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Aoyama, Reika

Kagei, Yoshiyuki
Faculty of Mathematics, Kyushu University

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Reika Aoyama
& Yoshiyuki Kagei

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Institute of Mathematics for Industry
Graduate School of Mathematics
Kyushu University
Fukuoka, JAPAN

Large time behavior of solutions to the compressible Navier-Stokes equations around a parallel flow in a cylindrical domain

Reika Aoyama and Yoshiyuki Kagei[1]

[1] Faculty of Mathematics,
Kyushu University,
Nishi-ku, Motooka 744,
Fukuoka 819-0395, Japan

Abstract

Stability of parallel flow of the compressible Navier-Stokes equation in a cylindrical domain is studied. It is shown that if the Reynolds and Mach numbers are sufficiently small, then parallel flow is asymptotically stable and the asymptotic leading part of the disturbances is described by a one dimensional viscous Burgers equation.

Keywords: Compressible Navier-Stokes equation, parallel flow, cylindrical domain, viscous Burgers equation asymptotic behavior

1 Introduction

This paper studies the large time behavior of solutions of the initial boundary value problem for the compressible Navier-Stokes equation

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div} v + \nabla p(\rho) = \rho g, \tag{1.2}$$

$$v|_{\partial D_*} = 0, \tag{1.3}$$

$$(\rho, v)|_{t=0} = (\rho_0, v_0) \tag{1.4}$$

in a cylindrical domain $\Omega_* = D_* \times \mathbf{R}$:

$$\Omega_* = \{x = (x', x_3);\ x' = (x_1, x_2) \in D_*, x_3 \in \mathbf{R}\}.$$

Here D_* is a bounded and connected domain in $\mathbf{R}^2$ with smooth boundary ∂D_*; $\rho = \rho(x,t)$ and $v = {}^T(v^1(x,t), v^2(x,t), v^3(x,t))$ denote the unknown density and velocity, respectively, at time $t \geq 0$ and position $x \in \Omega_*$; $p(\rho)$ is the pressure that is a smooth function of ρ and satisfies

$$p'(\rho_*) > 0$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that satisfy

$$\mu > 0, \quad \tfrac{2}{3}\mu + \mu' \geq 0;$$

and g is an external force of the form $g = {}^T\big(g^1(x'), g^2(x'), g^3(x')\big)$ with g^1 and g^2 satisfying

$$\big(g^1(x'), g^2(x')\big) = \big(\partial_{x_1}\Phi(x'), \partial_{x_2}\Phi(x')\big),$$

where Φ and g^3 are given smooth functions of x'. Here and in what follows ${}^T\cdot$ stands for the transposition.

Problem (1.1)-(1.3) has a stationary solution $\overline{u}_s = {}^T(\overline{\rho}_s(x'), \overline{v}_s(x'))$ which represents parallel flow. Here $\overline{\rho}_s$ is determined by

$$\begin{cases} \text{Const.} - \Phi(x') = \int_{\rho_*}^{\overline{\rho}_s} \frac{p'(\eta)}{\eta} d\eta, \\ \int_{D_*} \overline{\rho}_s - \rho_* dx' = 0, \end{cases}$$

while $\overline{v}_s$ takes the form

$$\overline{v}_s = {}^T\big(0, 0, \overline{v}_s^3(x')\big),$$

where $\overline{v}_s^3(x')$ is the solution of

$$\begin{cases} -\mu\Delta'\overline{v}_s^3 = \overline{\rho}_s g^3, \\ \overline{v}_s^3 \mid_{\partial D_*} = 0. \end{cases}$$

Here

$$\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2.$$

The purpose of this paper is to investigate the large time behavior of solutions to problem (1.1)-(1.4) when the initial value $(\rho, v)\mid_{t=0} = (\rho_0, v_0)$ is sufficiently close to the stationary solution $\overline{u}_s = {}^T(\overline{\rho}_s, \overline{v}_s)$.

Solutions of multi-dimensional compressible Navier-Stokes equations in unbounded domains exhibit interesting phenomenon, and detailed descriptions of large time behavior of solutions have been obtained. See, e.g., [6, 7, 16, 18, 19, 20, 21, 22, 24, 25] for the case of the whole space, half space and exterior domains. Besides these domains, infinite layers and cylindrical domains provide good subjects of the stability of flows, for example, the stability of parallel flows.

Concerning the large time behavior of solutions around parallel flow, the case of an n dimensional infinite layer $\mathbf{R}^{n-1} \times (0,1) = \{x = (x_h, x_n);\ x_h = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}, 0 < x_n < 1\}$ was studied in [10, 11, 15]. (See also [3, 4, 5] for the stability of time periodic parallel flow.) It was shown in [10, 11, 15] that if the Reynolds and Mach numbers are sufficiently small, then the parallel flow is stable under sufficiently small initial disturbances in some Sobolev space. Furthermore, in the case of $n \geq 3$, the disturbance $u(t)$ behaves like a solution of an $n-1$ dimensional linear heat equation as $t \to \infty$, whereas, in the case of $n = 2$, $u(t)$ behaves like a solution of a one dimensional viscous Burgers equation.

In the case of cylindrical domain Ω_*, Iooss and Padula [8] considered the linearized stability of parallel flow $\bar{u}_s$ under periodic boundary condition in x_3. It was proved in [8] that if the Reynolds number is suitably small, then the semigroup decays exponentially as $t \to \infty$, provided that the density-component has vanishing average over the basic period domain. Furthermore, the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line $\mathrm{Re}\lambda = -c$ for some number $c > 0$ consists of finite number of eigenvalues with finite multiplicities. As for the stability under local disturbances on Ω_*, i.e., disturbances which are non-periodic but decay at spatial infinity, the stability of the motionless state $\widetilde{u}_s = {}^T(\rho_*, 0)$ was studied in [17]; and it was shown in [17] that the disturbance decays in $L^2(\Omega_*)$ in the order $t^{-\frac{1}{4}}$ and its asymptotic leading part is given by a solution of a one dimensional linear heat equation if the initial disturbance is sufficiently small in $H^3(\Omega_*) \cap L^1(\Omega_*)$, where $H^3(\Omega_*)$ denotes the L^2-Sobolev space on Ω_* of order 3. (See also [9] for the analysis in $L^p(\Omega_*)$.)

In this paper we will consider the stability of parallel flow $\bar{u}_s$ under local disturbances on Ω_*. After introducing suitable non-dimensional variables, the equations for the disturbance $u = {}^T(\phi, w) = {}^T\big(\gamma^2(\rho - \rho_s), v - v_s\big)$ takes the following form:

$$\partial_t \phi + v_s^3 \partial_{x_3}\phi + \gamma^2 \mathrm{div}(\rho_s w) = f^0(\phi, w), \tag{1.5}$$

$$\begin{aligned} &\partial_t w - \tfrac{\nu}{\rho_s}\Delta w - \tfrac{\widetilde{\nu}}{\rho_s}\nabla \mathrm{div} w + \nabla\Big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi\Big) \\ &\quad + \tfrac{\nu\Delta' v_s^3}{\gamma^2\rho_s^2}\phi e_3 + v_s^3\partial_{x_3} w + (w' \cdot \nabla' v_s^3) e_3 = f(\phi, w), \end{aligned} \tag{1.6}$$

$$w \,|_{\partial\Omega} = 0, \tag{1.7}$$

$$(\phi, w)\,|_{t=0} = (\phi_0, w_0). \tag{1.8}$$

Here Ω_* is transformed to $\Omega = D \times \mathbf{R}$ with $|D| = 1$; $u_s = {}^T(\rho_s, v_s)$ and $P(\rho)$ denote the dimensionless parallel flow and pressure, respectively; ν, $\widetilde{\nu}$ and γ are the dimensionless parameters defined by

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \widetilde{\nu} = \frac{\mu + \mu'}{\rho_* \ell V}, \quad \gamma = \frac{\sqrt{p'(\rho_*)}}{V}$$

with the reference velocity V which measures the strength of $\overline{v_s}$; $e_3 = {}^T(0, 0, 1) \in \mathbf{R}^3$ and $\nabla' = {}^T(\partial_{x_1}, \partial_{x_2})$; $f^0(\phi, w)$ and $f(\phi, w)$ are the nonlinearities given by

$$f^0(\phi, w) = -\mathrm{div}(\phi w),$$

$$\begin{aligned} f(\phi, w) = &-w\cdot\nabla w + \tfrac{\nu\phi}{(\phi+\gamma^2\rho_s)\rho_s}\Big(-\Delta w + \tfrac{\Delta' v_s}{\gamma^2\rho_s}\phi\Big) - \tfrac{\widetilde{\nu}\phi}{(\phi+\gamma^2\rho_s)\rho_s}\nabla \mathrm{div} w \\ &+ \tfrac{\phi}{\gamma^2\rho_s}\nabla\Big(\tfrac{P'(\rho_s)\phi}{\gamma^2\rho_s}\Big) - \tfrac{1}{2\gamma^4\rho_s}\nabla\Big(P''(\rho_s)\phi^2\Big) + \widetilde{P}_3(\rho_s, \phi, \partial_{x'}\phi), \end{aligned}$$

where

$$\begin{aligned} \widetilde{P}_3(\rho_s, \phi, \partial_{x'}\phi) = &\tfrac{\phi^3}{\gamma^4(\phi+\gamma^2\rho_s)\rho_s^3}\nabla P(\rho_s) - \tfrac{1}{2\gamma^6\rho_s}\nabla\Big(\phi^3 P_3(\rho_s, \phi)\Big) \\ &+ \tfrac{\phi}{2\gamma^6\rho_s^2}\nabla\Big(P''(\rho_s)\phi^2 + \tfrac{1}{\gamma^2}\phi^3 P_3(\rho_s, \phi)\Big) \\ &- \tfrac{\phi^2}{\gamma^2(\phi+\gamma^2\rho_s)\rho_s^2}\nabla\Big(\tfrac{1}{\gamma^2}P'(\rho_s)\phi + \tfrac{1}{2\gamma^4}P''(\rho_s)\phi^2 + \tfrac{1}{2\gamma^6}\phi^3 P_3(\rho_s, \phi)\Big), \end{aligned}$$

with

$$P_3(\rho_s,\phi)=\int_0^1(1-\theta)^2P'''(\rho_s+\theta\gamma^{-2}\phi)d\theta.$$

See section 2.2 below for the definition of non-dimensional variables.

In [1, 2], spectral properties of the linearized semigroup e^{-tL} was studied in detail. It was proved that there exists a bounded projection P_0 satisfying $P_0e^{-tL}=e^{-tL}P_0$ such that if Reynolds and Mach numbers are sufficiently small, then, for the initial value $u_0={}^T(\phi_0,w_0)$, it holds that

$$\|e^{-tL}P_0u_0-[\mathcal{H}(t)\langle\phi_0\rangle]u^{(0)}(t)\|_{L^2(\Omega)}\le C(1+t)^{-\frac{3}{4}}\|u_0\|_{L^1(\Omega)}. \tag{1.9}$$

Here $u^{(0)}$ is some function of x'; $\langle\phi_0\rangle$ denotes the average of ϕ_0 over D, (thus, $\langle\phi_0\rangle$ is a function of $x_3\in\mathbf{R}$); and $\mathcal{H}(t)$ is the heat semigroup defined by

$$\mathcal{H}(t)=\mathcal{F}^{-1}e^{-(i\kappa_1\xi+\kappa_0\xi^2)t}\mathcal{F}$$

with some constants $\kappa_1\in\mathbf{R}$ and $\kappa_0>0$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbf{R}$. Furthermore, it was proved that the $(I-P_0)$-part of e^{-tL} satisfies the exponential decay estimate

$$\|e^{-tL}(I-P_0)u_0\|_{H^1(\Omega)}\le Ce^{-dt}\{\|u_0\|_{H^1(\Omega)\times\widetilde{H}^1(\Omega)}+t^{-\frac{1}{2}}\|w_0\|_{L^2(\Omega)}\} \tag{1.10}$$

for a positive constant d. Here $\widetilde{H}^1(\Omega)$ is the set of all locally H^1 functions in $L^2(\Omega)$ whose tangential derivatives near $\partial\Omega$ belong to $L^2(\Omega)$.

In this paper, based on the results on spectral properties of e^{-tL}, we investigate the nonlinear problem (1.5)–(1.8). We prove that if the initial disturbance $u_0={}^T(\phi_0,w_0)$ is sufficiently small, then the disturbance $u(t)$ exists globally in time and it satisfies

$$\|u(t)\|_{L^2(\Omega)}=O(t^{-\frac{1}{4}}) \tag{1.11}$$

$$\|u(t)-(\sigma u^{(0)})\|_{L^2(\Omega)}=O(t^{-\frac{3}{4}+\delta})\quad(\delta>0) \tag{1.12}$$

as $t\to\infty$. Here $\sigma=\sigma(x_3,t)$ satisfies the following one dimensional viscous Burgers equation

$$\partial_t\sigma-\kappa_0\partial_{x_3}^2\sigma+\kappa_1\partial_{x_3}\sigma+\kappa_2\partial_{x_3}(\sigma^2)=0$$

with initial value $\langle\phi_0\rangle$.

The proof of (1.11) and (1.12) is given by the factorization of $e^{-tL}P_0$, estimate (1.10) and the Matsumura-Nishida energy method. We decompose the disturbance $u(t)$ into its P_0 and $I-P_0$ parts as in [3, 11]. We then estimate the P_0-part by representing it in the form of variation of constants formula in terms of $e^{-tL}P_0$ and employ the factorization result of $e^{-tL}P_0$ obtained in [2]. For the $(I-P_0)$-part of $u(t)$, we employ the Matsumura-Nishida energy method. In contrast to [3, 11], we make use of the estimate (1.10) and combine it with the energy method. This simplifies the argument in [3, 11] where a complicated decomposition is also used in the energy method to estimate the $(I-P_0)$-part of $u(t)$. In this paper we do not need to use such a complicated decomposition of the $(I-P_0)$-part in the energy method due to (1.10).

This paper is organized as follows. In section 2 we first introduce notations and non-dimensional variables. We then state the existence of stationary solution which represents parallel flow. In section 3 we state our main result of this paper. In section 4 we state some spectral properties of e^{-tL} obtained in [2]. In section 5 we decompose the problem into the one for a coupled system of the P_0 and $I-P_0$ parts of $u(t)$. Sections 6 is devoted to estimating the P_0-part of the disturbance $u(t)$, while the $(I-P_0)$-part is estimated in section 7. Section 8 is devoted to the estimates for the nonlinearities. The proof of (1.12) is given in section 9.

2 Preliminaries

In this section we introduce notations throughout this paper. We then introduce non-dimensional variables and state the existence of stationary solution which represents parallel flow.

2.1 Notation

We first introduce some notations which will be used throughout the paper. For $1 \leq p \leq \infty$ we denote by $L^p(X)$ the usual Lebesgue space on a domain X and its norm is denoted by $\|\cdot\|_{L^p(X)}$. Let m be a nonnegative integer. $H^m(X)$ denotes the m th order L^2 Sobolev space on X with norm $\|\cdot\|_{H^m(X)}$. In particular, we write $L^2(X)$ for $H^0(X)$.

We denote by $C_0^m(X)$ the set of all C^m functions with compact support in X. $H_0^m(X)$ stands for the completion of $C_0^m(X)$ in $H^m(X)$. We denote by $H^{-1}(X)$ the dual space to $H_0^1(X)$ with norm $\|\cdot\|_{H^{-1}(X)}$.

We simply denote by $L^p(X)$ (resp., $H^m(X)$) the set of all vector fields $w = {}^T(w^1, w^2, w^3)$ on X and its norm is denoted by $\|\cdot\|_{L^p(X)}$ (resp., $\|\cdot\|_{H^m(X)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(X)$ and $w = {}^T(w^1, w^2, w^3) \in H^m(X)$, we define $\|u\|_{H^k(X)\times H^m(X)}$ by $\|u\|_{H^k(X)\times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^m(X)}$.

When $X = \Omega$ we abbreviate $L^p(\Omega)$ as L^p, and likewise, $H^m(\Omega)$ as H^m. The norm $\|\cdot\|_{L^p(\Omega)}$ is written as $\|\cdot\|_{L^p}$, and likewise, $\|\cdot\|_{H^m(\Omega)}$ as $\|\cdot\|_{H^m}$. The inner product of $L^2(\Omega)$ is denoted by

$$(f, g) = \int_\Omega f(x)g(x)dx, \quad f, g \in L^2(\Omega).$$

For $u_j = {}^T(\phi_j, w_j)$ $(j = 1, 2)$, we also define a weighted inner product $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \tfrac{1}{\gamma^2} \int_\Omega \phi_1 \phi_2 \tfrac{P'(\rho_s)}{\gamma^2 \rho_s} dx + \int_\Omega w_1 \cdot w_2 \rho_s dx,$$

where $\rho_s = \rho_s(x')$ is the density of the parallel flow u_s. As will be seen in Proposition 2.1 below, $\rho_s(x')$ and $\frac{P'(\rho_s(x'))}{\rho_s(x')}$ are strictly positive in D.

In the case $X = D$ we denote the norm of $L^p(D)$ by $|\cdot|_p$. The norm of $H^m(D)$ is denoted by $|\cdot|_{H^m}$, respectively. On D we will consider complex valued functions, and in this case, the inner product of $L^2(D)$ is denoted by

$$(f,g) = \int_D f(x')\overline{g(x')}dx', \quad f,g \in L^2(D).$$

Here $\overline{g}$ denotes the complex conjugate of g. For $u_j = {}^T(\phi_j, w_j)$ $(j=1,2)$, we also define a weighted inner product $\langle u_1, u_2\rangle$ by

$$\langle u_1, u_2\rangle = \tfrac{1}{\gamma^2}\int_D \phi_1\overline{\phi}_2 \tfrac{P'(\rho_s)}{\gamma^2\rho_s}dx' + \int_D w_1\cdot \overline{w}_2\rho_s dx'.$$

For $f \in L^1(D)$ we denote the mean value of f over D by $\langle f\rangle$:

$$\langle f\rangle = (f,1) = \frac{1}{|D|}\int_D f dx',$$

where $|D| = \int_D dx'$. For $u = {}^T(\phi, w) \in L^1(D)$ with $w = {}^T(w^1, w^2, w^3)$ we define $\langle u\rangle$ by

$$\langle u\rangle = \langle\phi\rangle + \langle w_1\rangle + \langle w_2\rangle + \langle w_3\rangle.$$

We set

$$[\![f(t)]\!]_k = \Big(\sum_{j=0}^{[\frac{k}{2}]}\|\partial_t^j f(t)\|^2_{H^{k-2j}}\Big)^{\frac{1}{2}},$$

$$|\!|\!|Df(t)|\!|\!|_k = \begin{cases} \|\partial_x f(t)\|_2, & k=0, \\ \Big([\![\partial_x f(t)]\!]_k^2 + [\![\partial_t f(t)]\!]_{k-1}^2\Big)^{\frac{1}{2}}, & k \geq 1. \end{cases}$$

We define a function space $Z(T)$ by

$$Z(T) = \big\{u = {}^T(\phi, w) \in C^0\big([0,T]; H^2\times(H^2\cap H_0^1)\big)\cap C^1\big([0,T];L^2\big);\ \|u\|_{Z(T)} < \infty\big\}$$

where

$$\|u\|_{Z(T)} = \sup_{0\leq t\leq T}[\![u(t)]\!]_2 + \Big(\int_0^T |\!|\!|Dw(t)|\!|\!|_2^2 dt\Big)^{\frac{1}{2}}.$$

Partial derivatives of a function u in x, x', x_3 and t are denoted by $\partial_x u$, $\partial_{x'}u$, $\partial_{x_3}u$ and $\partial_t u$. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

We denote the $n\times n$ identity matrix by I_n. We define 4×4 diagonal matrices Q_0 and $\widetilde{Q}$ by

$$Q_0 = \mathrm{diag}(1,0,0,0), \qquad \widetilde{Q} = \mathrm{diag}(0,1,1,1).$$

It then follows that for $u = {}^T(\phi, w)$ with $w = {}^T(w^1, w^2, w^3)$,

$$Q_0 u = \begin{pmatrix}\phi\\0\end{pmatrix}, \quad \widetilde{Q}u = \begin{pmatrix}0\\w\end{pmatrix}.$$

We denote the Fourier transform of $f = f(x_3)$ $(x_3 \in \mathbf{R})$ by $\widehat{f}$ or $\mathcal{F}[f]$:

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbf{R}} f(x_3)e^{-i\xi x_3}dx_3, \qquad \xi \in \mathbf{R}.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x_3) = (2\pi)^{-1} \int_{\mathbf{R}} f(\xi) e^{i\xi x_3} d\xi, \qquad x_3 \in \mathbf{R}.$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum by $\sigma(A)$.

We finally introduce a function space which consists of locally H^1 functions in $L^2(\Omega)$ whose tangential derivatives near ∂D belong to $L^2(\Omega)$. To do so, we first introduce a local curvilinear coordinate system. For any $\overline{x}_0' \in \partial D$, there exist a neighborhood $\widetilde{\mathcal{O}}_{\overline{x}_0}$ of $\overline{x}_0'$ and a smooth diffeomorphism map $\Psi = (\Psi_1, \Psi_2) : \widetilde{\mathcal{O}}_{\overline{x}_0'} \to B_1(0) = \{z' = (z_1, z_2) : |z'| < 1\}$ such that

$$\begin{cases} \Psi\big(\widetilde{\mathcal{O}}_{\overline{x}_0'} \cap D\big) = \{z' \in B_1(0) : z_1 > 0\}, \\ \Psi\big(\widetilde{\mathcal{O}}_{\overline{x}_0'} \cap \partial D\big) = \{z' \in B_1(0) : z_1 = 0\}, \\ \det \nabla_{x'} \Psi \neq 0 \quad \text{on} \quad \overline{\widetilde{\mathcal{O}}_{\overline{x}_0'} \cap D}. \end{cases}$$

By the tubular neighborhood theorem, there exist a neighborhood $\mathcal{O}_{\overline{x}_0'}$ of $\overline{x}_0'$ and a local curvilinear coordinate system $y' = (y_1, y_2)$ on $\mathcal{O}_{\overline{x}_0'}$ defined by

$$x' = y_1 a_1(y_2) + \Psi^{-1}(0, y_2) : \ \mathcal{R} \to \mathcal{O}_{\overline{x}_0'}, \tag{2.1}$$

where $\mathcal{R} = \{y' = (y_1, y_2) : |y_1| \le \widetilde{\delta}_1, \ |y_2| \le \widetilde{\delta}_2\}$ for some $\widetilde{\delta}_1, \widetilde{\delta}_2 > 0$; $a_1(y_2)$ is the unit inward normal to ∂D that is given by

$$a_1(y_2) = \frac{\nabla_{x'} \Psi_1}{|\nabla_{x'} \Psi_1|}.$$

Setting $y_3 = x_3$ we obtain

$$\nabla_x = e_1(y_2) \partial_{y_1} + J(y') e_2(y_2) \partial_{y_2} + e_3 \partial_{y_3},$$

$$\nabla_y = \begin{pmatrix} {}^T e_1(y_2) \\ \frac{1}{J(y')} {}^T e_2(y_2) \\ {}^T e_3 \end{pmatrix} \nabla_x,$$

where

$$e_1(y_2) = \begin{pmatrix} a_1(y_2) \\ 0 \end{pmatrix}, \quad e_2(y_2) = \begin{pmatrix} a_2(y_2) \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \tag{2.2}$$

$$J(y') = |\det \nabla_{x'} \Psi|, \quad a_2(y_2) = \frac{-\nabla_{x'}^{\perp} \Psi_1}{|\nabla_{x'}^{\perp} \Psi_1|}$$

with $\nabla_{x'}^{\perp} \Psi_1 = {}^T(-\partial_{x_2} \Psi_1, \, \partial_{x_1} \Psi_1)$. Note that ∂_{y_1} and ∂_{y_2} are the inward normal derivative and tangential derivative at $x' = \Psi^{-1}(0, y_2) \in \partial D \cap \mathcal{O}_{\overline{x}_0'}$, respectively. Let us denote the normal and tangential derivatives by ∂_n and ∂, i.e.,

$$\partial_n = \partial_{y_1}, \quad \partial = \partial_{y_2}.$$

Since ∂D is compact, there are bounded open sets $\mathcal{O}_m$ $(m = 1, \ldots, N)$ such that $\partial D \subset \cup_{m=1}^{N} \mathcal{O}_m$ and for each $m = 1, \ldots, N$, there exists a local curvilinear coordinate system $y' = (y_1, y_2)$ as defined in (2.1) with $\mathcal{O}_{\overline{x}_0'}$, Ψ and $\mathcal{R}$ replaced by $\mathcal{O}_m$, Ψ^m and $\mathcal{R}_m = \{y' = (y_1, y_2) : \ |y_1| < \widetilde{\delta}_1^m, |y_2| < \widetilde{\delta}_2^m\}$ for some $\widetilde{\delta}_1^m, \widetilde{\delta}_2^m > 0$. At last, we take an open set $\mathcal{O}_0 \subset D$ such that

$$\cup_{m=0}^{N} \mathcal{O}_m \supset D, \quad \overline{\mathcal{O}}_0 \cap \partial D = \emptyset.$$

We set a local coordinate $y' = (y_1, y_2)$ such that $y_1 = x_1$, $y_2 = x_2$ on $\mathcal{O}_0$. We note that if $h \in H^2(D)$, then $h \mid_{\partial D} = 0$ implies that $\partial^k h \mid_{\partial D \cap \mathcal{O}^m} = 0$ $(k = 0, 1)$.

Let us introduce a partition of unity $\{\chi_m\}_{m=0}^{N}$ subordinate to $\{\mathcal{O}_m\}_{m=0}^{N}$, satisfying

$$\sum_{m=0}^{N} \chi_m = 1 \text{ on } D, \quad \chi_m \in C_0^\infty(\mathcal{O}_m) \ (m = 0, 1, 2 \cdots, N).$$

We denote by $\widetilde{H}^1(\Omega)$ the set of all locally H^1 functions in $L^2(\Omega)$ whose tangential derivatives near $\partial\Omega$ belong to $L^2(\Omega)$, and its norm is denoted by $\|w\|_{\widetilde{H}^1(\Omega)}$:

$$\|w\|_{\widetilde{H}^1(\Omega)} = \|w\|_2 + \|\partial_{x_3} w\|_2 + \|\chi_0 \partial_{x'} w\|_2 + \sum_{m=1}^{N} \|\chi_m \partial w\|_2.$$

Note that $H_0^1(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$.

2.2 Non-dimensionalization and stationary solution

In this subsection we rewrite the problem into the one in a non-dimensional form and state the existence of stationary solution which represents parallel flow. Let k_0 be an integer satisfying $k_0 \geq 3$. We introduce the following non-dimensional variables:

$$x = \ell \widetilde{x}, \quad v = V \widetilde{v}, \quad \rho = \rho_* \widetilde{\rho}, \quad t = \tfrac{\ell}{V} \widetilde{t},$$
$$p = \rho_* V^2 \widetilde{P}, \quad \Phi = \tfrac{V^2}{\ell} \widetilde{\Phi}, \quad g^3 = \tfrac{V^2}{\ell} \widetilde{g}^3,$$
$$V = |\overline{v}_s^3|_{C_*^{k_0}(D_*)} = \sum_{k=0}^{k_0} \sup_{x' \in D_*} \ell^k |\partial_{x'}^k \overline{v}_s^3(x')|, \quad \ell = \Big(\int_{D_*} dx'\Big)^{\frac{1}{2}}.$$

The problem (1.1)-(1.3) is then transformed into the following non-dimensional problem on $\widetilde{\Omega} = \widetilde{D} \times \mathbf{R}$:

$$\partial_{\widetilde{t}} \widetilde{\rho} + \mathrm{div}_{\widetilde{x}} (\widetilde{\rho} \widetilde{v}) = 0, \tag{2.3}$$
$$\widetilde{\rho}(\partial_{\widetilde{t}} \widetilde{v} + \widetilde{v} \cdot \nabla_{\widetilde{x}} \widetilde{v}) - \nu \Delta_{\widetilde{x}} \widetilde{v} - (\nu + \nu') \nabla_{\widetilde{x}} \mathrm{div}_{\widetilde{x}} \widetilde{v} + \widetilde{P}'(\widetilde{\rho}) \nabla_{\widetilde{x}} \widetilde{\rho} = \widetilde{\rho} \widetilde{g}, \tag{2.4}$$
$$\widetilde{v} \mid_{\partial \widetilde{D}} = 0, \tag{2.5}$$
$$(\widetilde{\rho}, \widetilde{v}) \mid_{\widetilde{t}=0} = (\widetilde{\rho}_0, \widetilde{v}_0). \tag{2.6}$$

Here $\widetilde{D}$ is a bounded and connected domain in $\mathbf{R}^2$; $\widetilde{g} = {}^T\big(\partial_{\widetilde{x}_1} \widetilde{\Phi}, \partial_{\widetilde{x}_2} \widetilde{\Phi}, \widetilde{g}^3\big)$; and ν and ν' are non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}.$$

We also introduce a parameter γ:

$$\gamma = \sqrt{\widetilde{P}'(1)} = \tfrac{\sqrt{p'(\rho_*)}}{V}.$$

Note that the Reynolds and Mach numbers are given by $1/\nu$ and $1/\gamma$, respectively.

In what follows, for simplicity, we omit tildes of $\widetilde{x}$, $\widetilde{t}$, $\widetilde{v}$, $\widetilde{\rho}$, $\widetilde{g}$, $\widetilde{P}$, $\widetilde{\Phi}$, $\widetilde{D}$ and $\widetilde{\Omega}$ and write them as x, t, v, ρ, g, P, Φ, D and Ω. Observe that, due to the non-dimensionalization, we have

$$|D| = \int_D dx' = 1,$$

and thus,

$$\langle f \rangle = \int_D f(x')\, dx'.$$

Let us state the existence of a stationary solution which represents parallel flow.

Proposition 2.1. *If $\Phi \in C^{k_0}(\overline{D})$, $g^3 \in H^{k_0}(D)$ and $|\Phi|_{C^{k_0}}$ is sufficiently small, then* (2.3)-(2.5) *has a stationary solution $u_s = {}^T(\rho_s, v_s) \in C^{k_0}(\overline{D})$. Here ρ_s satisfies*

$$\begin{cases} \text{Const.} - \Phi(x') = \int_1^{\rho_s(x')} \frac{P'(\eta)}{\eta} d\eta, \\ \int_D \rho_s dx' = 1,\ \rho_1 < \rho_s(x') < \rho_2 \quad (\rho_1 < 1 < \rho_2), \end{cases}$$

for some constants $\rho_1, \rho_2 > 0$, and v_s is a function of the form $v_s = {}^T(0, 0, v_s^3)$ with $v_s^3 = v_s^3(x')$ being the solution of

$$\begin{cases} -\nu \Delta' v_s^3 = \rho_s g^3, \\ v_s^3 \mid_{\partial D} = 0. \end{cases}$$

Furthermore, $u_s = {}^T(\rho_s, v_s)$ satisfies the estimates:

$$|\rho_s(x') - 1|_{C^k} \le C|\Phi|_{C^k}(1 + |\Phi|_{C^k})^k,$$

$$|v_s^3|_{C^k} \le C|v_s^3|_{H^{k+2}} \le C|\Phi|_{C^k}(1 + |\Phi|_{C^k})^k |g^3|_{H^k}$$

for $k = 3, 4, \cdots, k_0$.

Proposition 2.1 can be proved in a similar manner to the proof of [23, Lemma 2.1].

Setting $\rho = \rho_s + \gamma^{-2}\phi$ and $v = v_s + w$ in (2.3)-(2.6) (without tildes), we arrive at the initial boundary value problem for the disturbance $u = {}^T(\phi, w)$ written in (1.5)-(1.8) in section 1.

3 Main result

In this section we state the main result of this paper. Hereafter we set

$$\widetilde{\nu} = \nu + \nu'.$$

Theorem 3.1. *There exist positive constant ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_0^2$ and $\frac{(\nu+\widetilde{\nu})\omega}{\nu} \leq \omega_0$, then the following assertions hold. There is a positive number ϵ_0 such that if $u_0 = {}^T(\phi_0, w_0)$ satisfies $u_0 \in H^2 \cap L^1$ with $w_0 \in H_0^1$ and $\|u_0\|_{H^2\cap L^1} \leq \epsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of* (1.5)-(1.8) *in $C^0([0,\infty); H^2 \times (H^2 \cap H_0^1)) \cap C^1([0,\infty); L^2)$; and the following estimates hold*

$$\|\partial_{x_3}^l u(t)\|_2 = \mathcal{O}(t^{-\frac{1}{4}-\frac{l}{2}}), \quad (l = 0, 1) \tag{3.1}$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = \mathcal{O}(t^{-\frac{3}{4}+\delta}) \quad (\forall \delta > 0) \tag{3.2}$$

as $t \to \infty$. Here $u^{(0)} = u^{(0)}(x')$ is a function given in Proposition 4.1 (iii) *below; and $\sigma = \sigma(x_3, t)$ is a function satisfying*

$$\begin{aligned} &\partial_t\sigma - \kappa_0\partial_{x_3}^2\sigma + \kappa_1\partial_{x_3}\sigma + \kappa_2\partial_{x_3}(\sigma^2) = 0, \\ &\sigma\mid_{t=0} = \int_D \phi_0(x', x_3)dx' \end{aligned} \tag{3.3}$$

with some constants $\kappa_0 > 0$ and $\kappa_1, \kappa_2 \in \mathbf{R}$.

As in [3, 11], Theorem 3.1 is proved by combining the local solvability (Proposition 5.1 below) and the appropriate a priori estimates. We will establish the necessary a priori estimates in sections 6-8.

To establish the a priori estimates, we will use the results on spectral properties of the linearized semigoup e^{-tL} which will be summarized in section 4. In section 5, we will decompose the problem into the one for a coupled system of the P_0 and $I - P_0$ parts of $u(t)$. The a priori estimates will then be derived in sections 6-8. The proof of (3.2) will be given in section 9.

4 Spectrum of the linearized operator

In this section we state some results on spectral properties of the linearized operator established in [1, 2] which will be used in the proof of Theorem 3.1.

We denote the linearized operator by L:

$$L = \begin{pmatrix} v_s^3\partial_{x_3} & \gamma^2\mathrm{div}(\rho_s\cdot) \\ \nabla\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\cdot\right) & -\frac{\nu}{\rho_s}\Delta I_3 - \frac{\nu+\nu'}{\rho_s}\nabla\mathrm{div} + v_s^3\partial_{x_3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu\Delta' v_s}{\gamma^2\rho_s^2} & e_3 \otimes (\nabla v_s^3) \end{pmatrix},$$

where L is considered as an operator on $L^2(\Omega)$ with domain

$$D(L) = \left\{u = {}^T(\phi, w) \in L^2(\Omega);\ w \in H_0^1(\Omega),\ Lu \in L^2(\Omega)\right\}.$$

Here, for $\mathbf{a} = {}^T(a_1, a_2, a_3)$ and $\mathbf{b} = {}^T(b_1, b_2, b_3)$, $\mathbf{a} \otimes \mathbf{b}$ is the 3×3 matrix (a_ib_j). See [1] for the details.

To investigate the spectrum of L, we consider the Fourier transform of (1.5)-(1.8) in x_3 variable with $f^0 = 0$ and $f = 0$, which takes the form

$$\partial_t \widehat{\phi} + i\xi v_s^3 \widehat{\phi} + \gamma^2 \nabla' \cdot (\rho_s \widehat{w}') + \gamma^2 i\xi \rho_s \widehat{w}^3 = 0, \tag{4.1}$$

$$\partial_t \widehat{w}' - \tfrac{\nu}{\rho_s}(\Delta' - \xi^2)\widehat{w}' - \tfrac{\nu+\nu'}{\rho_s}\nabla'(\nabla' \cdot \widehat{w}' + i\xi \widehat{w}^3) + \nabla'\big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\widehat{\phi}\big) + i\xi v_s^3 \widehat{w}' = 0, \tag{4.2}$$

$$\begin{aligned}\partial_t \widehat{w}^3 - \tfrac{\nu}{\rho_s}(\Delta' - \xi^2)\widehat{w}^3 - \tfrac{\nu+\nu'}{\rho_s} i\xi(\nabla' \cdot \widehat{w}' + i\xi \widehat{w}^3) + i\xi\big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\widehat{\phi}\big) + i\xi v_s^3 \widehat{w}^3 \\ + \tfrac{\nu\Delta' v_s^3}{\gamma^2\rho_s^2}\widehat{\phi} + \widehat{w}' \cdot \nabla' v_s^3 = 0,\end{aligned} \tag{4.3}$$

$$\widehat{w} \mid_{\partial D} = 0 \tag{4.4}$$

for $t > 0$, and

$$^T\big(\widehat{\phi}, \widehat{w}\big) \mid_{t=0} = {}^T\big(\widehat{\phi}_0, \widehat{w}_0\big) = \widehat{u}_0. \tag{4.5}$$

We thus arrive at the following problem

$$\frac{du}{dt} + \widehat{L}_\xi u = 0, \quad u \mid_{t=0} = u_0, \tag{4.6}$$

where $\xi \in \mathbf{R}$ is a parameter. Here $u = {}^T\big(\phi(x', t), w(x', t)\big) \in D\big(\widehat{L}_\xi\big)$ $(x' \in D,\ t > 0)$; $u_0 = {}^T(\phi_0(x'), w_0(x'))$ is a given initial value; and $\widehat{L}_\xi$ is the operator on $L^2(D)$ of the form

$$\begin{aligned}\widehat{L}_\xi \ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s}(\Delta' - |\xi|^2)I_2 - \frac{\nu+\nu'}{\rho_s}\nabla'\nabla'\cdot & -i\frac{\nu+\nu'}{\rho_s}\xi\nabla' \\ 0 & -i\frac{\nu+\nu'}{\rho_s}\xi\nabla'\cdot & -\frac{\nu}{\rho_s}(\Delta' - |\xi|^2) + \frac{\nu+\nu'}{\rho_s}|\xi|^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} i\xi v_s^3 & \gamma^2\nabla'(\rho_s\cdot) & i\gamma^2\rho_s\xi \\ \nabla'\big(\frac{P(\rho_s)}{\gamma^2\rho_s}\cdot\big) & i\xi v_s^3 I_2 & 0 \\ i\xi\frac{P(\rho_s)}{\gamma^2\rho_s} & 0 & i\xi v_s^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\nu\Delta' v_s^3}{\gamma^2\rho_s^2} & {}^T(\nabla' v_s^3) & 0 \end{pmatrix}\end{aligned}$$

with domain

$$D\big(\widehat{L}_\xi\big) = \big\{u = {}^T(\phi, w) \in L^2(D);\ w \in H_0^1(D),\ \widehat{L}_\xi u \in L^2(D)\big\}.$$

Note that $D\big(\widehat{L}_\xi\big) = D\big(\widehat{L}_0\big)$ for all $\xi \in \mathbf{R}$. Here we set $\widehat{L}_0 = \widehat{L}_\xi \mid_{\xi=0}$.

We also introduce the adjoint operator $\widehat{L}_\xi^*$ of $\widehat{L}_\xi$ with respect to the weighted inner product $\langle\cdot,\cdot\rangle$. The operator $\widehat{L}_\xi^*$ is given by

$$\begin{aligned}\widehat{L}_\xi^* \ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s}(\Delta' - |\xi|^2)I_2 - \frac{\nu+\nu'}{\rho_s}\nabla'\nabla'\cdot & -i\frac{\nu+\nu'}{\rho_s}\xi\nabla' \\ 0 & -i\frac{\nu+\nu'}{\rho_s}\xi\nabla'\cdot & -\frac{\nu}{\rho_s}(\Delta' - |\xi|^2) + \frac{\nu+\nu'}{\rho_s}|\xi|^2 \end{pmatrix} \\ &\quad - \begin{pmatrix} i\xi v_s^3 & \gamma^2\nabla'(\rho_s\cdot) & i\gamma^2\rho_s\xi \\ \nabla'\big(\frac{P(\rho_s)}{\gamma^2\rho_s}\cdot\big) & i\xi v_s^3 I_2 & 0 \\ i\xi\frac{P(\rho_s)}{\gamma^2\rho_s} & 0 & i\xi v_s^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{\gamma^2\nu\Delta' v_s^3}{P'(\rho_s)} \\ 0 & 0 & \nabla' v_s^3 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

with domain

$$D\big(\widehat{L}_\xi^*\big)=\big\{u={}^T(\phi,w)\in L^2(D);\, w\in H_0^1(D),\, \widehat{L}_\xi^* u\in L^2(D)\big\}.$$

Note that $D(\widehat{L}_\xi)=D(\widehat{L}_\xi^*)$ for any $\xi\in\mathbf{R}$.

Concerning the spectrum of $-\widehat{L}_\xi$ for $|\xi|\ll 1$, we have the following proposition, which was proved in [2, Theorem 4.5, 4.7] and [1, Lemma 4.1].

Proposition 4.1. (i) *There exist positive constants c_0, ν_1, γ_1, ω_1 and r_0 such that if $\nu\geq\nu_1$, $\frac{\gamma^2}{\nu+\widetilde{\nu}}\geq\gamma_1^2$ and $\omega\leq\omega_1$, then it holds that*

$$\sigma(-\widehat{L}_\xi)\cap\{\lambda:\,|\lambda|\leq\tfrac{c_0}{2}\}=\{\lambda_0(\xi)\}$$

for each ξ with $|\xi|\leq r_0$, where $\lambda_0(\xi)$ is a simple eigenvalue of $-\widehat{L}_\xi$ that has the form

$$\lambda_0(\xi)=-i\kappa_1\xi-\kappa_0\xi^2+\mathcal{O}\big(|\xi|^3\big)$$

as $|\xi|\to 0$. Here $\kappa_1\in\mathbf{R}$ and $\kappa_0>0$ are the numbers given by

$$\kappa_1=\langle v_s^3\phi^{(0)}+\gamma^2\rho_s w^{(0),3}\rangle=\mathcal{O}(1),$$

$$\kappa_0=\tfrac{\gamma^2}{\nu}\Big\{\alpha_0\big|(-\Delta')^{-\frac{1}{2}}\rho_s\big|_2^2+\mathcal{O}\big(\tfrac{1}{\gamma^2}\big)+\big(\tfrac{\nu}{\gamma^2}+\tfrac{1}{\nu^2}\big)\times\mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big)\Big\}.$$

Here $(-\Delta')$ is the Laplace operator on $L^2(D)$ under the zero Dirichlet boundary condition with domain

$$D(-\Delta')=H^2(D)\cap H_0^1(D).$$

(ii) *The eigenprojections $\widehat{\Pi}(\xi)$ and $\widehat{\Pi}^*(\xi)$ for the eigenvalues $\lambda_0(\xi)$ and $\overline{\lambda}_0(\xi)$ of $-\widehat{L}_\xi$ and $-\widehat{L}_\xi^*$ are given by*

$$\widehat{\Pi}(\xi)u=\langle u,u_\xi^*\rangle u_\xi,\quad \widehat{\Pi}^*(\xi)u=\langle u,u_\xi\rangle u_\xi^*,$$

respectively, where u_ξ and u_ξ^* are eigenfunctions for $\lambda_0(\xi)$ and $\overline{\lambda}_0(\xi)$, respectively, that satisfy

$$\langle u_\xi,u_\xi^*\rangle=1.$$

Furthermore, u_ξ and u_ξ^* are written in the form

$$u_\xi(x')=u^{(0)}(x')+i\xi u^{(1)}(x')+|\xi|^2u^{(2)}(x',\xi),$$
$$u_\xi^*(x')=u^{*(0)}(x')+i\xi u^{*(1)}(x')+|\xi|^2u^{*(2)}(x',\xi),$$

and the following estimates hold

$$|u|_{H^{k+2}}\leq C_{k,r_0}$$

for $u\in\{u_\xi,u_\xi^*,u^{(1)},u^{*(1)},u^{(2)},u^{*(2)}\}$ and $k=0,1,\cdots,k_0$ with a positive constant C_{k,r_0} depending on k and r_0. Here $u^{(0)}$ and $u^{(0)*}$ are eigenfunctions of $-\widehat{L}_0$ and $-\widehat{L}_0^*$ for the eigenvalue 0, respectively.

(iii) *The functions* $u^{(0)}$ *and* $u^{(0)*}$ *are given by*

$$u^{(0)} = {}^T(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = {}^T(0, 0, w^{(0),3})$$

and

$$u^{(0)*} = {}^T(\phi^{(0)*}, 0).$$

Here

$$\phi^{(0)}(x') = \alpha_0 \tfrac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))}, \quad \alpha_0 = \Big(\int_D \tfrac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))} dx' \Big)^{-1};$$

and $w^{(0),3}$ *is the solution of the following problem*

$$\begin{cases} -\Delta' w^{(0),3} = -\frac{1}{\gamma^2 \rho_s} \Delta' v_s^3 \phi^{(0)}, \\ w^{(0),3} \mid_{\partial D} = 0; \end{cases}$$

and

$$\phi^{(0)*}(x') = \tfrac{\gamma^2}{\alpha_0} \phi^{(0)}(x').$$

Furthermore, it holds that

$$\langle u_0, u_0^* \rangle = 1.$$

We next consider the spectral properties of the semigroup e^{-tL} generated by $-L$. We denote the characteristic function of the set $\{\xi \in \mathbf{R} : |\xi| \leq r_0\}$ by $\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi)$, i.e.,

$$\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) = \begin{cases} 1, & (0 \leq |\xi| \leq r_0), \\ 0, & (|\xi| > r_0), \end{cases} \qquad \text{for} \quad \xi \in \mathbf{R},$$

where r_0 is a positive constant given in Proposition 4.1.

We define the projections P_0 and P_∞ by

$$P_0 = \mathcal{F}^{-1} \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \widehat{\Pi}(\xi) \mathcal{F}$$

and

$$P_\infty = I - P_0.$$

Then P_0 and P_∞ satisfy

$$P_0 + P_\infty = I, \quad P^2 = P,$$

$$PL \subset LP, \quad P e^{-tL} = e^{-tL} P$$

for $P \in \{P_0, P_\infty\}$. The semigroup e^{-tL} has the following properties. See [2, Theorem 3.1].

Proposition 4.2. *If* $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2$ *and* $\omega \leq \omega_1$, *then* $e^{-tL} P_0$ *and* $e^{-tL} P_\infty$ *have the following properties.*

(i) *If* $u_0 = {}^T(\phi_0, w_0) \in L^1(\Omega) \cap L^2(\Omega)$, *then* $e^{-tL} P_0 u_0$ *satisfies the following estimates*

$$\| \partial_{x'}^k \partial_{x_3}^l e^{-tL} P_0 u_0 \|_2 \leq C_{k,l} (1 + t)^{-\frac{1}{4} - \frac{l}{2}} \| u_0 \|_1 \tag{4.7}$$

uniformly for $t \geq 0$ and for $k = 0, 1, \cdots, k_0$ and $l = 0, 1, \cdots$;

$$\|e^{-tL}P_0u_0 - [\mathcal{H}(t)\langle\phi_0\rangle]u^{(0)}\|_2 \leq Ct^{-\frac{3}{4}}\|u_0\|_1 \tag{4.8}$$

uniformly for $t > 0$. Here

$$\mathcal{H}(t)\langle\phi_0\rangle = \mathcal{F}^{-1}[e^{-(i\kappa_1\xi+\kappa_0\xi^2)t}\langle\widehat{\phi_0}\rangle],$$

where $u^{(0)} = u^{(0)}(x')$ is the function given in Proposition 4.1; and $\kappa_1 \in \mathbf{R}$ and $\kappa_0 > 0$ are the constants given in Proposition 4.1.

(ii) *If $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$, then there exists a constant $a_0 > 0$ such that $e^{-tL}P_\infty u_0$ satisfies*

$$\|e^{-tL}P_\infty u_0\|_{H^1} \leq Ce^{-a_0t}\big(\|u_0\|_{H^1\times\widetilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_2\big) \tag{4.9}$$

uniformly for $t \geq 0$.

More detailed information of the P_0-part of e^{-tL} is needed to analyze the nonlinear problem; and so we next give a factorization of $e^{-tL}P_0$ which was obtained in [2, Section 5].

Let us introduce operators related to u_ξ and u_ξ^*. We define the operators

$$\mathcal{T} : L^2(\mathbf{R}) \to L^2(\Omega), \quad \mathcal{P} : L^2(\Omega) \to L^2(\mathbf{R}), \quad \Lambda : L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

by

$$\mathcal{T}\sigma = \mathcal{F}^{-1}[\widehat{\mathcal{T}}_\xi\widehat{\sigma}], \qquad \widehat{\mathcal{T}}_\xi\widehat{\sigma} = \mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)u_\xi\widehat{\sigma};$$
$$\mathcal{P}u = \mathcal{F}^{-1}[\widehat{\mathcal{P}}_\xi\widehat{u}], \qquad \widehat{\mathcal{P}}_\xi\widehat{u} = \mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\langle\widehat{u}, u_\xi^*\rangle;$$
$$\Lambda\sigma = \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\lambda_0(\xi)\widehat{\sigma}]$$

for $u \in L^2(\Omega)$ and $\sigma \in L^2(\mathbf{R})$. It follows that

$$P_0 = \mathcal{T}\mathcal{P}, \quad \mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\widehat{\Pi}(\xi) = \widehat{\mathcal{T}}\widehat{\mathcal{P}},$$

$$P_0L \subset LP_0 = \Lambda P_0, \quad e^{-tL}P_0 = \mathcal{T}e^{t\Lambda}\mathcal{P}.$$

Note that $P_0e^{-tL} = e^{-tL}P_0$.

Here and in what follows we assume that

$$\nu \geq \nu_1, \quad \frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_1^2, \quad \omega \leq \omega_1,$$

where ν_1, γ_1 and ω_1 are the constants given in Proposition 4.1.

As for $\mathcal{T}$, we have the following proposition.

Proposition 4.3. ([2, Proposition 5.1]) *The operator $\mathcal{T}$ has the following properties:*

(i) $\partial_{x_3}^l\mathcal{T} = \mathcal{T}\partial_{x_3}^l$ *for $l = 0, 1, \cdots$.*

(ii) $\|\partial_{x'}^k\partial_{x_3}^l\mathcal{T}\sigma\|_2 \leq C\|\sigma\|_{L^2(\mathbf{R})}$ *for $k = 0, 1, \cdots k_0$, $l = 0, 1, \cdots$ and $\sigma \in L^2(\mathbf{R})$.*

(iii) $\mathcal{T}$ *is decomposed as*

$$\mathcal{T} = \mathcal{T}^{(0)} + \partial_{x_3}\mathcal{T}^{(1)} + \partial_{x_3}^2\mathcal{T}^{(2)},$$

where $\mathcal{T}^{(j)}\sigma = \mathcal{F}^{-1}[\widehat{\mathcal{T}}^{(j)}\widehat{\sigma}]$ $(j = 0, 1, 2)$ *with*

$$\begin{aligned}
\widehat{\mathcal{T}}^{(0)}\widehat{\sigma} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\widehat{\sigma}u^{(0)},\\
\widehat{\mathcal{T}}^{(1)}\widehat{\sigma} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\widehat{\sigma}u^{(1)}(\cdot),\\
\widehat{\mathcal{T}}^{(2)}\widehat{\sigma} &= -\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\widehat{\sigma}u^{(2)}(\cdot, \xi).
\end{aligned}$$

Here $\mathcal{T}^{(j)}$ $(j = 0, 1, 2)$ *satisfy estimates* (i) *and* (ii) *by replacing* $\mathcal{T}$ *with* $\mathcal{T}^{(j)}$.

As for $\mathcal{P}$, we have the following properties.

Proposition 4.4. ([2, Proposition 5.2]) *The operator* $\mathcal{P}$ *has the following properties:*

(i) $\partial_{x_3}^l\mathcal{P} = \mathcal{P}\partial_{x_3}^l$ *for* $l = 0, 1, \cdots$.

(ii) $\|\partial_{x_3}^l\mathcal{P}u\|_{L^2(\mathbf{R})} \le C\|u\|_2$ *for* $k = 0, 1, \cdots k_0$, $l = 0, 1, \cdots$ *and* $u \in L^2(\Omega)$. *Furthermore,* $\|\mathcal{P}u\|_{L^2(\mathbf{R})} \le C\|u\|_1$ *for* $u \in L^1(\Omega)$.

(iii) $\mathcal{P}$ *is decomposed as*

$$\mathcal{P} = \mathcal{P}^{(0)} + \partial_{x_3}\mathcal{P}^{(1)} + \partial_{x_3}^2\mathcal{P}^{(2)},$$

where $\mathcal{P}^{(j)}u = \mathcal{F}^{-1}[\widehat{\mathcal{P}}^{(j)}\widehat{u}]$ $(j = 0, 1, 2)$ *with*

$$\begin{aligned}
\widehat{\mathcal{P}}^{(0)}\widehat{u} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle \widehat{u}, u^{*(0)}\rangle = \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle Q_0\widehat{u}\rangle,\\
\widehat{\mathcal{P}}^{(1)}\widehat{u} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle \widehat{u}, u^{*(1)}\rangle,\\
\widehat{\mathcal{P}}^{(2)}\widehat{u} &= -\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle \widehat{u}, u^{*(2)}(\xi)\rangle.
\end{aligned}$$

Here $\mathcal{P}^{(j)}$ $(j = 0, 1, 2)$ *satisfy estimates* (i) *and* (ii) *by replacing* $\mathcal{P}$.

Concerning Λ, we have the following estimates for $e^{t\Lambda}$.

Proposition 4.5. ([2, Proposition 5.3]) *The operator* $e^{t\Lambda}$ *satisfies the following estimates.*

(i) $\|\partial_{x_3}^l e^{t\Lambda}\mathcal{P}u\|_{L^2(\mathbf{R})} \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u\|_1$,

(ii) $\|\partial_{x_3}^l e^{t\Lambda}\mathcal{P}^{(j)}u\|_{L^2(\mathbf{R})} \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u\|_1, \quad j = 0, 1, 2,$

(iii) $\|\partial_{x_3}^l(\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P}u\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{l+1}{2}}\|u\|_1$,

for $u \in L^1(\Omega)$ *and* $l = 0, 1, 2\cdots$.

We next consider the asymptotic behavior of $e^{t\Lambda}$. Let us define an operator $\mathcal{H}(t)$ by

$$\mathcal{H}(t)\sigma = \mathcal{F}^{-1}[e^{-(i\kappa_1\xi+\kappa_0\xi^2)t}\widehat{\sigma}]$$

for $\sigma \in L^2(\mathbf{R})$, where $\kappa_1 \in \mathbf{R}$ and $\kappa_0 > 0$ are given by Proposition 4.1. The asymptotic leading part of $e^{t\Lambda}$ is given by $\mathcal{H}(t)$. In fact, we have the following estimates.

Proposition 4.6. ([2, Proposition 5.8]) *For $u \in L^2(\Omega)$, we set $\sigma = \langle Q_0 u\rangle$. If $u \in L^1(\Omega)$, then there holds the estimate*

$$\|\partial_{x_3}^l \big(e^{t\Lambda}\mathcal{P}u - \mathcal{H}(t)\sigma\big)\|_{L^2(\mathbf{R})} \leq Ct^{-\frac{3}{4}-\frac{l}{2}}\|u\|_1 \quad (l = 0, 1, \cdots).$$

Observe that, since

$$e^{-tL}P_0 = \mathcal{T}e^{t\Lambda}\mathcal{P} = \mathcal{T}^{(0)}e^{t\Lambda}\mathcal{P} + (\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P},$$

we have (4.8) by Propositions 4.3 (iii), 4.5 (iii) and 4.6.

5 Decomposition of Problem

In this section we formulate the problem. The problem (1.5)-(1.8) is written as

$$\frac{du}{dt} + Lu = \boldsymbol{F}, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \tag{5.1}$$

Here $u = {}^T(\phi, w)$; $\boldsymbol{F} = \boldsymbol{F}(u)$ denotes the nonlinearity:

$$\boldsymbol{F} = {}^T(f^0(\phi, w), f(\phi, w)).$$

The local solvability in $Z(T)$ for (5.1) follows from [12].

Proposition 5.1. *If $u_0 = {}^T(\phi_0, w_0)$ satisfies the following conditions;*

(i) $u_0 \in H^2 \times (H^2 \cap H_0^1)$,

(ii) $-\frac{\gamma^2}{4}\rho_1 \leq \phi_0$,

then there exists a number $T_0 > 0$ depending on $\|u_0\|_{H^2}$ and ρ_1 such that the following assertions hold. Problem (5.1) *has a unique solution $u(t) \in Z(T)$ satisfying*

$$\phi(x, t) \geq -\tfrac{\gamma^2}{2}\rho_1 \quad \text{for} \quad \forall (x, t) \in \Omega \times [0, T_0];$$

and the following estimate holds

$$\|u\|_{Z(T)}^2 \leq C_0\{1 + \|u_0\|_{H^2}^2\}^\alpha \|u_0\|_{H^2}^2$$

for some positive constants C_0 and α.

Theorem 3.1 would follow if we would establish the a priori estimates of $u(t)$ in $Z(T)$ uniformly for T.

To obtain the appropriate a priori estimates, we decompose the solution u into its P_0 and P_∞ parts. Let us decompose the solution $u(t)$ of (5.1) as

$$u(t) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t),$$

where

$$\sigma_1(t) = \mathcal{P}u(t), \quad u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t), \quad u_\infty(t) = P_\infty u(t).$$

Note that $P_0 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t)$.

Since $u_1(t)$ is written as

$$u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t) = (\partial_{x_3}\mathcal{T}^{(1)} + \partial_{x_3}^2\mathcal{T}^{(2)})\sigma_1(t),$$

we see from Proposition 4.3 and Proposition 4.4 the following estimates for $\sigma_1(t)$ and $u_1(t)$.

Proposition 5.2. *Let $u(t)$ be a solution of* (5.1) *in $Z(T)$. Then there hold the estimates*

$$\|\partial_{x_3}^l \sigma_1(t)\|_2 \le C\|\partial_{x_3}\sigma_1(t)\|_2$$

for $1 \le l \le 3$; and

$$\|\partial_{x'}^k \partial_{x_3}^l \partial_t^m u_1(t)\|_2 \le C\{\|\partial_{x_3}\sigma_1(t)\|_2 + \|\partial_t \sigma_1(t)\|_2\}$$

for $1 \le k + l + 2m \le 3$.

We derive the equations for $\sigma_1(t)$ and $u_\infty(t)$.

Proposition 5.3. *Let $T > 0$ and assume that $u(t)$ is a solution of* (5.1) *in $Z(T)$. Then the following assertions hold.*

$$\sigma_1 \in \bigcap_{j=0}^{1} C^j([0,T] : H^2(\mathbf{R})), \quad u_\infty \in Z(T), \quad \phi_\infty \in C^1([0,T]; H^1).$$

Furthermore, σ_1 and u_∞ satisfy

$$\sigma_1(t) = e^{t\Lambda}\mathcal{P}u_0 + \int_0^T e^{(t-\tau)\Lambda}\mathcal{P}\boldsymbol{F}(\tau)d\tau; \tag{5.2}$$

and

$$\partial_t u_\infty + L u_\infty = \boldsymbol{F}_\infty, \quad w_\infty |_{\partial\Omega} = 0, \quad u_\infty |_{t=0} = u_{\infty,0}, \tag{5.3}$$

where $\boldsymbol{F}_\infty = P_\infty \boldsymbol{F}$ and $u_{\infty,0} = P_\infty u_0$.

Let $u(t)$ be a solution of (5.1) in $Z(T)$. From Proposition 5.2, we obtain

$$\begin{aligned}&\sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\{[\![u_1(\tau)]\!]_2 + [\![\partial_x u_1(\tau)]\!]_2\}\\ &\qquad \le C\sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\{\|\partial_{x_3}\sigma_1(\tau)\|_2 + \|\partial_\tau \sigma_1(\tau)\|_2\},\end{aligned}$$

and thus, the estimates for $u_1(t)$ follows from the ones for $\sigma_1(t)$. Therefore, as in [3], we introduce the quantity $M_1(t)$ defined by

$$M_1(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}}\|\sigma_1(\tau)\|_2 + \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\{\|\partial_{x_3}\sigma_1(\tau)\|_2 + \|\partial_\tau\sigma_1(\tau)\|_2\};$$

and we define the quantity $M(t) \ge 0$ by

$$M(t)^2 = M_1(t)^2 + \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{2}}E_\infty(\tau) \quad (t \in [0,T])$$

with

$$E_\infty(t) = [\![u_\infty(t)]\!]_2^2.$$

We define a quantity $D_\infty(t)$ for $u_\infty = {}^T(\phi_\infty, w_\infty)$ by

$$D_\infty(t) = |\!|\!|D\phi_\infty(t)|\!|\!|_1^2 + |\!|\!|Dw_\infty(t)|\!|\!|_2^2.$$

If we could show $M(t) \le C$ uniformly for $t \ge 0$, then Theorem 3.1 would follow. The uniform estimate for $M(t)$ is given by using the following estimates for $M_1(t)$ and $E_\infty(t)$.

Proposition 5.4. *There exist positive constants ν_0, γ_0 and ω_0 such that if $\nu \ge \nu_0$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \ge \gamma_0^2$ and $\omega \le \omega_0$, then the following assertions hold. There is a positive number ϵ_1 such that if a solution $u(t)$ of* (5.1) *in $Z(T)$ satisfies $\sup\limits_{0\le\tau\le t} [\![u(\tau)]\!]_2 \le \epsilon_1$ and $M(t) \le 1$ for $t \in [0,T]$, then the estimates*

$$M_1(t) \le C\{\|u_0\|_{L^1} + M(t)^2\} \tag{5.4}$$

and

$$\begin{aligned} &E_\infty(t) + \int_0^\infty e^{-a(t-\tau)} D_\infty(\tau)d\tau \\ &\le C\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^4 + \int_0^t e^{-a(t-\tau)}\mathcal{R}(\tau)d\tau\} \end{aligned} \tag{5.5}$$

hold uniformly for $t \in [0,T]$ with $C > 0$ independent of T. Here $a = a(\nu, \widetilde{\nu}, \gamma)$ is a positive constant; and $\mathcal{R}(t)$ is a function satisfying the estimate

$$\mathcal{R}(t) \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\} \tag{5.6}$$

provided that $\sup\limits_{0\le\tau\le t} [\![u(\tau)]\!]_2 \le \epsilon_2$ and $M(t) \le 1$.

Proposition 5.4 follows from Propositions 6.1, 7.1 and 8.1 below.

As in [3, 11], one can see from Propositions 5.1 and 5.4 that if $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then

$$M(t) \le C\|u_0\|_{H^2\cap L^1}$$

uniformly for $t \ge 0$, which proves Theorem 3.1.

6 Estimates for P_0-part of $u(t)$

In this section, we estimate the P_0-part of $u(t)$

$$P_0u(t) = (\sigma_1 u^{(0)})(t) + u_1(t),$$

where $\sigma_1(t) = \mathcal{P}u(t)$ and $u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t)$. We will prove the following estimate for $M_1(t)$.

Proposition 6.1. *Let $T > 0$ and assume that $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$ and $\omega \leq \omega_1$. Then there exists a positive constant ϵ independent of T such that if a solution $u(t)$ of* (5.1) *in $Z(T)$ satisfies $\sup\limits_{0\leq\tau\leq t} [\![u(\tau)]\!]_2 \leq \epsilon$ and $M(t) \leq 1$ for all $t \in [0,T]$, then the estimate*

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}$$

holds uniformly for $t \in [0,T]$, where C is a positive constant independent of T.

Let us prove Proposition 6.1. We decompose the nonlinearity $\boldsymbol{F}$ into

$$\boldsymbol{F} = \sigma_1^2 \boldsymbol{F}_1 + \boldsymbol{F}_2,$$

where

$$\begin{aligned} \boldsymbol{F}_1 &= \boldsymbol{F}_1(x') = -{}^T\Big(0, \ \ \tfrac{1}{2\gamma^4 \rho_s(x')} \nabla'\big\{P''(\rho_s(x'))\big(\phi^{(0)}(x')\big)^2\big\}, \ \ 0\Big), \\ \boldsymbol{F}_2 &= \boldsymbol{F} - \sigma_1^2 \boldsymbol{F}_1. \end{aligned}$$

Here $\sigma_1^2 \boldsymbol{F}_1(x')$ is the part of $\boldsymbol{F}$ containing only $\sigma_1^2(t)$ but not $\partial_{x_3}\sigma_1(t)$, $u_1(t)$, $u_\infty(t)$, $\sigma_1^3(t)$ and so on.

Before going further, we introduce a notation. For a function g we define $\langle g \rangle_0$ by

$$\langle g \rangle_0 = \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\langle \widehat{g} \rangle].$$

The nonlinearity $\boldsymbol{F}$ has the following properties.

Lemma 6.2. *There hold the following assertions.*

(i) $\langle Q_0 \boldsymbol{F} \rangle = -\partial_{x_3}\langle \phi w^3 \rangle$.

(ii) $\mathcal{P}\boldsymbol{F} = -\partial_{x_3}\langle \phi w^3 \rangle_0 + \partial_{x_3}\mathcal{P}^{(1)}\boldsymbol{F} + \partial_{x_3}^2 \mathcal{P}^{(2)}\boldsymbol{F}$.

Proof. As for (i), we see from integration by parts that $\langle \nabla' \cdot (\phi w') \rangle = 0$. It then follows that

$$\langle Q_0 \boldsymbol{F} \rangle = -\langle \mathrm{div}(\phi w) \rangle = -\langle \partial_{x_3}(\phi w^3) \rangle = -\partial_{x_3}\langle \phi w^3 \rangle.$$

We next prove (ii). From the definition of $\mathcal{P}^{(0)}$ and (i), there holds that

$$\mathcal{P}^{(0)}\boldsymbol{F} = \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\langle Q_0 \widehat{\boldsymbol{F}} \rangle] = \langle Q_0 \boldsymbol{F} \rangle_0 = -\partial_{x_3}\langle \phi w^3 \rangle_0.$$

We thus obtain (ii). This completes the proof. $\square$

Noting that $\|\sigma_1\|_\infty \leq C\|\sigma_1\|_2^{\frac{1}{2}}\|\partial_{x_3}\sigma_1\|_2^{\frac{1}{2}}$, one can obtain the following estimates by straightforward computations.

Lemma 6.3. *There exists a positive constant ϵ such that if a solution $u(t)$ of* (5.1) *in $Z(T)$ satisfies $\sup\limits_{0\leq\tau\leq t} [\![u(\tau)]\!]_2 \leq \epsilon$ and $M(t) \leq 1$ for all $t \in [0,T]$, then the following estimates hold for $t \in [0,T]$ with a positive constant C independent of T.*

(i) $\|\partial_{x_3}(\sigma_1^2(t))\|_1 \le C(1+t)^{-1}M(t)^2$.

(ii) $\|\partial_{x_3}\langle \phi w^3\rangle(t)\|_1 \le C(1+t)^{-1}M(t)^2$.

(iii) $\|\langle \phi w^3\rangle(t)\|_1 \le C(1+t)^{-\frac{1}{2}}M(t)^2$.

(iv) $\|\boldsymbol{F}(t)\|_1 \le C(1+t)^{-\frac{1}{2}}M(t)^2$.

(v) $\|\boldsymbol{F}_2(t)\|_1 \le C(1+t)^{-1}M(t)^2$.

(vi) $\|\boldsymbol{F}(t)\|_2 \le C(1+t)^{-\frac{3}{4}}M(t)^2$.

Proof of Proposition 6.1 We see from Proposition 4.5 that

$$\|\partial_{x_3}^l e^{t\Lambda}\mathcal{P}u_0\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u_0\|_1 \quad (l=0,1).$$

We next consider $\int_0^t e^{(t-\tau)\Lambda}\mathcal{P}\boldsymbol{F}(\tau)d\tau$. We write it as

$$\int_0^t e^{(t-\tau)\Lambda}\mathcal{P}\boldsymbol{F}(\tau)d\tau = \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right) e^{(t-\tau)\Lambda}\mathcal{P}\boldsymbol{F}(\tau)d\tau =: I_1(t) + I_2(t).$$

We see from Lemma 6.2 (ii) that

$$\begin{aligned} e^{(t-\tau)\Lambda}\mathcal{P}\boldsymbol{F}(\tau) &= e^{(t-\tau)\Lambda}\{-\partial_{x_3}\langle \phi w^3\rangle_0 + \partial_{x_3}\mathcal{P}^{(1)}\boldsymbol{F} + \partial_{x_3}^2\mathcal{P}^{(2)}\boldsymbol{F}\} \\ &= \partial_{x_3}e^{(t-\tau)\Lambda}\{-\langle \phi w^3\rangle_0 + \mathcal{P}^{(1)}\boldsymbol{F} + \partial_{x_3}\mathcal{P}^{(2)}\boldsymbol{F}\}. \end{aligned}$$

By Proposition 4.5 and Lemma 6.3 we then have

$$\begin{aligned} \|\partial_{x_3}^l I_1(t)\|_2 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{l}{2}}\big(\|\langle \phi w^3\rangle_0(\tau)\|_1 + \|\boldsymbol{F}(\tau)\|_1\big)d\tau \\ &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{l}{2}}(1+\tau)^{-\frac{1}{2}}d\tau M(t)^2 \\ &\le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}M(t)^2 \end{aligned}$$

for $l=0,1$. Applying Lemma 6.3 (ii) and (v) we have

$$\begin{aligned} \|\partial_{x_3}^l I_2(t)\|_2 &\le C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{1}{4}-\frac{l}{2}}(1+\tau)^{-1}d\tau M(t)^2 \\ &\le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}M(t)^2 \end{aligned}$$

for $l=0,1$. We thus obtain

$$\|\partial_{x_3}^l\sigma_1(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\{\|u_0\|_1 + M(t)^2\} \tag{6.1}$$

for $l=0,1$.

Let us estimate the time derivative. Since $\lambda_0(\xi) = -(i\kappa_1\xi+\kappa_0\xi^2+\mathcal{O}(\xi^3)) = \mathcal{O}(\xi)$, we obtain

$$\|\Lambda\sigma_1(t)\|_2 = \|\mathcal{F}^{-1}[\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\lambda_0(\xi)\widehat{\sigma}_1(t)]\|_2 \le C\|\partial_{x_3}\sigma_1(t)\|_2.$$

This, together with (5.2), (6.1) and Lemma 6.3, implies that

$$\|\partial_t\sigma_1(t)\|_2 \le C\{\|\partial_{x_3}\sigma_1(t)\|_2 + \|\boldsymbol{F}(t)\|_2\} \le C(1+t)^{-\frac{3}{4}}\{\|u_0\|_1 + M(t)^2\}. \tag{6.2}$$

By (6.1) and (6.2) we deduce the desired estimate. This completes the proof. $\square$

7 Estimates for P_∞-part of $u(t)$

In this section we derive the estimates for the P_∞-part of $u(t)$.

Throughout this section, we assume that $u(t)$ is a solution of (5.1) in $Z(T)$ for a given $T > 0$. We show the following estimate.

Proposition 7.1. *There exist positive constants $\nu_0\,(\geq \nu_1)$, $\gamma_0\,(\geq \gamma_1)$ and $\omega_0\,(\leq \omega_1)$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then*

$$
\begin{aligned}
&E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau)d\tau \\
&\qquad \leq C\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^4 + \int_0^t e^{-a(t-\tau)}\mathcal{R}(\tau)d\tau\}.
\end{aligned}
$$

uniformly for $t \in [0,T]$ with $C > 0$ independent of T.

Proposition 7.1 is proved by the estimate (4.9) for $e^{-tL}P_\infty$ and the Matsumura-Nishida energy method.

We introduce notations. In what follows C and C_j $(j = 1, 2, \cdots)$ denote various constants independent of T, ν, $\tilde{\nu}$ and γ, whereas, $C_{\nu\tilde{\nu}\gamma\cdots}$ denotes various constants which depends on $\nu, \tilde{\nu}, \gamma, \cdots$ but not on T.

We first establish the H^1-estimate for u_∞ which follows from the estimate (4.9) for $e^{-tL}P_\infty$.

Proposition 7.2. *There exist positive constants $\nu_0\,(\geq \nu_1)$, $\gamma_0\,(\geq \gamma_1)$ and $\omega_0\,(\leq \omega_1)$ such that if*

$$
\nu \geq \nu_0, \quad \frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2, \quad \omega \leq \omega_0; \tag{7.1}
$$

then, for any $0 < a < 2a_0$,

$$
\begin{aligned}
&\|u_\infty(t)\|_{H^1}^2 + \int_0^t e^{-a(t-\tau)}\|u_\infty(\tau)\|_{H^1}^2\,d\tau \\
&\leq \;\; C_{\nu\tilde{\nu}\gamma}\big\{e^{-at}\|u_{\infty,0}\|_{H^1}^2 + \sup_{0\leq\tau\leq t}\|\boldsymbol{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-a(t-\tau)}\|\boldsymbol{F}_\infty(\tau)\|_{H^1}^2\,d\tau\big\}.
\end{aligned}
$$

Proof. We write $u_\infty(t)$ as

$$
u_\infty(t) = e^{-tL}u_{\infty,0} + \int_0^t e^{-(t-\tau)L}\boldsymbol{F}_\infty(\tau)\,d\tau.
$$

Since $u_{\infty,0} \in H^1 \times H^1_0$, we see from (4.9) that

$$\begin{aligned}
\|u_\infty(t)\|_{H^1} &\le C\big\{e^{-a_0 t}\|u_{\infty,0}\|^2_{H^1} + \int_0^t e^{-a_0(t-\tau)}\|\boldsymbol{F}_\infty(\tau)\|_{H^1\times\widehat{H}^1}\,d\tau \\
&\qquad + \int_0^t e^{-a_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\boldsymbol{F}_\infty(\tau)\|_2\,d\tau\big\} \\
&\le C\big\{e^{-a_0 t}\|u_{\infty,0}\|^2_{H^1} + \sup_{0\le\tau\le t}\|\boldsymbol{F}_\infty(\tau)\|_2 \\
&\qquad + \int_0^t e^{-a_0(t-\tau)}\|\boldsymbol{F}_\infty(\tau)\|_{H^1\times\widehat{H}^1}\,d\tau\big\},
\end{aligned}$$

from which we have

$$\|u_\infty(t)\|^2_{H^1} \le C\big\{e^{-2a_0 t}\|u_{\infty,0}\|^2_{H^1} + \sup_{0\le\tau\le t}\|\boldsymbol{F}_\infty(\tau)\|^2_2 + \int_0^t e^{-a(t-\tau)}\|\boldsymbol{F}_\infty(\tau)\|^2_{H^1}\,d\tau\big\} \tag{7.2}$$

for any $0 < a < 2a_0$. Set $V(t) = \int_0^t e^{-\tilde{a}(t-\tau)}\|\boldsymbol{F}_\infty(\tau)\|^2_{H^1}\,d\tau$. Then $V(t)$ satisfies $dV/dt + \tilde{a}V = \|\boldsymbol{F}_\infty\|^2_{H^1}$ and $V(0) = 0$. It follows that $\int_0^t e^{-a(t-\tau)}V(t)\,d\tau \le \int_0^t e^{-a(t-\tau)}\|\boldsymbol{F}_\infty(\tau)\|^2_{H^1}\,d\tau$ for any $0 < a < \tilde{a}$. This, together with (7.2), yields the desired inequality. This completes the proof. □

We next derive the H^2 estimate for $u_\infty(t)$. In what follows we set

$$f^0_\infty = Q_0\boldsymbol{F}_\infty, \quad f_\infty = \widetilde{Q}\boldsymbol{F}_\infty$$

and

$$\dot{\phi}_\infty = \partial_t\phi_\infty + v^3_s\partial_{x_3}\phi_\infty + w\cdot\nabla\phi_\infty,$$

where

$$\widetilde{f}^0_\infty = f^0_\infty - w\cdot\nabla\phi_\infty.$$

Note that

$$\|\dot{\phi}_\infty\|_{H^1} \le C_{\nu\widetilde{\nu}\gamma}\big(\|u_\infty\|^2_{H^1\times H^2} + \|\widetilde{f}^0_\infty\|^2_{H^1}\big).$$

The following Propositions 7.3 – 7.6 can be proved in a similar manner in [1, Section 4]. So we here give outline of proof.

We begin with the L^2 energy estimates for $\partial_t u_\infty$ and $\partial^2_{x_3}u_\infty$.

Proposition 7.3. *Under the assumption* (7.1) *(with ν_0, γ_0 and ω_0^{-1} replaced by suitably larger ones), the following assertions hold.*

(i) *There exists positive constant c such that the following inequality holds:*

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big\{\tfrac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_t\phi_\infty\Big\|^2_2 + \big\|\sqrt{\rho_s}\partial_t w_\infty\big\|^2_2\Big\} \\
&+ \tfrac{1}{2}\nu\|\nabla\partial_t w_\infty\|^2_2 + \tfrac{1}{2}\widetilde{\nu}\|\mathrm{div}\partial_t w_\infty\|^2_2 + c\tfrac{\nu+\widetilde{\nu}}{\gamma^4}\|\partial_t\dot{\phi}_\infty\|^2_2 \\
&\le C_{\nu\widetilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2} + |A_1|.
\end{aligned} \tag{7.3}$$

Here

$$
\begin{aligned}
A_1 =&\tfrac{1}{2}\Big(|\partial_t\phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big) + \Big([\partial_t, w\cdot\nabla]\phi_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial_t\phi_\infty\Big) \\
&+ \Big(\partial_t\widetilde{f}^0_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial_t\phi_\infty\Big) + \Big(\partial_t f_\infty, \rho_s\partial_t w_\infty\Big) + Cc\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_t\widetilde{f}^0_\infty\|_2^2.
\end{aligned}
$$

(ii) *There exists positive constant b such that the following inequality holds:*

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big\{\tfrac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_{x_3}^2\phi_\infty\Big\|_2^2 + \|\sqrt{\rho_s}\partial_{x_3}^2 w_\infty\|_2^2\Big\} \\
&+\tfrac{1}{2}\nu\|\nabla\partial_{x_3}^2 w_\infty\|_2^2 + \tfrac{1}{2}\widetilde{\nu}\|\mathrm{div}\partial_{x_3}^2 w_\infty\|_2^2 + b\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_3}^2\dot{\phi}_\infty\|_2^2 \\
&\le C\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_3}^2\phi_\infty\|_2^2 + C_{\nu\widetilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2} + |A_{0,0,2}|.
\end{aligned}
\tag{7.4}
$$

Here

$$
\begin{aligned}
A_{0,0,2} =&\tfrac{1}{2}\Big(|\partial_{x_3}^2\phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big) + \Big([\partial_{x_3}^2, w\cdot\nabla]\phi_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial_{x_3}^2\phi_\infty\Big) \\
&+ \Big(\partial_{x_3}^2\widetilde{f}^0_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial_{x_3}^2\phi_\infty\Big) + \Big(\partial_{x_3} f_\infty, \partial_{x_3}(\rho_s\partial_{x_3}^2 w_\infty)\Big) + Cb\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_3}^2\widetilde{f}^0_\infty\|_2^2.
\end{aligned}
$$

Outline of Proof. We write L as

$$L = A + B,$$

where

$$
A = \begin{pmatrix} v_s^3\partial_{x_3} & \gamma^2\mathrm{div}(\rho_s\cdot) \\ \nabla\big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\cdot\big) & -\tfrac{\nu}{\rho_s}\Delta I_3 - \tfrac{\nu+\nu'}{\rho_s}\nabla\mathrm{div} + v_s^3\partial_{x_3} \end{pmatrix},
$$

$$
B = \begin{pmatrix} 0 & 0 \\ \tfrac{\nu\Delta' v_s}{\gamma^2\rho_s^2} & e_3\otimes(\nabla v_s^3) \end{pmatrix},
$$

Note that

$$\langle Au, u\rangle = \nu\|\nabla w\|_2^2 + \tilde{\nu}\|\mathrm{div}\, w\|_2^2. \tag{7.5}$$

In terms of A and B we rewrite problem (5.3) as

$$\partial_t u_\infty + Au_\infty = -Bu_\infty + \boldsymbol{F}_\infty. \tag{7.6}$$

Let j and k be nonnegative integers satisfying $j+2k=2$. We apply $\partial_{x_3}^j\partial_t^k$ to (7.6) and obtain

$$\partial_t\partial_{x_3}^j\partial_t^k u_\infty + A\partial_{x_3}^j\partial_t^k u_\infty = -B\partial_{x_3}^j\partial_t^k u_\infty + \partial_{x_3}^j\partial_t^k\boldsymbol{F}_\infty.$$

We take the weighted inner product $\langle\cdot,\cdot\rangle$ of this equation with $\partial_{x_3}^j\partial_t^k u_\infty$. One can then arrive at the desired estimates by using (7.5) and the relations

$$
\begin{aligned}
&\Big(\partial_{x_3}^j\partial_t^k(w\cdot\nabla\phi_\infty), \partial_{x_3}^j\partial_t^k\phi_\infty\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\Big) \\
=\ &\tfrac{1}{2}\Big(w, \nabla|\partial_{x_3}^j\partial_t^k\phi_\infty|^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\Big) + \Big([\partial_{x_3}^j\partial_t^k, w]\cdot\nabla\phi_\infty, \partial_{x_3}^j\partial_t^k\phi_\infty\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\Big) \\
=\ &-\tfrac{1}{2}\Big(|\partial_{x_3}^j\partial_t^k\phi_\infty|^2, \mathrm{div}\Big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\Big)\Big) + \Big([\partial_{x_3}^j\partial_t^k, w]\cdot\nabla\phi_\infty, \partial_{x_3}^j\partial_t^k\phi_\infty\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\Big)
\end{aligned}
$$

and

$$\partial_{x_3}^j \partial_t^k \dot{\phi}_\infty = -\gamma^2 \mathrm{div}\,(\rho_s \partial_{x_3}^j \partial_t^k w_\infty) + \partial_{x_3}^j \partial_t^k \tilde{f}_\infty^0.$$

We note that the lower order norm $\|u_\infty\|_{H^1\times H^2}^2$ arises from the lower order term Bu_∞. $\square$

We next state the interior estimate and the boundary estimates of the tangential derivatives.

Proposition 7.4. *Under the assumption* (7.1) *(with ν_0, γ_0 and ω_0^{-1} replaced by suitably larger ones), the following assertions hold.*

(i) *There exists positive constant b such that the estimate*

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big\{\tfrac{1}{\gamma^2}\Big\|\chi_0\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_{x'}^2\phi_\infty\Big\|_2^2 + \big\|\chi_0\sqrt{\rho_s}\partial_{x'}^2 w_\infty\big\|_2^2\Big\}\\
&+\tfrac{1}{2}\nu\|\chi_0\nabla\partial_{x'}^2 w_\infty\|_2^2 + \tfrac{1}{2}\tilde{\nu}\|\chi_0\mathrm{div}\partial_{x'}^2 w_\infty\|_2^2 + b\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_0\partial_{x'}^2\dot{\phi}_\infty\|_2^2\\
&\le \big(\epsilon + C\tfrac{\nu+\tilde{\nu}}{\gamma^4}\big)\|\partial_{x'}^2\phi_\infty\|_2^2 + C_{\epsilon\nu\tilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2}^2 + |A^{(0)}|
\end{aligned}\tag{7.7}$$

holds for any $\epsilon > 0$. Here

$$\begin{aligned}
A^{(0)} =&\tfrac{1}{2}\Big(|\partial_{x'}^2\phi_\infty|^2, \mathrm{div}\big(\chi_0^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big) + \Big([\partial_{x'}^2, w\cdot\nabla]\phi_\infty, \chi_0^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial_{x'}^2\phi_\infty\Big)\\
&+\Big(\partial_{x'}^2\tilde{f}_\infty^0, \chi_0^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial_{x'}^2\phi_\infty\Big) + \Big(\partial_{x'}f_\infty, \partial_{x'}(\chi_0^2\rho_s\partial_{x'}^2 w_\infty)\Big)\\
&+Cb\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_0\partial_{x'}^2\tilde{f}_\infty^0\|_2^2.
\end{aligned}$$

(ii) *Let $1 \le m \le N$. There exists positive constant b such that the estimate*

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big\{\tfrac{1}{\gamma^2}\Big\|\chi_m\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial^k\partial_{x_3}^j\phi_\infty\Big\|_2^2 + \big\|\chi_m\sqrt{\rho_s}\partial^k\partial_{x_3}^j w_\infty\big\|_2^2\Big\}\\
&+\tfrac{1}{2}\nu\|\chi_m\nabla\partial^k\partial_{x_3}^j w_\infty\|_2^2 + \tfrac{1}{2}\tilde{\nu}\|\chi_m\mathrm{div}\partial^k\partial_{x_3}^j w_\infty\|_2^2 + b\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_m\partial^k\partial_{x_3}^j\dot{\phi}_\infty\|_2^2\\
&\le \big(\epsilon + C\tfrac{1}{\gamma^2}\big)\|\partial_x^2\phi_\infty\|_2^2 + C_{\epsilon\nu\gamma}\|u_\infty\|_{H^1\times H^2} + |A_{0,k,j}^{(m)}|
\end{aligned}\tag{7.8}$$

holds for $(k, j) = (2, 0)$, $(1, 1)$ and any $\epsilon > 0$. Here

$$\begin{aligned}
A_{0,k,j}^{(m)} =&\tfrac{1}{2}\Big(|\partial^k\partial_{x_3}^j\phi_\infty|^2, \mathrm{div}\big(\chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\\
&+\Big([\partial^k\partial_{x_3}^j, w\cdot\nabla]\phi_\infty, \chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial^k\partial_{x_3}^j\phi_\infty\Big)\\
&+\Big(\partial^k\partial_{x_3}^j\tilde{f}_\infty^0, \chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}\partial^k\partial_{x_3}^j\phi_\infty\Big) + \Big(\partial^{k-1}\partial_{x_3}^j f_\infty, \partial(\chi_m^2\rho_s\partial^k\partial_{x_3}^j w_\infty)\Big)\\
&+Cb\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_m\partial^k\partial_{x_3}^j\tilde{f}_\infty^0\|_2^2.
\end{aligned}$$

Outline of Proof. Let us consider (ii). We apply $\partial^k\partial_{x_3}^j$ to (7.6) to have

$$\partial_t\partial^k\partial_{x_3}^j u_\infty + A\partial^k\partial_{x_3}^j u_\infty = -[\partial^k, A]\partial_{x_3}^j u_\infty - \partial^k B\partial_{x_3}^j u_\infty + \partial^k\partial_{x_3}^j \boldsymbol{F}_\infty.$$

Here $[\partial^k, A]$ denotes the commutator of ∂^k and A. We take the weighted inner product $\langle\cdot,\cdot\rangle$ of this equation with $\chi_m^2\partial^k\partial_{x_3}^j u_\infty$. One can then arrive at the desired estimate in (i) in a similar manner to the proof of Proposition 7.3. We note that the lower order norm $\|u_\infty\|^2_{H^1\times H^2}$ arises from the lower order term Bu_∞ and the term including the commutator $[\partial^k, A]$. The estimate in (i) can be obtained similarly. □

The normal derivatives of ϕ_∞ is estimated as follows.

Proposition 7.5. *Let $1 \le m \le N$. Under the assumption* (7.1) *(with ν_0, γ_0 and ω_0^{-1} replaced by suitably larger ones), there exists positive constant b such that the estimate*

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big(\tfrac{1}{\gamma^2}\Big\|\chi_m\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_n^{l+1}\partial^k\partial_{x_3}^j\phi_\infty\Big\|_2^2\Big) + \tfrac{1}{2}\tfrac{1}{\nu+\tilde{\nu}}\Big\|\chi_m\tfrac{P'(\rho_s)}{\gamma^2}\partial_n^{l+1}\partial^k\partial_{x_3}^j\phi_\infty\Big\|_2^2\\
&+ b\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_m\partial_n^{l+1}\partial^k\partial_{x_3}^j\dot{\phi}_\infty\|_2^2\\
&\le C\Big\{\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_x^2\phi_\infty\|_2^2 + \tfrac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_x w_\infty\|_2^2 + \tfrac{\nu^2}{\nu+\tilde{\nu}}\Big(\|\chi_m\partial_n^l\partial^k\partial_{x_3}^{j+2}w_\infty\|_2^2\\
&\quad + \|\chi_m\nabla\partial_n^l\partial^k\partial_{x_3}^{j+1}w_\infty\|_2^2 + \|\chi_m\nabla\partial_n^l\partial^{k+1}\partial_{x_3}^j w_\infty\|_2^2\Big)\Big\}\\
&\quad + C_{\nu\tilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2} + |A^{(m)}_{l+1,k,j}|
\end{aligned} \tag{7.9}$$

holds for $j,k,l \ge 0$ satisfying $j+k+l = 1$. Here

$$\begin{aligned}
A^{(m)}_{l+1,k,j} =& \tfrac{1}{2}\sum_{j+k+l=1}\Big|\Big(\big|\partial_n^{l+1}\partial^k\partial_{x_3}^j\phi_\infty\big|^2, \operatorname{div}\big(\chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big|\\
&+ C\sum_{j+k+l=1}\Big\|\chi_m[\partial_n^{l+1}\partial^k\partial_{x_3}^j, w\cdot\nabla]\phi_\infty\Big\|_2^2\\
&+ C(b+1)\big(\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_m\partial_n^{l+1}\partial^k\partial_{x_3}^j\tilde{f}^0_\infty\|_2^2 + \|f_\infty\|_{H^1}^2\big).
\end{aligned}$$

Outline of Proof. We transform a scalar field $p(x')$ on $D\cap\mathcal{O}_m$ as

$$\tilde{p}(y') = p(x') \quad \big(y' = \Psi^m(x'),\ x'\in D\cap\mathcal{O}_m\big),$$

where $\Psi^m(x')$ is a function given in section 2. Similarly we transform a vector field $v(x') = {}^T\big(v^1(x'), v^2(x'), v^3(x')\big)$ into $\tilde{v}(y') = {}^T\big(\tilde{v}^1(y'), \tilde{v}^2(y'), \tilde{v}^3(y')\big)$ as

$$v(x') = E(y')\tilde{v}(y')$$

where $E(y') = \big(e_1(y'), e_2(y'), e_3\big)$ with $e_1(y')$, $e_2(y')$ and e_3 given in section 2. From the proof of [1, Proposition 4.16], we have

$$\partial_\tau\partial_{y_1}\widetilde{\phi_\infty} + \tilde{v}_s^3\partial_{y_3}\partial_{y_1}\phi_\infty + \frac{\tilde{\rho}_s\widetilde{P}'(\tilde{\rho}_s)}{\nu+\tilde{\nu}}\partial_{y_1}\widetilde{\phi_\infty} = \tilde{\rho}_s h, \tag{7.10}$$

where

$$\begin{aligned}
h =& -\tfrac{\gamma^2}{\nu+\tilde{\nu}}\Big\{\partial_\tau\tilde{w}^1 + \nu\big(\mathrm{rot}_y\mathrm{rot}_y\tilde{w}\big)^1 + \tilde{\rho}_s\partial_{y_1}\Big(\tfrac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2\tilde{\rho}_s}\Big)\tilde{\phi} + \tfrac{\nu}{\gamma^2}\tilde{\rho}_s\big(\Delta_{y'}\tilde{v}_s\big)^1\tilde{\phi} + \tilde{\rho}_s\tilde{v}_s{}^3\partial_{y_3}\tilde{w}^1\Big\}\\
&-\Big\{\tfrac{1}{\tilde{\rho}_s}\partial_{y_1}\tilde{v}_s{}^3\partial_{y_3}\tilde{\phi} + \gamma^2\tfrac{1}{\tilde{\rho}_s}\partial_{y_1}\big(\mathrm{div}_y(\tilde{\rho}_s\tilde{w})\big) - \gamma^2\partial_{y_1}\mathrm{div}_y\tilde{w}\Big\}.
\end{aligned}$$

Here $\big(\mathrm{rot}_y\tilde{w}\big)^1$ denotes the $e_1(y')$ component of $\mathrm{rot}_y\tilde{w}$, and so on. We note that $\big(\mathrm{rot}_y\mathrm{rot}_y\tilde{w}\big)^1$ does not contain $\partial_{y_1}^2$. See the proof of [1, Proposition 4.16]. Applying $\partial_{y_1}^l\partial_{y_2}^k\partial_{x_3}^j$ to (7.10) and take the L^2 inner product with $\chi_m^2\frac{P'(\rho_s)}{\gamma^4\rho_s}\partial_{y_1}^{l+1}\partial_{y_2}^k\partial_{x_3}^j\phi_\infty$, one can obtain the desired result. □

Using the estimate for the Stokes system we have the following estimates.

Proposition 7.6. *Under the assumption* (7.1) *(with ν_0, γ_0 and ω_0^{-1} replaced by suitably larger ones), the following assertions hold.*

(i) *There holds the estimate*

$$
\begin{aligned}
&\tfrac{\nu^2}{\nu+\tilde{\nu}}\|\partial_x^3 w_\infty\|_2^2+\tfrac{1}{\nu+\tilde{\nu}}\|\partial_x^2\phi_\infty\|_2^2\\
&\le C\big\{\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_x^2\dot{\phi}_\infty\|_2^2+\tfrac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_x w_\infty\|_2^2+\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\widetilde{f}_\infty^0\|_{H^2}^2+\tfrac{1}{\nu+\tilde{\nu}}\|f_\infty\|_{H^1}^2\big\}\\
&+C_{\nu\tilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2}^2.
\end{aligned}
\tag{7.11}
$$

(ii) *Let $1\le m\le N$. There holds the estimate*

$$
\begin{aligned}
&\tfrac{\nu^2}{\nu+\tilde{\nu}}\|\chi_m\partial_x^2\partial w_\infty\|_2^2+\tfrac{1}{\nu+\tilde{\nu}}\|\chi_m\partial_x\partial\phi_\infty\|_2^2\\
&\le C\big\{\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_m\partial\partial_{x_3}\phi_\infty\|_2^2+\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\chi_m\partial_x\partial\dot{\phi}_\infty\|_2^2+\tfrac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_x w_\infty\|_2^2\\
&+\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\widetilde{f}_\infty^0\|_{H^2}^2+\tfrac{1}{\nu+\tilde{\nu}}\|f_\infty\|_{H^1}^2\big\}+C_{\nu\tilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2}^2.
\end{aligned}
\tag{7.12}
$$

(iii) *There holds the estimate*

$$
\begin{aligned}
&\tfrac{\nu^2}{\nu+\tilde{\nu}}\|\partial_x^2\partial_{x_3}w_\infty\|_2^2+\tfrac{1}{\nu+\tilde{\nu}}\|\partial_x\partial_{x_3}\phi_\infty\|_2^2\\
&\le C\big\{\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_x\partial_{x_3}\dot{\phi}_\infty\|_2^2+\tfrac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_x w_\infty\|_2^2+\tfrac{\nu+\tilde{\nu}}{\gamma^4}\|\widetilde{f}_\infty^0\|_{H^2}^2+\tfrac{1}{\nu+\tilde{\nu}}\|f_\infty\|_{H^1}^2\big\}\\
&+C_{\nu\tilde{\nu}\gamma}\|u_\infty\|_{H^1\times H^2}^2.
\end{aligned}
\tag{7.13}
$$

Outline of Proof. We write problem (5.3) as

$$\partial_t\phi_\infty+v_s^3\partial_{x_3}\phi_\infty+\gamma^2\mathrm{div}(\rho_s w_\infty)+w\cdot\nabla\phi_\infty=\tilde{f}_\infty^0, \tag{7.14}$$

$$
\begin{aligned}
&\partial_t w_\infty-\tfrac{\nu}{\rho_s}\Delta w_\infty-\tfrac{\tilde{\nu}}{\rho_s}\nabla\mathrm{div}w_\infty+\nabla\Big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\Big)\\
&\qquad+\tfrac{\nu\Delta' v_s^3}{\gamma^2\rho_s^2}\phi_\infty e_3+v_s^3\partial_{x_3}w_\infty+(w'_\infty\cdot\nabla' v_s^3)e_3=f_\infty,
\end{aligned}
\tag{7.15}
$$

$$w_\infty|_{\partial\Omega}=0, \tag{7.16}$$

$$(\phi_\infty,w_\infty)|_{t=0}=(\phi_{\infty,0},w_{\infty,0}). \tag{7.17}$$

By (7.14) and (7.15), we have

$$
\begin{aligned}
\mathrm{div}\,w_\infty &= G,\\
-\Delta w_\infty+\nabla\left(\tfrac{P'(\rho_s)}{\nu\gamma^2}\phi_\infty\right) &= F,\\
w_\infty|_{\partial\Omega} &= 0.
\end{aligned}
\tag{7.18}
$$

Here

$$G=\frac{1}{\gamma^2}\Big(\tilde{f}_\infty^0-\dot{\phi}_\infty-\gamma^2\mathrm{div}\,((\rho_s-1)w_\infty)\Big),$$

$$F=\frac{\rho_s}{\nu}f_\infty-\frac{\rho_s}{\nu}\Big\{\partial_t w_\infty-\tfrac{\tilde{\nu}}{\rho_s}\nabla G+\tfrac{\nu\Delta' v_s^3}{\gamma^2\rho_s^2}\phi_\infty e_3+v_s^3\partial_{x_3}w_\infty+(w'_\infty\cdot\nabla' v_s^3)e_3\Big\}.$$

Applying the estimate for the Stokes system, we have the estimate in (i). The estimates in (ii) and (iii) can be obtained similarly by applying $\chi_m \partial$ and ∂_{x_3} to (7.18), respectively. $\square$.

We are now in a position to prove Proposition 7.1.

Proof of Proposition 7.1 Let b_1 and b_2 be constants satisfying $b_1, b_2 > 1$. Define $\widetilde{\mathcal{E}}_2[u_\infty]$ by

$$
\begin{aligned}
\widetilde{\mathcal{E}}_2[u_\infty] =& \tfrac{1}{\gamma^2} \sum_{m=1}^{N} \Big\{ b_1 \Big(\Big\| \chi_m \sqrt{\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial^2 \phi_\infty \Big\|_2^2 + \Big\| \chi_m \sqrt{\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \partial_{x_3} \phi_\infty \Big\|_2^2 \Big) \\
& + \Big\| \chi_m \sqrt{\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \partial \phi_\infty \Big\|_2^2 + \Big\| \chi_m \sqrt{\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \partial_{x_3} \phi_\infty \Big\|_2^2 \Big\} \\
& + \tfrac{1}{\gamma^2} \Big(\Big\| \chi_0 \sqrt{\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'}^2 \phi_\infty \Big\|_2^2 + b_1 \Big\| \sqrt{\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x_3}^2 \phi_\infty \Big\|_2^2 \Big) \\
& + b_1 \sum_{m=1}^{N} \Big(\| \chi_m \sqrt{\rho_s} \partial^2 w_\infty \|_2^2 + \| \chi_m \sqrt{\rho_s} \partial \partial_{x_3} w_\infty \|_2^2 \Big) \\
& + \| \chi_0 \sqrt{\rho_s} \partial_{x'}^2 w_\infty \|_2^2 + b_1 \| \sqrt{\rho_s} \partial_{x_3}^2 w_\infty \|_2^2
\end{aligned}
$$

for $u_\infty = {}^T(\phi_\infty, w_\infty)$. We compute

$$
\begin{aligned}
& b_2 \Big[\sum_{m=1}^{N} \Big\{ b_1 \big\{ (7.8) \mid_{(k,j)=(2,0)} + (7.8) \mid_{(k,j)=(1,1)} \big\} \\
& + (7.9) \mid_{(l,k,j)=(0,1,0)} + (7.9) \mid_{(l,k,j)=(0,0,1)} \Big\} + (7.7) + b_1 (7.4) \Big] \\
& + \sum_{m=1}^{N} (7.12) + (7.13).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \tfrac{1}{2} \tfrac{d}{dt} b_2 \widetilde{\mathcal{E}}_2[u_\infty] + b b_2 \tfrac{\nu + \tilde{\nu}}{\gamma^4} \Big(\sum_{m=1}^{N} \| \chi_m \partial_x \partial \dot{\phi}_\infty \|_2^2 + \| \partial_x \partial_{x_3} \dot{\phi}_\infty \|_2^2 \Big) \\
& + \tfrac{\nu^2}{\nu + \tilde{\nu}} \Big(\sum_{m=1}^{N} \| \chi_m \partial_x^2 \partial w_\infty \|_2^2 + \| \partial_x^2 \partial_{x_3} w_\infty \|_2^2 \Big) \\
& + \tfrac{1}{\nu + \tilde{\nu}} \Big(\sum_{m=1}^{N} \| \chi_m \partial_x \partial \phi_\infty \|_2^2 + \| \chi_m \partial_x \partial_{x_3} \phi_\infty \|_2^2 \Big) \\
& + \tfrac{b_2}{2} \nu \Big\{ b_1 \sum_{m=1}^{N} \big(\| \chi_m \nabla \partial^2 w_\infty \|_2^2 + \| \chi_m \nabla \partial \partial_{x_3} w_\infty \|_2^2 \big) + \| \chi_0 \nabla \partial_{x'}^2 w_\infty \|_2^2 + b_1 \| \nabla \partial_{x_3}^2 w_\infty \|_2^2 \Big\} \\
& + \tfrac{b_2}{2} \tilde{\nu} \Big\{ b_1 \sum_{m=1}^{N} \big(\| \chi_m \mathrm{div} \partial^2 w_\infty \|_2^2 + \| \chi_m \mathrm{div} \partial \partial_{x_3} w_\infty \|_2^2 \big) + \| \chi_0 \mathrm{div} \partial_{x'}^2 w_\infty \|_2^2 \\
& + b_1 \| \mathrm{div} \partial_{x_3}^2 w_\infty \|_2^2 \Big\} + \tfrac{b_2}{2} \tfrac{1}{\nu + \tilde{\nu}} \sum_{m=1}^{N} \Big(\Big\| \chi_m \tfrac{P'(\rho_s)}{\gamma^2} \partial_n \partial \phi_\infty \Big\|_2^2 + \Big\| \chi_m \tfrac{P'(\rho_s)}{\gamma^2} \partial_n \partial_{x_3} \phi_\infty \Big\|_2^2 \Big) \\
& \le C_{b_1 b_2} \big\{ \big(\epsilon + \tfrac{1}{\gamma^2} + \tfrac{\nu + \tilde{\nu}}{\gamma^4} \big) \| \partial_x^2 \phi_\infty \|_2^2 + C_{\epsilon \nu \gamma \omega} \| u_\infty \|_{H^1 \times H^2}^2 + \tfrac{1}{\nu + \tilde{\nu}} \| \partial_t \partial_x w_\infty \|_2^2 + \mathcal{R}_0 \big\} \\
& + C_1 \Big\{ b_2 \tfrac{\nu^2}{\nu + \tilde{\nu}} \sum_{m=1}^{N} \big(\| \chi_m \nabla \partial^2 w_\infty \|_2^2 + \| \chi_m \nabla \partial \partial_{x_3} w_\infty \|_2^2 + \| \chi_m \nabla \partial_{x_3}^2 w_\infty \|_2^2 \big) \\
& + \tfrac{\nu + \tilde{\nu}}{\gamma^4} \Big(\sum_{m=1}^{N} \| \chi_m \partial_x \partial \dot{\phi}_\infty \|_2^2 + \| \partial_x \partial_{x_3} \dot{\phi}_\infty \|_2^2 \Big) \Big\}
\end{aligned}
$$

for any $\epsilon > 0$. Here

$$\mathcal{R}_0 = \sum_{m=1}^N (|A^{(m)}_{0,2,0}| + |A^{(m)}_{0,1,1}|) + |A^{(0)}| + |A_{0,0,2}| + \sum_{m=1}^N (|A^{(m)}_{1,1,0}| + |A^{(m)}_{1,0,1}|).$$

Fix $b_1 > 8C_1$ and $b_2 > 8\frac{C_1}{b}$. It then follows that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}b_2\widetilde{\mathcal{E}}_2[u_\infty] + b\tfrac{\nu+\widetilde{\nu}}{\gamma^4}\Big(\sum_{m=1}^N \|\chi_m\partial_x\partial\dot{\phi}_\infty\|_2^2 + \|\partial_x\partial_{x_3}\dot{\phi}_\infty\|_2^2\Big)\\
&+ \tfrac{\nu^2}{\nu+\widetilde{\nu}}\Big(\sum_{m=1}^N \|\chi_m\partial_x^2\partial w_\infty\|_2^2 + \|\partial_x^2\partial_{x_3}w_\infty\|_2^2\Big)\\
&+ \tfrac{1}{\nu+\widetilde{\nu}}\Big(\sum_{m=1}^N \|\chi_m\partial_x\partial\phi_\infty\|_2^2 + \|\chi_m\partial_x\partial_{x_3}\phi_\infty\|_2^2\Big)\\
&+ \tfrac{b_2}{2}I_1[w_\infty] + \tfrac{b_2}{2}\tfrac{1}{\nu+\widetilde{\nu}}\sum_{m=1}^N\Big(\Big\|\chi_m\tfrac{P'(\rho_s)}{\gamma^2}\partial_n\partial\phi_\infty\Big\|_2^2 + \Big\|\chi_m\tfrac{P'(\rho_s)}{\gamma^2}\partial_n\partial_{x_3}\phi_\infty\Big\|_2^2\Big)\\
&\le C_{b_1b_2}\big\{\big(\epsilon + \tfrac{1}{\gamma^2} + \tfrac{\nu+\widetilde{\nu}}{\gamma^4}\big)\|\partial_x^2\phi_\infty\|_2^2 + C_{\epsilon\nu\gamma\omega}\|u_\infty\|^2_{H^1\times H^2}\\
&\qquad\quad + \tfrac{1}{\nu+\widetilde{\nu}}\|\partial_t\partial_x w_\infty\|_2^2 + \mathcal{R}_0\big\}
\end{aligned} \tag{7.19}$$

for any $\epsilon > 0$. Here $I_1[w_\infty]$ is given by

$$\begin{aligned}
I_1[w_\infty] =&\nu\Big\{\sum_{m=1}^N \big(\|\chi_m\nabla\partial^2 w_\infty\|_2^2 + \|\chi_m\nabla\partial\partial_{x_3}w_\infty\|_2^2\big)\\
&+ \|\chi_0\nabla\partial^2_{x'}w_\infty\|_2^2 + \|\nabla\partial^2_{x_3}w_\infty\|_2^2\Big\}\\
&+ \widetilde{\nu}\Big\{\sum_{m=1}^N \big(\|\chi_m\mathrm{div}\partial^2 w_\infty\|_2^2 + \|\chi_m\mathrm{div}\partial\partial_{x_3}w_\infty\|_2^2\big)\\
&+ \|\chi_0\mathrm{div}\partial^2_{x'}w_\infty\|_2^2 + \|\mathrm{div}\partial^2_{x_3}w_\infty\|_2^2\Big\}.
\end{aligned}$$

Let b_3 and b_4 be constants satisfying $b_3, b_4 > 1$. Define $\mathcal{E}_2[u_\infty]$ by

$$\begin{aligned}
\mathcal{E}_2[u_\infty] =&b_2b_3b_4\widetilde{\mathcal{E}}_2[u_\infty] + b_4\tfrac{1}{\gamma^2}\sum_{m=1}^N\Big\|\chi_m\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_n^2\phi_\infty\Big\|_2^2\\
&+ \Big(\tfrac{1}{\gamma^2}\Big\|\sqrt{\tfrac{P'(\rho_s)}{\gamma^2\rho_s}}\partial_t\phi_\infty\Big\|_2^2 + \big\|\sqrt{\rho_s}\partial_t w_\infty\big\|_2^2\Big)
\end{aligned}$$

for $u_\infty = {}^T(\phi_\infty, w_\infty)$. We compute

$$b_4\Big\{b_3(7.19) + \sum_{m=1}^N (7.9)\mid_{(l,k,j)=(1,0,0)}\Big\} + (7.11) + b_5(7.3).$$

It follows that

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\mathcal{E}_2[u_\infty] + bb_4\tfrac{\nu+\widetilde{\nu}}{\gamma^4}\|\partial_x^2\dot{\phi}_\infty\|_2^2 + \tfrac{\nu^2}{\nu+\widetilde{\nu}}\|\partial_x^3 w_\infty\|_2^2 + \tfrac{1}{\nu+\widetilde{\nu}}\|\partial_x^2\phi_\infty\|_2^2\\
&+ b_3b_4\Big\{\tfrac{\nu^2}{\nu+\widetilde{\nu}}\Big(\sum_{m=1}^N \|\chi_m\partial_x^2\partial w_\infty\|_2^2 + \|\partial_x^2\partial_{x_3}w_\infty\|_2^2\Big)\\
&+ \tfrac{1}{\nu+\widetilde{\nu}}\Big(\sum_{m=1}^N \|\chi_m\partial_x\partial\phi_\infty\|_2^2 + \|\partial_x\partial_{x_3}\phi_\infty\|_2^2\Big)\Big\} + \tfrac{1}{2}b_2b_3b_4 I_1[w_\infty]\\
&+ \tfrac{b_4}{2}\tfrac{1}{\nu+\widetilde{\nu}}\sum_{m=1}^N \|\chi_m\tfrac{P'(\rho_s)}{\gamma^2}\partial_n\nabla\phi_\infty\|_2^2 + \tfrac{1}{2}\big\{\nu\|\nabla\partial_t w_\infty\|_2^2 + \widetilde{\nu}\|\mathrm{div}\partial_t w_\infty\|_2^2 + c\tfrac{\nu+\widetilde{\nu}}{\gamma^4}\|\partial_t\phi_\infty\|_2^2\big\}\\
&\le C_{b_1\cdots b_4}\big\{\big(\epsilon + \tfrac{1}{\gamma^2} + \tfrac{\nu+\widetilde{\nu}}{\gamma^4}\big)\|\partial_x^2\phi_\infty\|_2^2 + \tfrac{1}{\nu+\widetilde{\nu}}\|\partial_t\partial_x w_\infty\|_2^2 + C_{\epsilon\nu\gamma\omega}\|u_\infty\|_{H^1\times H^2}^2\\
&+ \mathcal{R}\big\} + C_2\big\{b_4\tfrac{\nu^2}{\nu+\widetilde{\nu}}\sum_{m=1}^N\big(\|\chi_m\partial_n\partial_{x_3}^2 w_\infty\|_2^2 + \|\chi_m\nabla\partial_n\partial_{x_3}w_\infty\|_2^2\\
&+ \|\chi_m\nabla\partial_n\partial w_\infty\|_2^2 + \tfrac{\nu+\widetilde{\nu}}{\gamma^4}\|\partial_x^2\dot{\phi}_\infty\|_2^2\big)\big\}
\end{aligned}
$$

for any $\epsilon > 0$. Here

$$\mathcal{R} = \mathcal{R}_0 + \sum_{m=1}^N |A_{2,0,0}^{(m)}| + |A_1|.$$

Fix b_3 and b_4 so large that $b_3 > 8C_2$ and $b_4 > 2\frac{C_2}{b}$. We assume that ν, $\widetilde{\nu}$ and γ also satisfy $\gamma^2 > 8C_{b_1\cdots b_4}$ and $\gamma^2 > 8C_{b_1\cdots b_4}(\nu+\widetilde{\nu})$. We take $\epsilon > 0$ sufficiently small such that $\epsilon < \frac{1}{8C_{b_1\cdots b_4}}\frac{1}{\nu+\widetilde{\nu}}$. It then follows that

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\mathcal{E}_2[u_\infty] + b\tfrac{\nu+\widetilde{\nu}}{\gamma^4}\|\partial_x^2\dot{\phi}_\infty\|_2^2 + \tfrac{1}{2}\tfrac{\nu^2}{\nu+\widetilde{\nu}}\|\partial_x^3 w_\infty\|_2^2 + \tfrac{1}{2}\tfrac{1}{\nu+\widetilde{\nu}}\|\partial_x^2\phi_\infty\|_2^2\\
&+ \tfrac{\nu^2}{\nu+\widetilde{\nu}}\Big(\sum_{m=1}^N \|\chi_m\partial_x^2\partial w_\infty\|_2^2 + \|\partial_x^2\partial_{x_3}w_\infty\|_2^2\Big)\\
&+ \tfrac{1}{\nu+\widetilde{\nu}}\Big(\sum_{m=1}^N \|\chi_m\partial_x\partial\phi_\infty\|_2^2 + \|\partial_x\partial_{x_3}\phi_\infty\|_2^2\Big)\\
&+ I_1[w_\infty] + \tfrac{1}{\nu+\widetilde{\nu}}\sum_{m=1}^N \|\chi_m\tfrac{P'(\rho_s)}{\gamma^2}\partial_n\nabla\phi_\infty\|_2^2\\
&+ \tfrac{1}{2}\big\{\nu\|\nabla\partial_t w_\infty\|_2^2 + \widetilde{\nu}\|\mathrm{div}\partial_t w_\infty\|_2^2 + c\tfrac{\nu+\widetilde{\nu}}{\gamma^4}\|\partial_t\phi_\infty\|_2^2\big\}\\
&\le \{C_{\epsilon\nu\gamma\omega}\|u_\infty\|_{H^1\times H^2}^2 + \mathcal{R}\}.
\end{aligned}
\tag{7.20}
$$

We thus see that there are positive constants c_1, c_2 and C_3 such that

$$
\begin{aligned}
&\tfrac{d}{dt}\mathcal{E}_2[u_\infty] + c_1\mathcal{E}_2[u_\infty]\\
&+ c_2\big(\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2\phi_\infty\|_2^2 + \|\dot{\phi}_\infty\|_{H^2} + \|\partial_t w_\infty\|_{H^1}^2 + \|\partial_t\phi_\infty\|_2^2\big)\\
&\le C_{\nu\widetilde{\nu}\gamma}(\|u_\infty\|_{H^1\times H^2} + \mathcal{R}).
\end{aligned}
$$

Since

$$\|\partial_x^2 w_\infty\|_2^2 \le \eta\|\partial_x^3 w_\infty\|_2^2 + C_\eta\|w_\infty\|_2^2$$

holds for any $\eta > 0$, taking η so small that $\eta < \frac{1}{2}\min\left\{\frac{c_2}{c_2+C_{\nu\tilde{\nu}\gamma}}, 1\right\}$, we obtain

$$
\begin{aligned}
&\tfrac{d}{dt}\mathcal{E}_2[u_\infty] + c_1\mathcal{E}_2[u_\infty] \\
&+ \frac{c_2}{2}\big(\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2\phi_\infty\|_2^2 + \|\dot{\phi}_\infty\|_{H^1}^2 + \|\partial_t w_\infty\|_{H^1}^2 + \|\partial_t\phi_\infty\|_2^2\big) \\
&\le 2C_{\nu\tilde{\nu}\gamma}(\|u_\infty\|_{H^1\times L^2}^2 + \mathcal{R}).
\end{aligned}
\tag{7.21}
$$

We see from (7.21) and Proposition 7.2 that there exist positive constants $\widetilde{c}_1$, $\widetilde{c}_2$ and $C_{\nu\tilde{\nu}\gamma}$ such that

$$
\begin{aligned}
&\mathcal{E}_2[u_\infty(t)] + \|u_\infty(t)\|_{H^1}^2 + \widetilde{c}_2\int_0^t e^{-\widetilde{c}_1(t-\tau)}\big(\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2\phi_\infty\|_2^2 + \|u_\infty\|_{H^1}^2 \\
&+ \|\dot{\phi}_\infty\|_{H^1}^2 + \|\partial_t u_\infty\|_{H^1\times L^2}^2\big)d\tau \\
&\le C_{\nu\tilde{\nu}\gamma}\Big\{e^{-\widetilde{c}_1 t}\big(\mathcal{E}_2[u_{\infty,0}] + \|u_{\infty,0}\|_{H^1}^2\big) + \sup_{0\le\tau\le t}\|\boldsymbol{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-\widetilde{c}_1(t-\tau)}\mathcal{R}d\tau\Big\}.
\end{aligned}
\tag{7.22}
$$

It remains to estimate the term $\|\partial_x^2 w_\infty(t)\|_2$. We write the second equation of (5.1) as

$$
-\nu\Delta w_\infty - \widetilde{\nu}\nabla\mathrm{div} w_\infty = J, \quad w_\infty\,|_{\partial\Omega} = 0,
$$

where

$$
J = -\rho_s\Big\{\partial_t w_\infty + \nabla\Big(\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\Big) + \tfrac{\nu\Delta' v_s^3}{\gamma^2\rho_s}\phi_\infty e_3 + v_s^3\partial_{x_3}w_\infty + (w_\infty'\cdot\nabla' v_s^3)e_3 - f_\infty\Big\}.
$$

Since $J \in L^2(\Omega)$, we obtain, by elliptic estimate,

$$
\|\partial_x^2 w_\infty\|_2^2 \le C(\|w_\infty\|_2^2 + \|J\|_2^2) \le C_{\nu\tilde{\nu}\gamma}(\mathcal{E}_2[u_\infty] + \|u_\infty\|_{H^1}^2 + \|f_\infty\|_2^2).
$$

From this, with (7.22), we see that

$$
\begin{aligned}
&\mathcal{E}_2[u_\infty(t)] + \|u_\infty(t)\|_{H^1}^2 + \|\partial_x^2 w_\infty(t)\|_2^2 + \widetilde{c}_2\int_0^t e^{-\widetilde{c}_1(t-\tau)}\big(\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2\phi_\infty\|_2^2 \\
&+ \|u_\infty\|_{H^1}^2 + \|\dot{\phi}_\infty\|_{H^1}^2 + \|\partial_t u_\infty\|_{H^1\times L^2}^2\big)d\tau \\
&\le C_{\nu\tilde{\nu}\gamma}\Big\{e^{-\widetilde{c}_1 t}\big(\mathcal{E}_2[u_{\infty,0}] + \|u_{\infty,0}\|_{H^1}^2\big) + \sup_{0\le\tau\le t}\|\boldsymbol{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-\widetilde{c}_1(t-\tau)}\mathcal{R}d\tau\Big\}.
\end{aligned}
\tag{7.23}
$$

As we will see in section 8 below, it holds that

$$
\sup_{0\le\tau\le t}\|\boldsymbol{F}_\infty(\tau)\|_2^2 \le C(1+t)^{-\frac{3}{2}}M(t)^4. \tag{7.24}
$$

Proposition 7.1 follows from (7.23) and (7.24). This completes the proof. □

8 Estimates of nonlinear terms

In this section we prove the estimate (7.24) and (5.6).

We first make an observation. By the Sobolev inequality we have

$$\|\phi(t)\|_\infty \le C[\![\phi(t)]\!]_{H^2} \le C_1[\![u(t)]\!]_2$$

for a positive constant C_1. It then follows that

$$\rho(x,t) = \rho_s(x') + \gamma^{-2}\phi(x,t) \ge \rho_1 - \gamma^{-2}\|\phi(t)\|_\infty \ge \rho_1 - C_1\gamma^{-2}[\![u(t)]\!]_2.$$

Fix a positive constant ϵ_s satisfying $\epsilon_s \le \frac{1}{4}\frac{\gamma^2\rho_1}{C_1}$. If $[\![u(t)]\!]_2 \le \epsilon_s$, then it holds that

$$\|\phi(t)\|_\infty \le \tfrac{1}{4}\gamma^2\rho_1, \quad \rho(x,t) \ge \tfrac{3}{4}\rho_1 > 0.$$

This implies that $\widetilde{Q}\boldsymbol{F}(t)$ is smooth whenever $[\![u(t)]\!]_2 \le \epsilon_s$.

We will show the following

Proposition 8.1. *If $[\![u(t)]\!]_2 \le \epsilon_s$ and $M(t) \le 1$, then*

$$\sup_{0\le\tau\le t} \|\boldsymbol{F}_\infty(\tau)\|_2^2 \le C(1+t)^{-\frac{3}{2}}M(t)^4, \tag{8.1}$$

$$\mathcal{R}(t) \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\} \tag{8.2}$$

To prove Proposition 8.1, we prepare several lemmas.

Lemma 8.2. (i) *For $2 \le p \le p \le \infty$. If j and k are nonnegative integers satisfying*

$$0 \le j < k, \quad k > j + n(\tfrac{1}{2} - \tfrac{1}{p}),$$

then there exists a positive constant C such that

$$\|\partial_x^j f\|_{L^p(\mathbf{R}^n)} \le \|f\|_{L^2(\mathbf{R}^2)}^{1-a}\|f\|_{H^k(\mathbf{R}^n)}^a.$$

Here $a = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p})$.

(ii) *For $2 \le p \le \infty$. If j and k are nonnegative integers satisfying*

$$0 \le j < k, \quad k > j + 3(\tfrac{1}{2} - \tfrac{1}{p}),$$

then there exists a positive constant C such that

$$\|\partial_x^j f\|_{L^p(\Omega)} \le C\|f\|_{H^k(\Omega)}.$$

(iii) *If $f \in H^1(\Omega)$ and $f = f(x_3)$ is independent of $x' \in D$, then it holds that*

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_{x_3} f\|_{L^2(\Omega)}^{\frac{1}{2}}$$

for a positive constant C.

The proof of Lemma 8.2 can be found, e.g., in [11, 16].

Lemma 8.3. (i) *For nonnegative integers m and m_k ($k=1,\cdots,l$) and multi index α_k ($k=1,\cdots,l$), we assume that*

$$m \ge [\tfrac{n}{2}], \quad 0 \le |\alpha_k| \le m_k \le 2+|\alpha_k| \quad (k=1,\cdots,l)$$

and

$$\sum_{k=1}^{l} m_k \ge 2(l-1) + \sum_{k=1}^{l} |\alpha_k|,$$

then the estimate holds

$$\big\| \prod_{k=1}^{l} \partial_x^{\alpha_k} f_k \big\|_2 \le C \prod_{k=1}^{l} \|f_k\|_{H^{m_k}}$$

for a positive constant C.

(ii) *For $1 \le k \le m$. We assume that $F(x,t,y)$ is a smooth function on $\Omega \times [0,\infty) \times I$ with a compact interval $I \subset \mathbf{R}$. For $|\alpha|+2j=k$ the estimates hold*

$$\begin{aligned}
&\big\| \big[\partial_x^\alpha \partial_t^j, F(x,t,f_1)\big] f_2 \big\|_2 \\
&\le \begin{cases} C_0(t,f_1(t))[\![f_2]\!]_{k-1} + C_1(t,f_1(t))\big\{1+|\!|\!|Df_1|\!|\!|_{m-1}^{|\alpha|+j-1}\big\}|\!|\!|Df_1|\!|\!|_{m-1}[\![f_2]\!]_k, \\ C_0(t,f_1(t))[\![f_2]\!]_{k-1} + C_1(t,f_1(t))\big\{1+|\!|\!|Df_1|\!|\!|_{m-1}^{|\alpha|+j-1}\big\}|\!|\!|Df_1|\!|\!|_{m}[\![f_2]\!]_{k-1}, \end{cases}
\end{aligned}$$

where

$$C_0(t,f_1(t)) = \sum_{(\beta,l)\le(\alpha,j),\,(\beta,l)\ne(0,0)} \sup_x \big| \big(\partial_x^\beta \partial_t^l F\big)\big(x,t,f_1(x,t)\big)\big|,$$

$$C_1(t,f_1(t)) = \sum_{(\beta,l)\le(\alpha,j),\,1\le p\le j+|\alpha|} \sup_x \big| \big(\partial_x^\beta \partial_t^l \partial_y^p F\big)\big(x,t,f_1(x,t)\big)\big|.$$

(iii) *For $m \ge 2$ the estimates hold*

$$\|f_1 \cdot f_2\|_{H^m} \le C_1 \|f_1\|_{H^m} \|f_2\|_{H^m}, \quad [\![f_1 \cdot f_2]\!]_m \le C_2 [\![f_1]\!]_m [\![f_2]\!]_m$$

for a positive constants C_1 and C_2.

See, e.g., [13, 16] for the proof of Lemma 8.3.

We begin with the following preliminary estimates for σ_1 and u_j ($j=1,\infty$).

Lemma 8.4. *We assume that $u(t) = {}^T(\phi(t), w(t)) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t)$ be a solution of (5.1) in $Z(T)$. The following estimates hold for all $t \in [0,T]$ with a positive constant C independent of T.*

(i) $\|\sigma_1(t)\|_2 \le C(1+t)^{-\frac{1}{4}} M(t)$,

(ii) $[\![u(t)]\!]_2 \le C(1+t)^{-\frac{1}{4}} M(t)$,

(iii) $|\!|\!|D\sigma_1(t)|\!|\!| \le C(1+t)^{-\frac{3}{4}} M(t)$,

(iv) $[\![u_j(t)]\!]_2 \le C(1+t)^{-\frac{3}{4}}M(t), \quad (j=1,\infty)$.

(v) $\|\sigma_1(t)\|_\infty \le C(1+t)^{-\frac{1}{2}}M(t)$,

(vi) $\|u_j(t)\|_\infty \le C(1+t)^{-\frac{3}{4}}M(t), \quad (j=1,\infty)$.

(vii) $\|u(t)\|_\infty \le C(1+t)^{-\frac{1}{2}}M(t)$.

Lemma 8.4 easily follows from Lemma 8.2 and the definition of $M(t)$.

Let us estimate the nonlinearities. For $Q_0\boldsymbol{F} = -\mathrm{div}(\phi w)$, we have the following estimates.

Proposition 8.5. *We assume that $u(t) = {}^T(\phi(t), w(t)) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t)$ be a solution of (5.1) in $Z(T)$. If $M(t) \le 1$ for all $t \in [0,T]$, then the estimates hold with a positive constant C independent of T.*

(i) $[\![\phi \mathrm{div} w]\!]_l \le \begin{cases} C(1+t)^{-\frac{5}{4}}M(t)^2 \quad (l=1), \\ C(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)|\!|\!|Dw_\infty(t)|\!|\!|_2 \quad (l=2). \end{cases}$

(ii) $\|w\cdot\nabla\phi_\infty\|_{H^1} \le C(1+t)^{-\frac{5}{4}}M(t)^2$.

(iii) $[\![w\cdot\nabla(\sigma_1\phi^{(0)} + \phi_1)]\!]_2 \le C(1+t)^{-\frac{5}{4}}M(t)^2$.

(iv)
$$\begin{aligned}
&\Big|\Big(|\partial_{x_3}^2\phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| + \Big|\Big(|\partial_{x'}^2\phi_\infty|^2, \mathrm{div}\big(\chi_0^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| \\
&+\Big|\Big(|\partial_t\phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| + \sum_{m=1}^N\Big\{\sum_{j+k=1}\Big|\Big(|\partial^{k+1}\partial_{x_3}^j\phi_\infty|^2, \mathrm{div}\big(\chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| \\
&+\sum_{j+k+l=1}\Big|\Big(\big|\partial_n^{l+1}\partial^k\partial_{x_3}^j\phi_\infty\big|^2, \mathrm{div}\big(\chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big|\Big\} \\
&\le C(1+t)^{-\frac{1}{2}}M(t)D_\infty(t).
\end{aligned}$$

(v)
$$\begin{aligned}
&\|[\partial_{x_3}, w\cdot\nabla]\phi_\infty\|_2 + \|\chi_0[\partial_{x'}^2, w\cdot\nabla]\phi_\infty\|_2 + \|[\partial_t, w\cdot\nabla]\phi_\infty\|_2 \\
&+\sum_{m=1}^N\Big\{\sum_{j+k=1}\Big\|\chi_m[\partial^{k+1}\partial_{x_3}^j, w\cdot\nabla]\phi_\infty\Big\|_2 + \sum_{j+k+l=1}\Big\|\chi_m[\partial_n^{l+1}\partial^k\partial_{x_3}^j, w\cdot\nabla]\phi_\infty\Big\|_2\Big\} \\
&\le C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)\sqrt{D_\infty(t)}\}.
\end{aligned}$$

(vi) $\|\partial_t(\phi w)\|_2 \le C(1+t)^{-1}M(t)^2$.

Proof. The estimates (i)-(iii) and (vi) can be proved by applying Lemmas 8.2 and 8.3 similarly to the proof of [11, Proposition 8.5]. As for (iv), we have

$$\begin{aligned}
&\Big|\Big(|\partial_{x_3}^2\phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| + \Big|\Big(|\partial_{x'}^2\phi_\infty|^2, \mathrm{div}\big(\chi_0^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| + \Big|\Big(|\partial_t\phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| \\
&\quad+\sum_{m=1}^N\Big\{\sum_{j+k=1}\Big|\Big(|\partial^{k+1}\partial_{x_3}^j\phi_\infty|^2, \mathrm{div}\big(\chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big| \\
&\qquad+\sum_{j+k+l=1}\Big|\Big(\big|\partial_n^{l+1}\partial^k\partial_{x_3}^j\phi_\infty\big|^2, \mathrm{div}\big(\chi_m^2\tfrac{P'(\rho_s)}{\gamma^4\rho_s}w\big)\Big)\Big|\Big\} \\
&\le C|\!|\!|D\phi_\infty|\!|\!|_1^2(\|w\|_\infty + \|\nabla w\|_\infty) \\
&\le C(1+t)^{-\frac{1}{2}}M(t)D_\infty(t).
\end{aligned}$$

We next consider (v). We observe that $[\partial^{k+1}\partial_{x_3}^j, w\cdot\nabla]\phi_\infty$ and $[\partial_n^{l+1}\partial^k\partial_{x_3}^j, w\cdot\nabla]\phi_\infty$ are written as a linear combination of terms of the forms $a[\partial_x^q, w]\nabla\phi_\infty$ and $(w\cdot\nabla a)\partial_x^q\phi_\infty$ with smooth function $a = a(x')$ and integer q satisfying $1 \le q \le 2$. Therefore, applying Lemma 8.3, we obtain the desired estimate. This completes the proof. $\square$

Let us consider the nonlinearity $\widetilde{Q}\boldsymbol{F} = {}^T(0, f)$. We write $\widetilde{Q}\boldsymbol{F} = {}^T(0, f)$ in the form

$$\widetilde{Q}\boldsymbol{F} = \widetilde{\boldsymbol{F}}_0 + \widetilde{\boldsymbol{F}}_1 + \widetilde{\boldsymbol{F}}_2 + \widetilde{\boldsymbol{F}}_3,$$

where $\widetilde{\boldsymbol{F}}_l = {}^T(0, h_l)$ $(l = 0, 1, 2, 3)$. Here

$$\begin{aligned} h_0 =& -w\cdot\nabla w + f_1(\rho_s,\phi)\Big(-\partial_{x_3}^2\sigma_1 w^{(0)} + \frac{\partial_{x'}^2 v_s}{\gamma^2\rho_s}(\phi_1+\phi_\infty)\Big) \\ &+ f_2(\rho_s,\phi)\big(-\partial_{x_3}^2\sigma_1 w^{(0)} - \partial_{x_3}\sigma_1\partial_{x'}w^{(0)}\big) \\ &+ f_{0,1}(x',\phi)\phi\sigma_1 + f_{0,2}(x',\phi)\partial_{x_3}\sigma_1 + f_{0,3}(x',\phi)\phi(\phi_1+\phi_\infty), \end{aligned}$$

$$\begin{aligned} h_1 &= -\mathrm{div}\big(f_1(\rho_s,\phi)\nabla(w_1+w_\infty)\big), \\ h_2 &= -\nabla\big(f_2(\rho_s,\phi)\mathrm{div}(w_1+w_\infty)\big) + \big(\mathrm{div}(w_1+w_\infty)\big)\nabla'\big(f_2(\rho_s,\phi)\big), \\ h_3 &= \nabla\big(f_3(x',\phi)\phi(\phi_1+\phi_\infty)\big) - (\phi_1+\phi_\infty)\nabla\big(f_3(x',\phi)\phi\big). \end{aligned}$$

Here $f_{0,l} = f_{0,l}(x',\phi)$ $(l = 1, 2, 3)$ and $f_3(x',\phi)$ are smooth functions of x' and ϕ.

Proposition 8.6. *We assume that $u(t)$ is a solution of (5.1) in $Z(T)$. If $\sup_{0\le\tau\le t}[\![u(\tau)]\!]_2 \le \epsilon_s$ and $M(t) \le 1$ for all $t \in [0, T]$, then the following estimates hold with a positive constant C independent of T.*

(i) $\|\widetilde{Q}\boldsymbol{F}(t)\|_2 \le C(1+t)^{-\frac{3}{4}}M(t)^2$.

(ii) $[\![h_0(t)]\!]_2 \le C\big\{(1+t)^{-\frac{3}{4}}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|\!|\!|Dw_\infty(t)|\!|\!|_2\big\}$.

(iii) $\|h_l(t)\|_{H^1} \le C\big\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|\!|\!|Dw_\infty(t)|\!|\!|_2\big\}, \quad (l = 1, 2, 3)$.

(iv) $\|\partial_t h_l(t)\|_2 \le C\big\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)|\!|\!|Dw_\infty(t)|\!|\!|_2\big\}, \quad (l = 1, 2, 3)$.

Proposition 8.6 can be proved in a similar manner to to the proof of [11, Proposition 8.6] and [3, Proposition 8.6].

Proof of Proposition 8.1 We first prove (8.1). We see from Proposition 8.5 and Proposition 8.6 that

$$\begin{aligned} \|Q_0\boldsymbol{F}\|_2 &\le C(1+t)^{-\frac{5}{4}}M(t)^2, \\ \|\widetilde{Q}\boldsymbol{F}\|_2 &\le C(1+t)^{-\frac{3}{4}}M(t)^2, \end{aligned}$$

and hence,

$$\|\widetilde{Q}\boldsymbol{F}_\infty\|_2^2 \le C\|\boldsymbol{F}\|_2^2 \le C(1+t)^{-\frac{3}{2}}M(t)^4.$$

This shows (8.1).

We next prove (8.2). We write

$$\mathcal{R} = \sum_{i=1}^{4} I_i,$$

where

$$\begin{aligned}
I_1 =& \Big| \Big(|\partial_{x_3}^2 \phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s} w\big)\Big)\Big| + \Big| \Big(|\partial_{x'}^2 \phi_\infty|^2, \mathrm{div}\big(\chi_0^2 \tfrac{P'(\rho_s)}{\gamma^4\rho_s} w\big)\Big)\Big| \\
&+ \Big| \Big(|\partial_t \phi_\infty|^2, \mathrm{div}\big(\tfrac{P'(\rho_s)}{\gamma^4\rho_s} w\big)\Big)\Big| + \sum_{m=1}^{N} \Big\{ \sum_{j+k=1} \Big| \Big(|\partial^{k+1}\partial_{x_3}^j \phi_\infty|^2, \mathrm{div}\big(\chi_m^2 \tfrac{P'(\rho_s)}{\gamma^4\rho_s} w\big)\Big)\Big| \\
&+ \sum_{j+k+l=1} \Big| \Big(\big|\partial_n^{l+1}\partial^k\partial_{x_3}^j \phi_\infty\big|^2, \mathrm{div}\big(\chi_m^2 \tfrac{P'(\rho_s)}{\gamma^4\rho_s} w\big)\Big)\Big| \Big\}, \\
I_2 =& \Big| \Big([\partial_{x_3}, w\cdot\nabla]\phi_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial_{x_3}\phi_\infty \Big)\Big| + \Big| \Big(\chi_0^2 [\partial_{x'}^2, w\cdot\nabla]\phi_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial_{x'}\phi_\infty \Big)\Big| \\
&+ \Big| \Big([\partial_t, w\cdot\nabla]\phi_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial_t\phi_\infty \Big)\Big| \\
&+ \sum_{m=1}^{N} \Big\{ \sum_{j+k=1} \Big| \Big(\chi_m^2 [\partial^{k+1}\partial_{x_3}^j, w\cdot\nabla]\phi_\infty, \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial^{k+1}\partial_{x_3}^j\phi_\infty \Big)\Big| \\
&+ \sum_{j+k+l=1} \big\| \chi_m^2 [\partial_n^{l+1}\partial^k\partial_{x_3}^j, w\cdot\nabla_y]\phi_\infty \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \big\|_2^2 \Big\}, \\
I_3 =& \Big| \Big(\partial_{x_3}^2 \widetilde{f}_\infty^0, \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial_{x_3}^2\phi_\infty \Big)\Big| + \Big| \Big(\partial_{x'}^2 \widetilde{f}_\infty^0, \chi_0^2 \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial_{x'}^2\phi_\infty \Big)\Big| + \Big| \Big(\partial_t \widetilde{f}_\infty^0, \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial_t\phi_\infty \Big)\Big| \\
&+ \sum_{m=1}^{N} \sum_{j+k=1} \Big| \Big(\partial^k\partial_{x_3}^j \widetilde{f}_\infty^0, \chi_m^2 \tfrac{P'(\rho_s)}{\gamma^4\rho_s} \partial^{k+1}\partial_{x_3}^j\phi_\infty \Big)\Big| + \|\widetilde{f}_\infty^0\|_{H^1}^2, \\
I_4 =& \big| \big(\partial_{x_3} f_\infty, \partial_{x_3}(\rho_s \partial_{x_3}^2 w_\infty)\big)\big| + \big| \big(\partial_{x'} f_\infty, \partial_{x'}(\chi_0^2 \rho_s \partial_{x'}^2 w_\infty)\big)\big| + \big| \big(\partial_t f_\infty, \rho_s \partial_t w_\infty\big)\big| \\
&+ \sum_{m=1}^{N} \sum_{j+k=1} \big| \big(\partial^k\partial_{x_3}^j f_\infty, \partial(\chi_m^2 \rho_s \partial^{k+1}\partial_{x_3}^j w_\infty)\big)\big| + \|f_\infty\|_2^2.
\end{aligned}$$

From Proposition 8.5 (iv), (v) and Lemma 8.4 we see that

$$I_1 \le C(1+t)^{-\frac{1}{2}} M(t) D_\infty(t),$$

$$\begin{aligned}
I_2 &\le C(1+t)^{-\frac{1}{4}} M(t) \sqrt{D_\infty}(t) [\![\phi_\infty]\!]_2 \\
&\le C(1+t)^{-1} M(t)^2 \sqrt{D_\infty}(t) \\
&\le C\big\{(1+t)^{-\frac{7}{4}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty(t)\big\}.
\end{aligned}$$

As for I_3 and I_4, we have

$$\begin{aligned}
I_3 + I_4 \le C\{ &\|\widetilde{f}_\infty^0\|_{H^2} \|\phi_\infty\|_{H^2} + \|f_\infty\|_{H^1} \|w_\infty\|_{H^2} \\
&+ \|\partial_t \widetilde{f}_\infty^0\|_2 \|\partial_t\phi_\infty\|_2 + \|\partial_t f_\infty\|_2 \|\partial_t w_\infty\|_2 \}.
\end{aligned}$$

Since $[\![Q_0 P_\infty \boldsymbol{F}]\!]_1 + \|\widetilde{Q} P_\infty \boldsymbol{F}\|_2 \le C[\![\boldsymbol{F}]\!]_1$, we find from Proposition 8.5 and Proposition 8.6 that

$$\|\widetilde{f}_\infty^0\|_{H^2} + \|f_\infty\|_{H^1} + \|\partial_t f_\infty\|_2 \le C\big\{(1+t)^{-\frac{3}{4}} M(t)^2 + (1+t)^{-\frac{1}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_2 \big\}.$$

It then follows from Lemma 8.4 that

$$\begin{aligned}&\|\widetilde{f}^0_\infty\|_{H^2}\|\phi_\infty\|_{H^2}+\|f_\infty\|_{H^1}\|w_\infty\|_{H^2}+\|\partial_t f_\infty\|_2\|\partial_t w_\infty\|_2\\&\le C\{(1+t)^{-\frac{3}{2}}M(t)^3+(1+t)^{-1}M(t)^2\sqrt{D_\infty(t)}\}.\end{aligned}$$

It remains to estimate $\|\partial_t\widetilde{f}^0_\infty\|_2\|\partial_t\phi_\infty\|_2$. Since

$$\partial_t\phi_\infty=-Q_0LP_\infty u+Q_0P_\infty\boldsymbol{F}.$$

we see from Lemma 8.3 and Proposition 8.5 (i) – (iii) that

$$\|\partial_t\phi_\infty\|_{H^1}\le C\{\|v_s^3\partial_{x_3}\phi_\infty\|_{H^1}+\|\partial_x w_\infty\|_{H^1}+\|Q_0\boldsymbol{F}_\infty\|_{H^1}\}\le C(1+t)^{-\frac{3}{4}}M(t).$$

This, together with Lemma 8.3 and Proposition 8.5 (i) – (iii), then yields

$$\|\partial_t\widetilde{f}^0_\infty\|_2\|\partial_t\phi_\infty\|_2\le C\big\{(1+t)^{-2}M(t)^3+(1+t)^{-\frac{5}{4}}M(t)^2\sqrt{D_\infty(t)}\big\},$$

and therefore, we have

$$I_3+I_4\le C\big\{(1+t)^{-\frac{3}{2}}M(t)^2+(1+t)^{-\frac{1}{2}}M(t)D_\infty(t)\big\}.$$

We thus conclude that

$$\mathcal{R}(t)\le C\big\{(1+t)^{-\frac{3}{2}}M(t)^2+(1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\big\}.$$

This completes the proof. □

9 Asymptotic behavior

In this section we prove the asymptotic behavior (3.2).

Since $M(t)\le C\|u_0\|_{H^2\cap L^1}$ for all $t\ge 0$, we see that

$$\|u(t)-(\sigma_1u^{(0)})(t)\|_2\le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^2\cap L^1}.$$

Therefore, to prove (3.2), it suffices to show the following

Proposition 9.1. *Let $\sigma=\sigma(x_3,t)$ be the solution of* (3.3) *with initial value $\sigma|_{t=0}=\langle\phi_0\rangle$. Assume that $\nu\ge\nu_0$, $\frac{\gamma^2}{\nu+\tilde{\nu}}\ge\gamma_0^2$ and $\omega\le\omega_0$. Then there exists $\epsilon>0$ such that if $\|u_0\|_{H^2\cap L^1}\le\epsilon$, then*

$$\|\sigma_1(t)-\sigma(t)\|_2\le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^2\cap L^1}\quad(\delta>0).$$

To prove Proposition (9.1), we prepare two lemmas.

In what follows we denote by $\sigma=\sigma(x_3,t)$ the solution of (3.3) with initial value $\sigma|_{t=0}=\sigma_0$.

It is well-known that $\sigma(t)$ has the following decay properties.

Lemma 9.2. *Assume that $\sigma(t)$ is a solution of (3.3) with $\sigma|_{t=0} = \sigma_0 \in H^1 \cap L^1$. Then*

$$\|\partial_{x_3}^l \sigma(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|\sigma_0\|_{H^1\cap L^1} \quad (l = 0, 1),$$
$$\|\sigma(t)\|_\infty \leq C(1+t)^{-\frac{1}{2}} \|\sigma_0\|_{H^1\cap L^1}.$$

We decompose $\mathcal{H}(t)$ into two parts. We define $\mathcal{H}_0(t)$ and $\mathcal{H}_\infty(t)$ by

$$\mathcal{H}_0(t) = \mathcal{F}^{-1}\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)e^{-(i\kappa_1\xi+\kappa_0\xi^2)t}\mathcal{F}, \quad \mathcal{H}_\infty(t) = \mathcal{H}(t) - \mathcal{H}_0(t).$$

Then $\mathcal{H}(t) = \mathcal{H}_0(t) + \mathcal{H}_\infty(t)$ and $\mathcal{H}_0(t)$ and $\mathcal{H}_\infty(t)$ have the following properties.

Lemma 9.3. *There hold the following estimates.*

$$\|\partial_{x_3}^l \mathcal{H}_0(t)\sigma_0\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|\sigma_0\|_1,$$

$$\|\partial_{x_3}^l \mathcal{H}_\infty(t)\sigma_0\|_2 \leq Ct^{-\frac{l}{2}}e^{-\frac{\kappa_0}{2}r_0^2 t} \|\sigma_0\|_2,$$

$$\|\partial_{x_3}^l (e^{t\Lambda}\sigma_0 - \mathcal{H}_0(t)\sigma_0)\|_2 \leq C(1+t)^{-\frac{3}{4}-\frac{l}{2}} \|\sigma_0\|_1.$$

Lemma 9.3 can be proved in a similar manner to the proof of [2, Proposition 5.8]; and we omit the proof.

We now prove Proposition 9.1.

Proof of Proposition 9.1. Let $\sigma_0 = \langle \phi_0 \rangle$. We define $N(t)$ by

$$N(t) = \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{3}{4}-\delta} \|\sigma_1(t) - \sigma(t)\|_{H^1}.$$

We write σ as

$$\sigma(t) = \mathcal{H}(t)\sigma_0 - \kappa_2 \int_0^t \mathcal{H}(t-\tau)\partial_{x_3}(\sigma^2)(\tau)d\tau. \tag{9.1}$$

As for $\sigma_1(t)$, by Lemma 6.2 (ii), we have

$$\begin{aligned}\mathcal{F}[\mathcal{P}\boldsymbol{F}] &= -i\xi\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\langle\widehat{\phi w^3}\rangle + \partial_{x_3}\mathcal{F}[\mathcal{P}^{(1)}\boldsymbol{F}] + \partial_{x_3}^2\mathcal{F}[\mathcal{P}^{(2)}\boldsymbol{F}] \\ &= -i\xi\kappa_{21}\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\widehat{(\sigma_1^2)} - i\xi\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\big(\langle\widehat{\phi w^3}\rangle - \langle\phi^{(0)}w^{(0),3}\rangle\widehat{(\sigma_1^2)}\big) \\ &\quad + \partial_{x_3}\mathcal{F}[\mathcal{P}^{(1)}(\sigma_1^2\boldsymbol{F}_1 + \boldsymbol{F}_2)] + \partial_{x_3}^2\mathcal{F}[\mathcal{P}^{(2)}\boldsymbol{F}],\end{aligned}$$

where $\kappa_{21} = \langle\phi^{(0)}w^{(0),3}\rangle$. Furthermore,

$$\begin{aligned}\mathcal{F}[\mathcal{P}^{(1)}(\sigma_1^2\boldsymbol{F}_1)] = \mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\big\langle\widehat{\sigma_1^2\boldsymbol{F}_1}, u^{*(1)}\big\rangle &= \mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\langle\boldsymbol{F}_1, u^{*(1)}\rangle\widehat{(\sigma_1^2)} \\ &= -\kappa_{22}\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\widehat{(\sigma_1^2)},\end{aligned}$$

where $\kappa_{22} = -\langle\boldsymbol{F}_1, u^{*(1)}\rangle$. We thus obtain

$$\begin{aligned}e^{(t-\tau)\Lambda}\mathcal{P}\boldsymbol{F} = &-\kappa_2 e^{(t-\tau)\Lambda}\partial_{x_3}(\sigma_1^2) - e^{(t-\tau)\Lambda}\partial_{x_3}\big\{\langle\phi w^3\rangle - \langle\phi^{(0)}w^{(0),3}\rangle\sigma_1^2\big\} \\ &+ e^{(t-\tau)\Lambda}J_4 + e^{(t-\tau)\Lambda}J_5.\end{aligned}$$

Here we set $\kappa_2 = \kappa_{21} + \kappa_{22}$,

$$J_4 = \partial_{x_3} \mathcal{P}^{(1)} \boldsymbol{F}_2 + \partial_{x_3}^2 \mathcal{P}^{(2)} \boldsymbol{F}_2,$$
$$J_5 = \partial_{x_3}^2 \mathcal{P}^{(2)} (\sigma_1^2 \boldsymbol{F}_1).$$

It then follows from (5.2) and (9.1) that $\sigma_1(t) - \sigma(t)$ is written as

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^{5} I_j(t),$$

where

$$\begin{aligned}
I_0(t) &= e^{t\Lambda} \mathcal{P} u_0 - \mathcal{H}(t)\sigma_0 + \kappa_2 \int_0^t \mathcal{H}_\infty(t-\tau)\partial_{x_3}(\sigma^2)(\tau)\, d\tau, \\
I_1(t) &= -\kappa_2 \int_0^t \mathcal{H}_0(t-\tau)\partial_{x_3}(\sigma_1^2 - \sigma^2) d\tau, \\
I_2(t) &= -\kappa_2 \int_0^t \big(e^{(t-\tau)\Lambda} - \mathcal{H}_0(t-\tau)\big)\partial_{x_3}(\sigma_1^2)(\tau)\, d\tau, \\
I_3(t) &= -\int_0^t \partial_{x_3} e^{(t-\tau)\Lambda} \big(\langle \phi w^3 \rangle - \langle \phi^{(0)} w^{(0),3} \rangle \sigma_1^2\big) d\tau, \\
I_j(t) &= \int_0^t e^{(t-\tau)\Lambda} J_j(\tau) d\tau, \quad (j = 4, 5).
\end{aligned}$$

We see from Proposition 4.6 and Lemmas 9.2, 9.3 that

$$\begin{aligned}
&\|I_0(t)\|_{H^1} \\
&\le C\Big\{(1+t)^{-\frac{3}{4}} \|u_0\|_{H^1\cap L^1} + \int_0^t (t-\tau)^{-\frac{l}{2}} e^{-\frac{\kappa_0}{2} r_0^2 (t-\tau)} \|\sigma\|_\infty \|\partial_{x_3}\sigma\|_2(\tau)\, d\tau\Big\} \\
&\le C\Big\{(1+t)^{-\frac{3}{4}} \|u_0\|_{H^1\cap L^1} + \int_0^t (t-\tau)^{-\frac{l}{2}} e^{-\frac{\kappa_0}{2} r_0^2 (t-\tau)} (1+\tau)^{-\frac{5}{4}}\, d\tau \|u_0\|_{H^1\cap L^1}^2\Big\} \\
&\le C(1+t)^{-\frac{3}{4}} \|u_0\|_{H^1\cap L^1} \{1 + \|u_0\|_{H^1\cap L^1}\}.
\end{aligned}$$

As for $I_1(t)$, we first observe

$$\|(\sigma_1^2 - \sigma^2)(t)\|_1 \le \|(\sigma_1 + \sigma)(t)\|_2 \|(\sigma_1 - \sigma)(t)\|_2 \le C(1+t)^{-1+\delta} N(t) \|u_0\|_{H^2\cap L^1}.$$

Since $\partial_{x_3}^k \mathcal{H}_0(t) = \mathcal{H}_0(t)\partial_{x_3}^k$ $(k = 0, 1)$, we see from Lemma 9.3 that

$$\begin{aligned}
\|\partial_{x_3}^k I_1(t)\|_2 &\le C \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-1+\delta} d\tau \|u_0\|_{H^2\cap L^1} N(t) \\
&\le C(1+t)^{-\frac{3}{4}+\delta} \|u_0\|_{H^2\cap L^1} N(t)
\end{aligned}$$

for $k = 0, 1$.

As for $I_2(t)$, we see from Lemma 9.3 that

$$\begin{aligned}
\|\partial_{x_3}^k I_2(t)\|_2 &\le CM(t)^2 \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-1} d\tau \\
&\le C(1+t)^{-\frac{3}{4}} \log(1+t) \|u_0\|_{H^2\cap L^1}^2
\end{aligned}$$

for $k=0,1$.

As for $I_3(t)$, since

$$
\begin{aligned}
&\|\langle \phi w^3\rangle(\tau)-\langle \phi^{(0)}w^{(0),3}(\tau)\rangle\sigma_1^2(\tau)\|_1\\
&\le C\big\{\|\sigma_1(\tau)\|_2\|u(\tau)-\sigma_1(\tau)u^{(0)}\|_2+\|u(t)\|_2\|u(\tau)-\sigma_1(\tau)u^{(0)}\|_2\big\}\\
&\le C(1+\tau)^{-1}M(t)^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|\partial_{x_3}^k I_3(t)\|_2 &\le CM(t)^2\int_0^t(1+\tau)^{-\frac34-\frac k2}(1+\tau)^{-1}d\tau\\
&\le C(1+t)^{-\frac34}\log(1+t)\|u_0\|_{H^2\cap L^1}^2
\end{aligned}
$$

for $k=0,1$.

By Proposition 4.5 and Lemma 6.3, $I_4(t)$ is estimated as

$$
\begin{aligned}
\|\partial_{x_3}^k I_4(t)\|_2 &= \|\int_0^t e^{(t-\tau)\Lambda}\partial_{x_3}\big(\mathcal{P}^{(1)}\boldsymbol{F}_2(\tau)+\partial_{x_3}\mathcal{P}^{(2)}\boldsymbol{F}_2(\tau)\big)d\tau\|_2M(t)^2\\
&\le C\int_0^t(1+t-\tau)^{-\frac34-\frac k2}(1+\tau)^{-1}d\tau\|u_0\|_{H^2\cap L^1}^2\\
&\le C(1+t)^{-\frac34}\log(1+t)\|u_0\|_{H^2\cap L^1}^2
\end{aligned}
$$

for $k=0,1$.

As for $I_5(t)$, since $\partial_{x_3}\mathcal{P}^{(2)}(\tau)=\mathcal{P}^{(2)}\partial_{x_3}$, we see from Lemma 6.3 that

$$
\begin{aligned}
\|\partial_{x_3}^k I_5(t)\|_2 &\le \Big\|\int_0^t e^{(t-\tau)\Lambda}\partial_{x_3}^{k+1}\mathcal{P}^{(2)}\big(\partial_{x_3}(\sigma_1^2)\boldsymbol{F}_1\big)(\tau)d\tau\Big\|_2\\
&\le C\Big\{\int_0^{\frac t2}(1+t-\tau)^{-\frac34-\frac k2}(1+\tau)^{-1}d\tau M(t)^2\\
&\le C(1+t)^{-\frac34}\log(1+t)\|u_0\|_{H^2\cap L^1}^2
\end{aligned}
$$

for $k=0,1$.

Therefore, we obtain

$$
\|(\sigma_1-\sigma)(t)\|_{H^1}\le C(1+t)^{-\frac34+\delta}\|u_0\|_{H^2\cap L^1}\{1+\|u_0\|_{H^2\cap L^1}+\|u_0\|_{H^2\cap L^1}^2+N(t)\}.
$$

It then follows that if $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then

$$
N(t)\le C\|u_0\|_{H^2\cap L^1}.
$$

We thus see that if $\|u_0\|_{H^2\cap L^1}\ll 1$, then

$$
\|\sigma_1(t)-\sigma(t)\|_2\le C(1+t)^{-\frac34+\delta}\|u_0\|_{H^2\cap L^1}
$$

This completes the proof. □

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