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Stability of time periodic solution of the Navier-Stokes equation on the half-space under oscillatory moving boundary condition

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Abstract

Navier-Stokes system on the half space with periodically oscillating boundary has a time periodic solution which depends on time variable and vertical variable only. It is proved that the time periodic solution is asymptotically stable when the Reynolds number is sufficiently small; and the decay estimates of the perturbations are established in the frameworks of both strong and weak solutions.

1 Introduction

This paper is concerned with time periodic problem for the incompressible Navier-Stokes equation:

$$\begin{cases} \rho_*(\partial_t v + v\cdot\nabla v) - \mu\Delta v + \nabla p = 0, \\ \mathrm{div} v = 0 \end{cases} \tag{1.1}$$

on the half-space

$$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n |\, x_n > 0,\, x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

Here $n \geq 2$; $v = {}^\top(v^1, \dots, v^n)$ and p are the unknown velocity field and pressure, respectively; $\rho_* > 0$ and $\mu > 0$ are the density and viscosity coefficients which are constants. We consider (1.1) under the boundary conditions

$$\begin{cases} v\,|_{x_n=0} = V(t)e_1, \\ v(x,t) \to 0\,(x_n \to \infty). \end{cases} \tag{1.2}$$

Here $e_1 = {}^\top(1, 0, \dots, 0) \in \mathbb{R}^n$; and $V(t)$ is a given periodic function of t with period $T > 0$ satisfying

$$V \in H^m_{per}(0, T)\,, \int_0^T V(t)dt = 0$$

for some $m > 1$, where $H^m_{per}(0, T)$ denotes the set of all T-periodic distributions whose *distribution derivatives up to m th order* belong to $L^2(0, T)$.

We are interested in the time evolution of solutions of (1.1)-(1.2) under appropriate initial condition. It is not difficult to see that (1.1)-(1.2) has a time periodic solution (v_{per}, p) of the form $v_{per}(x_n, t) = v^1_{per}(x_n, t)e_1$, $p = 0$, and v_{per} satisfies

$$\begin{aligned} &v_{per}(x_n, t + T) = v_{per}(x_n, t), \\ &|\partial^j_{x_n} v^1_{per}(x_n, t)| \le C a^j T^{m-\frac{1}{2}} \|V\|_{\dot{H}^m_{per}(0,T)} e^{-ax_n}\ (j = 0, 1). \end{aligned} \tag{1.3}$$

Here $a = (\frac{\rho_* \pi}{\mu T})^{\frac{1}{2}}$; C is a positive constant; and $\|V\|_{\dot{H}^m_{per}(0,T)}$ is the norm of V defined by

$$\|V\|_{\dot{H}^m_{per}(0,T)} = \left[\sum_{k\in\mathbb{Z},k\neq 0} \left(\frac{2k\pi}{T} \right)^{2m} |V_k|^2 \right]^{\frac{1}{2}},$$

where V_k ($k = \pm 1, \pm 2, \cdots$) denote the Fourier coefficients of $V(t)$, i.e., $V(t) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z},k\neq 0} V_k e^{i\frac{2k\pi}{T}t}$. In particular, when $V(t) = \frac{2}{\sqrt{T}} V \cos \frac{2\pi}{T} t$ with a constant $V \neq 0$, $v_{per}(x_n, t)$ can be written as $v_{per}(x_n, t) = \frac{2}{\sqrt{T}} V e^{-\sqrt{\frac{\rho_*\pi}{\mu T}} x_n} \cos(\frac{2\pi}{T} t - \sqrt{\frac{\rho_*\pi}{\mu T}} x_n)$ explicitly, and in this case, the problem is called as Stokes' second problem. As a first step to investigate the time evolution of solutions of (1.1)-(1.2), we consider the stability of the time periodic solution $v_{per}(x_n, t)$.

Time periodic problem for the incompressible Navier-Stokes equations has been extensively studied. See e.g., [7, 8, 9, 10, 11, 13] and references therein. As for time periodic problem on unbounded domains, Kozono and Nakao ([7]) constructed a time periodic solution $\overline{v}_{per}(x, t)$ of (1.1) with a time periodic external force $f(x, t)$ on the domains Ω being $\mathbb{R}^n$, $\mathbb{R}^n_+$ ($n \ge 3$) and exterior domains of $\mathbb{R}^n$ ($n \ge 4$). It was proved in [7] that if f is sufficiently small in $C([0, T]; L^p(\Omega) \cap L^n(\Omega))$ for some $1 < p < \infty$, then there exists a time periodic strong solution $\overline{v}_{per}(x, t) \in C([0, T]; L^p(\Omega) \cap L^n(\Omega))$. See also [8, 9]. Taniuchi ([11]) then studied the stability of the time periodic solution $\overline{v}_{per}(x, t)$ constructed in [7] and proved that $\overline{v}_{per}(x, t)$ is asymptotically stable under sufficiently small perturbations and the perturbation u satisfies the decay property

$$\|u(t)\|_{L^l} \le C t^{-\frac{n}{2}(\frac{1}{n} - \frac{1}{l})} \text{ for } n \le l \le 2n,$$

$$\lim_{t\to\infty}\|u(t)\|_{L^n}=0.$$

Yamazaki ([13]) extended the results of [7, 11]. It was shown in [13] that a time periodic solution $\overline{v}_{per}(x,t)$ exists in $C([0,T];L^{n,\infty}(\Omega))$ when the external force f is written in the form $f(t,x)=\operatorname{div}F(t,x)$ with F sufficiently small in $BC(\mathbb{R};(L^{\frac{n}{2},\infty}(\Omega))^{n^2})$ for the domains Ω considered in [7] including the case where Ω is an exterior domain of $\mathbb{R}^3$. Here $L^{p,q}(\Omega)$ denotes the Lorentz space on Ω. Furthermore, it was proved in [13] that the time periodic solution $\overline{v}_{per}(x,t)$ is asymptotically stable under sufficiently small perturbations and the perturbation u satisfies the decay properties

$$\|u(t)\|_{L^{p,\infty}}\le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} \text{ for } n<p<\infty,$$

$$\|u(t)\|_{L^{q,1}}\le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \text{ for } n<q<p.$$

The proofs of the stability in [11, 13] heavily rely on the fact that the time periodic solution $\overline{v}_{per}(x,t)$ decays as $|x|\to\infty$ which, roughly speaking, enables us to regard the lower order terms $(\overline{v}_{per}\cdot\nabla)u+(u\cdot\nabla)\overline{v}_{per}$ as a small perturbation of the Stokes operator. Since the time periodic solution $v_{per}(x_n,t)$ of (1.1)-(1.2) does not belong to any $L^p(\mathbb{R}^n_+)$ with $1\le p<\infty$, the methods in [11, 13] do not work well to investigate the stability of $v_{per}(x_n,t)$.

In this paper we will show that the time periodic solution v_{per} is asymptotically stable in the frameworks of strong and weak solutions when the number $\sqrt{\frac{\rho_*}{\mu}}T^m\|V\|_{\dot{H}^m_{per}(0,T)}$ is small enough. Here $\sqrt{\frac{\rho_*}{\mu}}T^m\|V\|_{\dot{H}^m_{per}(0,T)}$ is a non-dimensional number corresponding to the Reynolds number. More precisely, let $u={}^\top(u^1,\dots,u^n)=v-v_{per}$ be the perturbation of $v_{per}(x_n,t)$. Then the initial boundary value problem for the perturbation u is written as

$$\begin{cases}\rho_*(\partial_t u+u\cdot\nabla u+v^1_{per}\partial_{x_1}u+\partial_{x_n}v^1_{per}u^ne_1)-\mu\Delta u+\nabla p=0,\\ \operatorname{div}u=0,\\ u\,|_{x_n=0}=0,\\ u\,|_{t=0}=u_0.\end{cases}\tag{1.4}$$

In order to analyze (1.4), we first consider the linearized problem:

$$\begin{cases}\rho_*(\partial_t u+v^1_{per}\partial_{x_1}u+\partial_{x_n}v^1_{per}u^ne_1)-\mu\Delta u+\nabla p=0,\\ \operatorname{div}u=0,\\ u\,|_{x_n=0}=0,\\ u\,|_{t=s}=u_0,\end{cases}\tag{1.5}$$

or

$$\begin{cases}\partial_t u+\frac{\mu}{\rho_*}Au+P(v^1_{per}\partial_{x_1}u+\partial_{x_n}v^1_{per}u^ne_1)=0,\\ u|_{t=s}=u_0.\end{cases}\tag{1.6}$$

Here P denotes the Helmholtz projection and $A = -P\Delta u$ is the Stokes operator. The solution $u(t)$ of (1.6) is written as $u(t) = U(t,s)u_0$. Here and in what follows, $U(t,s)$ denotes the evolution operator for the linearized problem (1.6). We show that if $\sqrt{\frac{\rho_*}{\mu}}T^m\|V\|_{\dot{H}^m_{per}(0,T)}$ is sufficiently small, the evolution operator $U(t,s)$ satisfies

$$\|\partial_x^k U(t,s)u_0\|_{L^p(\mathbb{R}^n_+)} \le C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|u_0\|_{L^q(\mathbb{R}^n_+)}, \tag{1.7}$$

$$\|U(t,s)\partial_x u_0\|_{L^p(\mathbb{R}^n_+)} \le C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^q(\mathbb{R}^n_+)} \tag{1.8}$$

for $0 \le s < t < \infty$ and $1 < q \le p < \infty$ or $1 \le q < p \le \infty$. Furthermore, by using the oscillatory aspect of $v_{per}(x_n,t)$, we also prove that $U(t,s)$ is approximated by the Stokes semigroup for large $t-s \gg 1$; namely, we show that

$$\|\partial_x^k U(t,s)u_0 - \partial_x^k S(t-s)u_0\|_{L^p(\mathbb{R}^n_+)} \le C_\varepsilon(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)} \tag{1.9}$$

for $0 \le s < t < \infty$, and $1 < q \le p < \infty$ or $1 \le q < p \le \infty$, and an arbitrary positive number ε. Here $S(t)$ denotes the Stokes semigroup. Using (1.7) and (1.8) we prove the asymptotic stability of v_{per} under small initial perturbations in $L^n_\sigma(\mathbb{R}^n_+)$ in the framework of strong solutions based on the argument in [6], and, also, the asymptotic stability of v_{per} under arbitrary initial perturbations in $L^2_\sigma(\mathbb{R}^n_+)$ in the framework of weak solutions based on the arguments in [1, 2, 3], provided that $\sqrt{\frac{\rho_*}{\mu}}T^m\|V\|_{\dot{H}^m_{per}(0,T)}$ is small enough. See Theorems 3.2 and 3.3 bellow for precise statements.

To prove the L^p-L^q estimates (1.7)-(1.9), we write $U(t,s)$, after a suitable non-dimensionalization, as

$$U(t,s) = S(t-s) - \int_s^t S(t-\tau)PB(\tau)U(\tau,s)\,d\tau,$$

where $B(t)$ is the operator defined by $B(t)u = \frac{1}{\sqrt{\nu}}v^1_{per}(t)\partial_{x_1}u + \frac{1}{\sqrt{\nu}}\partial_{x_n}v^1_{per}(t)u^n e_1$ with $\frac{1}{\sqrt{\nu}} = \sqrt{\frac{\rho_*}{\mu}}T^m\|V\|_{\dot{H}^m_{per}(0,T)}$. We then show that $\int_s^t S(t-\tau)PB(\tau)U(\tau,s)\,d\tau$ is regarded as a small perturbation from the Stokes semigroup $S(t-s)$. In contrast to the case of [11, 13], a straightforward application of L^p-L^q estimates for $S(t-\tau)$ does not work well since v_{per} does not decay as $|x'| \to \infty$ although it decays exponentially as $x_n \to \infty$. To overcome this difficulty, we will decompose $U(t,s)$ into *low frequency* part $U_0(t,s)$ and *high frequency* part $U_\infty(t,s)$ as $U(t,s) = U_0(t,s) + U_\infty(t,s)$, and will prove the estimates

$$\|\partial_x^k U_0(t,s)u_0\|_{L^p(\mathbb{R}^n_+)} \le C(t-s)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q(\mathbb{R}^n_+)}, \tag{1.10}$$

$$\|\partial_x^k U_\infty(t,s)u_0\|_{L^p(\mathbb{R}^n_+)} \le C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)} \tag{1.11}$$

To prove (1.10)-(1.11), we will first establish the following anisotropic estimates for $S(t)$

$$\|\partial_x^k S_1(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|u_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}, \tag{1.12}$$

$$\|\partial_x^k S_2(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|u_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}, \tag{1.13}$$

$$\|\partial_x^k S_3(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{-ct}\|u_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}, \tag{1.14}$$

with $1<s\leq q\leq p\leq r<\infty$ or $1\leq s\leq q<p\leq r\leq\infty$, $k\in\mathbb{N}$, by using an integral representation of the Stokes semigroup $S(t)$ given in [5]. Here $S(t)$ is decomposed as $S(t)=S_1(t)+S_2(t)+S_3(t)$, where $S_1(t)$ is a *low frequency* part of $S(t)$, and $S_2(t)$ and $S_3(t)$ are *high frequency* parts of $S(t)$. See section 4 for the precise definition of $S_1(t)$, $S_2(t)$ and $S_3(t)$. Another key point in the proof is to use the estimates for $B(t)$

$$\|B(t)u\|_{L^p(\mathbb{R}^n_+)} \leq \frac{C}{\sqrt{\nu}}\|\partial_x^2 u\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}. \tag{1.15}$$

with $1<p\leq r\leq\infty$ or $1\leq p<r<\infty$, which is obtained by applying the (*one-dimensional*) *Hardy inequality* (see Proposition 2.3 below) based on the fact $u|_{x_n=0}=0$ and $\partial_{x_n}u^n|_{x_n=0}=0$, where the latter boundary condition comes from $\operatorname{div} u=0$. Due to the estimates (1.12)-(1.15), one can regard $\int_0^t S(t-\tau)PB(\tau)U(\tau,s)\,d\tau$ as a small pertubation of $S(t-s)$ to obtain the estimates (1.10) and (1.11).

This paper is organized as follows. In section 2, we introduce some notations and inequalities used in this paper. In section 3, we state the main results of this paper and rewrite problems (1.4), (1.5) and (1.6) in a non-dimensional form. In section 4, we establish anisotropic L^p-L^q type estimates for the Stokes semigroup. In section 5, we prove the estimates (1.7)-(1.9) for the evolution operator $U(t,s)$.

2 Preliminaries

In this section we first introduce some notations used in this paper. Let us introduce some function spaces. For $1\leq p\leq\infty$, $L^p(D)$ denotes the usual Lebesgue space over a domain D, and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let k be a nonnegative integer. We denote by $W^{k,p}(D)$ the k th order L^p Sobolev space over D, and its norm is denoted by $\|\cdot\|_{W^{k,p}(D)}$. For simplicity we denote by $L^p(D)$ (resp., $W^{k,p}(D)$) the set of all vector fields whose components belong to $L^p(D)$ (resp., $W^{k,p}(D)$).

We set

$$C_0^\infty(\mathbb{R}^n_+) = \{f \in C^\infty(\mathbb{R}^n_+) |\, f\,\text{has a compact support in}\,\mathbb{R}^n_+\},$$

$$C_{0,\sigma}^\infty(\mathbb{R}^n_+) = \{f = (f^1, f^2, \ldots, f^n) \in (C_0^\infty(\mathbb{R}^n_+))^n |\, \nabla \cdot f = 0\},$$

$$L^p_\sigma(\mathbb{R}^n_+) = \overline{C_{0,\sigma}^\infty(\mathbb{R}^n_+)}^{\|\cdot\|_{L^p(\mathbb{R}^n_+)}}.$$

$$W_0^{1,p}(\mathbb{R}^n_+) = \overline{C_0^\infty(\mathbb{R}^n_+)}^{\|\cdot\|_{W^{1,p}(\mathbb{R}^n_+)}}.$$

We denote by $L^p(\mathbb{R}_+; L^q(\mathbb{R}^{n-1}))$ the Bochner space on $\mathbb{R}_+ = (0,\infty)$ with values in $L^q(\mathbb{R}^{n-1})$, and its norm is denoted by $\|\cdot\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}$.

Let X be a Banach space. We denote by $BC([0,\infty); X)$ the set of all X-valued bounded continuous functions.

The Fourier transformation of f with respect to $x \in \mathbb{R}^n$ is denoted by $\mathcal{F}[f]$:

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,\ \xi \in \mathbb{R}^n,$$

and the inverse Fourier transformation of f with respect to $x \in \mathbb{R}^n$ by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)dx,\ \xi \in \mathbb{R}^n.$$

We denote the Fourier transformation of f with respect to $x' \in \mathbb{R}^{n-1}$ by $\mathcal{F}'[f]$ or $\hat{f}$:

$$\mathcal{F}'[f](\xi') = \hat{f}(\xi') = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} f(x')dx',\ \xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1},$$

and the inverse Fourier transformation of f with respect to $x' \in \mathbb{R}^{n-1}$ by $\mathcal{F}'^{-1}[f]$:

$$\mathcal{F}'^{-1}[f](\xi') = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} f(x')dx',\ \xi' \in \mathbb{R}^{n-1},$$

and, likewise, the Fourier transformation of f with respect to $x_n \in \mathbb{R}$ by $\mathcal{F}_n[f]$:

$$\mathcal{F}_n[f](\xi_n) = \int_{\mathbb{R}} e^{-ix_n\cdot\xi_n} f(x_n)dx_n,\ \xi_n \in \mathbb{R},$$

and the inverse Fourier transformation of f with respect to $x_n \in \mathbb{R}$ by $\mathcal{F}_n^{-1}[f]$:

$$\mathcal{F}_n^{-1}[f](\xi_n) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_n\cdot\xi_n} f(x_n)dx_n, \xi_n \in \mathbb{R}.$$

The convolution of f and g in $x' \in \mathbb{R}'$ is denoted by $f *' g$:

$$(f *' g)(x') = \int_{\mathbb{R}^{n-1}} f(y')g(x' - y')dy',$$

and the convolution of f and g in $x_n \in \mathbb{R}$ by $f *_n g$:

$$(f *_n g)(x_n) = \int_{\mathbb{R}} f(y_n)g(x_n - y_n)dy_n.$$

We define an extension operator E by

$$(Ef)(x', x_n) = \begin{cases} f(x', x_n), \; x_n > 0, \\ 0 \qquad\qquad , x_n < 0. \end{cases}$$

In this paper, we will often use the following well-known inequalities.

Proposition 2.1 (Jensen's inequality)**.** *Let $(X, M, \mu), (Y, N, \nu)$ be σ-finite, complete measure spaces and let $f(x, y)$ be a measurable function. If $0 < p \le q \le \infty$, then*

$$\| \|f(x, y)\|_{L^p(X)}\|_{L^q(Y)} \le \| \|f(x, y)\|_{L^q(Y)}\|_{L^p(X)}.$$

This inequality clearly holds for $p = q = \infty$, and, in other cases, the inequality follows from Minkowski's inequality for integrals $\| \int_X f(x, \cdot)\, d\mu(x)\|_{L^r(Y)} \le \int_X \|f(x, \cdot)\|_{L^r(Y)}\, d\mu(x)$ by taking $r = \frac{q}{p}$.

Proposition 2.2. (i) *Let a, b, c be numbers satisfying $0 \le a, c$ and $0 \le b < 1$, $\max\{a, c\} > 1$. Then it holds that*

$$\int_0^t (1 + t - \tau)^{-a}(1 + \tau^{-b})(1 + \tau)^{-c} d\tau \le C(1 + t)^{-\min\{a,c\}}.$$

(ii) *Let a, b, c be numbers satisfying $0 \le a, b < 1 < c$. Then it holds that*

$$\int_0^t (t - \tau)^{-a} e^{-(t-\tau)}(1 + \tau^{-b})(1 + \tau)^{-c} d\tau \le C(1 + t^{-\min\{a,b\}})(1 + t)^{-c}.$$

Proposition 2.2 can be proved by a straightforward computation.

The following inequalities are known as Hardy's and Hilbert's inequalities, respectively. See, e.g., [4, pp. 195–196].

Proposition 2.3 (Hardy's inequality)**.** *Let $1 < q \le \infty$. Then*

$$\left\| \frac{1}{x_n} \int_0^{x_n} f(y_n)dy_n \right\|_{L^q(\mathbb{R}_+)} \le \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}_+)}.$$

Proposition 2.4 (Hilbert's inequality). *Let $1 < p < \infty$ and define the operator T by*

$$Tf(x_n) = \int_0^\infty \frac{f(y_n)}{x_n + y_n} dy_n$$

for $x_n \in (0,\infty)$ and $f \in L^p(0,\infty)$. Then there exists a positive constant $C = C(p)$ such that

$$\|Tf\|_{L^p(0,\infty)} \le C\|f\|_{L^p(0,\infty)}.$$

In the end of this section we show the existence of time periodic solution v_{per}. We denote by $\dot{H}^m_{per}(0,T)$ the set of all T-periodic distributions with $\int_0^T f(t)\,dt = 0$ whose *distribution derivatives up to m th order* belong to $L^2(0,T)$, and the norm $\|f\|_{\dot{H}^m_{per}(0,T)}$ is defined by

$$\|f\|_{\dot{H}^m_{per}(0,T)} = \left[\sum_{k\in\mathbb{Z},k\ne0}\left(\frac{2k\pi}{T}\right)^{2m}|f_k|^2\right]^{\frac12},$$

where f_k $(k = \pm1, \pm2, \cdots)$ denote the Fourier coefficients of $f(t)$, i.e., $f(t) = \frac{1}{\sqrt{T}}\sum_{k\in\mathbb{Z},k\ne0} f_k e^{i\frac{2k\pi}{T}t}$.

Proposition 2.5. *Let $T > 0$. Assume that $V \in \dot{H}^m_{per}(0,T)$ for some $m > 1$. Then there exists a time periodic solution $v_{per}(x_n,t) = v^1_{per}(x_n,t)e_1$, $p = 0$ of (1.1)-(1.2) satisfying*

$$v_{per}(x_n,t+T) = v_{per}(x_n,t), \qquad \int_0^T v^1_{per}(x_n,t)\,dt = 0,$$

$$|\partial^j_{x_n} v^1_{per}(x_n,t)| \le Ca^jT^{m-\frac12}\|V\|_{\dot{H}^m_{per}(0,T)}e^{-ax_n},$$

$$|\partial^j_{x_n}(v^1_{per}(x_n,t) - v^1_{per}(x_n,s))| \le Ca^jT^{m-\gamma-\frac12}\|V\|_{\dot{H}^m_{per}(0,T)}e^{-ax_n}|t-s|^\gamma$$

for $j = 0,1$, where $a = (\frac{\rho_*\pi}{\mu T})^{\frac12}$; γ is a number satisfying $0 < \gamma < \min\{1, m - \frac{j+1}{2}\}$; and C is a positive constant depending only on m and γ.

Proof . We denote the Fourier series of $V(t)$ by $V(t) = \frac{1}{\sqrt{T}}\sum_{k\in\mathbb{Z},k\ne0} V_k e^{i\frac{2k\pi}{T}t}$, Since $V(t)$ is real valued, we require $V_{-k} = \overline{V_k}$, and hence, $V(t)$ is written as $V(t) = \frac{2}{\sqrt{T}}\sum_{k=1}^\infty \mathrm{Re}\left[V_k e^{i\frac{2k\pi}{T}t}\right]$. We define v^1_{per} by

$$v^1_{per}(x_n,t) = \frac{2}{\sqrt{T}}\sum_{k=1}^\infty e^{-a_kx_n}\mathrm{Re}\left[V_k e^{i\left(\frac{2k\pi}{T}t - a_kx_n\right)}\right].$$

Here $a_k = (\frac{\rho_* k\pi}{\mu T})^{\frac{1}{2}}$. It is not difficult to check that $v_{per} = v_{per}^1 e_1$, $p = 0$ is a T-time periodic function satisfying (1.1)-(1.2) with $\int_0^T v_{per}^1(x_n, t)\, dt = 0$. Moreover, we have

$$\begin{aligned}
&|\partial_{x_n}^j v_{per}^1(x_n, t)| \\
&= \frac{2e^{-a_1 x_n}}{\sqrt{T}} \left| \sum_{k=1}^{\infty} (-a_k)^j e^{-(a_k - a_1)x_n} \mathrm{Re}\left[(1+i)^j V_k e^{i\left(\frac{2k\pi}{T} t - a_k x_n\right)} \right] \right| \\
&\leq C a^j T^{m-\frac{1}{2}} \|V\|_{\dot{H}^m_{per}(0,T)} e^{-a_1 x_n}.
\end{aligned}$$

The estimate for $\partial_{x_n}^j (v_{per}^1(x_n, t) - v_{per}^1(x_n, s))$ can be obtained similarly. This completes the proof. $\square$

3 Main Results

In this section, we state the main results of this paper. We begin with L^p-L^q type estimates for the evolution operator $U(t, s)$ for the linearized problem (1.6).

Theorem 3.1. *Let $1 < q \leq p < \infty$ or $1 \leq q < p \leq \infty$. Suppose that $u_0 \in L^q_\sigma(\mathbb{R}^n_+)$. Then there exists $\delta_0 = \delta_0(q) > 0$ such that if $\sqrt{\frac{\rho_*}{\mu}} T^m \|V\|_{\dot{H}^m_{per}(0,T)} \leq \delta_0$, the following estimates hold with $k = 0, 1$:*

$$\|\partial_x^k U(t, s) u_0\|_{L^p(\mathbb{R}^n_+)} \leq C(t-s)^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \|u_0\|_{L^q(\mathbb{R}^n_+)},$$

$$\|U(t, s)\partial_x u_0\|_{L^p(\mathbb{R}^n_+)} \leq C(t-s)^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} \|u_0\|_{L^q(\mathbb{R}^n_+)}.$$

Here C is a positive constant depending only on n, p and q. Furthermore, for any ε satisfying $0 < \varepsilon < \frac{1}{2}$, there exists $\delta_0' = \delta_0'(q, \varepsilon) > 0$ such that if $\sqrt{\frac{\rho_*}{\mu}} T^m \|V\|_{\dot{H}^m_{per}(0,T)} \leq \delta_0'$, then for $k = 0, 1$, the following estimates hold.

$$\|\partial_x^k U(t, s) u_0 - \partial_x^k S(t-s) u_0\|_{L^p(\mathbb{R}^n_+)} \leq C(t-s)^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2} - \frac{1}{2} + \varepsilon} \|u_0\|_{L^q(\mathbb{R}^n_+)}.$$

Here $S(t)$ is the Stokes semigroup and C is a positive constant depending only on n, p, q and ε.

By using Theorem 3.1 and the standard method in the theory of Navier-Stokes equation, we can obtain the following results on the asymptotic stability of $v_{per}(x_n, t)$.

We first consider (1.4) in the form of the integral equation

$$u(t) = U(t, 0)u_0 - \int_0^t U(t, \tau) P(u \cdot \nabla u)(\tau)\, d\tau. \tag{3.1}$$

Theorem 3.2. (i) *Assume that* $\sqrt{\frac{\rho_*}{\mu}} T^m \|V\|_{\dot{H}^m_{per}(0,T)} \le \delta_0(n)$. *Then there exists* $\eta > 0$ *such that if* $u_0 \in L^n_\sigma(\mathbb{R}^n_+)$ *satisfies* $\|u_0\|_{L^n(\mathbb{R}^n_+)} \le \eta$, *there exists a unique global solution* $u \in BC([0,\infty); L^n_\sigma(\mathbb{R}^n_+))$ *of (3.1). Furthermore,* $u(t)$ *has the following properties:*

$$t^{\frac{1}{2}(1-\frac{n}{p})} u \in BC([0,\infty); L^p_\sigma(\mathbb{R}^n_+)), n \le p \le \infty,$$

$$t^{1-\frac{n}{2p}} \nabla u \in BC([0,\infty); L^p_\sigma(\mathbb{R}^n_+)), n \le p < \infty,$$

$$\|u(t)\|_{L^p(\mathbb{R}^n_+)} = o(t^{-\frac{1}{2}+\frac{n}{2p}})\,(t \to \infty), n \le p \le \infty,$$

$$\|\nabla u(t)\|_{L^p(\mathbb{R}^n_+)} = o(t^{-1+\frac{n}{2p}})\,(t \to \infty),\ n \le p < \infty.$$

(ii) *Moreover, let* $0 < \varepsilon < \frac{1}{2}$ *and let* $u_0 \in L^n_\sigma(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+)$ *for some* $1 < r < n$. *Assume that* $\sqrt{\frac{\rho_*}{\mu}} T^m \|V\|_{\dot{H}^m_{per}(0,T)} \le \delta_0'(r,\varepsilon)$. *Then*

$$\|u(t) - S(t)u_0\|_{L^r(\mathbb{R}^n_+)} = o(t^{-\frac{\alpha}{2}}).$$

Here α *is a positive number satisfying* $0 < \alpha < \min\{1-2\varepsilon, n-\frac{n}{r}, \frac{n}{r}-1\}$.

Theorem 3.2 (i) can be proved by combining the argument of [6] and Theorem 3.1. Furthermore, in a similar manner to that in [6], one can show

$$\|u(t) - U(t,0)u_0\|_{L^r(\mathbb{R}^n_+)} = o(t^{-\frac{\alpha}{2}}).$$

By Theorem 3.1 we have

$$\begin{aligned}
&\|u(t) - S(t)u_0\|_{L^r(\mathbb{R}^n_+)} \\
&\quad \le \|u(t) - U(t,0)u_0\|_{L^r(\mathbb{R}^n_+)} + \|U(t,0)u_0 - S(t)u_0\|_{L^r(\mathbb{R}^n_+)} \\
&\quad \le o(t^{-\frac{\alpha}{2}}) + Ct^{-\frac{1}{2}+\varepsilon} = o(t^{-\frac{\alpha}{2}}).
\end{aligned}$$

This proves Theorem 3.2 (ii).

We next consider the stability of $v_{per}(x_n,t)$ in the framework of weak solutions. We call u a *weak solution* of (1.4) with initial value $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ if u is weakly continuous on $[0,\infty)$ with values in $L^2_\sigma(\mathbb{R}^n_+)$ and u satisfies $\nabla u \in L^2_{loc}([0,\infty); L^2(\mathbb{R}^n_+))$, $u(0) = u_0$ and

$$\begin{aligned}
&(u(t),\varphi(t)) - (u(s),\varphi(s)) + \frac{\mu}{\rho_*}\int_s^t (\nabla u, \nabla\varphi)\,d\tau \\
= &\int_s^t (u,\varphi_t)\,d\tau - \int_s^t (v_{per}\otimes u + u \otimes v_{per} + u\otimes u, \nabla\varphi)\,d\tau
\end{aligned}$$

for all $\varphi \in C([0,\infty); (L^2_\sigma \cap W^{1,2}_0 \cap L^n)(\mathbb{R}^n_+)) \cap C^1([0,\infty); L^2_\sigma(\mathbb{R}^n_+))$. Here $(\cdot,\cdot)$ denotes the inner product of L^2.

Theorem 3.3. (i) *Let* $\sqrt{\frac{\rho_*}{\mu}} T^m \|V\|_{\dot{H}^m_{per}(0,T)} \le \delta_0(2)$. *Then, for each* $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$, *there is a weak solution* u *of* (1.4) *such that*

$$\|u(t)\|_{L^2(\mathbb{R}^n_+)} \to 0 \quad \text{as } t \to \infty.$$

(ii) *Furthermore, if* $\sqrt{\frac{\rho_*}{\mu}} T^m \|V\|_{\dot{H}^m_{per}(0,T)} \le \delta_0(r)$ *and* $u_0 \in L^r_\sigma(\mathbb{R}^n_+)$ *for some* $1 \le r < 2$, *then*

$$\|u(t)\|_{L^2(\mathbb{R}^n_+)} = o(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}).$$

Theorem 3.3 (i)-(ii) can be proved by combining the arguments of [1, 2, 3] and Theorem 3.1.

In the remaining of this paper, we will prove Theorem 3.1. To prove Theorem 3.1, it is convenient to consider a non-dimensional form of the problem. We introduce the following non-dimensional quantities:

$$x = V_* T \sqrt{\nu}\tilde{x}, \ t = T\tilde{t}, \ v = V_*\tilde{v}, \ v_{per} = V_*\tilde{v}_{per}, \ p = \rho_* V_*^2 \sqrt{\nu}\tilde{p},$$

where V_* is the reference velocity given by $V_* = T^{m-\frac{1}{2}} \|V\|_{\dot{H}^m_{per}(0,T)}$, and ν is the non-dimensional parameter defined by

$$\nu = \frac{\mu}{\rho_* T V_*^2} = \frac{\mu}{\rho_* T^{2m} \|V\|^2_{\dot{H}^m_{per}(0,T)}}.$$

Under this non-dimensionalization, $V(t)$ and $v_{per}(x_n, t)$ are transformed into $\tilde{V}(\tilde{t})$ and $\tilde{v}_{per}(\tilde{x}_n, \tilde{t})$ which are 1-periodic functions with the following properties.

Proposition 3.4. *Under the non-dimensionalization above,* $V(t)$ *and* $v_{per}(x_n, t)$ *are transformed into* $\tilde{V}(\tilde{t})$ *and* $\tilde{v}_{per}(\tilde{x}_n, \tilde{t})$ *which satisfy* $\tilde{V} \in \dot{H}^m_{per}(0,1)$ *with* $\|\tilde{V}\|_{\dot{H}^m_{per}(0,1)} = 1$ *and*

$$\tilde{v}^1_{per}(\tilde{x}_n, \tilde{t}) = 2\sum_{k=1}^{\infty} e^{-\tilde{a}_k \tilde{x}_n} \mathrm{Re}\left[\tilde{V}_k e^{i\left(2k\pi\tilde{t} - \tilde{a}_k\tilde{x}_n\right)}\right],$$

$$\tilde{v}_{per}(\tilde{x}_n, \tilde{t}+1) = \tilde{v}_{per}(\tilde{x}_n, \tilde{t}) \quad (\tilde{t} \in \mathbb{R}), \qquad \int_0^1 \tilde{v}_{per}(\tilde{x}_n, \tilde{t})\, d\tilde{t} = 0,$$

$$|\partial^j_{\tilde{x}_n} \tilde{v}^1_{per}(\tilde{x_n}, \tilde{t})| \le Ce^{-\sqrt{\pi}\tilde{x}_n},$$

$$|\partial^j_{\tilde{x}_n}(\tilde{v}^1_{per}(\tilde{x}_n, \tilde{t}) - \tilde{v}^1_{per}(\tilde{x}_n, \tilde{s}))| \le Ce^{-\sqrt{\pi}\tilde{x}_n}|\tilde{t} - \tilde{s}|^\gamma$$

for $j = 0, 1$, *where* $\tilde{a}_k = \sqrt{k\pi}$; $\tilde{V}_k$, $(k = 1, 2, 3, \cdots)$ *denote the Fourier coefficients of* $\tilde{V}(\tilde{t})$; C *is a positive constant independent of* μ *and* T; *and* γ *is a number satisfying* $0 < \gamma < \min\{1, m - \frac{j+1}{2}\}$.

Let $u = \tilde{v} - \tilde{v}_{per}$ denote the non-dimensional perturbation. Then, problem (1.4) is written, after omitting tildes, as

$$\begin{cases} \partial_t u - \Delta u + \frac{1}{\sqrt{\nu}} u \cdot \nabla u + \frac{1}{\sqrt{\nu}} v_{per}^1 \partial_{x_1} u + \frac{1}{\sqrt{\nu}} \partial_{x_n} v_{per}^1 u^n e_1 + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u \mid_{x_n=0} = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{3.2}$$

and the linearized problems (1.5) and (1.6) are written, by omitting tildes, as

$$\begin{cases} \partial_t u - \Delta u + \frac{1}{\sqrt{\nu}} v_{per}^1 \partial_{x_1} u + \frac{1}{\sqrt{\nu}} \partial_{x_n} v_{per}^1 u^n e_1 + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u \mid_{x_n=0} = 0, \\ u \mid_{t=s} = u_0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_t u + Au + PB(t)u = 0, \\ u|_{t=s} = u_0, \end{cases} \tag{3.4}$$

respectively. Here $B(t)$ denotes the lower order term given by

$$B(t)u = \frac{1}{\sqrt{\nu}} v_{per}^1(x_n, t) \partial_{x_1} u + \frac{1}{\sqrt{\nu}} \partial_{x_n} v_{per}^1(x_n, t) u^n e_1.$$

In what follows we will denote the evolution operator for the linearized problem (3.4) by $U(t,s)$. By Proposition 3.4 (cf., Proposition 2.5) one can show the existence of the evolution operator for (3.4) on $L^p_\sigma(\mathbb{R}^n_+)$ $(1 < p < \infty)$. (See [12, Chap. 5].) To prove Theorem 3.1, we will consider (3.4) instead of (1.6).

4 Anisotropic L^p-L^q type estimates for the Stokes semigroup

In this section we first introduce a solution formula for the nonstationary Stokes system on the half-space given in [5]. We then establish anisotropic L^p-L^q type estimates for the Stokes semigroup.

We consider the nonstationary Stokes system:

$$\begin{cases} \partial_t v - \Delta v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v \mid_{x_n=0} = 0, \\ v \mid_{t=0} = P v_0. \end{cases} \tag{4.1}$$

From [5, p. 245] the solution $v(t) = S(t)v_0$ of (4.1) is written as

$$S(t)v_0 = \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} e^{-|\xi|^2 t} G(\xi', \xi_n, x_n, y_n)\hat{v}_0(\xi', y_n)d\xi_n dy_n\right].$$

Here

$$G(\xi', \xi_n, x_n, y_n) = \sum_{k=1}^{5} G_k(\xi', \xi_n, x_n, y_n),$$

where

$$\begin{aligned} G_1(\xi', \xi_n, x_n, y_n) &= g_1(\xi', \xi_n)e^{i\xi_n(x_n - y_n)}, \\ G_2(\xi', \xi_n, x_n, y_n) &= g_2(\xi', \xi_n)e^{i\xi_n(x_n + y_n)}, \\ G_3(\xi', \xi_n, x_n, y_n) &= g_3(\xi', \xi_n)e^{i\xi_n x_n}e^{-|\xi'|y_n}, \\ G_4(\xi', \xi_n, x_n, y_n) &= g_4(\xi', \xi_n)e^{-|\xi'|x_n}e^{i\xi_n y_n}, \\ G_5(\xi', \xi_n, x_n, y_n) &= g_5(\xi', \xi_n)e^{-|\xi'|x_n}e^{-|\xi'|y_n}, \end{aligned}$$

with $g_j(\xi', \xi_n)$ $(j = 1, 2, 3, 4, 5)$ given by

$$g_1(\xi', \xi_n) = I_n - \frac{\xi^\top \xi}{|\xi|^2},$$

$$\begin{aligned} &g_2(\xi', \xi_n) \\ &\quad = -I_n + \frac{1}{|\xi'|(i\xi_n + |\xi'|)} \\ &\qquad \times \begin{pmatrix} i\xi_n|\xi'|\frac{\xi'^\top\xi'}{|\xi|^2} + \xi_n^2\frac{\xi'^\top\xi'}{|\xi|^2} + \xi'^\top\xi' & -i\xi_n^2\frac{\xi'|\xi'|}{|\xi|^2} + \xi_n\frac{\xi'|\xi'|^2}{|\xi|^2} \\ i\xi_n^2\frac{\xi'|\xi'|}{|\xi|^2} - \xi_n\frac{\xi'|\xi'|^2}{|\xi|^2} & i\xi_n\frac{|\xi'|^3}{|\xi|^2} + \xi_n^2\frac{|\xi'|^2}{|\xi|^2} + i\xi_n|\xi'| \end{pmatrix}, \end{aligned}$$

$$g_3(\xi', \xi_n) = \frac{2\xi_n}{|\xi'|(i\xi_n + |\xi'|)}\begin{pmatrix} -\xi_n\frac{\xi'^\top\xi'}{|\xi|^2} & i\xi_n\frac{\xi'|\xi'|}{|\xi|^2} \\ \frac{|\xi'|^{2\,T}\xi'}{|\xi|^2} & -i\frac{|\xi'|^3}{|\xi|^2} \end{pmatrix},$$

$$g_4(\xi', \xi_n) = \frac{2\xi_n}{|\xi'|(i\xi_n + |\xi'|)}\begin{pmatrix} -\xi_n\frac{\xi'^\top\xi'}{|\xi|^2} & -\frac{\xi'|\xi'|^2}{|\xi|^2} \\ -i\xi_n\frac{|\xi'|^\top\xi'}{|\xi|^2} & -i\frac{|\xi'|^3}{|\xi|^2} \end{pmatrix},$$

$$g_5(\xi', \xi_n) = \frac{2\xi_n}{i\xi_n + |\xi'|}\begin{pmatrix} i\frac{\xi'^\top\xi'}{|\xi|^2} & \frac{\xi'|\xi'|}{|\xi|^2} \\ -\frac{|\xi'|^\top\xi'}{|\xi|^2} & i\frac{|\xi'|^2}{|\xi|^2} \end{pmatrix}$$

and I_n being the $n \times n$ identity matrix.

One can show that $S(t) = e^{-tA}P$ on $L^p(\mathbb{R}^n_+)$, where P denotes the Helmholtz projection, $A = -P\Delta$ is the Stokes operator and e^{-tA} denotes the Stokes semigroup. Namely, $S(t)$ coincides with the Stokes semigroup on $L^p_\sigma(\mathbb{R}^n_+)$.

We will establish anisotropic L^p-L^q type estimates for low and high frequency parts of $S(t)$. We introduce a decomposition of $S(t)$ into its low and high frequency parts. Let $\chi_0'(\xi')$, $\chi_\infty'(\xi')$, $\chi_0^n(\xi_n)$ and $\chi_\infty^n(\xi_n)$ be cut-off functions satisfying

$$\chi_0'(\xi') \in C_0^\infty(\mathbb{R}^{n-1}),\ 0 \le \chi_0'(\xi') \le 1,\ \chi_0'(\xi') = \begin{cases} 1 & (|\xi'| \le 1), \\ 0 & (|\xi'| \ge 2), \end{cases}$$

$$\chi_\infty'(\xi') = 1 - \chi_0'(\xi'),$$

$$\chi_0^n(\xi_n) \in C_0^\infty(\mathbb{R}),\ 0 \le \chi_0^n(\xi_n) \le 1,\ \chi_0^n(\xi_n) = \begin{cases} 1 & (|\xi_n| \le 1), \\ 0 & (|\xi_n| \ge 2), \end{cases}$$

$$\chi_\infty^n(\xi_n) = 1 - \chi_0^n(\xi_n).$$

Moreover, let $\tilde{\chi}_0^n(\xi_n)$ and $\tilde{\chi}_\infty^n(\xi_n)$ be cut-off functions satisfying

$$\tilde{\chi}_0^n(\xi_n) \in C_0^\infty(\mathbb{R}),\ 0 \le \tilde{\chi}_0^n(\xi_n) \le 1,\ \tilde{\chi}_0^n(\xi_n) = \begin{cases} 1 & (|\xi_n| \le 2), \\ 0 & (|\xi_n| \ge 3), \end{cases}$$

$$\tilde{\chi}_\infty^n(\xi_n) \in C_0^\infty(\mathbb{R}),\ 0 \le \tilde{\chi}_\infty^n(\xi_n) \le 1,\ \tilde{\chi}_\infty^n(\xi_n) = \begin{cases} 0 & (|\xi_n| \le \frac{1}{2}), \\ 1 & (|\xi_n| \ge 1). \end{cases}$$

We decompose $S(t)$ as

$$S(t)v_0 = S_1(t)v_0 + S_2(t)v_0 + S_3(t)v_0,$$

where

$$S_1(t)v_0 = \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2 t}G(\xi', \xi_n, x_n, y_n)\hat{v}_0(\xi', y_n)\hat{v}_0 d\xi_n dy_n\right],$$

$$S_2(t)v_0 = \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_0'(\xi')\chi_\infty^n(\xi_n)e^{-|\xi|^2 t}G(\xi', \xi_n, x_n, y_n)\hat{v}_0(\xi', y_n)\hat{v}_0 d\xi_n dy_n\right],$$

$$S_3(t)v_0 = \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_\infty'(\xi')e^{-|\xi|^2 t}G(\xi', \xi_n, x_n, y_n)\hat{v}_0(\xi', y_n)\hat{v}_0 d\xi_n dy_n\right].$$

We have the following anisotropic L^p-L^q type estimates for $S_j(t)$ $(j = 1, 2, 3)$ which are the main results of this section.

Theorem 4.1. *Let p, q, r and s satisfy $1 < s \le q \le p \le r < \infty$ or $1 \le s \le q < p \le r \le \infty$ and let $k \in \mathbb{N}$. Then*

$$\begin{aligned}
\text{(i)}\ \|\partial_x^k S_1(t)v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\text{(ii)}\ \|\partial_x^k S_2(t)v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\text{(iii)}\ \|\partial_x^k S_3(t)v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\text{(iv)}\ \|S_1(t)\partial_x v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\text{(v)}\ \|S_2(t)\partial_x v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\text{(vi)}\ \|S_3(t)\partial_x v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}$$

Here C and c are positive constants depending only on n, p, q, r, s and k.

In order to prove Theorem 4.1, we first prepare some auxiliary inequalities. We then prove Theorem 4.1 (i)-(iii) for $s > 1$ (Lemma 4.4). We finally prove Theorem 4.1 (i)-(iii) for $s = 1$ and (iv)-(vi) by the duality argument.

We will use the following inequalities.

Lemma 4.2. *Let $1 \le p \le \infty$. Then*

(i) $$\|\mathcal{F}_n^{-1}[\chi_\infty^n(\xi_n)e^{-\xi_n^2 t}]\|_{L^p(\mathbb{R})} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})}e^{-ct},$$

(ii) $$\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')e^{-|\xi'|^2 t}]\|_{L^p(\mathbb{R}^{n-1})} \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})}e^{-ct}.$$

where C and c are positive constants depending only on n, p and q.

Let p, q and r be a number satisfying p, q, $r \ge 1$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + 1$ and let $k \in \mathbb{N}$. Then

(iii) $$\|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)(i\xi_n)^k e^{-\frac{1}{2}\xi_n^2 t}]\|_{L^r(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}},$$

(iv) $$\|\mathcal{F}'^{-1}[\chi_0'(\xi')|\xi'|^k e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^r(\mathbb{R}^{n-1})} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}.$$

where C is a positive constant depending only on n, p, q, r and k.

Proof . We give a proof of (ii) and (iv) only, since (i) and (iii) can be proved similarly. We fix ε satisfying $0 < \varepsilon < 1$. Since

$$\begin{aligned}
&\mathcal{F}'^{-1}[\chi'_\infty(\xi')e^{-|\xi'|^2 t}]\\
= \ &\mathcal{F}'^{-1}[|\xi'|^{\frac{\varepsilon}{2}-1}e^{-\frac{1}{3}|\xi'|^2 t}] *' \mathcal{F}'^{-1}[|\xi'|^{\frac{\varepsilon}{2}-1}e^{-\frac{1}{3}|\xi'|^2 t}] *' \mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2 t}],
\end{aligned}$$

we have

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')e^{-|\xi'|^2t}]\|_{L^1(\mathbb{R}^{n-1})}\\
\leq\ &\|\mathcal{F}'^{-1}[|\xi'|^{\frac{\varepsilon}{2}-1}e^{-\frac{1}{3}|\xi'|^2t}]\|^2_{L^1(\mathbb{R}^{n-1})}\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}]\|_{L^1(\mathbb{R}^{n-1})}\\
\leq\ &Ct^{-\frac{1}{2}(\varepsilon-2)}\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}]\|_{L^1(\mathbb{R}^{n-1})}.
\end{aligned}$$

Since

$$e^{ix'\cdot\xi'}=\sum_{j=1}^{n-1}\frac{-ix_j}{|x'|^2}\frac{\partial e^{ix'\cdot\xi'}}{\partial\xi_j},$$

by integration by parts, we have

$$\begin{aligned}
&\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}](x')\\
&\quad=\sum_{|\beta|=n}\left(\frac{ix'}{|x'|^2}\right)^\beta\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}}e^{ix'\cdot\xi'}\partial^\beta_{\xi'}(\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t})d\xi'.
\end{aligned}$$

Noting that $|\partial^\beta_{\xi'}(\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t})|\leq C|\xi'|^{-(n-2)-\varepsilon}e^{-\frac{1}{6}t}e^{-\frac{1}{6}|\xi'|^2t}$, we have

$$\begin{aligned}
|\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}](x')| &\leq C|x'|^{-n}e^{-\frac{1}{6}t}\int_{\mathbb{R}^{n-1}}\frac{1}{|\xi'|^{n-2+\varepsilon}}e^{-\frac{1}{6}|\xi'|^2t}d\xi'\\
&\leq C|x'|^{-n}t^{\frac{n-2+\varepsilon}{2}}t^{-\frac{n-1}{2}}e^{-\frac{1}{6}t}\\
&= C|x'|^{-n}t^{\frac{-1+\varepsilon}{2}}e^{-\frac{1}{6}t}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}]\|_{L^1(|x'|\geq t^{\frac{1}{2}})} &\leq Ct^{\frac{-1+\varepsilon}{2}}e^{-\frac{1}{6}t}\int_{|x'|\geq t^{\frac{1}{2}}}|x'|^{-n}dx'\\
&= Ct^{\frac{1}{2}(\varepsilon-2)}e^{-\frac{1}{6}t}. \qquad (4.2)
\end{aligned}$$

Since

$$|\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}](x')|\leq C\int_{\mathbb{R}^{n-1}}|\xi'|^{2-\varepsilon}e^{-\frac{1}{6}|\xi'|^2t}d\xi'e^{-\frac{1}{6}t},$$

we have

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')|\xi'|^{2-\varepsilon}e^{-\frac{1}{3}|\xi'|^2t}]\|_{L^1(|x'|\leq t^{\frac{1}{2}})}\\
\leq\ &C\left(\int_{|x'|\leq t^{\frac{1}{2}}}dx'\right)\left(\int_{\mathbb{R}^{n-1}}|\xi'|^{2-\varepsilon}e^{-\frac{1}{6}|\xi'|^2t}d\xi'\right)e^{-\frac{1}{6}t}\\
\leq\ &Ct^{\frac{1}{2}(\varepsilon-2)}e^{-\frac{1}{6}t}. \qquad (4.3)
\end{aligned}$$

It then follows from (4.2) and (4.3) that

$$\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')e^{-|\xi'|^2 t}]\|_{L^1(\mathbb{R}^{n-1})} \leq Ce^{-\frac{1}{6}t}.$$

Applying Young's inequality we see that

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')e^{-|\xi'|^2 t}]\|_{L^p(\mathbb{R}^{n-1})} \\
&\leq \|\mathcal{F}'^{-1}[e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^p(\mathbb{R}^{n-1})}\|\mathcal{F}'^{-1}[\chi'_\infty(\xi')e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^1(\mathbb{R}^{n-1})} \\
&\leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})}e^{-\frac{1}{6}t}.
\end{aligned}$$

This proves (ii).

Let us next prove (iv). From Young's inequality we have

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_0(\xi')|\xi'|^k e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^r(\mathbb{R}^{n-1})} \\
&\leq \|\mathcal{F}'^{-1}[\chi'_0(\xi')|\xi'|^k]\|_{L^r(\mathbb{R}^{n-1})}\|\mathcal{F}'^{-1}[e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^1(\mathbb{R}^{n-1})} \leq C_1
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_0(\xi')|\xi'|^k e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^r(\mathbb{R}^{n-1})} \\
&\leq \|\mathcal{F}'^{-1}[\chi'_0(\xi')]\|_{L^r(\mathbb{R}^{n-1})}\|\mathcal{F}'^{-1}[|\xi'|^k e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^1(\mathbb{R}^{n-1})} \\
&\leq C_2 t^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}.
\end{aligned}$$

Setting $C_0 = \max\{C_1(1+\varepsilon)^{\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})+\frac{k}{2}}, C_2(1+\frac{1}{\varepsilon})^{\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})+\frac{k}{2}}\}$ with $\varepsilon > 0$, we obtain, for $t \leq \varepsilon$,

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_0(\xi')|\xi'|^k e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^r(\mathbb{R}^{n-1})} \\
&\leq C_1(1+\varepsilon)^{\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})+\frac{k}{2}}(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \\
&\leq C_0(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}},
\end{aligned}$$

and for $t \geq \varepsilon$,

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi'_0(\xi')|\xi'|^k e^{-\frac{1}{2}|\xi'|^2 t}]\|_{L^r(\mathbb{R}^{n-1})} \\
&\leq C_2\left(1+\frac{1}{\varepsilon}\right)^{\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})+\frac{k}{2}}(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \\
&\leq C_0(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}.
\end{aligned}$$

This completes the proof. □

We will also use the following estimates.

Lemma 4.3. *Let $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ be a multi-index. Then*

(i) $$|\partial_{\xi'}^{\alpha'} g_i(\xi', \xi_n)| \le C_{\alpha'} |\xi'|^{-|\alpha'|},\ i = 1, 2, 3, 4, 5,$$

(ii) $$|\partial_{\xi_n}^{m} g_i(\xi', \xi_n)| \le C_m |\xi_n|^{-m},\ m \in \mathbb{N},\ i = 1, 2, 3, 4, 5,$$

(iii) $$|\mathcal{F}_n^{-1} g_i(\xi', \xi_n)| \le C|\xi'| e^{-|\xi'||x_n|},\ i = 3, 4, 5.$$

It is not difficult to show Lemma 4.3 (i) and (ii) by the Leibniz rule, and (iii) is proved by using the relation $\mathcal{F}_n[\frac{1}{|\xi|^2}] = \pi \frac{e^{-|\xi'||x_n|}}{|\xi'|}$.

To prove Theorem 4.1, we write $S_1(t)$, $S_2(t)$ and $S_3(t)$ as follows:

$$S_1(t)v_0 = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t) + I_5(x,t),$$

$$S_2(t)v_0 = J_1(x,t) + J_2(x,t) + J_3(x,t) + J_4(x,t) + J_5(x,t),$$
$$S_3(t)v_0 = K_1(x,t) + K_2(x,t) + K_3(x,t) + K_4(x,t) + K_5(x,t),$$

where

$$I_j = \mathcal{F'}^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n) e^{-|\xi|^2 t} G_j(\xi', \xi_n, x_n, y_n)\hat{v}_0 d\xi_n dy_n\right],$$

$$J_j = \mathcal{F'}^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_0'(\xi')\chi_\infty^n(\xi_n) e^{-|\xi|^2 t} G_j(\xi', \xi_n, x_n, y_n)\hat{v}_0 d\xi_n dy_n\right],$$

$$K_j = \mathcal{F'}^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_\infty'(\xi') e^{-|\xi|^2 t} G_j(\xi', \xi_n, x_n, y_n)\hat{v}_0 d\xi_n dy_n\right]$$

for $j = 1, 2, 3, 4, 5$.

We first prove Theorem 4.1 (i)-(iii) for $s > 1$.

Proof of Theorem 4.1 (i)-(iii) for $\mathbf{s > 1}$. Theorem 4.1 (i)-(iii) for $s > 1$ are consequences of the following Lemma 4.4.

Lemma 4.4. *Let p, q, r and s satisfy $1 < s \le q \le p \le r < \infty$ or $1 < s \le q < p \le r \le \infty$ and let $k \in \mathbb{N}$. Then, for $j = 1, 2, 3, 4, 5$,*

$$\begin{aligned}
\|\partial_{x_n}^k I_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\|\partial_{x_n}^k J_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\|\partial_{x_n}^k K_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} &\le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}$$

Here C is a positive constant depending only on n, p, q, r, s and k.

Proof of Lemma 4.4 for $\mathbf{j = 1, 2}$. We here give a proof for $j = 2$ only since the case $j = 1$ can be proved similarly. In the proof we often use the relation $\mathcal{F} = \mathcal{F}'\mathcal{F}_n = \mathcal{F}_n\mathcal{F}'$, $\mathcal{F}^{-1} = \mathcal{F}'^{-1}\mathcal{F}_n^{-1} = \mathcal{F}_n^{-1}\mathcal{F}'^{-1}$. Note that $\chi_0^n(\xi_n) = \tilde{\chi}_0^n(\xi_n)\chi_0^n(\xi_n)$. We write $\partial_{x_n}^k I_2$ as

$$
\begin{aligned}
&\partial_{x_n}^k I_2(x)\\
&\quad= 2\pi\partial_{x_n}^k \mathcal{F}^{-1}[\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2 t}g_2(\xi',\xi_n)\mathcal{F}_n^{-1}[E\hat{v}_0]]\\
&\quad= 2\pi\mathcal{F}^{-1}[\chi_0'(\xi')\chi_0^n(\xi_n)(i\xi_n)^k e^{-|\xi|^2 t}g_2(\xi',\xi_n)\mathcal{F}_n^{-1}[E\hat{v}_0]]\\
&\quad= 2\pi\mathcal{F}_n^{-1}[\tilde{\chi}_0^n(\xi_n)(i\xi_n)^k e^{-\frac{1}{2}\xi_n^2 t}\\
&\qquad\quad\times\mathcal{F}'^{-1}[\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}g_2(\xi',\xi_n)\mathcal{F}_n^{-1}[E\hat{v}_0]]]\\
&\quad= 2\pi\mathcal{F}_n^{-1}[\tilde{\chi}_0^n(\xi_n)(i\xi_n)^k e^{-\frac{1}{2}\xi_n^2 t}]\\
&\qquad\quad *_n\mathcal{F}^{-1}[g_2(\xi',\xi_n)\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}\mathcal{F}_n^{-1}[E\hat{v}_0]].
\end{aligned}
$$

Let σ be a number satisfying $\frac{1}{\sigma} = \frac{1}{r} - \frac{1}{p} + 1$. From Young's inequality and Lemma 4.2, we have

$$
\begin{aligned}
&\|\partial_{x_n}^k I_2\|_{L^r(\mathbb{R}_+)}\\
&\quad\le C\|\mathcal{F}_n^{-1}[\tilde{\chi}_0^n(\xi_n)(i\xi_n)^k e^{-\frac{1}{2}\xi_n^2 t}]\|_{L^\sigma(\mathbb{R})}\\
&\qquad\qquad\times\|\mathcal{F}^{-1}[g_2(\xi',\xi_n)\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}\mathcal{F}_n^{-1}[E\hat{v}_0]]\|_{L^p(\mathbb{R})}\\
&\quad\le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\\
&\qquad\qquad\times\|\mathcal{F}^{-1}[g_2(\xi',\xi_n)\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}\mathcal{F}_n^{-1}[E\hat{v}_0]]\|_{L^p(\mathbb{R})}.
\end{aligned}
$$

Applying Proposition 2.1, Lemma 4.3 and the Fourier multiplier theorem, we have

$$
\begin{aligned}
&\|\partial_{x_n}^k I_2\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\\
&\quad\le \|\,\|\partial_{x_n}^k I_2\|_{L^r(\mathbb{R}_+)}\|_{L^p(\mathbb{R}^{n-1})}\\
&\quad\le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\\
&\qquad\qquad\times\|\mathcal{F}^{-1}[g_2(\xi',\xi_n)\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}\mathcal{F}_n^{-1}[E\hat{v}_0]]\|_{L^p(\mathbb{R}^n)}\\
&\quad\le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\|\mathcal{F}^{-1}[\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}\mathcal{F}_n^{-1}[E\hat{v}_0]]\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

We next take m in such a way that $\frac{1}{p} = \frac{1}{s} + \frac{1}{m} - 1$. Since

$$
\begin{aligned}
&\mathcal{F}^{-1}[\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2 t}e^{-\frac{1}{2}\xi_n^2 t}\mathcal{F}_n^{-1}[E\hat{v}_0]]\\
&\quad= \mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\frac{1}{2}\xi_n^2 t}] *_n \mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[E\hat{v}_0]]],
\end{aligned}
$$

we apply Young's inequality, Lemma 4.2 and Proposition 2.1 to obtain

$$
\begin{aligned}
&(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\|\mathcal{F}^{-1}[\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi'|^2t}e^{-\frac{1}{2}\xi_n^2t}\mathcal{F}_n^{-1}[E\hat{v}_0]]\|_{L^p(\mathbb{R}^n)}\\
&\quad\le\ C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\Big\|\,\|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\frac{1}{2}\xi_n^2t}]\|_{L^m(\mathbb{R})}\\
&\qquad\qquad\times\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[E\hat{v}_0]]]\|_{L^s(\mathbb{R})}\Big\|_{L^p(\mathbb{R}^{n-1})}\\
&\quad\le\ C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\Big\|\,\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[E\hat{v}_0]]]\|_{L^s(\mathbb{R})}\Big\|_{L^p(\mathbb{R}^{n-1})}\\
&\quad\le\ C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\Big\|\,\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[E\hat{v}_0]]]\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^s(\mathbb{R})}.
\end{aligned}
$$

Let l be a number satisfying $\frac{1}{p}=\frac{1}{q}+\frac{1}{l}-1$. Since

$$
\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[E\hat{v}_0]]]=\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}]*'\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[Ev_0]],
$$

by applying Young's inequality and Lemma 4.2, we have

$$
\begin{aligned}
&(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\Big\|\,\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[E\hat{v}_0]]]\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^s(\mathbb{R})}\\
&\quad\le\ C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\Big\|\,\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2t}]\|_{L^l(\mathbb{R}^{n-1})}\\
&\qquad\qquad\times\|\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[Ev_0]]\|_{L^q(\mathbb{R}^{n-1})}\Big\|_{L^s(\mathbb{R})}\\
&\quad\le\ C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\big\|\,\|\mathcal{F}_n^{-1}[\mathcal{F}_n^{-1}[Ev_0]]\|_{L^q(\mathbb{R}^{n-1})}\big\|_{L^s(\mathbb{R})}\\
&\quad\le\ C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}
$$

We thus obtain

$$
\|\partial_{x_n}^k I_2(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
$$

Similarly, applying Lemma 4.2 we can see that

$$
\begin{aligned}
\|\partial_{x_n}^k J_2\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\ &\le\ Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},\\
\|\partial_{x_n}^k K_2\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\ &\le\ Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}
$$

This completes the proof. □

We next consider I_3, J_3 and K_3.

Proof of Lemma 4.4 for j = 3. We write $\partial_{x_n}^k I_3$ as

$$\begin{aligned}
&\partial_{x_n}^k I_3(x)\\
= \ & \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)(i\xi_n)^k e^{-|\xi|^2 t} e^{i\xi_n x_n} e^{-|\xi'|y_n} g_3(\xi',\xi_n)\hat{v}_0 d\xi_n dy_n\right]\\
= \ & \mathcal{F}_n^{-1}\left[\chi_0^n(\xi_n)(i\xi_n)^k e^{-\xi_n^2 t}\int_0^\infty \mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n} g_3(\xi',\xi_n)\hat{v}_0] dy_n\right]\\
= \ & \mathcal{F}_n^{-1}[\chi_0^n(\xi_n)(i\xi_n)^k e^{-\xi_n^2 t}]\\
&\quad *_n \left(\int_0^\infty \mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0] dy_n\right).
\end{aligned}$$

Let m be a number satisfying $\frac{1}{m} = \frac{1}{r} - \frac{1}{s} + 1$. We have

$$\begin{aligned}
&\|\partial_{x_n}^k I_3(x',\cdot)\|_{L^r(\mathbb{R}_+)}\\
\leq \ & \|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)(i\xi_n)^k e^{-\xi_n^2 t}]\|_{L^m(\mathbb{R})}\\
&\times \left\|\int_0^\infty \mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0] dy_n\right\|_{L^s(\mathbb{R})}\\
\leq \ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\left\|\int_0^\infty \mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0] dy_n\right\|_{L^s(\mathbb{R})}.
\end{aligned}$$

Since $s \leq p \leq r$, we apply Proposition 2.1 to obtain

$$\begin{aligned}
&\|\partial_{x_n}^k I_3\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\\
\leq \ & \left\| \|\partial_{x_n}^k I_3\|_{L^r(\mathbb{R}_+)}\right\|_{L^p(\mathbb{R}^{n-1})}\\
\leq \ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\\
&\times \left\|\left\|\int_0^\infty \mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0] dy_n\right\|_{L^s(\mathbb{R})}\right\|_{L^p(\mathbb{R}^{n-1})}\\
\leq \ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}\\
&\times \left\|\int_0^\infty \left\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0]\right\|_{L^p(\mathbb{R}^{n-1})} dy_n\right\|_{L^s(\mathbb{R})}.
\end{aligned}$$

Choose l such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{l} - 1$. The Young inequality and Lemma 4.2 then yield

$$\begin{aligned}
&\|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0]\|_{L^p(\mathbb{R}^{n-1})}\\
= \ & \|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}] *' \mathcal{F}'^{-1}[e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0]\|_{L^p(\mathbb{R}^{n-1})}\\
\leq \ & \|\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|^2 t}]\|_{L^l(\mathbb{R}^{n-1})}\|\mathcal{F}'^{-1}[e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0]\|_{L^q(\mathbb{R}^{n-1})}\\
\leq \ & C(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})}\|\mathcal{F}'^{-1}[e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0]\|_{L^q(\mathbb{R}^{n-1})}.
\end{aligned}$$

Since

$$|\partial_{\xi'}^{\alpha'}(e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)])| \leq C_{\alpha'}|\xi'|e^{-|\xi'|(x_n+y_n)}|\xi'|^{-|\alpha'|} \leq C_{\alpha'}\frac{1}{x_n+y_n}|\xi'|^{-|\alpha'|},$$

by applying the Fourier multiplier theorem, we have

$$\|\mathcal{F}'^{-1}[e^{-|\xi'|y_n}\mathcal{F}_n^{-1}[g_3(\xi',\xi_n)]\hat{v}_0]\|_{L^q(\mathbb{R}^{n-1})} \leq \frac{C}{x_n+y_n}\|v_0\|_{L^q(\mathbb{R}^{n-1})}.$$

This, together with Proposition 2.4, implies that

$$\begin{aligned}
&\|\partial_{x_n}^k I_3\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\left\|\int_0^\infty \frac{1}{x_n+y_n}\|v_0\|_{L^q(\mathbb{R}^{n-1})}dy_n\right\|_{L^s(\mathbb{R})} \\
\leq\ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}$$

Similarly, we can obtain

$$\|\partial_{x_n}^k J_3\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},$$

$$\|\partial_{x_n}^k K_3\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.$$

This completes the proof. □

Let us next estimate $\partial_{x_n}^k I_4$.

Proof of Lemma 4.4 for $\mathbf{j=4}$. We first observe that

$$\chi_0^n(\xi_n)e^{-\xi_n^2 t} = \mathcal{F}_n[\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}]] = \int_{\mathbb{R}} e^{-i\xi_n z_n}\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)dz_n.$$

Set

$$I_0(x',x_n,t,\xi_n) = \mathcal{F}'^{-1}\left[\chi_0'(\xi')e^{-|\xi'|x_n}e^{-|\xi'|^2 t}(-|\xi'|)^k g_4(\xi',\xi_n)\hat{v}_0\right].$$

Then $\partial_{x_n}^k I_4$ is written as

$$\begin{aligned}
&\partial_{x_n}^k I_4(x) \\
=\ & \frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}}\chi_0^n(\xi_n)e^{-\xi_n^2 t}e^{i\xi_n y_n} \\
&\qquad\times\mathcal{F}'^{-1}[\chi_0'(\xi')e^{-|\xi'|x_n}e^{-|\xi'|^2 t}(-|\xi'|)^k g_4(\xi',\xi_n)\hat{v}_0]d\xi_n dy_n \\
=\ & \frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}}\chi_0^n(\xi_n)e^{-\xi_n^2 t}e^{i\xi_n y_n}I_0(x',x_n,t,\xi_n)d\xi_n dy_n \\
=\ & \frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}}\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)\left(\int_{\mathbb{R}}e^{i\xi_n(y_n-z_n)}I_0(x',x_n,t,\xi_n)d\xi_n\right)dz_n dy_n,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\|\partial_{x_n}^k I_4(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} \qquad (4.4)\\
&\leq C\int_0^\infty \int_{\mathbb{R}} |\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)| \\
&\qquad \times \left\| \int_{\mathbb{R}} e^{i\xi_n(y_n-z_n)} I_0(x',x_n,t,\xi_n)d\xi_n \right\|_{L^p(\mathbb{R}^{n-1})} dz_n dy_n.
\end{aligned}$$

Let us estimate $\| \int_{\mathbb{R}} e^{i\xi_n(y_n-z_n)} I_0(x',x_n,t,\xi_n)d\xi_n\|_{L^p(\mathbb{R}^{n-1})}$. By setting

$$\hat{G}(\xi',x_n) = 2\pi\mathcal{F}_n^{-1}[g_4(\xi',\cdot)](x_n),$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}} e^{i\xi_n(y_n-z_n)} I_0(x',x_n,t,\xi_n)d\xi_n \\
&= \mathcal{F}'^{-1}\left[\chi_0'(\xi')e^{-|\xi'|^2 t}(-|\xi'|)^k e^{-|\xi'|x_n}\hat{G}(\xi',y_n-z_n)\hat{v}_0\right] \\
&= \mathcal{F}'^{-1}\left[\chi_0'(\xi')e^{-|\xi'|^2 t}(-|\xi'|)^k |\xi'|^{\frac{1}{s}-\frac{1}{r}}\right] \\
&\qquad *' \mathcal{F}'^{-1}\left[|\xi'|^{-\frac{1}{s}+\frac{1}{r}} e^{-|\xi'|x_n}\hat{G}(\xi',y_n-z_n)\hat{v}_0\right].
\end{aligned}$$

Since

$$|\partial_{\xi'}^{\alpha'}(G(\xi',y_n-z_n))| \leq C_{\alpha'}|\xi'|e^{-|\xi'||y_n-z_n|}|\xi'|^{-|\alpha'|},$$

by choosing m such that $\frac{1}{m} = \frac{1}{r} - \frac{1}{s} + 1$, we have

$$\begin{aligned}
\left|\partial_{\xi'}^{\alpha'}(|\xi'|^{-\frac{1}{s}+\frac{1}{r}} e^{-|\xi'|x_n}\hat{G}(\xi',y_n-z_n))\right| &\leq C_{\alpha'}|\xi'|^{\frac{1}{m}} e^{-|\xi'|(x_n+|y_n-z_n|)}|\xi'|^{-|\alpha'|} \\
&\leq C_{\alpha'}\frac{1}{(x_n+|y_n-z_n|)^{\frac{1}{m}}}|\xi'|^{-|\alpha'|}.
\end{aligned}$$

Let l be a number satisfying $\frac{1}{l} = \frac{1}{p} - \frac{1}{q} + 1$. Applying Proposition 4.2 and the Fourier multiplier theorem, we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} e^{i\xi_n(y_n-z_n)} I_0(x',x_n,t,\xi_n)d\xi_n \right\|_{L^p(\mathbb{R}^{n-1})} \\
&\leq \Big\| \mathcal{F}'^{-1}\left[\chi_0'(\xi')e^{-|\xi'|^2 t}(-|\xi'|)^k |\xi'|^{\frac{1}{s}-\frac{1}{r}}\right] \\
&\qquad *' \mathcal{F}'^{-1}\left[|\xi'|^{-\frac{1}{s}+\frac{1}{r}} e^{-|\xi'|x_n}\hat{G}(\xi',y_n-z_n)\hat{v}_0\right]\Big\|_{L^p(\mathbb{R}^{n-1})} \\
&\leq \left\| \mathcal{F}'^{-1}\left[\chi_0'(\xi')e^{-|\xi'|^2 t}(-|\xi'|)^k |\xi'|^{\frac{1}{s}-\frac{1}{r}}\right]\right\|_{L^l(\mathbb{R}^{n-1})} \\
&\qquad \times \left\| \mathcal{F}'^{-1}\left[|\xi'|^{-\frac{1}{s}+\frac{1}{r}} e^{-|\xi'|x_n}\hat{G}(\xi',y_n-z_n)\hat{v}_0\right]\right\|_{L^q(\mathbb{R}^{n-1})} \\
&\leq C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \frac{1}{(x_n+|y_n-z_n|)^{\frac{1}{m}}}\|v_0\|_{L^q(\mathbb{R}^{n-1})}.
\end{aligned}$$

It then follows from (4.4) that

$$
\begin{aligned}
&\|\partial_{x_n}^k I_4(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})}\\
\le\ & C\int_0^\infty\int_{\mathbb{R}}|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)|\\
&\times\left\|\int_{\mathbb{R}}e^{i\xi_n(y_n-z_n)}I_0(x',x_n,t,\xi_n)d\xi_n\right\|_{L^p(\mathbb{R}^{n-1})}dz_n dy_n\\
\le\ & C(1+t)^{-\frac12(\frac1s-\frac1r)-\frac{n-1}{2}(\frac1q-\frac1p)-\frac k2}\\
&\times\int_{\mathbb{R}}|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)|\int_0^\infty\frac{1}{(x_n+|y_n-z_n|)^{\frac1m}}\|v_0\|_{L^q(\mathbb{R}^{n-1})}dy_n dz_n\\
\le\ & C(1+t)^{-\frac12(\frac1s-\frac1r)-\frac{n-1}{2}(\frac1q-\frac1p)-\frac k2}\int_{\mathbb{R}}|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)|\\
&\times\left(\int_0^\infty\|v_0\|^s_{L^q(\mathbb{R}^{n-1})}dy_n\right)^{\frac{1}{m'}}\left(\int_0^\infty\frac{1}{x_n+|y_n-z_n|}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}dy_n\right)^{\frac1m}dz_n\\
=\ & C(1+t)^{-\frac12(\frac1s-\frac1r)-\frac{n-1}{2}(\frac1q-\frac1p)-\frac k2}\|v_0\|^{\frac{s}{m'}}_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}\\
&\times\int_{\mathbb{R}}|\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)|\left(\int_0^\infty\frac{1}{x_n+|y_n-z_n|}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(y_n)dy_n\right)^{\frac1m}dz_n.
\end{aligned}
$$

Setting $f_{z_n}(w_n)=\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(z_n+w_n)$ and $g_{z_n}(w_n)=\|Ev_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(z_n-w_n)$ we obtain

$$
\begin{aligned}
&\int_0^\infty\frac{1}{x_n+|y_n-z_n|}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(y_n)dy_n\\
&=\underbrace{\int_{z_n}^\infty\frac{1}{x_n+|y_n-z_n|}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(y_n)dy_n}_{w_n=y_n-z_n}\\
&\quad+\underbrace{\int_0^{z_n}\frac{1}{x_n+|y_n-z_n|}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(y_n)dy_n}_{w_n=z_n-y_n}\\
&=\int_0^\infty\frac{1}{x_n+w_n}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(z_n+w_n)dw_n\\
&\quad+\int_0^{z_n}\frac{1}{x_n+w_n}\|v_0\|^{m-\frac{m}{m'}s}_{L^q(\mathbb{R}^{n-1})}(z_n-w_n)dw_n\\
&=\int_0^\infty\frac{1}{x_n+w_n}f_{z_n}(w_n)dw_n+\int_0^\infty\frac{1}{x_n+w_n}g_{z_n}(w_n)dw_n.
\end{aligned}
$$

Noting that $\frac{r}{m}(m - \frac{m}{m'}s) = s$ and $\frac{r}{m} \geq 1$, we have

$$
\begin{aligned}
&\left\| \left(\int_0^\infty \frac{1}{x_n + |y_n - z_n|} \|v_0\|_{L^q(\mathbb{R}^{n-1})}^{m - \frac{m}{m'}s} dy_n \right)^{\frac{1}{m}} \right\|_{L^r(\mathbb{R}_+)} \\
&= \left\| \int_0^\infty \frac{1}{x_n + |y_n - z_n|} \|v_0\|_{L^q(\mathbb{R}^{n-1})}^{m - \frac{m}{m'}s} dy_n \right\|_{L^{\frac{r}{m}}(\mathbb{R}_+)}^{\frac{1}{m}} \\
&\leq C(\|f_{z_n}\|_{L^{\frac{r}{m}}(\mathbb{R}_+)} + \|g_{z_n}\|_{L^{\frac{r}{m}}(\mathbb{R}_+)})^{\frac{1}{m}} \\
&\leq C\| \|v_0\|_{L^q(\mathbb{R}^{n-1})}^{m-\frac{m}{m'}s} \|_{L^{\frac{r}{m}}(\mathbb{R}_+)}^{\frac{1}{m}} \\
&\leq C\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{r}}.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&\|\partial_{x_n}^k I_4\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{m'}} \|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{r}} \\
&\times \int_{\mathbb{R}_{z_n}} |\mathcal{F}_n^{-1}[\chi_0^n(\xi_n)e^{-\xi_n^2 t}](z_n)| dz_n \\
\leq\ & C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}
$$

We can prove the desired estimate of J_4 and K_4 by a similar calculation as that for I_4. This completes the proof. □

We finally consider I_5, J_5 and K_5.

Proof of Lemma 4.4 for j = 5. We observe that

$$|g_5(\xi', \xi_n)| \leq C\frac{|\xi'|^2}{|\xi|^2} \in L^1_{\xi_n}(\mathbb{R}).$$

Set $\widetilde{G}(\xi', t) = \int_{\mathbb{R}} \chi_0^n(\xi_n) e^{-\xi_n^2 t} g_5(\xi', \xi_n) d\xi_n$. We write $\partial_{x_n}^k I_5$ as

$$
\begin{aligned}
&\partial_{x_n}^k I_5(x) \\
=\ & \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)(-|\xi'|)^k e^{-|\xi|^2 t} g_5(\xi', \xi_n) e^{-|\xi'|x_n} e^{-|\xi'|y_n} \hat{v}_0 d\xi_n dy_n\right] \\
=\ & \frac{1}{2\pi}\int_0^\infty \mathcal{F}'^{-1}\left[\chi_0'(\xi')(-|\xi'|)^k e^{-|\xi'|^2 t} e^{-|\xi'|(x_n+y_n)} \widetilde{G}(\xi', t)\hat{v}_0\right] dy_n \\
=\ & \frac{1}{2\pi}\int_0^\infty \mathcal{F}'^{-1}\left[\chi_0'(\xi')(-|\xi'|)^k |\xi'|^{\frac{1}{s}-\frac{1}{r}} e^{-|\xi'|^2 t}\right] \\
&\qquad *' \mathcal{F}'^{-1}\left[|\xi'|^{-\frac{1}{s}+\frac{1}{r}} e^{-|\xi'|(x_n+y_n)} \widetilde{G}(\xi', t)\hat{v}_0\right] dy_n.
\end{aligned}
$$

Choosing l such that $\frac{1}{l}=\frac{1}{p}-\frac{1}{q}+1$, we have

$$
\begin{aligned}
&\|\partial_{x_n}^k I_5(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})}\\
&\leq \frac{1}{2\pi}\int_0^\infty \left\|\mathcal{F}'^{-1}[\chi_0'(\xi')(-|\xi'|)^k|\xi'|^{\frac{1}{s}-\frac{1}{r}}e^{-|\xi'|^2 t}]\right\|_{L^r(\mathbb{R}^{n-1})}\\
&\qquad\times\left\|\mathcal{F}'^{-1}[|\xi'|^{-\frac{1}{s}+\frac{1}{r}}e^{-|\xi'|(x_n+y_n)}\widetilde{G}(\xi',t)\hat{v}_0]\right\|_{L^q(\mathbb{R}^{n-1})}dy_n\\
&\leq C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\\
&\qquad\times\int_0^\infty\left\|\mathcal{F}'^{-1}[|\xi'|^{-\frac{1}{q}+\frac{1}{p}}e^{-|\xi'|(x_n+y_n)}\widetilde{G}(\xi',t)\hat{v}_0]\right\|_{L^q(\mathbb{R}^{n-1})}dy_n.
\end{aligned}
$$

We choose m satisfying $\frac{1}{m}=\frac{1}{p}-\frac{1}{q}+1$. Since

$$
|\partial_{\xi'}^{\alpha'}(\widetilde{G}(\xi',t))|=\left|\int_{\mathbb{R}}\chi_0^n(\xi_n)e^{-\xi_n^2 t}\partial_{\xi'}^{\alpha'}(g_5(\xi',\xi_n))d\xi_n\right|\leq C|\xi'||\xi'|^{-|\alpha'|},
$$

we have

$$
\begin{aligned}
\left|\partial_{\xi'}^{\alpha'}(|\xi'|^{-\frac{1}{s}+\frac{1}{r}}e^{-|\xi'|(x_n+y_n)}\widetilde{G}(\xi',t))\right| &\leq C|\xi'|^{1-\frac{1}{s}+\frac{1}{r}}e^{-|\xi'|(x_n+y_n)}|\xi'|^{-|\alpha'|}\\
&\leq C\frac{1}{(x_n+y_n)^{\frac{1}{m}}}|\xi'|^{-|\alpha'|}.
\end{aligned}
$$

Hence, by applying the Fourier multiplier theorem, we see that

$$
\begin{aligned}
&\int_0^\infty\left\|\mathcal{F}'^{-1}[|\xi'|^{-\frac{1}{q}+\frac{1}{p}}e^{-|\xi'|(x_n+y_n)}\widetilde{G}(\xi',t)\hat{v}_0]\right\|_{L^q(\mathbb{R}^{n-1})}dy_n\\
&\leq C\int_0^\infty\frac{1}{(x_n+y_n)^{\frac{1}{m}}}\|v_0\|_{L^q(\mathbb{R}^{n-1})}(y_n)dy_n\\
&\leq C\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{m'}}\left(\int_0^\infty\frac{1}{x_n+y_n}\|v_0\|_{L^q(\mathbb{R}^{n-1})}^{m-\frac{m}{m'}s}dy_n\right)^{\frac{1}{m}}.
\end{aligned}
$$

An application of Proposition 2.4 then yields

$$
\begin{aligned}
&\|\partial_{x_n}^k I_5\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\\
&\leq C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{m'}}\\
&\qquad\times\left\|\left(\int_0^\infty\frac{1}{x_n+y_n}\|v_0\|_{L^q(\mathbb{R}^{n-1})}^{m-\frac{m}{m'}s}dy_n\right)^{\frac{1}{m}}\right\|_{L^r(\mathbb{R}_+)}\\
&\leq C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{m'}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}^{\frac{s}{r}}\\
&= C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.
\end{aligned}
$$

For $J_5(x)$ and $K_5(x)$ we can show the desire estimate by a similar calculation as that for I_5. This completes the proof. $\square$

By Lemma 4.4 we have

$$\|\partial_{x_n}^k S_1(t)v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},$$

$$\|\partial_{x_n}^k S_2(t)v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},$$

$$\|\partial_{x_n}^k S_3(t)v_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le Ct^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{-ct}\|v_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.$$

Similarly, we can show the desire estimates for $\partial_{x'}^k S_j(t)v_0$ $(j = 1,2,3)$. Theorem 4.1 (i)-(iii) are thus proved for $1 < s \le q \le p \le r < \infty$ or $1 < s \le q < p \le r \le \infty$.

Proof of Theorem 4.1 (i)-(iii) for s = 1 and (iv)-(vi). To prove Theorem 4.1 (i)-(iii) for $s = 1$ and (iv)-(vi), we employ the following lemma.

Lemma 4.5. *Let $1 < p \le \infty$ and let $u_0 \in L^p(\mathbb{R}^n_+)$ and $v_0 \in L^{p'}(\mathbb{R}^n_+)$. Then*

$$(S_i(t)u_0, v_0) = (u_0, S_i(t)v_0)$$

for $i = 1,2,3$.

Proof of Lemma 4.5. We denote by $G^*(\xi', \xi_n, x_n, y_n)$ the adjoint matrix of $G(\xi', \xi_n, y_n, x_n)$. Then the desired relation follows from the fact $G^*(\xi', -\xi_n, x_n, y_n) = G(\xi', \xi_n, y_n, x_n)$. In fact, we have

$$\begin{aligned}
&(S_1(t)u_0, v_0)\\
=\ & \left(\mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}G(\xi',\xi_n,x_n,y_n)\hat{u}_0(\xi',y_n)d\xi_n dy_n\right], v_0\right)\\
=\ & \left(\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}G(\xi',\xi_n,x_n,y_n)\hat{u}_0(\xi',y_n)d\xi_n dy_n, \hat{v}_0\right)\\
=\ & \int_0^\infty\int_{\mathbb{R}^{n-1}_{\xi'}}\left[\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}G(\xi',\xi_n,x_n,y_n)\hat{u}_0(\xi',y_n)d\xi_n dy_n\right]\\
&\qquad\qquad \times\overline{\hat{v}_0}(\xi',x_n)d\xi' dx_n\\
=\ & \int_0^\infty\int_{\mathbb{R}^{n-1}_{\xi'}}\hat{u}_0(\xi',y_n)\\
&\times\left[\overline{\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}\ G^*(\xi',\xi_n,x_n,y_n)\hat{v}_0(\xi',x_n)d\xi_n dx_n}\right]d\xi' dy_n\\
=\ & \left(\hat{u}_0, \frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}\ G^*(\xi',\xi_n,x_n,y_n)\hat{v}_0(\xi',x_n)d\xi_n dx_n\right)\\
=\ & \left(u_0, \mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} \chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}\ G^*(\xi',\xi_n,x_n,y_n)\hat{v}_0(\xi',x_n)d\xi_n dx_n\right]\right).
\end{aligned}$$

Changing the variable ξ_n to $-\xi_n$, we have

$$\begin{aligned}
&\underbrace{\mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}}\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}\,G^*(\xi',\xi_n,x_n,y_n)\hat{v}_0(\xi',x_n)d\xi_n dx_n\right]}_{\xi_n\to-\xi_n}\\
=\;&\mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}}\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}\,G^*(\xi',-\xi_n,x_n,y_n)\hat{v}_0(\xi',x_n)d\xi_n dx_n\right]\\
=\;&\mathcal{F}'^{-1}\left[\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}}\chi_0'(\xi')\chi_0^n(\xi_n)e^{-|\xi|^2t}G(\xi',\xi_n,y_n,x_n)\hat{v}_0(\xi',x_n)d\xi_n dx_n\right]\\
=\;&S_1(t)v_0.
\end{aligned}$$

We thus obtain

$$(S_1(t)u_0,v_0)=(u_0,S_1(t)v_0).$$

The case $j=2,3$ can be shown in the same way. The proof of Lemma 4.5 is complete. □

Let us show Theorem 4.1 (i)-(iii) for $s=1$. Let $\varphi\in C_0^\infty(\mathbb{R}^n_+)$ and $k\in\mathbb{N}$. Then it is easy to see that

$$\|S_1(t)\partial_x^k\varphi\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|\varphi\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}$$

for $1<s\le q\le p\le r<\infty$ or $1<s\le q<p\le r\le\infty$ since $\partial_{x_n}^k\varphi|_{x_n=0}=0$. By Lemma 4.5 we have

$$\begin{aligned}
&|(\partial_x^k S_1(t)u_0,\varphi)|\\
&\quad=\;|(u_0,S_1(t)\partial_x^k\varphi)|\\
&\quad\le\;\|u_0\|_{L^1(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}\|S_1(t)\partial_x^k\varphi\|_{L^\infty(\mathbb{R}_+;L^{q'}(\mathbb{R}^{n-1}))}\\
&\quad\le\;C(1+t)^{-\frac{1}{2}(1-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|u_0\|_{L^1(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}\|\varphi\|_{L^{r'}(\mathbb{R}_+;L^{p'}(\mathbb{R}^n_+))}.
\end{aligned}$$

We thus obtain

$$\|\partial_x^k S_1(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^n_+))}\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|u_0\|_{L^1(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}.$$

The estimates for $S_2(t),S_3(t)$ can be shown by a similar method. As for Theorem 4.1 (iv)-(vi), we observe that $S_j(t)\varphi|_{x_n=0}=$ for $j=1,2,3$, and hence, $(S_j(t)\partial_x v_0,\varphi)=-(v_0,\partial_x S_j(t)\varphi)$. The desired results are thus obtained by the duality argument as above. This completes the proof. □

Remark 4.6. *In a similar manner to above, one can obtain the estimate for the time derivative:*

$$\|\partial_t S_1(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\le C(1+t)^{-\frac{1}{2}(\frac{1}{s}-\frac{1}{r})-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-1}\|u_0\|_{L^s(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))},$$

which will be used later.

5 Estimates for evolution operator $U(t,s)$

In this section we give a proof of Theorem 3.1. In addition to the anisotropic L^p-L^q estimates for the Stokes semigroup (Theorem 4.1), the following estimates for $B(t)u$ plays a key role to establish the L^p-L^q estimates for $U(t,s)$.

Lemma 5.1. *Let p, q and r satisfy $1 < p \leq q \leq r \leq \infty$ or $1 \leq p \leq q < r < \infty$ and let $k = 1, 2$. Then*

$$\|B(t)u\|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq C\frac{1}{\sqrt{\nu}}\|\partial_x^k u\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}.$$

Here C is a positive constant depending only on n, p, q and r.

Proof. We here give a proof for $k = 2$, since the case $k = 1$ can be proved similarly. Since $\operatorname{div} u = 0$ and $u|_{x_n=0} = 0$, we have $\partial_{x_n}u^n|_{x_n=0} = 0$. Therefore, $u(x)$ and $u^n(x)$ are written as

$$\begin{aligned} u(x', x_n) &= \int_0^{x_n} \partial_{y_n}u(x', y_n)dy_n, \\ u^n(x', x_n) &= \int_0^{x_n}\int_0^{y_n} \partial_{z_n}^2 u^n(x', z_n)dz_n dy_n = \int_0^{x_n}(x_n - z_n)\partial_{z_n}^2 u^n(x', z_n)dz_n. \end{aligned}$$

Let m be a number satisfying $\frac{1}{q} = \frac{1}{m} + \frac{1}{r}$. It then follows from Proposition 2.3 and Proposition 3.4 that

$$\begin{aligned} &\left\|\frac{1}{\sqrt{\nu}}v_{per}^1\partial_{x_1}u\right\|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\ &= \left\|\frac{1}{\sqrt{\nu}}v_{per}^1\int_0^{x_n}\partial_{x_1}\partial_{y_n}u_1(x', y_n)dy_n\right\|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\ &\leq \left\|\frac{1}{\sqrt{\nu}}v_{per}^1\int_0^{x_n}\|\partial_{x_1}\partial_{y_n}u(x', y_n)\|_{L^p(\mathbb{R}^{n-1})}\,dy_n\right\|_{L^q(\mathbb{R}_+)} \\ &\leq \frac{1}{\sqrt{\nu}}\|x_n v_{per}^1\|_{L^m(\mathbb{R}_+)}\left\|\frac{1}{x_n}\int_0^{x_n}\|\partial_{x_1}\partial_{y_n}u_1(x', y_n)\|_{L^p(\mathbb{R}^{n-1})}dy_n\right\|_{L^r(\mathbb{R}_+)} \\ &\leq C\frac{1}{\sqrt{\nu}}\|\partial_x^2 u\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}, \end{aligned}$$

$$
\begin{aligned}
&\left\| \frac{1}{\sqrt{\nu}} \partial_{x_n} v^1_{per} u^n e_1 \right\|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
&= \left\| \frac{1}{\sqrt{\nu}} \partial_{x_n} v^1_{per} \int_0^{x_n} (x_n - z_n) \partial^2_{z_n} u^n(x', z_n) dz_n e_1 \right\|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
&\leq \left\| \frac{1}{\sqrt{\nu}} \partial_{x_n} v^1_{per} \int_0^{x_n} (x_n - z_n) \| \partial^2_{z_n} u^n(x', z_n) \|_{L^p(\mathbb{R}^{n-1})} dz_n e_1 \right\|_{L^q(\mathbb{R}_+)} \\
&\leq \left\| \frac{1}{\sqrt{\nu}} x_n^2 \partial_{x_n} v^1_{per} \frac{1}{x_n} \int_0^{x_n} \| \partial^2_{z_n} u^n(x', z_n) \|_{L^p(\mathbb{R}^{n-1})} dz_n \right\|_{L^q(\mathbb{R}_+)} \\
&\leq \frac{1}{\sqrt{\nu}} \| x_n^2 \partial_{x_n} v^1_{per} \|_{L^m(\mathbb{R}^n_+)} \left\| \frac{1}{x_n} \int_0^{x_n} \| \partial^2_{z_n} u^n(x', z_n) \|_{L^p(\mathbb{R}^{n-1})} dz_n \right\|_{L^r(\mathbb{R}_+)} \\
&\leq C \frac{1}{\sqrt{\nu}} \| \partial^2_{z_n} u^n \|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}.
\end{aligned}
$$

We thus obtain

$$
\| B(t) u \|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq C \frac{1}{\sqrt{\nu}} \| \partial^2_x u \|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}.
$$

This completes the proof. □

To obtain L^p-L^q type estimates for $U(t,s)$, we decompose the evolution operator $U(t,s)$ into low frequency part and high frequency part:

$$
U(t,s) = U_0(t,s) + U_\infty(t,s).
$$

Here $U_j(t,s) = \mathcal{F}'^{-1} \hat{\chi}'_j(\xi') \mathcal{F}' U(t,s)$, $j = 0, \infty$, with $\hat{\chi}'_j(\xi')$ being the cut-off functions given in section 4. Theorem 3.1 is a consequence of the following estimates for $U_0(t,s)$ and $U_\infty(t,s)$.

Theorem 5.2. *Let $1 < q \leq p < \infty$ or $1 \leq q < p \leq \infty$. Then for $0 < \varepsilon < \frac{1}{2}$ there exists $\delta_0 = \delta_0(q, \varepsilon) > 0$ such that if $\frac{1}{\sqrt{\nu}} \leq \delta_0$, the following estimates hold with $k = 0, 1$.*

(i) $\| \partial^k_x U_0(t,s) u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} (1+t-s)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$,

(ii) $\| U_0(t,s) \partial_x u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} (1+t-s)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$,

(iii) $\| \partial^k_x U_\infty(t,s) u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} (1+t-s)^{-\frac{1}{2}+\varepsilon} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$,

(iv) $\| U_\infty(t,s) \partial_x u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} (1+t-s)^{-\frac{1}{2}+\varepsilon} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$,

(v) $\| \partial^k_x U(t,s) u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$,

(vi) $\| U(t,s) \partial_x u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$.

(vii) $\| \partial^k_x U(t,s) u_0 - \partial^k_x S(t-s) u_0 \|_{L^p(\mathbb{R}^n_+)} \leq C (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon} \| u_0 \|_{L^q(\mathbb{R}^n_+)}$.

Here C is a positive constant depending only on n, p, q and ε.

In the remaining of this section we give a proof of Theorem 5.2. We will first prepare some auxiliary estimates for $U(t,s)$ (Lemmas 5.3, 5.4), and then prove the estimates in Theorem 5.2 for a restricted range of p and q (Lemma 5.5). We will finally extend the range of p and q to complete the proof.

Proof of Theorem 5.2. To prove Theorem 5.2, we write the solution of (3.4) as

$$U(t,s)u_0 = S(t-s)u_0 - \int_s^t S(t-\tau)B(\tau)U(\tau,s)u_0 d\tau,$$

and define $U_1(t,s)$, $U_2(t,s)$, $U_3(t,s)$ by

$$U_j(t,s)u_0 = S_j(t-s)u_0 - \int_s^t S_j(t-\tau)B(\tau)U(\tau,s)u_0 d\tau \quad (j=1,2),$$

and

$$U_3(t,s)u_0 = U_\infty(t,s).$$

It then follows that

$$U(t,s) = U_1(t,s) + U_2(t,s) + U_3(t,s),$$

$$U_0(t,s) = U_1(t,s) + U_2(t,s),$$

and

$$U_j(t,s)u_0 = S_j(t-s)u_0 - \int_s^t S_j(t-\tau)B(\tau)(U_1(\tau,s)+U_2(\tau,s)+U_3(\tau,s))u_0 d\tau$$

for $j=1,2,3$.

We begin with the following estimates for $U_j(t,s)$ $(j=1,2,3)$.

Lemma 5.3. *Let p and r satisfy $1<p<r\le\infty$ or $1\le p<r<\infty$. Then there exists a constant $\delta_1=\delta_1(p,r)>0$ such that if $\frac{1}{\sqrt{\nu}}\le\delta_1$, the following estimates hold.*

$$\|\partial_x^k U_1(t,s)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le C(1+t-s)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\|u_0\|_{L^p(\mathbb{R}^n_+)},$$

for $k=0,1,2$, and

$$\|\partial_x^k U_j(t,s)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le C(1+(t-s)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}})(1+t-s)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1}\|u_0\|_{L^p(\mathbb{R}^n_+)},$$

for $j=2,3$ and $k=0,1$. Here C is a positive constant depending only on n, p and r.

Similarly, we have the following

Lemma 5.4. *Let p and r satisfy $1 < p < r \le \infty$ or $1 \le p < r < \infty$. Then there exists a constant $\delta_1 = \delta_1(p, r) > 0$ such that if $\frac{1}{\sqrt{\nu}} \le \delta_1$, the following estimates hold.*

$$\|U_1(t,s)\partial_x u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le C(1+t-s)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}}\|u_0\|_{L^p(\mathbb{R}^n_+)},$$

$$\|U_j(t,s)\partial_x u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le C(1+(t-s)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})(1+t-s)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1}\|u_0\|_{L^p(\mathbb{R}^n_+)},$$

for $j = 2, 3$. Here C is a positive constant depending only on n, p and r.

We first prove Lemma 5.3.

Proof of Lemma 5.3. It suffices to prove Lemma 5.3 for the case $s = 0$. We set $u_j(t) = U_j(t, 0)u_0$, $j = 1, 2, 3$. We introduce the quantity $M_1(t)$ defined by

$$\begin{aligned} M_1(t) &= \sup_{0<\tau\le t}\Big\{\sum_{k=0}^{2}(1+\tau)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+\frac{k}{2}}\|\partial_x^k u_1(\tau)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\ &+ \sum_{j=2}^{3}\sum_{k=0}^{1}(1+\tau^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}})^{-1}(1+\tau)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+1}\|\partial_x^k u_j(\tau)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}\Big\}. \end{aligned}$$

Let us prove

$$M_1(t) \le C\|u_0\|_{L^p(\mathbb{R}^n_+)} + C\delta M_1(t), \tag{5.1}$$

where $\delta = \frac{1}{\sqrt{\nu}}$ and C is a positive constant depending only on n, p, and r.

We write u_j $(j = 1, 2, 3)$ as

$$u_j(t) = S_j(t)u_0 - I_j(t),\ j = 1, 2, 3,$$

where

$$I_j(t) = \int_0^t S_j(t-\tau)B(\tau)(u_1+u_2+u_3)(\tau)d\tau.$$

By Theorem 4.1, we have

$$\|\partial_x^k S_1(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}\|u_0\|_{L^p(\mathbb{R}^n_+)} \tag{5.2}$$

for $k = 0, 1, 2$ and

$$\|\partial_x^k S_j(t)u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \le t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}}e^{-ct}\|u_0\|_{L^p(\mathbb{R}^n_+)} \tag{5.3}$$

for $j = 2, 3$, $k = 0, 1$. As for $I_1(t)$, by Theorem 4.1, Lemma 5.1 and Proposition 2.2, we have

$$\begin{aligned}
&\|\partial_x^k I_1(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & \int_0^t \|\partial_x^k S_1(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} d\tau \\
\leq\ & C\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} \|B(\tau)(u_1+u_2+u_3)\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} \\
&\qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} (1+\tau^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1} d\tau M_1(t) \\
\leq\ & C\delta(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} M_1(t).
\end{aligned}$$

We thus obtain

$$(1+t)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+\frac{k}{2}} \|\partial_x^k u_1(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^n_+))} \leq C\|u_0\|_{L^p(\mathbb{R}^n_+)} + C\delta M_1(t). \tag{5.4}$$

Similarly, by Theorem 4.1, Lemma 5.1 and Proposition 2.2, we have for $j = 2, 3$,

$$\begin{aligned}
&\|\partial_x^k I_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & \int_0^t \|\partial_x^k S_j(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} d\tau \\
\leq\ & C\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} e^{-c(t-\tau)} \|B(\tau)(u_1+u_2+u_3)\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} e^{-c(t-\tau)} \\
&\qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}} e^{-c(t-\tau)} (1+\tau^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1} d\tau M_1(t) \\
\leq\ & C\delta(1+t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1} M_1(t).
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
&(1+t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{k}{2}})^{-1}(1+t)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+1} \|\partial_x^k u_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & C\|u_0\|_{L^p(\mathbb{R}^n_+)} + C\delta M_1(t)
\end{aligned} \tag{5.5}$$

for $j = 2, 3$. Therefore, from (5.2)-(5.5) we obtain (5.1). It then follows from (5.1) that there exists $\delta_1 > 0$ such that if $\delta \leq \delta_1$, then

$$M_1(t) \leq C\|u_0\|_{L^p(\mathbb{R}^n_+)}.$$

This proves Lemma 5.3. □

We next prove Lemma 5.4.

Proof of Lemma 5.4. It suffices to prove Lemma 5.4 for the case $s = 0$. We set $v_j(t) = U_j(t, 0)\partial_{x_n} u_0,\ j = 1, 2, 3$, and introduce the quantity $M_2(t)$ defined by

$$\begin{aligned} M_2(t) &= \sup_{0 \leq \tau \leq t} \Big\{ \sum_{k=0}^{1} (1+\tau)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+\frac{1}{2}+\frac{k}{2}} \|\partial_x^k v_1(\tau)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\ &+ \sum_{j=2}^{3} (1+\tau^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})^{-1} (1+\tau)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+1} \|v_j(\tau)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \Big\}. \end{aligned}$$

As in the proof of Lemma 5.3 we only need to show

$$M_2(t) \leq C\|u_0\|_{L^p(\mathbb{R}^n_+)} + C\delta M_2(t). \tag{5.6}$$

To prove (5.6), we first observe that the following estimates hold uniformly for $t \geq 0$ and $\delta > 0$:

$$\begin{aligned} &\|v_{per} \otimes (v_1 + v_2 + v_3)\|_{L^p(\mathbb{R}^n_+)} + \|(v_1 + v_2 + v_3) \otimes v_{per}\|_{L^p(\mathbb{R}^n_+)} \\ &\leq C\delta(\|\partial_x v_1\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} + \|v_2\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} + \|v_3\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}). \end{aligned} \tag{5.7}$$

These estimates can be shown as in the proof of Lemma 5.1.

We write v_j ($j = 1, 2, 3$) as

$$v_j(t) = S_j(t)\partial_{x_n} u_0 - J_j(t),\ j = 1, 2, 3,$$

where

$$J_j(t) = \int_0^t S_j(t-\tau)B(\tau)(v_1 + v_2 + v_3)(\tau)d\tau.$$

By Theorem 4.1, we have

$$\|\partial_x^k S_1(t)\partial_{x_n} u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{L^p(\mathbb{R}^n_+)} \tag{5.8}$$

for $k = 0, 1$, and

$$\|S_j(t)\partial_{x_n} u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \leq Ct^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} e^{-ct} \|u_0\|_{L^p(\mathbb{R}^n_+)} \tag{5.9}$$

for $j = 2, 3$.

As for $J_1(t)$, by Theorem 4.1, (5.7) and Proposition 2.2, we have

$$
\begin{aligned}
& \|\partial_x^k J_1(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & \int_0^t \|\partial_x^k S_1(t-\tau)(\mathrm{div}(v_{per} \otimes \sum_{j=1}^3 v_j) + \mathrm{div}(\sum_{j=1}^3 v_j \otimes v_{per}))\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} d\tau \\
\leq\ & C\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}-\frac{k}{2}} \|v_{per} \otimes \sum_{j=1}^3 v_j + \sum_{j=1}^3 v_j \otimes v_{per}\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta \int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}-\frac{k}{2}} \\
& \qquad \times (\|\partial_x v_1\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|v_j\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta \int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}-\frac{k}{2}} (1+\tau^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1} d\tau M_2(t) \\
\leq\ & C\delta(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}-\frac{k}{2}} M_2(t).
\end{aligned}
$$

We thus obtain

$$
(1+t)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+\frac{1}{2}+\frac{k}{2}} \|\partial_x^k v_1(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^n_+))} \leq C\|u_0\|_{L^p(\mathbb{R}^n_+)} + C\delta M_2(t). \tag{5.10}
$$

Similarly, by Theorem 4.1, (5.7) and Proposition 2.2, we have, for $j = 2, 3$,

$$
\begin{aligned}
& \|J_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
\leq\ & \int_0^t \|S_2(t-\tau)(\mathrm{div}(v_{per} \otimes \sum_{j=1}^3 v_j) + \mathrm{div}(\sum_{j=1}^3 v_j \otimes v_{per}))\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} d\tau \\
\leq\ & C\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} e^{-c(t-\tau)} \|v_{per} \otimes \sum_{j=1}^3 v_j + \sum_{j=1}^3 v_j \otimes v_{per}\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta \int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} e^{-c(t-\tau)} \\
& \qquad \times (\|\partial_x v_1\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|v_j\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta \int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} e^{-c(t-\tau)} (1+\tau^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1} d\tau M_2(t) \\
\leq\ & C\delta(1+t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-1} M_2(t).
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&(1+t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}})^{-1}(1+t)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})+1}\|v_j(t)\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))} \\
&\leq C\|u_0\|_{L^p(\mathbb{R}^n_+)} + C\delta M_2(t)
\end{aligned}
\tag{5.11}
$$

for $j=2,3$. Therefore from (5.8)-(5.11) we have (5.6). The desired result of Lemma 5.4 now follows from (5.6) as in the proof of Lemma 5.3. This completes the proof. □

By using Theorem 4.1, Lemmas 5.1 and 5.3, we first show the estimates in Theorem 5.2 for a restricted range of p and q.

Lemma 5.5. *Let $0<\varepsilon<\frac{1}{2}$. Suppose that $1\leq q<\infty$ and $q\leq p\leq \frac{2n-1}{2n-2}q$ or $\frac{n}{2\varepsilon}\leq q<\infty$ and $p=\infty$. If $\frac{1}{\sqrt{\nu}}\leq \delta_0(q)$, then the estimates in Theorem 5.2 hold true. Here $\delta_0(q):=\delta_1(q,\frac{2n-1}{n-1}q)$ is the constant given in Lemma 5.3 and Lemma 5.4.*

Remark 5.6. *Let r be the number satisfying $\frac{1}{r}=\frac{n-1}{(2n-1)q}$. Then the condition $1\leq q<\infty$ and $q\leq p\leq\frac{2n-1}{2n-2}q$ implies that $\frac{n}{2}(\frac{1}{q}-\frac{1}{p})\leq\frac{1}{2}(\frac{1}{q}-\frac{1}{r})<\frac{1}{2}$. On the other hand, the condition $\frac{n}{2\varepsilon}\leq q<\infty$ and $p=\infty$ implies that $\frac{n}{2q}\leq\frac{1}{2}(\frac{1}{q}-\frac{1}{r})+\varepsilon$.*

Proof of Lemma 5.5. We first give a proof of estimates (i) and (iii). It suffices to prove for the case $s=0$. We set $u_j(t)=U_j(t,0)u_0$ ($j=1,2,3$), and write u_j ($j=1,2,3$) as

$$
u_j(t)=S_j(t)u_0-I_j(t),\ j=1,2,3,
$$

where

$$
I_j(t)=\int_0^t S_j(t-\tau)B(\tau)(u_1+u_2+u_3)(\tau)d\tau.
$$

By Theorem 4.1, we have

$$
\|\partial_x^k S_1(t)u_0\|_{L^p(\mathbb{R}^n_+)}\leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}\|u_0\|_{L^q(\mathbb{R}^n_+)}
$$

for $k=0,1,2$ and

$$
\|\partial_x^k S_2(t)u_0\|_{L^p(\mathbb{R}^n_+)}\leq t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}
$$

and

$$
\|\partial_x^k S_3(t)u_0\|_{L^p(\mathbb{R}^n_+)}\leq t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t)^{-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}
$$

for $k=0,1$.

We first assume $1 \le q < \infty$ and $q \le p \le \frac{2n-1}{2n-2}q$. It then follows from Theorem 4.1, Lemmas 5.1, 5.3 and Proposition 2.2 that

$$
\begin{aligned}
&\|\partial_x^k I_1(t)\|_{L^p(\mathbb{R}^n_+)} \\
\le\ & \int_0^t \|\partial_x^k S_1(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\le\ & C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\le\ & C\delta\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \\
& \qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}) d\tau \\
\le\ & C\delta\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} (1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\le\ & C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \|u_0\|_{L^q(\mathbb{R}^n_+)}
\end{aligned}
$$

for $k = 0, 1, 2$. Similarly, we have

$$
\begin{aligned}
&\|\partial_x^k I_2(t)\|_{L^p(\mathbb{R}^n_+)} \\
\le\ & \int_0^t \|\partial_x^k S_2(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\le\ & C\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{-c(t-\tau)} \|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\le\ & C\delta\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{-c(t-\tau)} \\
& \qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}) d\tau \\
\le\ & C\delta\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{-c(t-\tau)} (1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\le\ & C(1+t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}})(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1} \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\le\ & Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} (1+t)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} \|u_0\|_{L^q(\mathbb{R}^n_+)} \qquad (5.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\partial_x^k I_3(t)\|_{L^p(\mathbb{R}^n_+)} \\
\leq\ & \int_0^t \|\partial_x^k S_3(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^p(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{-c(t-\tau)} \|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta\int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{-c(t-\tau)} \\
&\qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta\int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{-c(t-\tau)} (1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\leq\ & C(1+t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}})(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1} \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\leq\ & Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t)^{-\frac{1}{2}} \|u_0\|_{L^q(\mathbb{R}^n_+)} \qquad (5.13)
\end{aligned}
$$

for $k=0,1$. Hence, estimates (i) and (iii) hold for $1 \leq q < \infty$ and $q \leq p \leq \frac{2n-1}{2n-2}q$.

We next assume $\frac{n}{2\varepsilon} \leq q < \infty$ and $p=\infty$. Since $\frac{n}{2q}+\frac{k}{2} \leq \frac{1}{2}(\frac{1}{q}-\frac{1}{r})+1$, we see from Theorem 4.1, Lemmas 5.1, 5.3 and Proposition 2.2 that

$$
\begin{aligned}
&\|\partial_x^k I_1(t)\|_{L^\infty(\mathbb{R}^n_+)} \\
\leq\ & \int_0^t \|\partial_x^k S_1(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^\infty(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\int_0^t (1+t-\tau)^{-\frac{n}{2q}-\frac{k}{2}} \|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta\int_0^t (1+t-\tau)^{-\frac{n}{2q}-\frac{k}{2}} \\
&\qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta\int_0^t (1+t-\tau)^{-\frac{n}{2q}-\frac{k}{2}} (1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\leq\ & C(1+t)^{-\frac{n}{2q}-\frac{k}{2}} \|u_0\|_{L^q(\mathbb{R}^n_+)}
\end{aligned}
$$

for $k=0,1$. Since $\frac{n}{2q}+1-\varepsilon \le \frac{1}{2}(\frac{1}{q}-\frac{1}{r})+1 \le \frac{n}{2q}+1$, we similarly have

$$\begin{aligned}
&\|\partial_x^2 I_1(t)\|_{L^\infty(\mathbb{R}^n_+)} \\
\le\ & \int_0^t \|\partial_x^2 S_1(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^\infty(\mathbb{R}^n_+)} d\tau \\
\le\ & C\int_0^t (1+t-\tau)^{-\frac{n}{2q}-1}\|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\le\ & C\delta\int_0^t (1+t-\tau)^{-\frac{n}{2q}-1} \\
&\quad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}+\sum_{j=2}^3\|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))})d\tau \\
\le\ & C\delta\int_0^t (1+t-\tau)^{-\frac{n}{2q}-1}(1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau\|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\le\ & C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\le\ & C(1+t)^{-\frac{n}{2q}-1+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}$$

We thus obtain

$$\|\partial_x^2 u_1(t)\|_{L^p(\mathbb{R}^n_+)} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)} \tag{5.14}$$

for $1 \le q < \infty$ and $q \le p \le \frac{2n-1}{2n-2}q$ or $\frac{n}{2\varepsilon} \le q < \infty$ and $p=\infty$. Since $\frac{n}{2q}+1-\varepsilon \le \frac{1}{2}(\frac{1}{q}-\frac{1}{r})+1$, we have

$$\begin{aligned}
&\|\partial_x^k I_2(t)\|_{L^\infty(\mathbb{R}^n_+)} \\
\le\ & \int_0^t \|\partial_x^k S_2(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^\infty(\mathbb{R}^n_+)} d\tau \\
\le\ & C\int_0^t (t-\tau)^{-\frac{1}{2q}-\frac{k}{2}}e^{-c(t-\tau)}\|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\le\ & C\delta\int_0^t (t-\tau)^{-\frac{1}{2q}-\frac{k}{2}}e^{-c(t-\tau)} \\
&\quad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}+\sum_{j=2}^3\|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))})d\tau \\
\le\ & C\delta\int_0^t (t-\tau)^{-\frac{1}{2q}-\frac{k}{2}}e^{-c(t-\tau)}(1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau\|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\le\ & \begin{cases} C(1+t^{-\frac{1}{2q}})(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1},\ k=0, \\ C(1+t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1},\ k=1 \end{cases} \\
\le\ & Ct^{-\frac{1}{2q}-\frac{k}{2}}(1+t)^{-\frac{n-1}{2q}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}
\end{aligned} \tag{5.15}$$

and

$$
\begin{aligned}
& \|\partial_x^k I_3(t)\|_{L^\infty(\mathbb{R}^n_+)} \\
\leq\ & \int_0^t \|\partial_x^k S_3(t-\tau)B(\tau)(u_1+u_2+u_3)\|_{L^\infty(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\int_0^t (t-\tau)^{-\frac{n}{2q}-\frac{k}{2}} e^{-c(t-\tau)} \|B(\tau)(u_1+u_2+u_3)\|_{L^q(\mathbb{R}^n_+)} d\tau \\
\leq\ & C\delta\int_0^t (t-\tau)^{-\frac{n}{2q}-\frac{k}{2}} e^{-c(t-\tau)} \\
& \qquad \times(\|\partial_x^2 u_1\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} + \sum_{j=2}^3 \|\partial_x u_j\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}) d\tau \\
\leq\ & C\delta\int_0^t (t-\tau)^{-\frac{n}{2q}-\frac{k}{2}} e^{-c(t-\tau)} (1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} d\tau \|u_0\|_{L^q(\mathbb{R}^n_+)} \\
\leq\ & C(1+t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} \\
\leq\ & Ct^{-\frac{n}{2q}-\frac{k}{2}}(1+t)^{-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}
\end{aligned}
\tag{5.16}
$$

for $k = 0, 1$. We thus obtain estimates (i) and (iii) for $\frac{n}{2\varepsilon} \leq q < \infty$ and $p = \infty$, and the proof of (i) and (iii) is complete.

Estimates (ii) and (iv) can be proved similarly. Estimates (v) and (vi) are simple consequences of (i)-(iv).

Let us next prove (vii). We first estimate $\partial_t u_1(t)$. Since

$$
\partial_t u_1(t) = \partial_t S_1(t)u_0 - B(t)(\sum_{j=1}^3 u_j(t)) - \int_0^t \partial_t S_1(t-\tau)B(\tau)(\sum_{j=1}^3 u_j(\tau))d\tau,
$$

applying Remark 4.6, Lemma 5.1, Lemma 5.3 and Proposition 2.2, we have

$$
\begin{aligned}
&\|\partial_t u_1(t)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} \\
&\quad \le \; C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|u_0\|_{L^q(\mathbb{R}^n_+)} + C\|\partial_x^2 u_1(t)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} \\
&\qquad +\sum_{j=2}^{3}\|\partial_x u_j(t)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} \\
&\qquad +C\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|B(\tau)(\sum_{j=1}^{3} u_j(\tau))\|_{L^q(\mathbb{R}^n_+)}d\tau \\
&\quad \le \; C(1+t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|u_0\|_{L^q(\mathbb{R}^n_+)} \\
&\qquad +C\delta\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1} \\
&\qquad\qquad \times(\|\partial_x^2 u_1(\tau)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))} + \sum_{j=2}^{3}\|\partial_x u_j(\tau)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))})d\tau \\
&\quad \le \; C(1+t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|u_0\|_{L^q(\mathbb{R}^n_+)} \\
&\qquad +C\delta\int_0^t (1+t-\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}(1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}d\tau\|u_0\|_{L^q(\mathbb{R}^n_+)} \\
&\quad \le \; C(1+t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|u_0\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}
$$

We now estimate $u_1(t)-S_1(t)u_0$. We set

$$V_{per}(x_n,t)=\int_0^t v_{per}(x_n,\tau)d\tau.$$

Noting that $\int_0^1 v^1_{per}(x_n,t)\,dt=0$, we have

$$\partial_t V_{per}(x_n,t)=v_{per}(x_n,t), \quad V_{per}(x_n,t+1)=V_{per}(x_n,t)$$

and

$$|V_{per}(x_n,t)|\le Ce^{-cx_n}.$$

We also set $\tilde{B}(t)u(t)=\mathrm{div}(V_{per}\otimes u+u\otimes V_{per})(t)$. Then

$$\partial_x^k u_1(t)-\partial_x^k S_1(t)u_0 \;=\; -(I_1+I_2+I_3),$$

where

$$I_j=\int_0^t \partial_x^k S_1(t-\tau)B(\tau)u_j(\tau)d\tau, \quad j=1,2,3.$$

As for I_1, we write it as

$$I_1 = \int_0^t \partial_x^k S_1(t-\tau)\mathrm{div}(\partial_\tau V_{per} \otimes u_1 + u_1 \otimes \partial_\tau V_{per})(\tau)d\tau.$$

By duality and integration by parts, we see that

$$I_1 = I_1^1 + I_1^2 + I_1^3 + I_1^4,$$

where

$$\begin{aligned} I_1^1 = \partial_x^k \tilde{B}(t)u_1(t), \quad I_1^2 &= -\partial_x^k S_1(t)\tilde{B}(0)u_1(0), \\ I_1^3 &= \int_0^t \partial_x^k(\partial_t S_1)(t-\tau)\tilde{B}(\tau)u_1(\tau)d\tau, \\ I_1^4 &= -\int_0^t \partial_x^k S_1(t-\tau)\tilde{B}(\tau)\partial_\tau u_1(\tau)d\tau. \end{aligned}$$

By using (5.14), Theorem 4.1, Remark 4.6, Lemmas 5.1, 5.3 and Proposition 2.2, we obtain

$$\|I_1^1\|_{L^p(\mathbb{R}_+^n)} \leq C\|\partial_x^2 u_1(t)\|_{L^p(\mathbb{R}_+^n)} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1+\varepsilon}\|u_0\|_{L^q(\mathbb{R}_+^n)},$$

$$\begin{aligned} \|I_1^2\|_{L^p(\mathbb{R}_+^n)} &\leq \|\partial_x^k S_1(t)\mathrm{div}(V_{per}\otimes u + u \otimes V_{per})(0)\|_{L^p(\mathbb{R}_+^n)} \\ &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}\|u_1(0)\|_{L^q(\mathbb{R}_+^n)} \\ &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^q(\mathbb{R}_+^n)}, \end{aligned}$$

$$\begin{aligned} &\|I_1^3\|_{L^p(\mathbb{R}_+^n)} \\ &\leq \int_0^t \|\partial_x^k(\partial_t S_1)(t-\tau)\tilde{B}(\tau)u_1(\tau)\|_{L^p(\mathbb{R}_+^n)}d\tau \\ &\leq C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1-\frac{k}{2}}\|\tilde{B}(\tau)u_1(\tau)\|_{L^q(\mathbb{R}_+^n)}d\tau \\ &\leq C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1-\frac{k}{2}}\|\partial_x^2 u_1(\tau)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}d\tau \\ &\leq C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1-\frac{k}{2}}(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}d\tau\|u_0\|_{L^q(\mathbb{R}_+^n)} \\ &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}_+^n)}, \end{aligned}$$

and

$$
\begin{aligned}
&\|I_1^4\|_{L^p(\mathbb{R}^n_+)}\\
&\quad\le \int_0^t \|\partial_x^k S_1(t-\tau)\mathrm{div}(V_{per}\otimes\partial_\tau u_1+\partial_\tau u_1\otimes V_{per})(\tau)\|_{L^p(\mathbb{R}^n_+)}d\tau\\
&\quad\le C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}\\
&\qquad\qquad\times\|(V_{per}\otimes\partial_\tau u_1+\partial_\tau u_1\otimes V_{per})(\tau)\|_{L^q(\mathbb{R}^n_+)}d\tau\\
&\quad\le C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}\|\partial_\tau u_1(\tau)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}d\tau\\
&\quad\le C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}d\tau\|u_0\|_{L^q(\mathbb{R}^n_+)}\\
&\quad\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}
$$

We thus have $\|I_1\|_{L^p(\mathbb{R}^n_+)}\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}$.

Let us next estimate I_j for $j=2,3$. Applying Theorem 4.1, Lemmas 5.1, 5.3 and Proposition 2.2, we have

$$
\begin{aligned}
&\|I_j\|_{L^p(\mathbb{R}^n_+)}\\
&\le \int_0^t \|\partial_x^k S_1(t-\tau)\mathrm{div}(v_{per}\otimes u_j+u_j\otimes v_{per})(\tau)\|_{L^p(\mathbb{R}^n_+)}d\tau\\
&\le C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}\|\partial_x u_j(\tau)\|_{L^r(\mathbb{R}_+;L^q(\mathbb{R}^{n-1}))}d\tau\\
&\le C\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}-\frac{k}{2}}(1+\tau^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}})(1+\tau)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}d\tau\|u_0\|_{L^q(\mathbb{R}^n_+)}\\
&\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}
\end{aligned}
$$

for $j=2,3$.

Consequently, we obtain

$$\|\partial_x^k u_1(t)-\partial_x^k S_1(t)u_0\|_{L^p(\mathbb{R}^n_+)}\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}.$$

By (5.12), (5.13), (5.15) and (5.16), we have

$$
\begin{aligned}
&\|\partial_x^k u_2(t)-\partial_x^k S_2(t)u_0\|_{L^p(\mathbb{R}^n_+)}+\|\partial_x^k u_3(t)-\partial_x^k S_3(t)u_0\|_{L^p(\mathbb{R}^n_+)}\\
&\quad\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t)^{-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}
$$

We thus obtain

$$\|\partial_x^k u(t)-\partial_x^k S(t)u_0\|_{L^p(\mathbb{R}^n_+)}\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^q(\mathbb{R}^n_+)}.$$

This completes the proof. □

Remark 5.7. *$U_0(t,s)$ and $U_\infty(t,s)$ are not evolution operators. More precisely, the relations*

$$U_j(t,s) = U_j(t,r)U_j(r,s) \quad (j = 0, \infty)$$

do not necessarily hold for $0 \le s < r < t$. However, the following assertion holds true. Let $\tilde{\chi}'_j(\xi')$ for $j = 0, \infty$ be defined by

$$\tilde{\chi}'_0(\xi') \in C_0^\infty(\mathbb{R}^{n-1}),\ 0 \le \tilde{\chi}'_0(\xi') \le 1,\ \tilde{\chi}'_0(\xi') = \begin{cases} 1 & (|\xi'| \le 2), \\ 0 & (|\xi'| \ge 3), \end{cases}$$

$$\tilde{\chi}'_\infty(\xi') \in C_0^\infty(\mathbb{R}^{n-1}),\ 0 \le \tilde{\chi}'_\infty(\xi') \le 1,\ \tilde{\chi}'_\infty(\xi') = \begin{cases} 0 & (|\xi'| \le \frac{1}{2}), \\ 1 & (|\xi'| \ge 1). \end{cases}$$

We set $\widetilde{U}_j(t,s) = \mathcal{F}'^{-1}\tilde{\chi}'_j(\xi')\mathcal{F}'U(t,s)$ $(j = 0, \infty)$. Then, since $U_j(t,s) = \mathcal{F}'^{-1}\chi'_j(\xi')\mathcal{F}'U(t,s)$, we have

$$U_j(t,s) = \widetilde{U}_j(t,r)U_j(r,s)$$

for $j = 0, \infty$. It is clear that $\widetilde{U}_j(t,s)$ satisfies the same estimates as those for $U_j(t,s)$ with obvious modifications of constants $\delta_0(p,q)$ and C.

Final step of the proof of Theorem 5.2. We finally extend the range of p and q in Lemma 5.5 to complete the proof of Theorem 5.2.

Let p and q satisfy $1 \le q < p < \infty$ and let $\delta_0(q)$ be the number given in Lemma 5.5. We fix ε so that $0 < \varepsilon < \frac{1}{2}$.

When $\frac{n}{2\varepsilon} \le q$, applying Lemma 5.5 and the interpolation inequality, we have

$$\begin{aligned}
& \|\partial_x^k U_0(t,s)u_0\|_{L^p(\mathbb{R}^n_+)} \\
\le\ & \|\partial_x^k U_0(t,s)u_0\|^{\alpha}_{L^\infty(\mathbb{R}^n_+)} \|\partial_x^k U_0(t,s)u_0\|^{1-\alpha}_{L^q(\mathbb{R}^n_+)} \\
\le\ & \{C(t-s)^{-\frac{1}{2q}-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2q}}\|u_0\|_{L^q(\mathbb{R}^n_+)}\}^{\alpha}\{C(t-s)^{-\frac{k}{2}}\|u_0\|_{L^q(\mathbb{R}^n_+)}\}^{1-\alpha} \\
\le\ & C(t-s)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q(\mathbb{R}^n_+)},
\end{aligned}$$

where $\alpha = q(\frac{1}{q} - \frac{1}{p})$.

We next consider the case $q < \frac{n}{2\varepsilon}$. Let η be the integer such that $\frac{n}{2\varepsilon} \le (\frac{2n-1}{2n-2})^\eta q$ and q_m be the sequence such that

$$q_m = \left(\frac{2n-1}{2n-2}\right)^m q,\ m = 0, 1, 2, \ldots, \eta.$$

We choose $\delta_0 = \min\{\delta_0(q_0), \delta_0(q_1), \dots, \delta_0(q_\eta)\}$. This δ_0 depends only on q.

We first assume $p \le q_\eta$. Let l be the integer such that $q_{l-1} \le p \le q_l$ ($1 \le l \le \eta$). This l depends only on n, p and q. We set $t_m = s + \frac{m(t-s)}{l}$, $m = 0, 1, \dots, l$. It then follows from Lemma 5.5 that

$$\begin{aligned}
&\|\partial_x^k U_0(t,s)u_0\|_{L^p(\mathbb{R}^n_+)} \\
= \ & \|\partial_x^k \widetilde{U}_0(t, t_{l-1}) U_0(t_{l-1}, s) u_0\|_{L^p(\mathbb{R}^n_+)} \\
\le \ & C(t-s)^{-\frac{1}{2}(\frac{1}{q_{l-1}}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q_{l-1}}-\frac{1}{p})}\|U_0(t_{l-1},s)u_0\|_{L^{q_{l-1}}(\mathbb{R}^n_+)} \\
= \ & C(t-s)^{-\frac{1}{2}(\frac{1}{q_{l-1}}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q_{l-1}}-\frac{1}{p})} \\
& \quad \times \|\widetilde{U}_0(t_{l-1}, t_{l-2}) U_0(t_{l-2}, s) u_0\|_{L^{q_{l-1}}(\mathbb{R}^n_+)} \\
\le \ & C(t-s)^{-\frac{1}{2}(\frac{1}{q_{l-2}}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q_{l-2}}-\frac{1}{p})}\|U_0(t_{l-2},s)u_0\|_{L^{q_{l-2}}(\mathbb{R}^n_+)}.
\end{aligned}$$

Repeating this argument, we have

$$\begin{aligned}
&\|\partial_x^k U_0(t,s)u_0\|_{L^p(\mathbb{R}^n_+)} \\
&\le \ C(t-s)^{-\frac{1}{2}(\frac{1}{q_0}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q_0}-\frac{1}{p})}\|U_0(t_0,s)u_0\|_{L^{q_0}(\mathbb{R}^n_+)} \\
&= \ C(t-s)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}$$

Note that $t_m - t_{m-1} = \frac{t-s}{l}$, $m = 1, \dots, l$. When $p = \infty$, we have

$$\begin{aligned}
&\|\partial_x^k U_0(t,s)u_0\|_{L^\infty(\mathbb{R}^n_+)} \\
= \ & \left\|\partial_x \widetilde{U}_0\left(t, \frac{t+s}{2}\right) U_0\left(\frac{t+s}{2}, s\right) u_0\right\|_{L^\infty(\mathbb{R}^n_+)} \\
\le \ & C(t-s)^{-\frac{1}{2}(\frac{1}{q_\eta}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q_\eta}-\frac{1}{p})}\left\|U_0\left(\frac{t+s}{2}, s\right) u_0\right\|_{L^{q_\eta}(\mathbb{R}^n_+)} \\
\le \ & C(t-s)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}(1+t-s)^{-\frac{n-1}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q(\mathbb{R}^n_+)}
\end{aligned}$$

for $1 \le q < \frac{n}{2\varepsilon}$ and $k = 0, 1$.

When $q_\eta < p < \infty$, we obtain the desire estimates by using the interpolation estimate. Estimates for $U_\infty(t,s)$ can be shown similarly.

Let us next prove (vii). By Lemma 5.5, we have

$$\|\partial_x^k U(t,s)u_0 - \partial_x^k S(t-s)u_0\|_{L^p(\mathbb{R}^n_+)} \le C(t-s)^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon}\|u_0\|_{L^l(\mathbb{R}^n_+)}$$

for $q \le l < \infty$ and $l \le p \le \frac{2n-1}{2n-2}l$, or $\max\{q, \frac{n}{2\varepsilon}\} \le l < \infty$ and $p = \infty$. It then follows that

$$
\begin{aligned}
&\|\partial_x^k U(t,s)u_0 - \partial_x^k S(t-s)u_0\|_{L^p(\mathbb{R}^n_+)} \\
&\leq \left\| \partial_x^k U\left(t, \frac{t+s}{2}\right) \left(U\left(\frac{t+s}{2}, s\right) - S\left(\frac{t+s}{2} - s\right)\right) u_0 \right\|_{L^p(\mathbb{R}^n_+)} \\
&\quad + \left\| \partial_x^k \left(U\left(t, \frac{t+s}{2}\right) - S\left(t - \frac{t+s}{2}\right)\right) S\left(\frac{t+s}{2} - s\right) u_0 \right\|_{L^p(\mathbb{R}^n_+)} \\
&\leq C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \left\| U\left(\frac{t+s}{2}, s\right) - S\left(\frac{t+s}{2} - s\right) u_0 \right\|_{L^q(\mathbb{R}^n_+)} \\
&\quad + C(t-s)^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon} \left\| S\left(\frac{t+s}{2} - s\right) u_0 \right\|_{L^l(\mathbb{R}^n_+)} \\
&\leq C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}-\frac{1}{2}+\varepsilon} \|u_0\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}
$$

The proof of Theorem 5.2 is complete. □

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