not empty then $\mathscr{B}$ is a boundary of $\mathscr{A}$.*

Proposition 4.6: *If $m \leq \rho(n-1)$ then $\mathscr{C} \not\subset \mathscr{S}$, where $\rho(n-1)$ is a Hurwitz-Radon number.*

Proof: Let $(\mathbf{A}_1; \ldots; \mathbf{A}_\ell; \mathbf{E}_{n-1})$ be an $(n-1) \times (n-1) \times m$ absolutely nonsingular tensor. Put $\mathbf{B}_k = \mathbf{Diag}(a_k, \mathbf{A}_k)$ for $1 \leq k \leq \ell$ and $\mathbf{B}_m = \mathbf{Diag}(1, \mathbf{E}_{n-1}) = \mathbf{E}_n$, and $\mathbf{B} = (\mathbf{B}_1; \ldots; \mathbf{B}_\ell)$. Then it is easy to see that $\mathbf{B} \in \mathscr{C}$ and $|\sum_{k=1}^{\ell} x_k\mathbf{B}_k - x_m\mathbf{B}_m| = 0$ implies $x_m = \sum_{k=1}^{\ell} a_kx_k$. Therefore $V(\mathbf{B}) = \{a(1,0,\ldots,0)^T \in \mathbb{R}^n \mid a \in \mathbb{R}\}$. In particular $\mathbf{B} \notin \mathscr{S}$. $\square$

5. Proof of Theorem 1.1

In this section we show Theorem 1.1.

In the space of homogeneous polynomials in m variables, there exists a proper Zariski closed subset S such that if a polynomial does not belong to S then it is irreducible [7, Theorem 7], since $m \geq 3$. Let $P(m,n)$ be the set of homogeneous polynomials in m variables $x_1, \ldots, x_m$ with real coefficients of degree n such that the coefficient of x_m^n is one. Its dimension is $\binom{m+n-1}{m-1} - 1$. Let I_ℓ be a nonempty Zariski open subset of $P(m,n)$ such that any polynomial of I_ℓ is irreducible. Note that $|-\mathbf{M}(\boldsymbol{x}, \mathbf{Y})| \in P(m,n)$.

A polynomial $f \in P(m,n)$ is *hyperbolic* with respect to a vector $\boldsymbol{e} \in \mathbb{R}^m$ if both $f(\boldsymbol{e}) \neq 0$ and for all vectors $\boldsymbol{v} \in \mathbb{R}^m$, the univariate polynomial $t \mapsto f(\boldsymbol{v} - t\boldsymbol{e})$ has all real roots.

Theorem 5.1 ([8])**:** *A homogeneous polynomial f of $P(3,n)$ is hyperbolic with respect to the vector $\boldsymbol{e} = (0,0,1)^T$ and satisfies $f(\boldsymbol{e}) = 1$ if and only if there exist $n \times n$ symmetric matrices $\mathbf{A}$, $\mathbf{B}$ such that f is given by*

$$f(x,y,z) = |x\mathbf{A} + y\mathbf{B} + z\mathbf{E}_n|.$$

Proposition 5.2: *The set*

$$\{\mathbf{Y} \in \mathbb{R}^{n\times n\times \ell} \mid |-\mathbf{M}(\boldsymbol{x},\mathbf{Y})| \in I_\ell\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$.

Proof: Let $f_\ell \colon \mathbb{R}^{n\times n\times \ell} \to P(m,n)$ be a map which sends $\mathbf{Y}$ to $|-\mathbf{M}(\boldsymbol{x},\mathbf{Y})|$ and put

$$\mathscr{I}_\ell := \{\mathbf{Y} \in \mathbb{R}^{n\times n\times \ell} \mid f_\ell(\mathbf{Y}) \in I_\ell\}.$$

Since I_ℓ is a Zariski open set, $\mathscr{I}_\ell$ is a Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$. Then it suffices to show that $\mathscr{I}_\ell$ is not empty. For $(\mathbf{Z}_1;\mathbf{Z}_2) \in \mathbb{R}^{n\times n\times 2}$, if the polynomial

$$|-\mathbf{M}((x_1,x_2,x_3)^T,(\mathbf{Z}_1;\mathbf{Z}_2))|$$

is irreducible, then for any $(\mathbf{Y}_3;\dots;\mathbf{Y}_\ell) \in \mathbb{R}^{n\times n\times(\ell-2)}$, the polynomial

$$|-\mathbf{M}(\boldsymbol{x},(\mathbf{Z}_1;\mathbf{Z}_2;\mathbf{Y}_3;\dots;\mathbf{Y}_\ell)|$$

is also irreducible. Therefore, it suffices to show the assertion only for $m = 3$.

The set of hyperbolic homogeneous polynomials of degree n contains an open subset [10]. Then there is an irreducible homogenious polynomial f of $P(3,n)$ which is hyperbolic with respect to $(e_1,e_2,e_3)^T$. Since $f(e_1,e_2,e_3) \neq 0$, it holds that $(e_1,e_2,e_3)^T \neq \mathbf{0}$. Thus there is an irreducible homogeneous polynomial $g \in P(3,n)$ such that g is hyperbolic with respect to $(0,0,1)^T$ and $g(0,0,1) = 1$. By Theorem 5.1, $g(x_1,x_2,x_3)$ is described as

$$|x_1\mathbf{Z}_1 + x_2\mathbf{Z}_2 + x_3\mathbf{E}_n| = |-\mathbf{M}((x_1,x_2,x_3)^T,(-\mathbf{Z}_1;-\mathbf{Z}_2))|$$

for some $n\times n$ matrices $\mathbf{Z}_1,\mathbf{Z}_2$. Thus, $\mathscr{I}_2$ is nonempty. This completes the proof. □

Let $\tilde{\boldsymbol{x}} = (x_1,\dots,x_\ell)^T$ for $\boldsymbol{x} = (x_1,\dots,x_\ell,x_m)^T$, and put

$$\psi(\boldsymbol{x},\mathbf{Y}) := \begin{pmatrix} (-1)^{n+1}|\mathbf{M}(\boldsymbol{x},\mathbf{Y})_{n,1}| \\ (-1)^{n+2}|\mathbf{M}(\boldsymbol{x},\mathbf{Y})_{n,2}| \\ \vdots \\ (-1)^{n+n}|\mathbf{M}(\boldsymbol{x},\mathbf{Y})_{n,n}| \end{pmatrix} \in \mathbb{R}^n, \quad \tilde{\boldsymbol{x}} \otimes \psi(\boldsymbol{x},\mathbf{Y}) := \begin{pmatrix} x_1\psi(\boldsymbol{x},\mathbf{Y}) \\ x_2\psi(\boldsymbol{x},\mathbf{Y}) \\ \vdots \\ x_\ell\psi(\boldsymbol{x},\mathbf{Y}) \end{pmatrix} \in \mathbb{R}^p,$$

and

$$U(\mathbf{Y}) := \langle \tilde{\boldsymbol{u}} \otimes \psi(\boldsymbol{u},\mathbf{Y}) \mid |\mathbf{M}(\boldsymbol{u},\mathbf{Y})| = 0,\ \boldsymbol{u} \in \mathbb{R}^m \rangle.$$

Lemma 5.3: *If* $\dim U(\mathbf{Y}) = p$*, then* $\mathbf{Y} \in \mathscr{M}$.

Proof: Let $\dim U(\mathbf{Y}) = p$. Then there are $\boldsymbol{u}_j = (u_{1j},\dots,u_{mj})^T \in U(\mathbf{Y})$ for $1 \leq j \leq p$ such that

$$\mathbf{B} = (\tilde{\boldsymbol{u}}_1 \otimes \psi(\boldsymbol{u}_1,\mathbf{Y}),\dots,\tilde{\boldsymbol{u}}_p \otimes \psi(\boldsymbol{u}_p,\mathbf{Y}))$$

is nonsingular. Note that $\mathbf{M}(\boldsymbol{u}_j, \mathbf{Y})\psi(\boldsymbol{u}_j, \mathbf{Y}) = \mathbf{0}$ for $1 \le j \le p$ and

$$\mathbf{B} = \begin{pmatrix} \mathbf{AD}_1 \\ \vdots \\ \mathbf{AD}_\ell \end{pmatrix},$$

where $\mathbf{A} = (\psi(\boldsymbol{u}_1, \mathbf{Y}), \dots, \psi(\boldsymbol{u}_p, \mathbf{Y}))$ and $\mathbf{D}_k = \mathbf{Diag}(u_{k1}, \cdots, u_{kp})$ for $1 \le k \le \ell$. Thus $\mathbf{Y} \in \mathscr{M}$. □

For an $n \times \ell$ matrix $\mathbf{C} = (\boldsymbol{c}_1, \dots \boldsymbol{c}_\ell)$, we put

$$g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C}) := \begin{vmatrix} \mathbf{M}(\boldsymbol{x}, \mathbf{Y})^{<n} \\ \sum_{k=1}^{\ell} x_k \boldsymbol{c}_k^T \end{vmatrix},$$

where $\mathbf{M}(\boldsymbol{x}, \mathbf{Y})^{<n}$ is the $(n-1) \times n$ matrix obtained from $\mathbf{M}(\boldsymbol{x}, \mathbf{Y})$ by removing the n-th row.

Lemma 5.4: *Let $\mathbf{C} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_\ell)$ be an $n \times \ell$ matrix. The following claims are equivalent.*

(1) $\dim U(\mathbf{Y}) = p$.
(2) $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C}) = 0$ *for any* $\boldsymbol{a} \in \mathbb{R}^m$ *with* $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| = 0$ *implies* $\mathbf{C} = \mathbf{O}$.

Proof: Let $\mathbf{C} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_\ell)$ be an $n \times \ell$ matrix. Put $\boldsymbol{d} = (\boldsymbol{c}_1^T, \dots, \boldsymbol{c}_\ell^T)^T \in \mathbb{R}^p$. The inner product of this vector $\boldsymbol{d}$ with $\tilde{\boldsymbol{a}} \otimes \psi(\boldsymbol{a}, Y)$ is equal to $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C})$. Therefore $\boldsymbol{d}$ belongs to the orthogonal complement of $U(\mathbf{Y})$ if and only if $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C}) = 0$ for any $\boldsymbol{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| = 0$. Thus the assertion holds. □

For any i and k with $1 \le i \le n-1$ and $1 \le k \le n$, let $s_i^{(k)}$ be an elementary symmetric polynomial of degree i with variables $\alpha_1, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n$. Put

$$\mathbf{S}_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ s_1^{(1)} & s_1^{(2)} & \dots & s_1^{(n)} \\ s_2^{(1)} & s_2^{(2)} & \dots & s_2^{(n)} \\ \vdots & \vdots & & \vdots \\ s_{n-1}^{(1)} & s_{n-1}^{(2)} & \dots & s_{n-1}^{(n)} \end{pmatrix}.$$

Lemma 5.5: *The determinant $|\mathbf{S}_n|$ of the $n \times n$ matrix $\mathbf{S}_n$ is equal to*

$$\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j).$$

In particular, if $\alpha_1, \dots, \alpha_n$ are distinct from each other, then $\mathbf{S}_n$ is nonsingular.

Proof: For any i and k with $1 \le i \le n-1$ and $2 \le k \le n-1$, let $t_i^{(k-1)}$ be an elementary symmetric polynomial of degree i with variables $\alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n$. For $1 \le i \le n-1$ and $1 \le k \le n$, we have $s_i^{(k)} - s_i^{(1)} = (\alpha_1 - \alpha_k) t_{i-1}^{(k-1)}$. Then

$$|\mathbf{S}_n| = \prod_{2 \le k \le n} (\alpha_1 - \alpha_k) \begin{vmatrix} 1 & 1 & \dots & 1 \\ t_1^{(1)} & t_1^{(2)} & \dots & t_1^{(n-1)} \\ \vdots & \vdots & & \vdots \\ t_{n-2}^{(1)} & t_{n-2}^{(2)} & \dots & t_{n-2}^{(n-1)} \end{vmatrix}.$$

Therefore we have the assertion by induction on n. □

The following lemma is obtained straightforwardly.

Lemma 5.6:

$$\begin{vmatrix} \alpha_1+z & & & & a_1 \\ & \alpha_2+z & & & a_2 \\ & & \ddots & & \vdots \\ & & & \alpha_n+z & a_n \\ b_1 & b_2 & \dots & b_n & 0 \end{vmatrix} = -(z^{n-1}, z^{n-2}, \dots, 1)\mathbf{S}_n \begin{pmatrix} a_1b_1 \\ a_2b_2 \\ \vdots \\ a_nb_n \end{pmatrix}.$$

Proof: We see the left hand of the equation is equal to

$$\begin{aligned} & -\sum_{k=1}^{n} a_k b_k \frac{\prod_{1\le i\le n}(\alpha_i+z)}{\alpha_k+z} \\ &= -\sum_{k=1}^{n} a_k b_k \left(\sum_{i=1}^{n} s_{i-1}^{(k)}\right) z^{n-i} \\ &= -\sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_k b_k s_{i-1}^{(k)}\right) z^{n-i} \\ &= -(z^{n-1}, z^{n-2}, \dots, 1) \begin{pmatrix} \sum_{k=1}^{n} a_k b_k \\ \sum_{k=1}^{n} a_k b_k s_1^{(k)} \\ \vdots \\ \sum_{k=1}^{n} a_k b_k s_{n-1}^{(k)} \end{pmatrix}. \end{aligned}$$

□

Corollary 5.7: *Let $\alpha_1,\dots,\alpha_{n-1}$ be distinct complex numbers, $a_1,\dots,a_{n-1}$ nonzero complex numbers, and $b_1,\dots,b_{n-1}$ complex numbers. If*

$$\begin{vmatrix} \mathbf{Diag}(\alpha_1,\dots,\alpha_{n-1})+z\mathbf{E}_{n-1} & \boldsymbol{a} \\ \boldsymbol{b}^T & 0 \end{vmatrix} = 0$$

for any $z\in\mathbb{R}$, then $\boldsymbol{b}=\mathbf{0}$, where $\boldsymbol{a}=(a_1\dots,a_{n-1})^T$ and $\boldsymbol{b}=(b_1,\dots,b_{n-1})^T$.

Proof: Since $\mathbf{S}_n \begin{pmatrix} a_1b_1 \\ a_2b_2 \\ \vdots \\ a_nb_n \end{pmatrix} = \mathbf{0}$ and $\mathbf{S}_n$ is nonsingular, we have $(a_1b_1,\dots,a_nb_n)=\mathbf{0}^T$. □

The set

$$\mathscr{U}_1 = \{Y\in\mathbb{R}^{n\times n\times\ell} \mid |\mathbf{M}(\boldsymbol{x},\mathbf{Y})| \text{ is irreducible}\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times\ell}$ (see Proposition 5.2). Let W be the subset of $\mathbb{R}^{n\times n}$ consisting of matrices $\begin{pmatrix} \mathbf{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & e \end{pmatrix}$ such that all eigenvalues of $\mathbf{A}$ are distinct over the complex number field and every element of the vector $\mathbf{P}^{-1}\boldsymbol{b}$ is nonzero complex number where $\mathbf{A}\in\mathbb{R}^{(n-1)\times(n-1)}$, $\mathbf{P}\in\mathbb{C}^{(n-1)\times(n-1)}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. Note that the validity of the condition that every element of the vector $\mathbf{P}^{-1}\boldsymbol{b}$ is nonzero is independent of the choice of **P**. We put

$$\mathscr{U}_2 := \{(\mathbf{Y}_1;\ldots;\mathbf{Y}_\ell) \in \mathbb{R}^{n\times n\times \ell} \mid \mathbf{Y}_k \in W, 1 \le k \le \ell\}.$$

The set $\mathscr{U}_2$ is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times\ell}$ and $\mathscr{U} := \mathscr{U}_1 \cap \mathscr{U}_2$ is also.

Lemma 5.8: *Let $\mathbf{Y} \in \mathscr{U}_2$ and $\boldsymbol{d}_1,\ldots,\boldsymbol{d}_\ell \in \mathbb{R}^{n-1}$. If*

$$\begin{vmatrix} \mathbf{M}(\boldsymbol{a},\mathbf{Y})^{<n} & \\ \sum_{k=1}^{\ell} a_k \boldsymbol{d}_k^T & 0 \end{vmatrix} = 0$$

for any $\boldsymbol{a} = (a_1,\ldots,a_m)^T \in \mathbb{R}^m$, then $\boldsymbol{d}_1 = \cdots = \boldsymbol{d}_\ell = \mathbf{0}$.

Proof: Let $1 \le k \le \ell$. Take $a_k = 1$ and $a_j = 0$ for $1 \le j \le \ell$, $j \ne k$ and put $\mathbf{Y}_k = \begin{pmatrix} \mathbf{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & e \end{pmatrix}$, where $\mathbf{A}$ is an $(n-1)\times(n-1)$ matrix. Since $\mathbf{Y}_k \in W$, there are a matrix $\mathbf{P} \in \mathbb{C}^{(n-1)\times(n-1)}$ and distinct complex numbers $\alpha_1,\ldots,\alpha_{n-1}$ such that

$$\mathbf{Diag}(\mathbf{P},1)^{-1}\begin{pmatrix} (\mathbf{Y}_k - a_m\mathbf{E}_n)^{<n} & \\ \boldsymbol{d}_k^T & 0 \end{pmatrix}\mathbf{Diag}(\mathbf{P},1) = \begin{pmatrix} \mathbf{Diag}(\alpha_1,\ldots,\alpha_{n-1}) - a_m\mathbf{E}_{n-1} & \mathbf{P}^{-1}\boldsymbol{b} \\ \boldsymbol{d}_k^T\mathbf{P} & 0 \end{pmatrix}$$

and every element of $\mathbf{P}^{-1}\boldsymbol{b}$ is nonzero. Then we have $\boldsymbol{d}_k^T\mathbf{P} = \mathbf{0}^T$ by Corollary 5.7 and thus $\boldsymbol{d}_k = \mathbf{0}$. □

The following lemma is essential for the proof of Theorem 1.1.

Lemma 5.9: *$\mathscr{U} \cap \mathscr{C} \subset \mathscr{M}$. In particular, $\overline{\mathscr{C}} \subset \overline{\mathscr{M}}$ holds.*

Proof: Let $\mathbf{Y} \in \mathscr{U} \cap \mathscr{C}$ and fix it. There exists $\boldsymbol{a} = (a_1,\ldots,a_\ell,a_m)^T$ such that $|\mathbf{M}(\boldsymbol{a},\mathbf{Y})| < 0$. Then there is an open neighborhood U of $(a_1,\ldots,a_\ell)^T$ and a mapping $\mu : U \to \mathbb{R}$ such that

$$|\mathbf{M}(\begin{pmatrix} \boldsymbol{y} \\ \mu(\boldsymbol{y}) \end{pmatrix},\mathbf{Y})| = 0$$

for any $\boldsymbol{y} \in U$. Thus $|\mathbf{M}(\boldsymbol{x},\mathbf{Y})| = 0$ determines an $(m-1)$-dimensional algebraic set. Let $\mathbf{C}$ be an $n\times\ell$ matrix. Now suppose that $g(\boldsymbol{a},\mathbf{Y},\mathbf{C}) = 0$ holds for any $\boldsymbol{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\boldsymbol{a},\mathbf{Y})| = 0$. We show that $g(\boldsymbol{x},\mathbf{Y},\mathbf{C})$ is zero as a polynomial over elements of $\boldsymbol{x}$. As a contrary, assume that $g(\boldsymbol{x},\mathbf{Y},\mathbf{C})$ is not zero. The degree of $g(\boldsymbol{x},\mathbf{Y},\mathbf{C})$ corresponding to the m-th element of $\boldsymbol{x}$ is less than n which is that of $|\mathbf{M}(\boldsymbol{x},\mathbf{Y})|$. Furthermore, since $|\mathbf{M}(\boldsymbol{x},\mathbf{Y})|$ is irreducible, $|\mathbf{M}(\boldsymbol{x},\mathbf{Y})|$ and $g(\boldsymbol{x},\mathbf{Y},\mathbf{C})$ are coprime. Then there are polynomials $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x}) \in \mathbb{R}[x_1,\ldots,x_\ell,x_m]$ and a nonzero polynomial $h(\tilde{\boldsymbol{x}}) \in \mathbb{R}[x_1,\ldots,x_\ell]$ such that

$$f_1(\boldsymbol{x})|\mathbf{M}(\boldsymbol{x},\mathbf{Y})| + f_2(\boldsymbol{x})g(\boldsymbol{x},\mathbf{Y},\mathbf{C}) = h(\tilde{\boldsymbol{x}})$$

as a polynomial over elements of $\boldsymbol{x}$, by Euclidean algorithm. However, we can take $\boldsymbol{b} \in U$ so that $h(\boldsymbol{b}) \ne 0$. Then the above equation does not hold at $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{b} \\ \mu(\boldsymbol{b}) \end{pmatrix}$. Hence $g(\boldsymbol{x},\mathbf{Y},\mathbf{C})$ must be the zero polynomial over elements of $\boldsymbol{x}$. Let $\boldsymbol{c}_k^T = (c_{1k},\ldots,c_{nk})$. By seeing the coefficient of $x_m^{n-1}x_k$, we get $c_{nk} = 0$ for $1 \le k \le \ell$. Therefore $\mathbf{C} = \mathbf{O}$ by Lemma 5.8. By Lemmas 5.4 and 5.3 we get $\mathbf{Y} \in \mathscr{M}$. Therefore $\mathscr{U} \cap \mathscr{C}$ is a subset of $\mathscr{M}$. Since $\mathscr{C}$ is open and $\mathscr{U}$ is dense, we have $\overline{\mathscr{C}} = \overline{\mathscr{C} \cap \mathscr{U}}$ and then $\overline{\mathscr{C}}$ is a subset of $\overline{\mathscr{M}}$. □

Theorem 5.10: *$\overline{\mathscr{S}} = \overline{\mathscr{M}} = \overline{\mathscr{C}}$ holds.*

Proof : We have $\overline{\mathscr{M}} \subset \overline{\mathscr{S}}$ by Proposition 3.2. By Propositions 4.1 and 4.4, the set $\mathscr{S}$ is a subset of $\overline{\mathscr{C}}$ and then $\overline{\mathscr{S}} \subset \overline{\mathscr{C}}$. Therefore $\overline{\mathscr{S}} = \overline{\mathscr{M}} = \overline{\mathscr{C}}$ by Lemma 5.9. □

Proof of Theorem 1.1: For almost all $\mathbf{Y} \in \mathscr{A}$, $\operatorname{rank} X(\mathbf{Y}) = p+1$ by Theorem 3.4. Since $\mathscr{A}$ is an open set, if $\mathscr{A}$ is not an empty set, then $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p) = \{p, p+1\}$ ([13, Theorem 3.4]). Suppose that $\mathscr{A}$ is empty. Then $\overline{\mathscr{M}} = \mathbb{R}^{n\times n\times \ell}$ and the closure of the set consisting of all $n \times p \times m$ tensors equivalent to $X(\mathbf{Y})$ for some $\mathbf{Y} \in \mathscr{M}$ is $\mathbb{R}^{n\times p\times m}$. Recall that any tensor $X(\mathbf{Y})$ for $\mathbf{Y} \in \mathscr{M}$ has rank p. By Theorem 2.1, p is the maximal typical rank of $\mathbb{R}^{n\times p\times m}$. Therefore,

$$\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p) = \mathrm{typical\_rank}_{\mathbb{R}}(n,p,m) = \{p\}$$

holds.

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