Typical ranks of m×n×(m-1)n tensors with 3≤m≤n over the real number field

Sumi, Toshio
Faculty of Arts and Science, Kyushu University

Miyazaki, Mitsuhiro
Department of Mathematics, Kyoto University of Education

Sakata, Toshio
Faculty of Design, Kyushu University

出版情報：Linear and Multilinear Algebra. 63 (5), pp.940-955, 2015. Taylor & Francis
バージョン：
権利関係：

Linear and Multilinear Algebra
Vol. 00, No. 00, Month 200x, 1–14

RESEARCH ARTICLE

Typical ranks of $m \times n \times (m-1)n$ tensors with $3 \leq m \leq n$ over the real number field

Toshio Sumi[a*] , Mitsuhiro Miyazaki[b], and Toshio Sakata[c]

[a]*Kyushu University, Faculty of Arts and Science, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN*; [b]*Kyoto University of Education, Department of Mathematics, 1 Fujinomoricho, Fukakusa, Fushimi-ku, Kyoto, 612-8522, JAPAN*; [c]*Kyushu University, Faculty of Design, 4-9-1 Shiobaru, Minami-ku, Fukuoka, 815-8540, JAPAN*

()

Let $3 \leq m \leq n$. We study typical ranks of $m \times n \times (m-1)n$ tensors over the real number field. Let ρ be the Hurwitz-Radon function defined as $\rho(n) = 2^b + 8c$ for nonnegative integers a, b, c such that $n = (2a+1)2^{b+4c}$ and $0 \leq b < 4$. If $m \leq \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has two typical ranks $(m-1)n, (m-1)n+1$. In this paper, we show that the converse is also true: if $m > \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has only one typical rank $(m-1)n$.

Keywords: tensor; 3-way array; typical rank; Hurwitz-Radon number; perfect

AMS Subject Classification: 15A69, 14P10, 14Q20

1. Introduction

For positive integers m, n, p, an $m \times n \times p$ tensor is an element of tensor product of vector spaces V_1, V_2, V_3 of dimension m, n, p, respectively. If we choose bases of their vector spaces, we can identify a tensor with a 3-way array

$$(a_{ijk})_{1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq p}.$$

Hitchcock [6] defined the rank of a tensor. An $m \times n \times p$ tensor of form

$$(x_i y_j z_k)_{1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq p}$$

is called a rank one tensor. The rank of a tensor T, denoted by $\operatorname{rank} T$, is defined as the minimal number of rank one tensors which describe T as a sum. The rank depends on the base field. For example there is a $2 \times 2 \times 2$ tensor over the real number field whose rank is 3 but is 2 as a tensor over the complex number field.

Throughout this paper, we assume that the base field is the real number field $\mathbb{R}$. Let $\mathbb{R}^{m \times n \times p}$ be the set of $m \times n \times p$ tensors with Euclidean topology. A number r is a typical rank of $m \times n \times p$ tensors if the set of tensors with rank r contains a nonempty open semi-algebraic set of $\mathbb{R}^{m \times n \times p}$ (see Theorem 2.1). We denote by $\operatorname{typical\_rank}_{\mathbb{R}}(m, n, p)$

*Corresponding author. Email: sumi@artsci.kyushu-u.ac.jp

ISSN: 0308-1087 print/ISSN 1563-5139 online

DOI: 10.1080/03081080xxxxxxxxx
http://www.informaworld.com

the set of typical ranks of $\mathbb{R}^{m\times n\times p}$. If s (resp. t) is the minimal (resp. maximal) number of $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p)$, then

$$\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p)=[s,t]\cap\mathbb{Z},$$

the interval of all integers between s and t, including both, and s is equal to the generic rank of the set of $m\times n\times p$ tensors over the complex number field [5]. In the case where $m=2$, the set of typical ranks of $2\times n\times p$ tensor is well-known [15]:

$$\mathrm{typical\_rank}_{\mathbb{R}}(2,n,p)=\begin{cases}\{p\}, & n<p\le 2n\\ \{2n\}, & 2n<p\\ \{p,p+1\}, & n=p\ge 2\end{cases}$$

Suppose that $3\le m\le n$. If $p>(m-1)n$ then the set of typical ranks of $m\times n\times p$ tensors is just $\{\min(p,mn)\}$. If $p=(m-1)n$ then the set of typical ranks of $m\times n\times p$ tensor is $\{p\}$ or $\{p,p+1\}$ [14]. Until our paper [13], only a few cases where $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,(m-1)n)=\{(m-1)n,(m-1)n+1\}$ [3, 5] are known and we constructed infinitely many examples by using the concept of absolutely nonsingular tensors in [13]: If $m\le\rho(n)$ then $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p)=\{p,p+1\}$, where $\rho(n)$ is the Hurwitz-Radon number given by $\rho(n)=2^b+8c$ for nonnegative integers a,b,c such that $n=(2a+1)2^{b+4c}$ and $0\le b<4$.

n	1	2	3	4	5	6	7	8	9	10	11	12	16	32	64
$\rho(n)$	1	2	1	4	1	2	1	8	1	2	1	4	9	10	12

The purpose of this paper is to completely determine the set of typical ranks of $m\times n\times(m-1)n$ tensors:

Theorem 1.1 : *Let $3\le m\le n$ and $p=(m-1)n$. Then it holds*

$$\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p)=\begin{cases}\{p\}, & m>\rho(n)\\ \{p,p+1\}, & m\le\rho(n).\end{cases}$$

We denote an $m_1\times m_2\times m_3$ tensor (x_{ijk}) by $(\mathbf{X}_1;\ldots;\mathbf{X}_{m_3})$, where $\mathbf{X}_t=(x_{ijt})$ is an $m_1\times m_2$ matrix for each $1\le t\le m_3$. Let $3\le m\le n$ and $p=(m-1)n$. For an $n\times p\times m$ tensor $\mathbf{X}=(\mathbf{X}_1;\ldots;\mathbf{X}_{m-1};\mathbf{X}_m)$, let $\mathbf{H}(\mathbf{X})$ and $\hat{\mathbf{H}}(\mathbf{X})$ be a $p\times p$ matrix and an $mn\times p$ matrix respectively defined as follows.

$$\mathbf{H}(\mathbf{X})=\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \vdots\\ \mathbf{X}_{m-1}\end{pmatrix},\quad \hat{\mathbf{H}}(\mathbf{X})=\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \vdots\\ \mathbf{X}_{m}\end{pmatrix}$$

Let

$$\mathscr{R}=\{\mathbf{X}\in\mathbb{R}^{n\times p\times m}\mid \mathbf{H}(\mathbf{X})\text{ is nonsingular}\}.$$

This is a nonempty Zariski open set. For $\mathbf{X} = (\mathbf{X}_1; \ldots; \mathbf{X}_{m-1}; \mathbf{X}_m) \in \mathscr{R}$, we see

$$\hat{\mathbf{H}}(\mathbf{X})\mathbf{H}(\mathbf{X})^{-1} = \begin{pmatrix} \mathbf{E}_n & & & \\ & \mathbf{E}_n & & \\ & & \ddots & \\ & & & \mathbf{E}_n \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_{m-1} \end{pmatrix},$$

where $\mathbf{E}_n$ is the $n \times n$ identity matrix and $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{m-1}) = \mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}$. Note that $\operatorname{rank} \mathbf{X} \geq p$ for $\mathbf{X} \in \mathscr{R}$. Let h be an isomorphism from the set of $n \times p$ matrices to $\mathbb{R}^{n \times n \times (m-1)}$ given by

$$(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{m-1}) \mapsto (\mathbf{Y}_1; \mathbf{Y}_2; \ldots; \mathbf{Y}_{m-1}).$$

Then $h(\mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}) \in \mathbb{R}^{n \times n \times (m-1)}$. We consider the following subsets of $\mathbb{R}^{n \times n \times (m-1)}$. For $\mathbf{Y} = (\mathbf{Y}_1; \mathbf{Y}_2; \ldots; \mathbf{Y}_{m-1}) \in \mathbb{R}^{n \times n \times (m-1)}$ and $\boldsymbol{a} = (a_1, \ldots, a_{m-1}, a_m)^T \in \mathbb{R}^m$, let

$$\mathbf{M}(\boldsymbol{a}, \mathbf{Y}) = \sum_{k=1}^{m-1} a_k \mathbf{Y}_k - a_m \mathbf{E}_n$$

and set

$$\mathscr{C} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times (m-1)} \mid |\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| < 0 \text{ for some } \boldsymbol{a} \in \mathbb{R}^m\}$$

and

$$\mathscr{A} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times (m-1)} \mid |\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| > 0 \text{ for all } \boldsymbol{a} \neq \mathbf{0}\}.$$

Here, $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})|$ is the determinant of the matrix $\mathbf{M}(\boldsymbol{a}, \mathbf{Y})$. The subsets $\mathscr{C}$ and $\mathscr{A}$ are open sets in Euclidean topology and $\overline{\mathscr{C}} \cup \overline{\mathscr{A}} = \mathbb{R}^{n \times n \times (m-1)}$. In [13], we show that $\mathscr{A}$ is not empty if and only if $m \leq \rho(n)$ and that $\operatorname{rank} \mathbf{X} > p$ for any $\mathbf{X} \in \mathscr{R}$ with $h(\mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}) \in \mathscr{A}$. In this paper, we show that there exists an open subset $\mathscr{F}$ of $\mathscr{C}$ such that $\overline{\mathscr{F}} = \overline{\mathscr{C}}$ and $\operatorname{rank} \mathbf{X} = p$ for any $\mathbf{X} \in \mathscr{R}$ with $h(\mathbf{X}_m \mathbf{H}(\mathbf{X})^{-1}) \in \mathscr{F}$.

2. Typical rank

Due to [11, 14] and others, a number r is a typical rank of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ if the subset of tensors of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ of rank r has nonzero volume. De Silva and Lim [4] studied the space of real tensors and tensor rank with Euclidean topology and real semi-algebraic sets. They [4, Theorem 6.1] showed that for a positive integer r, the set of tensors with rank r is a semi-algebraic set by using the Tarski-Seidenberg principle (cf. [1]). In the sence, a number r is a typical rank if the set of tensors with rank r contains an open semi-algebraic set.

For $\boldsymbol{x} = (x_1, \ldots, x_{m_1})^T \in \mathbb{C}^{m_1}$, $\boldsymbol{y} = (y_1, \ldots, y_{m_2})^T \in \mathbb{C}^{m_2}$, and $\boldsymbol{z} = (z_1, \ldots, z_{m_3})^T \in \mathbb{C}^{m_3}$, we denote $(x_i y_j z_k) \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ by $\boldsymbol{x} \otimes \boldsymbol{y} \otimes \boldsymbol{z}$. Let $f_t \colon (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^t \to \mathbb{C}^{m_1 \times m_2 \times m_3}$ be a map given by

$$f_t(\boldsymbol{x}_{1,1}, \boldsymbol{x}_{1,2}, \boldsymbol{x}_{1,3}, \ldots, \boldsymbol{x}_{t,1}, \boldsymbol{x}_{t,2}, \boldsymbol{x}_{t,3}) = \sum_{\ell=1}^{t} \boldsymbol{x}_{\ell,1} \otimes \boldsymbol{x}_{\ell,2} \otimes \boldsymbol{x}_{\ell,3}.$$

Let S be a subset of $\mathbb{R}^{m_1\times m_2\times m_3}$. S is called semi-algebraic if it is a finite Boolean combination (that is, a finite composition of disjunctions, conjunctions and negations) of sets of the form

$$\{(a_{ijk})\in\mathbb{R}^{m_1\times m_2\times m_3}\mid f(a_{111},\dots,a_{m_1,m_2,m_3})>0\} \tag{1}$$

and

$$\{(a_{ijk})\in\mathbb{R}^{m_1\times m_2\times m_3}\mid g(a_{111},\dots,a_{m_1,m_2,m_3})=0\},$$

where f and g are polynomials in $m_1m_2m_3$ indeterminates $x_{111},\dots,x_{m_1,m_2,m_3}$ over $\mathbb{R}$. Then S is an open semi-algebraic set if and only if it is expressed as a finite Boolean combinations of sets of the form (1), and it is a dense open semi-algebraic set if and only if it is a Zariski open set, that is, expressed as

$$\{(a_{ijk})\in\mathbb{R}^{m_1\times m_2\times m_3}\mid g(a_{111},\dots,a_{m_1,m_2,m_3})\neq 0\}.$$

Theorem 2.1 ([5, Theorem 7.1])**:** *The space $\mathbb{R}^{m_1\times m_2\times m_3}$, $m_1,m_2,m_3\in\mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $\mathscr{O}_1,\dots,\mathscr{O}_M$ satisfying the following properties.*

(1) $\mathbb{R}^{m_1\times m_2\times m_3}\smallsetminus\cup_{i=1}^M\mathscr{O}_i$ is a closed semi-algebraic set $\mathbb{R}^{m_1\times m_2\times m_3}$ of dimension less than $m_1m_2m_3$.
(2) Each $\mathbf{T}\in\mathscr{O}_i$ has rank r_i for $i=1,\dots,M$.
(3) The number $\min(r_1,\dots,r_M)$ is equal to the generic rank $\operatorname{grank}(m_1,m_2,m_3)$ of $\mathbb{C}^{m_1\times m_2\times m_3}$, that is, the minimal $t\in\mathbb{N}$ such that the closure of the image of f_t is equal to $\mathbb{C}^{m_1\times m_2\times m_3}$.
(4) $\operatorname{mtrank}(m_1,m_2,m_3):=\max(r_1,\dots,r_M)$ is the minimal $t\in\mathbb{N}$ such that the closure of $f_t((\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{m_3})^t)$ is equal to $\mathbb{R}^{m_1\times m_2\times m_3}$.
(5) For each integer $r\in[\operatorname{grank}(m_1,m_2,m_3),\operatorname{mtrank}(m_1,m_2,m_3)]$, there exists $r_i=r$ for some integer $i\in[1,M]$.

Definition 2.2: A positive integer r is called a typical rank of $\mathbb{R}^{m_1\times m_2\times m_3}$ if

$$r\in[\operatorname{grank}(m_1,m_2,m_3),\operatorname{mtrank}(m_1,m_2,m_3)].$$

Put

$$\operatorname{typical\_rank}_{\mathbb{R}}(m_1,m_2,m_3)=[\operatorname{grank}(m_1,m_2,m_3),\operatorname{mtrank}(m_1,m_2,m_3)]\cap\mathbb{Z}.$$

This is the set of all typical ranks of $\mathbb{R}^{m_1\times m_2\times m_3}$.

We state basic facts. Some of them are seen in [4].

Proposition 2.3: *Let r be a positive integer and $\mathscr{L}$ a nonempty open set of $\mathbb{R}^{m_1\times m_2\times m_3}$. If every tensor of $\mathscr{L}$ has rank r, then r is a typical rank of $\mathbb{R}^{m_1\times m_2\times m_3}$.*

Proof: Let $\mathscr{O}_1,\dots,\mathscr{O}_M$ be open connected disjoint semi-algebraic sets as in Theorem 2.1. Since $\dim(\mathbb{R}^{m_1\times m_2\times m_3}\smallsetminus\cup_{i=1}^M\mathscr{O}_i)<m_1m_2m_3$, there exists $i\in[1,M]$ such that $\mathscr{L}\cap\mathscr{O}_i$ is not empty. □

Proposition 2.4: *Let $m_1,m_2,m_3,m_4\in\mathbb{N}$ with $m_3<m_4$. Then*

$$\operatorname{grank}(m_1,m_2,m_3)\le\operatorname{grank}(m_1,m_2,m_4)$$

and

$$\operatorname{mtrank}(m_1, m_2, m_3) \leq \operatorname{mtrank}(m_1, m_2, m_4).$$

Proof: Let $\mathscr{L}$ be the nonempty Zariski open subset of $\mathbb{C}^{m_1 \times m_2 \times m_4}$ consisting of all tensors of rank $\operatorname{grank}(m_1, m_2, m_4)$ and put

$$\mathscr{H} = \{(\mathbf{Y}_1; \mathbf{Y}_2; \ldots; \mathbf{Y}_{m_3}) \in \mathbb{C}^{m_1 \times m_2 \times m_3} \mid (\mathbf{Y}_1; \mathbf{Y}_2; \ldots; \mathbf{Y}_{m_4}) \in \mathscr{L}\}.$$

Then $\mathscr{H}$ is a nonempty Zariski open set of $\mathbb{C}^{m_1 \times m_2 \times m_3}$. For the subset $\mathscr{L}'$ of $\mathbb{C}^{m_1 \times m_2 \times m_3}$ consisting of all tensors of rank $\operatorname{grank}(m_1, m_2, m_3)$, the intersection $\mathscr{H} \cap \mathscr{L}'$ is a nonempty Zariski open set. Since $\operatorname{rank} \mathbf{Y} \leq \operatorname{rank}(\mathbf{Y}; \mathbf{X})$ for $\mathbf{Y} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ and $(\mathbf{Y}; \mathbf{X}) \in \mathbb{C}^{m_1 \times m_2 \times m_4}$, we see

$$\operatorname{grank}(m_1, m_2, m_3) \leq \operatorname{grank}(m_1, m_2, m_4).$$

Next, take an open semi-algebraic set $\mathscr{H}$ of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ consisting of tensors of rank $\operatorname{mtrank}(m_1, m_2, m_3)$. Then there are $s \in \operatorname{typical\_rank}_{\mathbb{R}}(m_1, m_2, m_4)$ and an open semi-algebraic set $\mathscr{O}$ of $\mathbb{R}^{m_1 \times m_2 \times m_4}$ consisting of tensors of rank s such that $\{(\mathbf{A}; \mathbf{B}) \mid \mathbf{A} \in \mathscr{H},\ \mathbf{B} \in \mathbb{R}^{m_1 \times m_2 \times (m_4 - m_3)}\} \cap \mathscr{O} \neq \varnothing$. Thus

$$\operatorname{mtrank}(m_1, m_2, m_3) \leq s \leq \operatorname{mtrank}(m_1, m_2, m_4).$$

□

The action of $\mathrm{GL}(m) \times \mathrm{GL}(n) \times \mathrm{GL}(p)$ on $\mathbb{R}^{m \times n \times p}$ is given as follows. Let $\mathbf{P} = (p_{ij}) \in \mathrm{GL}(n)$, $\mathbf{Q} = (q_{ij}) \in \mathrm{GL}(m)$, and $\mathbf{R} = (r_{ij}) \in \mathrm{GL}(p)$. The tensor $(b_{ijk}) = (\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot (a_{ijk})$ is defined as

$$b_{ijk} = \sum_{s=1}^{m} \sum_{t=1}^{n} \sum_{u=1}^{p} p_{is} q_{jt} r_{ku} a_{stu}.$$

Therefore,

$$(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \cdot (\mathbf{A}_1; \ldots; \mathbf{A}_p) = (\sum_{u=1}^{p} r_{1u} \mathbf{P} \mathbf{A}_u \mathbf{Q}^T; \ldots; \sum_{u=1}^{p} r_{pu} \mathbf{P} \mathbf{A}_u \mathbf{Q}^T).$$

Definition 2.5: Two tensors $\mathbf{A}$ and $\mathbf{B}$ is called *equivalent* if there exists $g \in \mathrm{GL}(m) \times \mathrm{GL}(n) \times \mathrm{GL}(p)$ such that $\mathbf{B} = g \cdot \mathbf{A}$.

Proposition 2.6 (cf. [4, Lemma 2.3])**:** *If two tensors are equivalent, then they have the same rank.*

A $1 \times m_2 \times m_3$ tensor $\mathbf{T}$ is an $m_2 \times m_3$ matrix and $\operatorname{rank} \mathbf{T}$ is equal to the matrix rank. The following three propositions are well-known. The former two propositions easily follow from the definition.

Proposition 2.7: *Let $m_1, m_2, m_3 \in \mathbb{N}$ with $2 \leq m_1 \leq m_2 \leq m_3$. If $m_1 m_2 \leq m_3$, then typical rank of $\mathbb{R}^{m_1 \times m_2 \times m_3}$ is only one integer $m_1 m_2$.*

Proposition 2.8: *Let $\mathbf{X} = (x_{ijk})$ be an $m_1 \times m_2 \times m_3$ tensor. For an $m_2 \times m_1 \times m_3$ tensor $\mathbf{Y} = (x_{jik})$ and an $m_1 \times m_3 \times m_2$ tensor $\mathbf{Z} = (x_{ikj})$, it holds that*

$$\operatorname{rank} \mathbf{X} = \operatorname{rank} \mathbf{Y} = \operatorname{rank} \mathbf{Z}.$$

A rank decomposition implies the following proposition.

Proposition 2.9 (cf. [2, 14])**:** *An $m_1 \times m_2 \times m_3$ tensor $(\mathbf{Y}_1;\dots;\mathbf{Y}_{m_3})$ has rank less than or equal to r if and only if there are an $m_1 \times r$ matrix $\mathbf{P}$, an $r \times m_2$ matrix $\mathbf{Q}$, and $r \times r$ diagonal matrices $\mathbf{D}_1,\dots,\mathbf{D}_{m_3}$ such that $\mathbf{Y}_k = \mathbf{P}\mathbf{D}_k\mathbf{Q}$ for $1 \le k \le m_3$.*

For an integer $2 \le m < n < 2m$, the number n is an only typical rank of $\mathbb{R}^{m\times n\times 2}$. Indeed, it is known that

Theorem 2.10 ([9])**:** *Let $2 \le m < n$. There is an open dense semi-algebraic set $\mathscr{O}$ of $\mathbb{R}^{m\times n\times 2}$ of which any tensor is equivalent to $((\mathbf{E}_m,\mathbf{O});(\mathbf{O},\mathbf{E}_m))$ which has rank* $\min(n,2m)$.

Furthermore, by Proposition 2.4, $\mathrm{typical\_rank}_{\mathbb{R}}(m,m,2)$ is equal to either $\{m\}$ or $\{m,m+1\}$. Let $\mathscr{L}$ be an open subset of $\mathbb{R}^{m\times m\times 2}$ consisting of $(\mathbf{A};\mathbf{B})$ such that $\mathbf{A}$ is an $m\times m$ nonsingular matrix and all eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are distinct and contain non-real numbers. For $m \ge 2$, the set $\mathscr{L}$ is not empty and any tensor of $\mathscr{L}$ has rank $m+1$ (cf. [12, Theorem 4.6]) and therefore $\mathrm{typical\_rank}_{\mathbb{R}}(m,m,2) = \{m,m+1\}$ by Proposition 2.3.

Theorem 2.11 ([14, Result 2])**:** *Let $m,n,\ell \in \mathbb{N}$ with $3 \le m \le n \le u$. If $(m-1)n < u < mn$, typical rank of $\mathbb{R}^{m\times n\times u}$ is only one integer u.*

In fact we can construct a nonempty Zariski open subset consisting of tensors with rank $\le u$. We omit the construction.

Corollary 2.12: *Let $3 \le m \le n$. Then the set of typical ranks of $m\times n\times(m-1)n$ tensors is either $\{(m-1)n\}$ or $\{(m-1)n,(m-1)n+1\}$.*

Proof: The typical rank of $\mathbb{R}^{m\times n\times((m-1)n+1)}$ is only $(m-1)n+1$ by Theorem 2.11 and the minimal typical rank of $\mathbb{R}^{m\times n\times(m-1)n}$ is equal to $(m-1)n$, since it is equal to the generic rank of $\mathbb{C}^{m\times n\times(m-1)n}$. Thus the assertion follows from Proposition 2.4. □

3. Characterization

From now on, let $3 \le m \le n$, $\ell = m-1$ and $p = (m-1)n$. For an $n\times n\times \ell$ tensor $(\mathbf{Y}_1;\dots;\mathbf{Y}_\ell)$, consider an $n\times p\times m$ tensor $\mathbf{X}(\mathbf{Y}_1,\dots,\mathbf{Y}_\ell) = (\mathbf{X}_1;\dots;\mathbf{X}_m)$ given by

$$\begin{pmatrix}\mathbf{X}_1\\ \vdots\\ \mathbf{X}_m\end{pmatrix} = \begin{pmatrix}\mathbf{E}_n & & & \\ & \mathbf{E}_n & & \\ & & \ddots & \\ & & & \mathbf{E}_n \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_\ell\end{pmatrix}. \tag{2}$$

Note that $\operatorname{rank}\mathbf{X}(\mathbf{Y}_1,\dots,\mathbf{Y}_\ell) \ge p$, since $\operatorname{rank}\mathbf{X}(\mathbf{Y}_1,\dots,\mathbf{Y}_\ell)$ is greater than or equal to the rank of the $mn\times p$ matrix (2). Generically, an $m\times n\times p$ tensor is equivalent to a tensor of type as $\mathbf{X}(\mathbf{Y}_1,\dots,\mathbf{Y}_\ell)$.

We denote by $\mathscr{M}$ the set of tensors $\mathbf{Y} = (\mathbf{Y}_1;\dots;\mathbf{Y}_\ell) \in \mathbb{R}^{n\times n\times \ell}$ such that there exist an $m\times p$ matrix (x_{ij}) and an $n\times p$ matrix $\mathbf{A} = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_p)$ such that

$$(x_{1j}\mathbf{Y}_1 + \cdots + x_{m-1,j}\mathbf{Y}_{m-1} - x_{mj}\mathbf{E}_n)\boldsymbol{a}_j = \mathbf{0} \tag{3}$$

for $1 \le j \le p$ and

$$\mathbf{B} := \begin{pmatrix} \mathbf{AD}_1 \\ \vdots \\ \mathbf{AD}_\ell \end{pmatrix} \tag{4}$$

is nonsingular, where $\mathbf{D}_k = \mathbf{Diag}(x_{k1}, \cdots, x_{kp})$ for $1 \le k \le \ell$.

Lemma 3.1: $\operatorname{rank} \mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = p$ *if and only if* $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathscr{M}$.

Proof: Suppose that $\operatorname{rank}\mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = p$. There are an $n \times p$ matrix $\mathbf{A}$, a $p \times p$ matrix $\mathbf{Q}$ and $p \times p$ diagonal matrices $\mathbf{D}_i$ such that $\mathbf{X}_k = \mathbf{AD}_k\mathbf{Q}$ for $k = 1, \ldots, m$. Since

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_\ell \end{pmatrix} = \mathbf{E}_p = \begin{pmatrix} \mathbf{AD}_1 \\ \vdots \\ \mathbf{AD}_\ell \end{pmatrix} \mathbf{Q},$$

$\mathbf{B}$ is nonsingular. Then $(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell)\mathbf{B} = \mathbf{AD}_m$ implies that $\sum_{k=1}^{\ell} \mathbf{Y}_k\mathbf{AD}_k = \mathbf{AD}_m$. Therefore, the j-th column vector $\boldsymbol{a}_j$ of $\mathbf{A}$ satisfies (3). Therefore $(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) \in \mathscr{M}$. It is easy to see that the converse is also true. □

For an $n \times n \times \ell$ tensor $\mathbf{Y} = (\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell)$, we put

$$V(\mathbf{Y}) = \{\boldsymbol{a} \in \mathbb{R}^n \mid \sum_{k=1}^{\ell} x_k \mathbf{Y}_k \boldsymbol{a} = x_m \boldsymbol{a} \text{ for some } (x_1, \ldots, x_m)^T \neq \mathbf{0}\}.$$

The set $V(\mathbf{Y})$ is not a vector subspace of $\mathbb{R}^n$. Let $\hat{V}(\mathbf{Y})$ be the smallest vector subspace of $\mathbb{R}^n$ including $V(\mathbf{Y})$. Let

$$\mathscr{S} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid \dim \hat{V}(\mathbf{Y}) = n\}.$$

Proposition 3.2: $\mathscr{M} \subset \mathscr{S}$ *holds.*

Proof: Let $\mathbf{Y} \in \mathscr{M}$. Consider the matrix $\mathbf{B}$ in (4) for any $m \times p$ matrix (x_{ij}) and any $n \times p$ matrix $\mathbf{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$ satisfying the equation (3). By column operations, $\mathbf{B}$ is transformed to a $p \times p$ matrix having a form

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{O}_{n, p-\dim \hat{V}(\mathbf{Y})} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

where $\mathbf{P}_{11}$ is an $n \times \dim \hat{V}(\mathbf{Y})$ submatrix of $\mathbf{A}$. Since $\mathbf{B}$ is nonsingular, $\mathbf{P}_{11}$ is also nonsingular, which implies that $\dim \hat{V}(\mathbf{Y}) = n$. □

By Corollary 2.12, Lemma 3.1 and Proposition 3.2, we have the following

Proposition 3.3: *If* $\operatorname{rank} \mathbf{X}(\mathbf{Y}) = p$ *then* $Y \in \mathscr{S}$. *In particular,* $\overline{\mathscr{S}} \neq \mathbb{R}^{n \times n \times \ell}$ *implies that* $\operatorname{typical\_rank}_{\mathbb{R}}(m, n, p) = \{p, p+1\}$.

Theorem 3.4 ([13])**:** *If* $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell; \mathbf{E}_n)$ *is an absolutely nonsingular tensor, then it holds that* $\operatorname{rank} \mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) > p$.

Here $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell; \mathbf{Y}_m)$ is called an absolutely nonsingular tensor if $|\sum_{k=1}^{m} x_k \mathbf{Y}_k| = 0$ implies $(x_1, \ldots, x_m)^T = \mathbf{0}$. Therefore,

Proposition 3.5: $\dim \hat{V}(\mathbf{Y}) = 0$ *if and only if* $(\mathbf{Y}; \mathbf{E}_n)$ *is an* $n \times n \times m$ *absolutely nonsingular tensor.*

Note that there exists an $n \times n \times m$ absolutely nonsingular tensor if and only if m is less than or equal to the Hurwitz-Radon number $\rho(n)$ [13].

Proposition 3.6: *Let* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be* $n \times n \times m$ *tensors. Suppose* $(\mathbf{P},\mathbf{Q},\mathbf{R}) \cdot \mathbf{Y} = \mathbf{Z}$ *for* $(\mathbf{P},\mathbf{Q},\mathbf{R}) \in \mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(m)$. *Then* $V(\mathbf{Y}) = \mathbf{Q}^T V(\mathbf{Z}) = \{\mathbf{Q}^T \boldsymbol{y} \mid \boldsymbol{y} \in V(\mathbf{Z})\}$. *In particular,* $\dim \hat{V}(\mathbf{Z}) = \dim \hat{V}(\mathbf{Y})$.

Proof: Suppose that $\sum_{k=1}^m x_k \mathbf{Z}_k \boldsymbol{y} = \mathbf{0}$. Then from the definition of the action, it follows that

$$\sum_{k=1}^m d_k \sum_{u=1}^m r_{ku} \mathbf{P}\mathbf{Y}_u \mathbf{Q}^T \boldsymbol{y} = \mathbf{P}(\sum_{u=1}^m (\sum_{k=1}^m d_k r_{ku} \mathbf{Y}_u))\mathbf{Q}^T \boldsymbol{y} = \mathbf{0}.$$

Thus $\mathbf{Q}^T \boldsymbol{y} \in V(\mathbf{Y})$. □

Corollary 3.7: *$\mathscr{S}$ is closed under the equivalence relation.*

The closure of the set of all $n \times p \times m$ tensors equivalent to $\mathbf{X}(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell)$ for some $\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell$ is $\mathbb{R}^{n \times p \times m}$. Furthermore, the following claim holds. Let $\mathscr{H}$ be the set of $n \times p \times m$ tensors $(\mathbf{X}_1; \ldots; \mathbf{X}_m)$ such that $\mathbf{A} = (\mathbf{X}_1^T, \ldots, \mathbf{X}_\ell^T)$ is a nonsingular $p \times p$ matrix and $(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell)$ given by $(\mathbf{Y}_1, \ldots, \mathbf{Y}_\ell) = \mathbf{A}^{-1}\mathbf{X}_m$ lies in $\mathscr{M}$. Any tensor of $\mathscr{H}$ has rank p. If $\mathscr{M}$ is dense in $\mathbb{R}^{n \times n \times \ell}$ then $\mathscr{H}$ is dense in $\mathbb{R}^{n \times p \times m}$.

4. Classes of $n \times n \times \ell$ tensors

We separate $\mathbb{R}^{n \times n \times \ell}$ into three classes $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{B}$ as follows. Let $\mathscr{A}$ be the set of tensors Y such that $(\mathbf{Y}; \mathbf{E}_n)$ is absolutely nonsingular. By Proposition 3.5, we have the following

Proposition 4.1: $\mathscr{A} \cap \mathscr{S} = \varnothing$.

From now on, we use symbols $x_1, \ldots, x_\ell, x_m$ as indeterminates over $\mathbb{R}$. For $\mathbf{Y} = (\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathbb{R}^{n \times n \times \ell}$, we define the $n \times n$ matrix with entries in $\mathbb{R}[x_1, \ldots, x_\ell, x_m]$ as follows.

$$\mathbf{M}(\boldsymbol{x}, \mathbf{Y}) = \sum_{k=1}^{\ell} x_k \mathbf{Y}_k - x_m \mathbf{E}_n$$

Note that fixing $a_1, \ldots, a_\ell$, the determinant $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})|$ is positive for $a_m \ll 0$, where $\boldsymbol{a} = (a_1, \ldots, a_\ell, a_m)^T$. Set

$$\mathscr{C} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| < 0 \text{ for some } \boldsymbol{a} \in \mathbb{R}^m\}.$$

Note that $\mathscr{C}$ is not empty, and if n is not congruent to 0 modulo 4 then $\rho(n) \leq 2$ and thus $\mathscr{A}$ is empty since $m \geq 3$. Set $\mathscr{B} = \mathbb{R}^{n \times n \times \ell} \setminus (\mathscr{A} \cup \mathscr{C})$. In particular, if n is odd, then $\mathscr{C} = \mathbb{R}^{n \times n \times \ell}$, and otherwise $\mathscr{B}$ contains the zero tensor.

Proposition 4.2: *$\mathscr{A}$ and $\mathscr{C}$ are open subsets of* $\mathbb{R}^{n \times n \times \ell}$.

Recall that

$$\mathscr{A} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid |\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| > 0 \text{ for all } \boldsymbol{a} \neq \mathbf{0}\}.$$

Thus it holds

$$\mathscr{B} = \{\mathbf{Y} \in \mathbb{R}^{n \times n \times \ell} \mid \begin{array}{l} |\mathbf{M}(\boldsymbol{b}, \mathbf{Y})| = 0 \text{ for some } \boldsymbol{b} \neq \mathbf{0} \text{ and} \\ |\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| \geq 0 \text{ for all } \boldsymbol{a} \end{array}\}.$$

If n is even then $\mathscr{B}$ contains a nonzero tensor in general. We give an example.

Example 4.3 Let $m = 3$ and $n = 6$. Note that $\mathscr{A}$ is empty, since $\rho(6) = 2$. Let $\mathbf{Y} = (\mathbf{Y}_1; \mathbf{Y}_2)$ be a $6 \times 6 \times 2$ tensor given by

$$x_1\mathbf{Y}_1 + x_2\mathbf{Y}_2 = \begin{pmatrix} 0 & -x_2 & 0 & 0 & 0 & -x_1 \\ x_1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & x_1 & 0 & x_2 \\ -x_2 & 0 & 0 & 0 & x_1 & 0 \end{pmatrix}.$$

Then $|\mathbf{M}((a_1,a_2,a_3)^T, \mathbf{Y})| = a_3^2(a_1a_2 - a_3^2)^2 + (a_1^3 + a_2^3)^2 \geq 0$. The equality holds if $a_3 = 0$ and $a_1 = -a_2$. Thus $\mathbf{Y} \in \mathscr{B}$ and $\mathbf{Y} \notin \mathscr{S}$ since $\dim \hat{V}(\mathbf{Y}) = 1$. Let $\mathbf{X} = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ be a 6×6 matrix and let $\varepsilon \in \mathbb{R}$. If $x_1 = -1$ and $x_3 = \varepsilon c$, then

$$|\mathbf{M}((x_1,x_2,x_3)^T, (\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon\mathbf{X}))| = -\varepsilon^2(1+\varepsilon^2c^2)c((1-\varepsilon c)^2 - c(1+\varepsilon^2c^2)).$$

Thus, for any $\varepsilon \neq 0$, there is $c_0 > 0$ such that $(1-\varepsilon c_0)^2 - c_0(1+\varepsilon^2c_0^2)) > 0$ and then $|\mathbf{M}((-1,1,\varepsilon c_0)^T, (\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon\mathbf{X}))| < 0$, which implies that $(\mathbf{Y}_1; \mathbf{Y}_2 + \varepsilon\mathbf{X}) \in \mathscr{C}$.

Proposition 4.4: *$\mathscr{B}$ is a boundary of $\mathscr{C}$. In particular, $\mathbb{R}^{n\times n\times \ell}$ is a disjoint sum of $\mathscr{A}$ and the closure $\overline{\mathscr{C}}$ of $\mathscr{C}$.*

Proof: It suffices to show that $\mathscr{B} \subset \overline{\mathscr{C}}$ by Proposition 4.2. Let $\mathbf{Y} = (\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathscr{B}$. There are a nonzero vector $\boldsymbol{b} = (b_1, \ldots, b_\ell, b_m)^T \in \mathbb{R}^n$ with $|\mathbf{M}(\boldsymbol{b}, \mathbf{Y})| = 0$ and an element $g \in \mathrm{GL}(\ell)$ such that $g \cdot \mathbf{Y} = (\mathbf{Z}_1; \mathbf{Z}_2; \ldots; \mathbf{Z}_\ell)$ and $\mathbf{Z}_1 = \sum_{k=1}^{\ell} b_k\mathbf{Y}_k$. Then $|\mathbf{Z}_1 - b_m\mathbf{E}_n| = 0$. Take a sequence $\{\mathbf{Z}_1^{(u)}\}_{u\geq 1}$ such that $|\mathbf{Z}_1^{(u)} - b_m\mathbf{E}_n| < 0$ and $\lim_{u\to\infty} \mathbf{Z}_1^{(u)} = \mathbf{Z}_1$. Thus, $(\mathbf{Z}_1^{(u)}; \mathbf{Z}_2; \ldots; \mathbf{Z}_\ell) \in \mathscr{C}$ and then $g^{-1} \cdot (\mathbf{Z}_1^{(u)}; \mathbf{Z}_2; \ldots; \mathbf{Z}_\ell) \in \mathscr{C}$. Therefore, $\mathbf{Y} \in \overline{\mathscr{C}}$. □

Corollary 4.5: *If $\mathscr{A}$ is not empty then $\mathscr{B}$ is a boundary of $\mathscr{A}$.*

Proposition 4.6: *If $m \leq \rho(n-1)$ then $\mathscr{C} \not\subset \mathscr{S}$, where $\rho(n-1)$ is a Hurwitz-Radon number.*

Proof: Let $(\mathbf{A}_1; \ldots; \mathbf{A}_\ell; \mathbf{E}_{n-1})$ be an $(n-1)\times(n-1)\times m$ absolutely nonsingular tensor. Put $\mathbf{B}_k = \mathbf{Diag}(a_k, \mathbf{A}_k)$ for $1 \leq k \leq \ell$ and $\mathbf{B}_m = \mathbf{Diag}(1, \mathbf{E}_{n-1}) = \mathbf{E}_n$, and $\mathbf{B} = (\mathbf{B}_1; \ldots; \mathbf{B}_\ell)$. Then it is easy to see that $\mathbf{B} \in \mathscr{C}$ and $|\sum_{k=1}^{\ell} x_k\mathbf{B}_k - x_m\mathbf{B}_m| = 0$ implies $x_m = \sum_{k=1}^{\ell} a_kx_k$. Therefore $V(\mathbf{B}) = \{a(1,0,\ldots,0)^T \in \mathbb{R}^n \mid a \in \mathbb{R}\}$. In particular $\mathbf{B} \notin \mathscr{S}$. □

5. Proof of Theorem 1.1

In this section we show Theorem 1.1.

In the space of homogeneous polynomials in m variables, there exists a proper Zariski closed subset S such that if a polynomial does not belong to S then it is irreducible [7, Theorem 7], since $m \geq 3$. Let $P(m,n)$ be the set of homogeneous polynomials in m variables $x_1, \ldots, x_m$ with real coefficients of degree n such that the coefficient of x_m^n is one. Its dimension is $\binom{m+n-1}{m-1} - 1$. Let I_ℓ be a nonempty Zariski open subset of $P(m,n)$ such that any polynomial of I_ℓ is irreducible. Note that $|-\mathbf{M}(\boldsymbol{x}, \mathbf{Y})| \in P(m,n)$.

A polynomial $f \in P(m,n)$ is *hyperbolic* with respect to a vector $\boldsymbol{e} \in \mathbb{R}^m$ if both $f(\boldsymbol{e}) \neq 0$ and for all vectors $\boldsymbol{v} \in \mathbb{R}^m$, the univariate polynomial $t \mapsto f(\boldsymbol{v} - t\boldsymbol{e})$ has all real roots.

Theorem 5.1 ([8]) **:** *A homogeneous polynomial f of $P(3,n)$ is hyperbolic with respect to the vector $\boldsymbol{e}=(0,0,1)^T$ and satisfies $f(\boldsymbol{e})=1$ if and only if there exist $n\times n$ symmetric matrices* $\mathbf{A}$, $\mathbf{B}$ *such that f is given by*

$$f(x,y,z)=|x\mathbf{A}+y\mathbf{B}+z\mathbf{E}_n|.$$

Proposition 5.2: *The set*

$$\{\mathbf{Y}\in\mathbb{R}^{n\times n\times\ell}\mid |-\mathbf{M}(\boldsymbol{x},\mathbf{Y})|\in I_\ell\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times\ell}$.

Proof: Let $f_\ell\colon\mathbb{R}^{n\times n\times\ell}\to P(m,n)$ be a map which sends $\mathbf{Y}$ to $|-\mathbf{M}(\boldsymbol{x},\mathbf{Y})|$ and put

$$\mathscr{I}_\ell:=\{\mathbf{Y}\in\mathbb{R}^{n\times n\times\ell}\mid f_\ell(\mathbf{Y})\in I_\ell\}.$$

Since I_ℓ is a Zariski open set, $\mathscr{I}_\ell$ is a Zariski open subset of $\mathbb{R}^{n\times n\times\ell}$. Then it suffices to show that $\mathscr{I}_\ell$ is not empty. For $(\mathbf{Z}_1;\mathbf{Z}_2)\in\mathbb{R}^{n\times n\times 2}$, if the polynomial

$$|-\mathbf{M}((x_1,x_2,x_3)^T,(\mathbf{Z}_1;\mathbf{Z}_2))|$$

is irreducible, then for any $(\mathbf{Y}_3;\dots;\mathbf{Y}_\ell)\in\mathbb{R}^{n\times n\times(\ell-2)}$, the polynomial

$$|-\mathbf{M}(\boldsymbol{x},(\mathbf{Z}_1;\mathbf{Z}_2;\mathbf{Y}_3;\dots;\mathbf{Y}_\ell)|$$

is also irreducible. Therefore, it suffices to show the assertion only for $m=3$.

The set of hyperbolic homogeneous polynomials of degree n contains an open subset [10]. Then there is an irreducible homogenious polynomial f of $P(3,n)$ which is hyperbolic with respect to $(e_1,e_2,e_3)^T$. Since $f(e_1,e_2,e_3)\neq 0$, it holds that $(e_1,e_2,e_3)^T\neq\mathbf{0}$. Thus there is an irreducible homogeneous polynomial $g\in P(3,n)$ such that g is hyperbolic with respect to $(0,0,1)^T$ and $g(0,0,1)=1$. By Theorem 5.1, $g(x_1,x_2,x_3)$ is described as

$$|x_1\mathbf{Z}_1+x_2\mathbf{Z}_2+x_3\mathbf{E}_n|=|-\mathbf{M}((x_1,x_2,x_3)^T,(-\mathbf{Z}_1;-\mathbf{Z}_2))|$$

for some $n\times n$ matrices $\mathbf{Z}_1,\mathbf{Z}_2$. Thus, $\mathscr{I}_2$ is nonempty. This completes the proof. □

Let $\tilde{\boldsymbol{x}}=(x_1,\dots,x_\ell)^T$ for $\boldsymbol{x}=(x_1,\dots,x_\ell,x_m)^T$, and put

$$\psi(\boldsymbol{x},\mathbf{Y}):=\begin{pmatrix}(-1)^{n+1}|\mathbf{M}(\boldsymbol{x},\mathbf{Y})_{n,1}|\\(-1)^{n+2}|\mathbf{M}(\boldsymbol{x},\mathbf{Y})_{n,2}|\\\vdots\\(-1)^{n+n}|\mathbf{M}(\boldsymbol{x},\mathbf{Y})_{n,n}|\end{pmatrix}\in\mathbb{R}^n,\quad \tilde{\boldsymbol{x}}\otimes\psi(\boldsymbol{x},\mathbf{Y}):=\begin{pmatrix}x_1\psi(\boldsymbol{x},\mathbf{Y})\\x_2\psi(\boldsymbol{x},\mathbf{Y})\\\vdots\\x_\ell\psi(\boldsymbol{x},\mathbf{Y})\end{pmatrix}\in\mathbb{R}^p,$$

and

$$U(\mathbf{Y}):=\langle\tilde{\boldsymbol{u}}\otimes\psi(\boldsymbol{u},\mathbf{Y})\mid |\mathbf{M}(\boldsymbol{u},\mathbf{Y})|=0,\ \boldsymbol{u}\in\mathbb{R}^m\rangle.$$

Lemma 5.3: *If* $\dim U(\mathbf{Y})=p$, *then* $\mathbf{Y}\in\mathscr{M}$.

Proof: Let $\dim U(\mathbf{Y})=p$. Then there are $\boldsymbol{u}_j=(u_{1j},\dots,u_{mj})^T\in U(\mathbf{Y})$ for $1\le j\le p$ such that

$$\mathbf{B}=(\tilde{\boldsymbol{u}}_1\otimes\psi(\boldsymbol{u}_1,\mathbf{Y}),\dots,\tilde{\boldsymbol{u}}_p\otimes\psi(\boldsymbol{u}_p,\mathbf{Y}))$$

is nonsingular. Note that $\mathbf{M}(\boldsymbol{u}_j, \mathbf{Y})\psi(\boldsymbol{u}_j, \mathbf{Y}) = \mathbf{0}$ for $1 \le j \le p$ and

$$\mathbf{B} = \begin{pmatrix} \mathbf{AD}_1 \\ \vdots \\ \mathbf{AD}_\ell \end{pmatrix},$$

where $\mathbf{A} = (\psi(\boldsymbol{u}_1, \mathbf{Y}), \ldots, \psi(\boldsymbol{u}_p, \mathbf{Y}))$ and $\mathbf{D}_k = \mathbf{Diag}(u_{k1}, \cdots, u_{kp})$ for $1 \le k \le \ell$. Thus $\mathbf{Y} \in \mathscr{M}$. □

For an $n \times \ell$ matrix $\mathbf{C} = (\boldsymbol{c}_1, \ldots \boldsymbol{c}_\ell)$, we put

$$g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C}) := \begin{vmatrix} \mathbf{M}(\boldsymbol{x}, \mathbf{Y})^{<n} \\ \sum_{k=1}^{\ell} x_k \boldsymbol{c}_k^T \end{vmatrix},$$

where $\mathbf{M}(\boldsymbol{x}, \mathbf{Y})^{<n}$ is the $(n-1) \times n$ matrix obtained from $\mathbf{M}(\boldsymbol{x}, \mathbf{Y})$ by removing the n-th row.

Lemma 5.4: *Let $\mathbf{C} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell)$ be an $n \times \ell$ matrix. The following claims are equivalent.*

(1) $\dim U(\mathbf{Y}) = p$.
(2) $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C}) = 0$ *for any* $\boldsymbol{a} \in \mathbb{R}^m$ *with* $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| = 0$ *implies* $\mathbf{C} = \mathbf{O}$.

Proof: Let $\mathbf{C} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell)$ be an $n \times \ell$ matrix. Put $\boldsymbol{d} = (\boldsymbol{c}_1^T, \ldots, \boldsymbol{c}_\ell^T)^T \in \mathbb{R}^p$. The inner product of this vector $\boldsymbol{d}$ with $\tilde{\boldsymbol{a}} \otimes \psi(\boldsymbol{a}, Y)$ is equal to $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C})$. Therefore $\boldsymbol{d}$ belongs to the orthogonal complement of $U(\mathbf{Y})$ if and only if $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C}) = 0$ for any $\boldsymbol{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| = 0$. Thus the assertion holds. □

For any i and k with $1 \le i \le n-1$ and $1 \le k \le n$, let $s_i^{(k)}$ be an elementary symmetric polynomial of degree i with variables $\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. Put

$$\mathbf{S}_n = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ s_1^{(1)} & s_1^{(2)} & \ldots & s_1^{(n)} \\ s_2^{(1)} & s_2^{(2)} & \ldots & s_2^{(n)} \\ \vdots & \vdots & & \vdots \\ s_{n-1}^{(1)} & s_{n-1}^{(2)} & \ldots & s_{n-1}^{(n)} \end{pmatrix}.$$

Lemma 5.5: *The determinant $|\mathbf{S}_n|$ of the $n \times n$ matrix $\mathbf{S}_n$ is equal to*

$$\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j).$$

In particular, if $\alpha_1, \ldots, \alpha_n$ are distinct from each other, then $\mathbf{S}_n$ is nonsingular.

Proof: For any i and k with $1 \le i \le n-1$ and $2 \le k \le n-1$, let $t_i^{(k-1)}$ be an elementary symmetric polynomial of degree i with variables $\alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. For $1 \le i \le n-1$ and $1 \le k \le n$, we have $s_i^{(k)} - s_i^{(1)} = (\alpha_1 - \alpha_k) t_{i-1}^{(k-1)}$. Then

$$|\mathbf{S}_n| = \prod_{2 \le k \le n} (\alpha_1 - \alpha_k) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ t_1^{(1)} & t_1^{(2)} & \ldots & t_1^{(n-1)} \\ \vdots & \vdots & & \vdots \\ t_{n-2}^{(1)} & t_{n-2}^{(2)} & \ldots & t_{n-2}^{(n-1)} \end{vmatrix}.$$

Therefore we have the assertion by induction on n. □

The following lemma is obtained straightforwardly.

Lemma 5.6:

$$\begin{vmatrix} \alpha_1+z & & & & a_1 \\ & \alpha_2+z & & & a_2 \\ & & \ddots & & \vdots \\ & & & \alpha_n+z & a_n \\ b_1 & b_2 & \dots & b_n & 0 \end{vmatrix} = -(z^{n-1}, z^{n-2}, \dots, 1)\mathbf{S}_n \begin{pmatrix} a_1b_1 \\ a_2b_2 \\ \vdots \\ a_nb_n \end{pmatrix}.$$

Proof: We see the left hand of the equation is equal to

$$\begin{aligned} &-\sum_{k=1}^{n} a_k b_k \frac{\prod_{1\le i\le n}(\alpha_i+z)}{\alpha_k+z} \\ &= -\sum_{k=1}^{n} a_k b_k \left(\sum_{i=1}^{n} s_{i-1}^{(k)}\right) z^{n-i} \\ &= -\sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_k b_k s_{i-1}^{(k)}\right) z^{n-i} \\ &= -(z^{n-1}, z^{n-2}, \dots, 1) \begin{pmatrix} \sum_{k=1}^{n} a_k b_k \\ \sum_{k=1}^{n} a_k b_k s_1^{(k)} \\ \vdots \\ \sum_{k=1}^{n} a_k b_k s_{n-1}^{(k)} \end{pmatrix}. \end{aligned}$$

□

Corollary 5.7: *Let $\alpha_1, \dots, \alpha_{n-1}$ be distinct complex numbers, $a_1, \dots, a_{n-1}$ nonzero complex numbers, and $b_1, \dots, b_{n-1}$ complex numbers. If*

$$\begin{vmatrix} \mathbf{Diag}(\alpha_1, \dots, \alpha_{n-1}) + z\mathbf{E}_{n-1} & \boldsymbol{a} \\ \boldsymbol{b}^T & 0 \end{vmatrix} = 0$$

for any $z \in \mathbb{R}$, then $\boldsymbol{b} = \mathbf{0}$, where $\boldsymbol{a} = (a_1 \dots, a_{n-1})^T$ and $\boldsymbol{b} = (b_1, \dots, b_{n-1})^T$.

Proof: Since $\mathbf{S}_n \begin{pmatrix} a_1b_1 \\ a_2b_2 \\ \vdots \\ a_nb_n \end{pmatrix} = \mathbf{0}$ and $\mathbf{S}_n$ is nonsingular, we have $(a_1b_1, \dots, a_nb_n) = \mathbf{0}^T$. □

The set

$$\mathscr{U}_1 = \{Y \in \mathbb{R}^{n\times n\times \ell} \mid |\mathbf{M}(\boldsymbol{x}, \mathbf{Y})| \text{ is irreducible}\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$ (see Proposition 5.2). Let W be the subset of $\mathbb{R}^{n\times n}$ consisting of matrices $\begin{pmatrix} \mathbf{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & e \end{pmatrix}$ such that all eigenvalues of $\mathbf{A}$ are distinct over the complex number field and every element of the vector $\mathbf{P}^{-1}\boldsymbol{b}$ is nonzero complex number where $\mathbf{A} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\mathbf{P} \in \mathbb{C}^{(n-1)\times(n-1)}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. Note that the validity of the condition that every element of the vector $\mathbf{P}^{-1}\boldsymbol{b}$ is nonzero is

independent of the choice of $\mathbf{P}$. We put

$$\mathscr{U}_2 := \{(\mathbf{Y}_1; \ldots; \mathbf{Y}_\ell) \in \mathbb{R}^{n\times n\times \ell} \mid \mathbf{Y}_k \in W, 1 \le k \le \ell\}.$$

The set $\mathscr{U}_2$ is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$ and $\mathscr{U} := \mathscr{U}_1 \cap \mathscr{U}_2$ is also.

Lemma 5.8: *Let $\mathbf{Y} \in \mathscr{U}_2$ and $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_\ell \in \mathbb{R}^{n-1}$. If*

$$\begin{vmatrix} \mathbf{M}(\boldsymbol{a}, \mathbf{Y})^{<n} \\ \sum_{k=1}^{\ell} a_k \boldsymbol{d}_k^T \ \ 0 \end{vmatrix} = 0$$

for any $\boldsymbol{a} = (a_1, \ldots, a_m)^T \in \mathbb{R}^m$, then $\boldsymbol{d}_1 = \cdots = \boldsymbol{d}_\ell = \mathbf{0}$.

Proof: Let $1 \le k \le \ell$. Take $a_k = 1$ and $a_j = 0$ for $1 \le j \le \ell$, $j \ne k$ and put $\mathbf{Y}_k = \begin{pmatrix} \mathbf{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & e \end{pmatrix}$, where $\mathbf{A}$ is an $(n-1)\times(n-1)$ matrix. Since $\mathbf{Y}_k \in W$, there are a matrix $\mathbf{P} \in \mathbb{C}^{(n-1)\times(n-1)}$ and distinct complex numbers $\alpha_1, \ldots, \alpha_{n-1}$ such that

$$\mathbf{Diag}(\mathbf{P}, 1)^{-1} \begin{pmatrix} (\mathbf{Y}_k - a_m \mathbf{E}_n)^{<n} & \\ \boldsymbol{d}_k^T & 0 \end{pmatrix} \mathbf{Diag}(\mathbf{P}, 1) = \begin{pmatrix} \mathbf{Diag}(\alpha_1, \ldots, \alpha_{n-1}) - a_m \mathbf{E}_{n-1} & \mathbf{P}^{-1}\boldsymbol{b} \\ \boldsymbol{d}_k^T \mathbf{P} & 0 \end{pmatrix}$$

and every element of $\mathbf{P}^{-1}\boldsymbol{b}$ is nonzero. Then we have $\boldsymbol{d}_k^T \mathbf{P} = \mathbf{0}^T$ by Corollary 5.7 and thus $\boldsymbol{d}_k = \mathbf{0}$. □

The following lemma is essential for the proof of Theorem 1.1.

Lemma 5.9: *$\mathscr{U} \cap \mathscr{C} \subset \mathscr{M}$. In particular, $\overline{\mathscr{C}} \subset \overline{\mathscr{M}}$ holds.*

Proof: Let $\mathbf{Y} \in \mathscr{U} \cap \mathscr{C}$ and fix it. There exists $\boldsymbol{a} = (a_1, \ldots, a_\ell, a_m)^T$ such that $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| < 0$. Then there is an open neighborhood U of $(a_1, \ldots, a_\ell)^T$ and a mapping $\mu : U \to \mathbb{R}$ such that

$$|\mathbf{M}(\begin{pmatrix} \boldsymbol{y} \\ \mu(\boldsymbol{y}) \end{pmatrix}, \mathbf{Y})| = 0$$

for any $\boldsymbol{y} \in U$. Thus $|\mathbf{M}(\boldsymbol{x}, \mathbf{Y})| = 0$ determines an $(m-1)$-dimensional algebraic set. Let $\mathbf{C}$ be an $n \times \ell$ matrix. Now suppose that $g(\boldsymbol{a}, \mathbf{Y}, \mathbf{C}) = 0$ holds for any $\boldsymbol{a} \in \mathbb{R}^m$ with $|\mathbf{M}(\boldsymbol{a}, \mathbf{Y})| = 0$. We show that $g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C})$ is zero as a polynomial over elements of $\boldsymbol{x}$. As a contrary, assume that $g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C})$ is not zero. The degree of $g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C})$ corresponding to the m-th element of $\boldsymbol{x}$ is less than n which is that of $|\mathbf{M}(\boldsymbol{x}, \mathbf{Y})|$. Furthermore, since $|\mathbf{M}(\boldsymbol{x}, \mathbf{Y})|$ is irreducible, $|\mathbf{M}(\boldsymbol{x}, \mathbf{Y})|$ and $g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C})$ are coprime. Then there are polynomials $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x}) \in \mathbb{R}[x_1, \ldots, x_\ell, x_m]$ and a nonzero polynomial $h(\tilde{\boldsymbol{x}}) \in \mathbb{R}[x_1, \ldots, x_\ell]$ such that

$$f_1(\boldsymbol{x})|\mathbf{M}(\boldsymbol{x}, \mathbf{Y})| + f_2(\boldsymbol{x}) g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C}) = h(\tilde{\boldsymbol{x}})$$

as a polynomial over elements of $\boldsymbol{x}$, by Euclidean algorithm. However, we can take $\boldsymbol{b} \in U$ so that $h(\boldsymbol{b}) \ne 0$. Then the above equation does not hold at $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{b} \\ \mu(\boldsymbol{b}) \end{pmatrix}$. Hence $g(\boldsymbol{x}, \mathbf{Y}, \mathbf{C})$ must be the zero polynomial over elements of $\boldsymbol{x}$. Let $\boldsymbol{c}_k^T = (c_{1k}, \ldots, c_{nk})$. By seeing the coefficient of $x_m^{n-1} x_k$, we get $c_{nk} = 0$ for $1 \le k \le \ell$. Therefore $\mathbf{C} = \mathbf{O}$ by Lemma 5.8. By Lemmas 5.4 and 5.3 we get $\mathbf{Y} \in \mathscr{M}$. Therefore $\mathscr{U} \cap \mathscr{C}$ is a subset of $\mathscr{M}$. Since $\mathscr{C}$ is open and $\mathscr{U}$ is dense, we have $\overline{\mathscr{C}} = \overline{\mathscr{C} \cap \mathscr{U}}$ and then $\overline{\mathscr{C}}$ is a subset of $\overline{\mathscr{M}}$. □

Theorem 5.10: *$\overline{\mathscr{S}} = \overline{\mathscr{M}} = \overline{\mathscr{C}}$ holds.*

Proof : We have $\overline{\mathscr{M}} \subset \overline{\mathscr{S}}$ by Proposition 3.2. By Propositions 4.1 and 4.4, the set $\mathscr{S}$ is a subset of $\overline{\mathscr{C}}$ and then $\overline{\mathscr{S}} \subset \overline{\mathscr{C}}$. Therefore $\overline{\mathscr{S}} = \overline{\mathscr{M}} = \overline{\mathscr{C}}$ by Lemma 5.9. □

Proof of Theorem 1.1: For almost all $\mathbf{Y} \in \mathscr{A}$, $\operatorname{rank} X(\mathbf{Y}) = p+1$ by Theorem 3.4. Since $\mathscr{A}$ is an open set, if $\mathscr{A}$ is not an empty set, then $\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p) = \{p, p+1\}$ ([13, Theorem 3.4]). Suppose that $\mathscr{A}$ is empty. Then $\overline{\mathscr{M}} = \mathbb{R}^{n\times n\times \ell}$ and the closure of the set consisting of all $n \times p \times m$ tensors equivalent to $X(\mathbf{Y})$ for some $\mathbf{Y} \in \mathscr{M}$ is $\mathbb{R}^{n\times p\times m}$. Recall that any tensor $X(\mathbf{Y})$ for $\mathbf{Y} \in \mathscr{M}$ has rank p. By Theorem 2.1, p is the maximal typical rank of $\mathbb{R}^{n\times p\times m}$. Therefore,

$$\mathrm{typical\_rank}_{\mathbb{R}}(m,n,p) = \mathrm{typical\_rank}_{\mathbb{R}}(n,p,m) = \{p\}$$

holds.

Acknowledgements

The authors are grateful to the anonymous referees for variable comments. One pointed out the paper [8] which shortens the proof of Proposition 5.2. All authors were supported partially by JSPS KAKENHI Grant Number 24540131.

References

[1] J. Bochnak, M. Coste, and M.F. Roy, Real algebraic geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 36, Springer-Verlag, Berlin 1998, translated from the 1987 French original, Revised by the authors.
[2] P. Bürgisser, M. Clausen, and M.A. Shokrollahi, Algebraic complexity theory, Grundlehren der Mathematischen Wissenschaften, Vol. 315, Springer-Verlag, Berlin 1997, with the collaboration of Thomas Lickteig.
[3] P. Comon, J.M.F. ten Berge, L. De Lathauwer, and J. Castaing, *Generic and typical ranks of multi-way arrays*, Linear Algebra Appl. 430 (2009), pp. 2997–3007.
[4] V. de Silva and L.H. Lim, *Tensor rank and the ill-posedness of the best low-rank approximation problem*, SIAM J. Matrix Anal. Appl. 30 (2008), pp. 1084–1127.
[5] S. Friedland, *On the generic and typical ranks of 3-tensors*, Linear Algebra Appl. 436 (2012), pp. 478–497.
[6] F.L. Hitchcock, *The expression of a tensor or a polyadic as a sum of products*, Jour. Math. Phys. 6 (1927), pp. 164–189.
[7] E. Kaltofen, *Effective Noether irreducibility forms and applications*, J. Comput. Syst. Sci. 50 (1995), pp. 274–295.
[8] A.S. Lewis, P.A. Parrilo, and M.V. Ramana, *The Lax conjecture is true*, Proc. Amer. Math. Soc. 133 (2005), pp. 2495–2499 (electronic),
[9] M. Miyazaki, T. Sumi, and T. Sakata, *Tensor rank determination problem*, in *International conference Non Linear Theory and its Applications 2009, Proceedings CD*, 2009, pp. 391–394.
[10] W. Nuij, *A note on hyperbolic polynomials*, Math. Scand. 23 (1968), pp. 69–72 (1969).
[11] V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. 52/53 (1983), pp. 645–685.
[12] T. Sumi, M. Miyazaki, and T. Sakata, *Rank of 3-tensors with 2 slices and Kronecker canonical forms*, Linear Algebra Appl. 431 (2009), pp. 1858–1868.
[13] T. Sumi, T. Sakata, and M. Miyazaki, *Typical ranks for $m \times n \times (m-1)n$ tensors with $m \leq n$*, Linear Algebra Appl. 438 (2013), pp. 953–958.
[14] J.M.F. ten Berge, *The typical rank of tall three-way arrays*, Psychometrika 65 (2000), pp. 525–532.
[15] J.M.F. ten Berge and H.A.L. Kiers, *Simplicity of core arrays in three-way principal component analysis and the typical rank of $p \times q \times 2$ arrays*, Linear Algebra Appl. 294 (1999), pp. 169–179.