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Large time behavior of solutions to the compressible Navier-Stokes equations around a parallel flow in a cylindrical domain

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Abstract
Stability of parallel flow of the compressible Navier-Stokes equation in a cylindrical domain is studied. It is shown that if the Reynolds and Mach numbers are sufficiently small, then parallel flow is asymptotically stable and the asymptotic leading part of the disturbances is described by a one dimensional viscous Burgers equation.

Keywords: Compressible Navier-Stokes equation, parallel flow, cylindrical domain, viscous Burgers equation asymptotic behavior

1 Introduction

This paper studies the large time behavior of solutions of the initial boundary value problem for the compressible Navier-Stokes equation

\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \text{div} v + \nabla p(\rho) &= \rho g, \\
v|_{\partial D_s} &= 0, \\
(\rho, v)|_{t=0} &= (\rho_0, v_0)
\end{align*}

in a cylindrical domain \(\Omega_s = D_s \times \mathbb{R}\):

\[\Omega_s = \{x = (x', x_3); x' = (x_1, x_2) \in D_s, x_3 \in \mathbb{R}\}.\]

Here \(D_s\) is a bounded and connected domain in \(\mathbb{R}^2\) with smooth boundary \(\partial D_s\); \(\rho = \rho(x, t)\) and \(v = (v^1(x, t), v^2(x, t), v^3(x, t))\) denote the unknown density and velocity, respectively, at time \(t \geq 0\) and position \(x \in \Omega_s\); \(p(\rho)\) is the pressure that is a smooth function of \(\rho\) and satisfies

\[p'(\rho_*) > 0\]
for a given positive constant \( \rho_*; \mu \) and \( \mu' \) are the viscosity coefficients that satisfy
\[
\mu > 0, \quad \frac{2}{3}\mu + \mu' \geq 0;
\]
and \( g \) is an external force of the form \( g = T(g^1(x'), g^2(x'), g^3(x')) \) with \( g^1 \) and \( g^2 \) satisfying
\[
(g^1(x'), g^2(x')) = (\partial_x \Phi(x'), \partial_x^2 \Phi(x')),
\]
where \( \Phi \) and \( g^3 \) are given smooth functions of \( x' \). Here and in what follows \( T \) stands for the transposition.

Problem (1.1)-(1.3) has a stationary solution \( u_s = T(p_s(x'), v_s(x')) \) which represents parallel flow. Here \( p_s \) is determined by
\[
\begin{cases}
\text{Const.} - \Phi(x') = \int_{\Omega} p_s \nu \left( \frac{x'}{\eta} \right) d\eta, \\
\int_{\partial \Omega} p_s - \rho_* dx' = 0,
\end{cases}
\]
while \( v_s \) takes the form
\[
v_s = T(0, 0, v_3^s(x')),
\]
where \( v_3^s(x') \) is the solution of
\[
\begin{cases}
-\mu \Delta' v_3^s = \nabla g^3, \\
v_3^s \big|_{\partial \Omega} = 0.
\end{cases}
\]
Here
\[
\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2.
\]

The purpose of this paper is to investigate the large time behavior of solutions to problem (1.1)-(1.4) when the initial value \( (\rho, v) \big|_{t=0} = (\rho_0, v_0) \) is sufficiently close to the stationary solution \( u_s = T(p_s, v_s) \).

Solutions of multi-dimensional compressible Navier-Stokes equations in unbounded domains exhibit interesting phenomenon, and detailed descriptions of large time behavior of solutions have been obtained. See, e.g., [6, 7, 16, 18, 19, 20, 21, 24, 25] for the case of the whole space, half space and exterior domains. Besides these domains, infinite layers and cylindrical domains provide good subjects of the stability of flows, for example, the stability of parallel flows.

Concerning the large time behavior of solutions around parallel flow, the case of an \( n \) dimensional infinite layer \( \mathbb{R}^{n-1} \times (0, 1) = \{ x = (x_h, x_n); x_h = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, 0 < x_n < 1 \} \) was studied in [10, 11, 15]. (See also [3, 4, 5] for the stability of time periodic parallel flow.) It was shown in [10, 11, 15] that if the Reynolds and Mach numbers are sufficiently small, then the parallel flow is stable under sufficiently small initial disturbances in some Sobolev space. Furthermore, in the case of \( n \geq 3 \), the disturbance \( u(t) \) behaves like a solution of an \( n - 1 \) dimensional linear heat equation as \( t \to \infty \), whereas, in the case of \( n = 2 \), \( u(t) \) behaves like a solution of a one dimensional viscous Burgers equation.

In the case of cylindrical domain \( \Omega_* \), Iooss and Padula [8] considered the linearized stability of parallel flow \( \bar{u}_s \) under periodic boundary condition in \( x_3 \). It was
proved in [8] that if the Reynolds number is suitably small, then the semigroup decays exponentially as \( t \to \infty \), provided that the density-component has vanishing average over the basic period domain. Furthermore, the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line \( \text{Re}\lambda = -c \) for some number \( c > 0 \) consists of finite number of eigenvalues with finite multiplicities. 

As for the stability under local disturbances on \( \Omega_s \), i.e., disturbances which are non-periodic but decay at spatial infinity, the stability of the motionless state \( \tilde{\mathbf{u}}_s = T(\rho_s, 0) \) was studied in [17]; and it was shown in [17] that the disturbance decays in \( L^2(\Omega_s) \) in the order \( t^{-\frac{1}{2}} \) and its asymptotic leading part is given by a solution of a one dimensional linear heat equation if the initial disturbance is sufficiently small in \( H^3(\Omega_s) \cap L^4(\Omega_s) \), where \( H^3(\Omega_s) \) denotes the \( L^2\)-Sobolev space on \( \Omega_s \) of order 3. (See also [9] for the analysis in \( L^p(\Omega_s) \).)

In this paper we will consider the stability of parallel flow \( \tilde{\mathbf{u}}_s \) under local disturbances on \( \Omega_s \). After introducing suitable non-dimensional variables, the equations for the disturbance \( \mathbf{u} = T(\phi, w) - T(\gamma^2(\rho - \rho_s), v - v_s) \) takes the following form:

\[
\begin{align*}
\partial_t \phi + v^2 \partial_{x_3} \phi + \gamma^2 \text{div}(\rho_\omega w) &= f^0(\phi, w), \\
\partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\nu}{\rho_s} \nabla \text{div} w + \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) + v^2 \partial_{x_3} w + (w' \cdot \nabla v^3) e_3 &= f(\phi, w), \\
w_{|\partial \Omega} &= 0, \\
(\phi, w)_{|t=0} &= (\phi_0, w_0).
\end{align*}
\]

Here \( \Omega_s \) is transformed to \( \Omega = D \times \mathbb{R} \) with \( |D| = 1 \); \( u_s = T(\rho_s, v_s) \) and \( P(\rho) \) denote the dimensionless parallel flow and pressure, respectively; \( \nu, \nu' \) and \( \gamma \) are the dimensionless parameters defined by

\[
\nu = \frac{\mu}{\rho_s \ell V}, \quad \nu' = \frac{\mu + \mu'}{\rho_s \ell V}, \quad \gamma = \frac{\sqrt{P'(\rho_s)}}{V}
\]

with the reference velocity \( V \) which measures the strength of \( \tau^3_0 \); \( e_3 = T(0, 0, 1) \in \mathbb{R}^3 \) and \( \nabla' = T(\partial_{x_1}, \partial_{x_2}) \); \( f^0(\phi, w) \) and \( f(\phi, w) \) are the non-linearities given by

\[
f^0(\phi, w) = -\text{div}(\phi w), \quad f(\phi, w) = -w \cdot \nabla w + \frac{\nu^2}{(\phi + \gamma^2 \rho_s)\rho_s} \left( -\Delta w + \frac{\Delta v^2}{\gamma^2 \rho_s} \phi \right) - \frac{\nu}{(\phi + \gamma^2 \rho_s)\rho_s} \nabla \text{div} w + \frac{\phi}{\gamma^2 \rho_s} \nabla \left( P'(\rho_s) \phi \right) - \frac{1}{2 \gamma^2 \rho_s} \nabla \left( P''(\rho_s) \phi^2 \right) + \tilde{P}_3(\rho_s, \phi, \partial_{x'} \phi),
\]

where

\[
\tilde{P}_3(\rho_s, \phi, \partial_{x'} \phi) = \frac{\phi^3}{\gamma^3 (\phi + \gamma^2 \rho_s) \rho_s^2} \nabla P(\rho_s) - \frac{\phi^3}{2 \gamma^2 \rho_s} \nabla \left( \phi^3 P_3(\rho_s, \phi) \right) + \frac{\phi^2}{2 \gamma^2 \rho_s^2} \nabla \left( \frac{1}{\gamma^2} P'(\rho_s) \phi + \frac{1}{2 \gamma^2} P''(\rho_s) \phi^2 + \frac{1}{2 \gamma^3} \phi^3 P_3(\rho_s, \phi) \right).
\]
with
\[ P_3(\rho_s, \phi) = \int_0^1 (1 - \theta)^2 P''(\rho_s + \theta^{-2} \phi) d\theta. \]

See section 2.2 below for the definition of non-dimensional variables.

In [1, 2], spectral properties of the linearized semigroup \( e^{-tL} \) was studied in detail. It was proved that there exists a bounded projection \( P_0 \) satisfying \( P_0 e^{-tL} = e^{-tL} P_0 \) such that if Reynolds and Mach numbers are sufficiently small, then, for the initial value \( u_0 = T(\phi_0, w_0) \), it holds that
\[ \| e^{-tL} P_0 u_0 - [H(t) \langle \phi_0 \rangle] u^{(0)}(t) \|_{L^2(\Omega)} \leq C (1 + t)^{-\frac{3}{2}} \| u_0 \|_{L^1(\Omega)}. \]  \hfill (1.9)

Here \( u^{(0)} \) is some function of \( x' \); \( \langle \phi_0 \rangle \) denotes the average of \( \phi_0 \) over \( D \), (thus, \( \langle \phi_0 \rangle \) is a function of \( x_3 \in \mathbb{R} \)); and \( H(t) \) is the heat semigroup defined by
\[ H(t) = \mathcal{F}^{-1} e^{-((\kappa_1 \xi_1 + \kappa_0 \xi_2^2) - \mathcal{F}} \]
with some constants \( \kappa_1 \in \mathbb{R} \) and \( \kappa_0 > 0 \), where \( \mathcal{F} \) denotes the Fourier transform on \( \mathbb{R} \). Furthermore, it was proved that the \((I - P_0)\)-part of \( e^{-tL} \) satisfies the exponential decay estimate
\[ \| e^{-tL} (I - P_0) u_0 \|_{H^1(\Omega)} \leq C e^{-dt} \{ \| u_0 \|_{H^1(\Omega) \times H^{-1}(\Omega)} + t^{-\frac{1}{4}} \| w_0 \|_{L^2(\Omega)} \} \]  \hfill (1.10)
for a positive constant \( d \). Here \( \tilde{H}^1(\Omega) \) is the set of all locally \( H^1 \) functions in \( L^2(\Omega) \) whose tangential derivatives near \( \partial \Omega \) belong to \( L^2(\Omega) \).

In this paper, based on the results on spectral properties of \( e^{-tL} \), we investigate the nonlinear problem (1.5)–(1.8). We prove that if the initial disturbance \( u_0 = T(\phi_0, w_0) \) is sufficiently small, then the disturbance \( u(t) \) exists globally in time and it satisfies
\[ \| u(t) \|_{L^2(\Omega)} = O(t^{-\frac{1}{2}}) \]  \hfill (1.11)
\[ \| u(t) - (\sigma u^{(0)}(t) \|_{L^2(\Omega)} = O(t^{-\frac{3}{4} + \delta}) \quad (\delta > 0) \]  \hfill (1.12)
as \( t \to \infty \). Here \( \sigma = \sigma(x_3, t) \) satisfies the following one dimensional viscous Burgers equation
\[ \partial_t \sigma - \kappa_0 \partial_{x_3}^2 \sigma + \kappa_1 \partial_{x_3} \sigma + \kappa_2 \partial_{x_3}(\sigma^2) = 0 \]
with initial value \( \langle \phi_0 \rangle \).

The proof of (1.11) and (1.12) is given by the factorization of \( e^{-tL} P_0 \), estimate (1.10) and the Matsumura-Nishida energy method. We decompose the disturbance \( u(t) \) into its \( P_0 \) and \( I - P_0 \) parts as in [3, 11]. We then estimate the \( P_0\)-part by representing it in the form of variation of constants formula in terms of \( e^{-tL} P_0 \) and employ the factorization result of \( e^{-tL} P_0 \) obtained in [2]. For the \((I - P_0)\)-part of \( u(t) \), we employ the Matsumura-Nishida energy method. In contrast to [3, 11], we make use of the estimate (1.10) and combine it with the energy method. This simplifies the argument in [3, 11] where a complicated decomposition is also used in the energy method to estimate the \((I - P_0)\)-part of \( u(t) \). In this paper we do not need to use such a complicated decomposition of the \((I - P_0)\)-part in the energy method due to (1.10).
This paper is organized as follows. In section 2 we first introduce notations and non-dimensional variables. We then state the existence of stationary solution which represents parallel flow. In section 3 we state our main result of this paper. In section 4 we decompose the problem into the one for a coupled system of the $P_0$ and $I - P_0$ parts of $u(t)$. Sections 6 is devoted to estimating the $P_0$-part of the disturbance $u(t)$, while the $(I - P_0)$-part is estimated in section 7. Section 8 is devoted to the estimates for the nonlinearities. The proof of (1.12) is given in section 9.

2 Preliminaries

In this section we introduce notations throughout this paper. We then introduce non-dimensional variables and state the existence of stationary solution which represents parallel flow.

2.1 Notation

We first introduce some notations which will be used throughout the paper. For $1 \leq p \leq \infty$ we denote by $L^p(X)$ the usual Lebesgue space on a domain $X$ and its norm is denoted by $\| \cdot \|_{L^p(X)}$. Let $m$ be a nonnegative integer. $H^m(X)$ denotes the $m$ th order $L^2$ Sobolev space on $X$ with norm $\| \cdot \|_{H^m(X)}$. In particular, we write $L^2(X)$ for $H^0(X)$.

We denote by $C(0)^m(X)$ the set of all $C^m$ functions with compact support in $X$. $H^m_0(X)$ stands for the completion of $C(0)^m(X)$ in $H^m(X)$. We denote by $H^{-1}(X)$ the dual space to $H^1_0(X)$ with norm $\| \cdot \|_{H^{-1}(X)}$.

We simply denote by $L^p(X)$ (resp., $H^m(X)$) the set of all vector fields $w = T(w^1, w^2, w^3)$ on $X$ and its norm is denoted by $\| \cdot \|_{L^p(X)}$ (resp., $\| \cdot \|_{H^m(X)}$). For $u = T(\phi, w)$ with $\phi \in H^k(X)$ and $w = T(w^1, w^2, w^3) \in H^m(X)$, we define $\|u\|_{H^k(X) \times H^m(X)}$ by $\|u\|_{H^k(X) \times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^m(X)}$.

When $X = \Omega$ we abbreviate $L^p(\Omega)$ as $L^p$, and likewise, $H^m(\Omega)$ as $H^m$. The norm $\| \cdot \|_{L^p(\Omega)}$ is written as $\| \cdot \|_{L^p}$, and likewise, $\| \cdot \|_{H^m(\Omega)}$ as $\| \cdot \|_{H^m}$. The inner product of $L^2(\Omega)$ is denoted by

$$(f, g) = \int_{\Omega} f(x)g(x)dx, \ f, g \in L^2(\Omega).$$

For $u_j = T(\phi_j, w_j)$ ($j = 1, 2$), we also define a weighted inner product $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = 1/\gamma^2 \int_{\Omega} \phi_1 \phi_2 P_\gamma(\rho_s)dx + \int_{\Omega} w_1 \cdot w_2 \rho_s dx,$$

where $\rho_s = \rho_s(x')$ is the density of the parallel flow $u_s$. As will be seen in Proposition 2.1 below, $\rho_s(x')$ and $P_\gamma(\rho_s(x'))$ are strictly positive in $D$.

In the case $X = D$ we denote the norm of $L^p(D)$ by $\| \cdot \|_p$. The norm of $H^m(D)$ is denoted by $\| \cdot \|_{H^m}$, respectively. On $D$ we will consider complex valued functions,
and in this case, the inner product of $L^2(D)$ is denoted by

$$(f, g) = \int_D f(x') \overline{g(x')} dx', \quad f, g \in L^2(D).$$

Here $\overline{g}$ denotes the complex conjugate of $g$. For $u_j = T(\phi_j, w_j)$ ($j = 1, 2$), we also define a weighted inner product $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \frac{1}{T} \int_D \phi_1 \overline{\phi_2} \frac{P^T}{\rho_2} dx' + \int_D w_1 \cdot \overline{w_2} \rho_2 dx'.$$

For $f \in L^1(D)$ we denote the mean value of $f$ over $D$ by $\langle f \rangle$:

$$\langle f \rangle = (f, 1) = \frac{1}{|D|} \int_D f dx',$$

where $|D| = \int_D dx'$. For $u = T(\phi, w) \in L^1(D)$ with $w = T(w^1, w^2, w^3)$ we define $\langle u \rangle$ by

$$\langle u \rangle = \langle \phi \rangle + \langle w_1 \rangle + \langle w_2 \rangle + \langle w_3 \rangle.$$

We set

$$\|f(t)\|_k = \left( \frac{1}{2} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \|\partial_t^j f(t)\|_{H^{k-2j}}^2 \right)^{\frac{1}{2}},$$

$$\|Df(t)\|_k = \begin{cases} \|\partial_x f(t)\|_2, & k = 0, \\ \left( \|\partial_x f(t)\|_k^2 + \|\partial_t f(t)\|_{k-1}^2 \right)^{\frac{1}{2}}, & k \geq 1. \end{cases}$$

We define a function space $Z(T)$ by

$$Z(T) = \{ u = T(\phi, w) \in C^0([0, T]; H^2 \times (H^2 \cap H^1_0)) \cap C^1([0, T]; L^2): \|u\|_{Z(T)} < \infty \}$$

where

$$\|u\|_{Z(T)} = \sup_{0 \leq t \leq T} \|u(t)\|_2 + \left( \int_0^T ||Dw(t)||_{H^1}^2 dt \right)^{\frac{1}{2}}.$$

Partial derivatives of a function $u$ in $x$, $x'$, $x_3$ and $t$ are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_3} u$ and $\partial_t u$. We also write higher order partial derivatives of $u$ in $x$ as $\partial_x^k u = (\partial_x^k u; |\alpha| = k)$.

We denote the $n \times n$ identity matrix by $I_n$. We define $4 \times 4$ diagonal matrices $Q_0$ and $\tilde{Q}$ by

$$Q_0 = \operatorname{diag}(1, 0, 0, 0), \quad \tilde{Q} = \operatorname{diag}(0, 1, 1, 1).$$

It then follows that for $u = T(\phi, w)$ with $w = T(w^1, w^2, w^3)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We denote the Fourier transform of $f = f(x_3)$ ($x_3 \in \mathbb{R}$) by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x_3) e^{-i \xi x_3} dx_3, \quad \xi \in \mathbb{R}.$$
The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x_3) = (2\pi)^{-1} \int_{\mathbb{R}} f(\xi) e^{ix_3 \xi} \, d\xi, \quad x_3 \in \mathbb{R}.$$ 

We denote the resolvent set of a closed operator $A$ by $\rho(A)$ and the spectrum by $\sigma(A)$.

We finally introduce a function space which consists of locally $H^1$ functions in $L^2(\Omega)$ whose tangential derivatives near $\partial D$ belong to $L^2(\Omega)$. To do so, we first introduce a local curvilinear coordinate system. For any $\overline{\Omega}_0 \subset \partial D$, there exist a neighborhood $\overline{\Omega}_0$ of $\overline{\Omega}_0$ and a smooth diffeomorphism map $\Psi = (\Psi_1, \Psi_2) : \overline{\Omega}_0 \rightarrow B_1(0) = \{z' = (z_1, z_2) : |z'| < 1\}$ such that

$$\begin{align*}
\Psi(\overline{\Omega}_0') \cap D &= \{z' \in B_1(0) : z_1 > 0\}, \\
\Psi(\overline{\Omega}_0' \cap \partial D) &= \{z' \in B_1(0) : z_1 = 0\}, \\
\det \nabla_x \Psi \neq 0 & \text{ on } \overline{\Omega}_0' \cap D.
\end{align*} \tag{2.1}$$

By the tubular neighborhood theorem, there exist a neighborhood $\mathcal{O}_{\overline{\Omega}_0'}$ of $\overline{\Omega}_0'$ and a local curvilinear coordinate system $y' = (y_1, y_2)$ on $\mathcal{O}_{\overline{\Omega}_0}$ defined by

$$x' = y_1 a_1(y_2) + \Psi^{-1}(0, y_2) : \mathcal{R} \rightarrow \mathcal{O}_{\overline{\Omega}_0'},$$

where $\mathcal{R} = \{y' = (y_1, y_2) : |y_1| \leq \tilde{\delta}_1, |y_2| \leq \tilde{\delta}_2\}$ for some $\tilde{\delta}_1, \tilde{\delta}_2 > 0$; $a_1(y_2)$ is the unit inward normal to $\partial D$ that is given by

$$a_1(y_2) = \frac{\nabla_x \Psi_1}{|\nabla_x \Psi_1|}.$$ 

Setting $y_3 = x_3$ we obtain

$$\nabla_x = e_1(y_2) \partial_{y_1} + J(y') e_2(y_2) \partial_{y_2} + e_3 \partial_{y_3},$$

$$\nabla_y = \begin{pmatrix}
\frac{1}{J(y')} e_2(y_2) \\
-rac{1}{J(y')} e_1(y_2)
\end{pmatrix} \nabla_x,$$

where

$$e_1(y_2) = \begin{pmatrix} a_1(y_2) \\ 0 \end{pmatrix}, \quad e_2(y_2) = \begin{pmatrix} a_2(y_2) \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \tag{2.2}$$

$$J(y') = |\det \nabla_x \Psi|, \quad a_2(y_2) = \frac{-\nabla_x \Psi_1}{|\nabla_x \Psi_1|}$$

with $\nabla_x \Psi_1 = \mathcal{T}(\partial_{y_2} \Psi_1, \partial_{y_1} \Psi_1)$. Note that $\partial_{y_1}$ and $\partial_{y_2}$ are the inward normal derivative and tangential derivative at $x' = \Psi^{-1}(0, y_2) \in \partial D \cap \mathcal{O}_{\overline{\Omega}_0'}$, respectively. Let us denote the normal and tangential derivatives by $\partial_n$ and $\partial_t$, i.e.,

$$\partial_n = \partial_{y_1}, \quad \partial_t = \partial_{y_2}.$$
Since \( \partial D \) is compact, there are bounded open sets \( \mathcal{O}_m \) \((m = 1, \ldots, N)\) such that 
\( \partial D \subset \bigcup_{m=1}^{N} \mathcal{O}_m \) and for each \( m = 1, \ldots, N \), there exists a local curvilinear coordinate system \( y' = (y_1, y_2) \) as defined in (2.1) with \( \mathcal{O}_{\mathcal{E}_0}, \Psi \) and \( \mathcal{R} \) replaced by \( \mathcal{O}_m, \Psi^m \) and \( \mathcal{R}_m \). The problem (1.1)-(1.3) is then transformed into the following non-dimensional problem

\[
\begin{align*}
\mathcal{E} \mathcal{O}_m & \supset D, \quad \mathcal{O}_0 \cap \partial D = \emptyset.
\end{align*}
\]

We set a local coordinate \( y' = (y_1, y_2) \) such that \( y_1 = x_1, y_2 = x_2 \) on \( \mathcal{O}_0 \). We note that if \( h \in H^2(D) \), then \( h \mid_{\partial D} = 0 \) implies that \( \partial^k h \mid_{\partial D \cap \mathcal{O}_m} = 0 \) \((k = 0, 1)\).

Let us introduce a partition of unity \( \{\chi_m\}_{m=0}^{N} \) subordinate to \( \{\mathcal{O}_m\}_{m=0}^{N} \), satisfying

\[
\sum_{m=0}^{N} \chi_m = 1 \text{ on } D, \quad \chi_m \in \mathcal{C}_0^\infty(\mathcal{O}_m) \ (m = 0, 1, 2, \ldots, N).
\]

We denote by \( \tilde{H}^1(\Omega) \) the set of all locally \( H^1 \) functions in \( L^2(\Omega) \) whose tangential derivatives near \( \partial \Omega \) belong to \( L^2(\Omega) \), and its norm is denoted by \( \|w\|_{\tilde{H}^1(\Omega)} \):

\[
\|w\|_{\tilde{H}^1(\Omega)} = \|w\|_2 + \|\partial_{x_3} w\|_2 + \|\chi_0 \partial_{x^*} w\|_2 + \sum_{m=1}^{N} \|\chi_m \partial w\|_2.
\]

Note that \( H^1_0(\Omega) \) is dense in \( \tilde{H}^1(\Omega) \).

### 2.2 Non-dimensionalization and stationary solution

In this subsection we rewrite the problem into the one in a non-dimensional form and state the existence of stationary solution which represents parallel flow. Let \( k_0 \) be an integer satisfying \( k_0 \geq 3 \). We introduce the following non-dimensional variables:

\[
\begin{align*}
x = \ell \tilde{x}, \quad v = V \tilde{v}, \quad \rho = \rho_s \tilde{\rho}, \quad t = \frac{\ell}{V} \tilde{t}, \\
p = \rho_s V^2 \tilde{P}, \quad \Phi = \frac{V}{\ell} \tilde{\Phi}, \quad \tilde{g} = \frac{V^2}{\ell} \tilde{g}, \\
V = |\nabla\tilde{z}|_{\mathcal{C}^3_0(D_s)} = \sum_{k=0}^{k_0} \sup_{x' \in D_s} \ell^k |\partial_x^k \nabla^3(\tilde{x}')|, \quad \ell = \left( \int_{D_s} dx' \right)^{\frac{1}{2}}.
\end{align*}
\]

The problem (1.1)-(1.3) is then transformed into the following non-dimensional problem on \( \tilde{\Omega} = \tilde{D} \times \mathbb{R} \):

\[
\begin{align}
\partial_t \tilde{\rho} + \text{div}_x (\rho \tilde{v}) &= 0, \\
\tilde{\rho} \left( \partial_t \tilde{v} + \tilde{v} \cdot \nabla_x \tilde{v} \right) - \nu \Delta_x \tilde{v} - (\nu + \nu') \nabla_x \text{div}_x \tilde{v} + \tilde{P}'(\tilde{\rho}) \nabla_x \tilde{p} &= \tilde{\rho} \tilde{g}, \\
\tilde{v} \mid_{\partial \tilde{D}} &= 0, \\
(\tilde{\rho}, \tilde{v}) \mid_{t=0} &= (\tilde{\rho}_0, \tilde{v}_0).
\end{align}
\]

Here \( \tilde{D} \) is a bounded and connected domain in \( \mathbb{R}^2 \); \( \tilde{g} = \tau(\partial_{\tilde{x}_1} \tilde{\Phi}, \partial_{\tilde{x}_2} \tilde{\Phi}, \tilde{g}') \); and \( \nu \) and \( \nu' \) are non-dimensional parameters:

\[
\nu = \frac{\mu}{\rho_s \ell V}, \quad \nu' = \frac{\mu'}{\rho_s \ell V}.
\]
We also introduce a parameter $\gamma$:

$$
\gamma = \sqrt{\frac{\bar{P}'}{(\bar{\rho}')}} = \sqrt{\frac{P'(\rho_s)}{\bar{v}}},
$$

Note that the Reynolds and Mach numbers are given by $1/\nu$ and $1/\gamma$, respectively.

In what follows, for simplicity, we omit tildes of $\bar{x}$, $\bar{t}$, $\bar{v}$, $\bar{\rho}$, $\bar{g}$, $\bar{P}$, $\bar{\Phi}$, $\bar{D}$ and $\bar{\Omega}$ and write them as $x$, $t$, $v$, $\rho$, $g$, $P$, $\Phi$ and $\Omega$. Observe that, due to the non-dimensionalization, we have

$$
|D| = \int_D dx' = 1,
$$

and thus,

$$
(\langle f \rangle) = \int_D f(x') dx'.
$$

Let us state the existence of a stationary solution which represents parallel flow.

**Proposition 2.1.** If $\Phi \in C^{k_0}(\bar{D})$, $g^3 \in H^{k_0}(D)$ and $|\Phi|_{C^{k_0}}$ is sufficiently small, then (2.3)-(2.5) has a stationary solution $u_s = \tau(\rho_s, v_s) \in C^{k_0}(\bar{D})$. Here $\rho_s$ satisfies

$$
\begin{cases}
\text{Const.} - \Phi(x') = \int_1^{\rho_s(x')} \frac{P'(\eta)}{\bar{v}} d\eta, \\
\int_D \rho_s dx' = 1, \; \rho_1 < \rho_s(x') < \rho_2 \quad (\rho_1 < 1 < \rho_2),
\end{cases}
$$

for some constants $\rho_1, \rho_2 > 0$, and $v_s$ is a function of the form $v_s = \tau(0, 0, v_3^3)$ with $v_3^3 = v_3^3(x')$ being the solution of

$$
\begin{cases}
-\nu \Delta v_3^3 = \rho_s g^3, \\
v_3^3 |_{\partial D} = 0.
\end{cases}
$$

Furthermore, $u_s = \tau(\rho_s, v_s)$ satisfies the estimates:

$$
|\rho_s(x') - 1|_{C^{k_0}} \leq C|\Phi|_{C^{k_0}}(1 + |\Phi|_{C^{k_0}})^k,
$$

$$
|v_3^3|_{C^{k_0}} \leq C|v_3^3|_{H^{k+2}} \leq C|\Phi|_{C^{k_0}}(1 + |\Phi|_{C^{k_0}})^k|g^3|_{H^k}
$$

for $k = 3, 4, \ldots, k_0$.

Proposition 2.1 can be proved in a similar manner to the proof of [23, Lemma 2.1].

Setting $\rho = \rho_s + \gamma^{-2}\phi$ and $v = v_s + w$ in (2.3)-(2.6) (without tildes), we arrive at the initial boundary value problem for the disturbance $u = \tau(\phi, w)$ written in (1.5)-(1.8) in section 1.
3 Main result

In this section we state the main result of this paper. Hereafter we set

\[ \tilde{\nu} = \nu + \nu'. \]

**Theorem 3.1.** There exist positive constant \( \nu_0, \gamma_0 \) and \( \omega_0 \) such that if \( \nu \geq \nu_0, \gamma^2 + \frac{\nu^2}{\nu + \nu'} \geq \gamma_0 \) and \( \frac{(\nu + \nu') \omega}{\nu} \leq \omega_0 \), then the following assertions hold. There is a positive number \( \epsilon_0 \) such that if \( u_0 = T(\phi_0, w_0) \) satisfies \( u_0 \in H^2 \cap L^1 \) with \( w_0 \in H^1_0 \) and \( \| u_0 \|_{H^2 \cap L^1} \leq \epsilon_0 \), then there exists a unique global solution \( u(t) = T(\phi(t), w(t)) \) of (1.5)-(1.8) in \( C^0([0, \infty); H^2 \times (H^2 \cap H^1_0)) \cap C^1([0, \infty); L^2) \); and the following estimates hold

\[ \| \partial_t u(t) \|_2 = O(t^{-\frac{3}{2} - \frac{1}{2}}), \quad (l = 0, 1) \]  
\[ \| u(t) - (\sigma u^{(0)})(t) \|_2 = O(t^{-\frac{3}{2} + \delta}) \quad (\forall \delta > 0) \]

as \( t \to \infty \). Here \( u^{(0)} = u^{(0)}(x') \) is a function given in Proposition 4.1 (iii) below; and \( \sigma = \sigma(x_3, t) \) is a function satisfying

\[ \partial_t \sigma - \kappa_0 \partial^2_{x_3} \sigma + \kappa_1 \partial_{x_3} \sigma + \kappa_2 \partial_{x_3} (\sigma^2) = 0, \]
\[ \sigma|_{t=0} = \int_D \phi_0(x', x_3) dx' \]  

(3.3)

with some constants \( \kappa_0 > 0 \) and \( \kappa_1, \kappa_2 \in \mathbb{R} \).

As in [3, 11], Theorem 3.1 is proved by combining the local solvability (Proposition 5.1 below) and the appropriate a priori estimates. We will establish the necessary a priori estimates in sections 6-8.

To establish the a priori estimates, we will use the results on spectral properties of the linearized semigroup \( e^{-tL} \) which will be summarized in section 4. In section 5, we will decompose the problem into the one for a coupled system of the linearized semigroup \( e^{-tL} \) and the appropriate a priori estimates. We will establish the necessary a priori estimates in sections 6-8. The a priori estimates will then be derived in sections 6-8. The proof of (3.2) will be given in section 9.

4 Spectrum of the linearized operator

In this section we state some results on spectral properties of the linearized operator established in [1, 2] which will be used in the proof of Theorem 3.1.

We denote the linearized operator by \( L \):

\[ L = \begin{pmatrix} \frac{\nu^2}{\gamma + \rho_s} \partial_{x_3} & \frac{\gamma^2 \div (\rho_s \Delta I_3) - \frac{\nu}{\rho_s} \Delta I_3 - \frac{\nu + \nu'}{\rho_s} \nabla \div + \frac{\nu}{\gamma - \rho_s^2} \partial_{x_3}}{\gamma + \rho_s} & 0 \\ \frac{\nu}{\gamma - \rho_s^2} \partial_{x_3} + \frac{\nu}{\gamma - \rho_s^2} \partial_{x_3} & e_3 \otimes (\nabla v^3) \end{pmatrix}, \]

where \( L \) is considered as an operator on \( L^2(\Omega) \) with domain

\[ D(L) = \{ u = T(\phi, w) \in L^2(\Omega) ; w \in H^1_0(\Omega), Lu \in L^2(\Omega) \}. \]

Here, for \( a = T(a_1, a_2, a_3) \) and \( b = T(b_1, b_2, b_3) \), \( a \otimes b \) is the \( 3 \times 3 \) matrix \((a_ib_j)\). See [1] for the details.
To investigate the spectrum of $L_0$, we consider the Fourier transform of (1.5)-(1.8) in $x_3$ variable with $f^0 = 0$ and $f = 0$, which takes the form

$$\partial_t \hat{\phi} + i\xi v_3^3 \hat{\phi} + \gamma^2 \nabla' \cdot (\rho_s \hat{w}') + \gamma^2 i\xi \rho_s \hat{w}^3 = 0,$$

(4.1)

$$\partial_t \hat{w}' - \frac{\nu}{\rho_s} (\Delta' - \xi^2) \hat{w}' - \frac{\nu + \nu'}{\rho_s} \nabla' (\nabla' \cdot \hat{w}' + i\xi \hat{w}^3) + \nabla' \left( \frac{p(\rho_s)}{\gamma^2 \rho_s} \hat{\phi} \right) + i\xi v_3^3 \hat{w}' = 0,$$

(4.2)

$$\partial_t \hat{w}^3 - \frac{\nu}{\rho_s} (\Delta' - \xi^2) \hat{w}^3 - \frac{\nu + \nu'}{\rho_s} i\xi (\nabla' \cdot \hat{w}' + i\xi \hat{w}^3) + i\xi \left( \frac{p(\rho_s)}{\gamma^2 \rho_s} \hat{\phi} \right) + i\xi v_3^3 \hat{w}^3 + \frac{\nu \Delta v_3^3}{\gamma^2 \rho_s} \hat{\phi} + \hat{w}' \cdot \nabla' v_3^3 = 0,$$

(4.3)

$$\hat{w} |_{\partial D} = 0$$

(4.4)

for $t > 0$, and

$$T(\hat{\phi}, \hat{w}) |_{t=0} = T(\hat{\phi}_0, \hat{w}_0) = \hat{u}_0.$$  

(4.5)

We thus arrive at the following problem

$$\frac{du}{dt} + \hat{L}_\xi u = 0, \quad u |_{t=0} = u_0,$$

(4.6)

where $\xi \in \mathbb{R}$ is a parameter. Here $u = T(\phi(x', t), w(x', t)) \in D(\hat{L}_\xi)$ ($x' \in D, t > 0$); $u_0 = T(\phi_0(x'), w_0(x'))$ is a given initial value; and $\hat{L}_\xi$ is the operator on $L^2(D)$ of the form

$$\hat{L}_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s} (\Delta' - |\xi|^2) I_2 - \frac{\nu + \nu'}{\rho_s} \nabla' \nabla' & -i\nu + \nu' \xi \nabla' \\ 0 & -i\nu + \nu' \xi \nabla' & -\nu \rho_s (\Delta' - |\xi|^2) + \frac{\nu + \nu'}{\rho_s} |\xi|^2 \end{pmatrix}$$

$$+ \begin{pmatrix} i\xi v_3^3 & \gamma^2 \nabla' (\rho_s) & i\gamma^2 \rho_s \xi \\ \nabla' \left( \frac{p(\rho_s)}{\gamma^2 \rho_s} \right) & i\xi v_3^3 I_2 & 0 \\ i\xi \frac{p(\rho_s)}{\gamma^2 \rho_s} & 0 & i\xi v_3^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\nu \Delta v_3^3}{\gamma^2 \rho_s} \nabla T(\nabla' v_3^3) & 0 \end{pmatrix}$$

with domain

$$D(\hat{L}_\xi) = \{ u = T(\phi, w) \in L^2(D); \ w \in H^1_0(D), \ \hat{L}_\xi u \in L^2(D) \}.$$  

Note that $D(\hat{L}_\xi) = D(\hat{L}_0)$ for all $\xi \in \mathbb{R}$. Here we set $\hat{L}_0 = \hat{L}_\xi |_{\xi=0}$.

We also introduce the adjoint operator $\hat{L}^*_\xi$ of $\hat{L}_\xi$ with respect to the weighted inner product $\langle \cdot, \cdot \rangle$. The operator $\hat{L}^*_\xi$ is given by

$$\hat{L}^*_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s} (\Delta' - |\xi|^2) I_2 - \frac{\nu + \nu'}{\rho_s} \nabla' \nabla' & -i\nu + \nu' \xi \nabla' \\ 0 & -i\nu + \nu' \xi \nabla' & -\nu \rho_s (\Delta' - |\xi|^2) + \frac{\nu + \nu'}{\rho_s} |\xi|^2 \end{pmatrix}$$

$$- \begin{pmatrix} i\xi v_3^3 & \gamma^2 \nabla' (\rho_s) & i\gamma^2 \rho_s \xi \\ \nabla' \left( \frac{p(\rho_s)}{\gamma^2 \rho_s} \right) & i\xi v_3^3 I_2 & 0 \\ i\xi \frac{p(\rho_s)}{\gamma^2 \rho_s} & 0 & i\xi v_3^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\nu \Delta v_3^3}{\gamma^2 \rho_s} \\ 0 & \nabla T(\nabla' v_3^3) & 0 \end{pmatrix}$$

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with domain
\[ D(\hat{L}_\xi) = \{ u = T(\phi, w) \in L^2(D); w \in H_0^1(D), \hat{L}_\xi^* u \in L^2(D) \}. \]

Note that \( D(\hat{L}_\xi) = D(\hat{L}_\xi^*) \) for any \( \xi \in \mathbb{R} \).

Concerning the spectrum of \(-\hat{L}_\xi \) for \( |\xi| \ll 1 \), we have the following proposition, which was proved in [2, Theorem 4.5, 4.7] and [1, Lemma 4.1].

**Proposition 4.1.** (i) There exist positive constants \( c_0, \nu_1, \gamma_1, \omega_1 \) and \( r_0 \) such that if \( \nu \geq \nu_1, \frac{\gamma^2}{\nu + \nu} \geq \gamma_1^2 \) and \( \omega \leq \omega_1 \), then it holds that
\[ \sigma(-\hat{L}_\xi) \cap \{ \lambda : |\lambda| \leq \frac{\omega}{2} \} = \{ \lambda_0(\xi) \} \]
for each \( \xi \) with \( |\xi| \leq r_0 \), where \( \lambda_0(\xi) \) is a simple eigenvalue of \(-\hat{L}_\xi \) that has the form
\[ \lambda_0(\xi) = -i \kappa_1 \xi - \kappa_0 \xi^2 + O(|\xi|^3) \]
as \( |\xi| \to 0 \). Here \( \kappa_1 \in \mathbb{R} \) and \( \kappa_0 > 0 \) are the numbers given by
\[ \kappa_1 = \langle v_0^2 \phi(0) + \gamma^2 \rho_0 w(0), 3 \rangle = O(1), \]
\[ \kappa_0 = \frac{\gamma^2}{\nu} \left\{ a_0 \left( -\Delta' \right)^{-\frac{\nu}{2}} \rho_0 \right\}^2 + O(\frac{1}{\gamma}) + \left( \frac{\xi}{\gamma} + \frac{1}{\nu} \right) \times O(\frac{\nu + \nu}{\gamma}) \} \].

Here \((-\Delta')\) is the Laplace operator on \( L^2(D) \) under the zero Dirichlet boundary condition with domain
\[ D(-\Delta') = H^2(D) \cap H_0^1(D). \]

(ii) The eigenprojections \( \hat{\Pi}(\xi) \) and \( \hat{\Pi}^*(\xi) \) for the eigenvalues \( \lambda_0(\xi) \) and \( \lambda_0(\xi) \) of 
\(-\hat{L}_\xi \) and \(-\hat{L}_\xi^* \) are given by
\[ \hat{\Pi}(\xi) u = \langle u, u_\xi \rangle u_\xi, \quad \hat{\Pi}^*(\xi) u = \langle u, u_\xi \rangle u_\xi, \]
respectively, where \( u_\xi \) and \( u_\xi^* \) are eigenfunctions for \( \lambda_0(\xi) \) and \( \lambda_0(\xi) \), respectively, that satisfy
\[ \langle u_\xi, u_\xi^* \rangle = 1. \]
Furthermore, \( u_\xi \) and \( u_\xi^* \) are written in the form
\[ u_\xi(x') = u^{(0)}(x') + i \xi u^{(1)}(x') + |\xi|^2 u^{(2)}(x', \xi), \]
\[ u_\xi^*(x') = u^{(0)}(x') + i \xi u^{(1)}(x') + |\xi|^2 u^{(2)}(x', \xi), \]
and the following estimates hold
\[ |u|_{H^{k+2}} \leq C_{k,r_0} \]
for \( u \in \{ u_\xi, u_\xi^*, u^{(1)}, u^{(1)}, u^{(2)}, u^{(2)} \} \) and \( k = 0, 1, \ldots, k_0 \) with a positive constant \( C_{k,r_0} \) depending on \( k \) and \( r_0 \). Here \( u^{(0)} \) and \( u^{(0)*} \) are eigenfunctions of \(-\hat{L}_0 \) and \(-\hat{L}_0^* \) for the eigenvalue 0, respectively.
(iii) The functions $u^{(0)}$ and $u^{(0)*}$ are given by
\[ u^{(0)} = T(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = T(0, 0, w^{(0)}), \]
and
\[ u^{(0)*} = T(\phi^{(0)*}, 0). \]
Here
\[ \phi^{(0)}(x') = \alpha_0 \frac{\gamma^2 \rho_s(x')}{\mathcal{F}(\rho_s(x'))}, \quad \alpha_0 = \left( \int_D \frac{\gamma^2 \rho_s(x')}{\mathcal{F}(\rho_s(x'))} dx' \right)^{-1}; \]
and $w^{(0)}$ is the solution of the following problem
\[ \begin{cases} -\Delta' w^{(0)} & = -\frac{1}{\gamma^2 \rho_s} \Delta' v^{(0)}, \\ w^{(0)} & = 0 \text{ on } \partial D; \end{cases} \]
and
\[ \phi^{(0)*}(x') = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x'). \]
Furthermore, it holds that
\[ \langle u_0, u_0^* \rangle = 1. \]

We next consider the spectral properties of the semigroup $e^{-tL}$ generated by $-L$. We denote the characteristic function of the set \( \{ \xi \in \mathbb{R} : |\xi| \leq r_0 \} \) by \(1_{\{|\eta| \leq r_0\}}(\xi)\), i.e.,
\[ 1_{\{|\eta| \leq r_0\}}(\xi) = \begin{cases} 1, & (0 \leq |\xi| \leq r_0), \\ 0, & (|\xi| > r_0), \end{cases} \text{ for } \xi \in \mathbb{R}, \]
where $r_0$ is a positive constant given in Proposition 4.1.

We define the projections $P_0$ and $P_\infty$ by
\[ P_0 = \mathcal{F}^{-1} 1_{\{|\eta| \leq r_0\}}(\xi) \hat{\Pi}(\xi) \mathcal{F} \]
and
\[ P_\infty = I - P_0. \]
Then $P_0$ and $P_\infty$ satisfy
\[ P_0 + P_\infty = I, \quad P^2 = P, \]
\[ PL \subset LP, \quad Pe^{-tL} = e^{-tL} P \]
for $P \in \{P_0, P_\infty\}$. The semigroup $e^{-tL}$ has the following properties. See [2, Theorem 3.1].

**Proposition 4.2.** If $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu + \nu} \geq \gamma_1^2$ and $\omega \leq \omega_1$, then $e^{-tL}P_0$ and $e^{-tL}P_\infty$ have the following properties.

(i) If $u_0 = T(\phi_0, w_0) \in L^1(\Omega) \cap L^2(\Omega)$, then $e^{-tL}P_0u_0$ satisfies the following estimates
\[ \|\partial^k_x \partial^l_x e^{-tL} P_0 u_0\|_2 \leq C_{k,l}(1 + t)^{-\frac{1}{2} - \frac{l}{2}} \|u_0\|_1 \quad (4.7) \]
uniformly for \( t \geq 0 \) and for \( k = 0, 1, \cdots, k_0 \) and \( l = 0, 1, \cdots \);
\[
\| e^{-tL} P_0 u_0 - [\mathcal{H}(t) \langle \phi_0 \rangle] u^{(0)} \|_2 \leq Ct^{-\frac{3}{4}} \| u_0 \|_1
\]  
(4.8)
uniformly for \( t > 0 \). Here
\[
\mathcal{H}(t) \langle \phi_0 \rangle = \mathcal{F}^{-1}[e^{-(\kappa_1 t + \nu_0 t^2)}(\hat{\phi}_0)],
\]
where \( u^{(0)} = u^{(0)}(x') \) is the function given in Proposition 4.1; and \( \kappa_1 \in \mathbb{R} \) and \( \kappa_0 > 0 \) are the constants given in Proposition 4.1.

(ii) If \( u_0 \in H^1(\Omega) \times \tilde{H}^1(\Omega) \), then there exists a constant \( a_0 > 0 \) such that \( e^{-tL} P_\infty u_0 \) satisfies
\[
\| e^{-tL} P_\infty u_0 \|_{H^1} \leq C e^{-a_0 t} \left( \| u_0 \|_{H^1 \times \tilde{H}^1} + t^{-\frac{1}{2}} \| u_0 \|_2 \right)
\]  
(4.9)
uniformly for \( t \geq 0 \).

More detailed information of the \( P_0 \)-part of \( e^{-tL} \) is needed to analyze the non-linear problem; and so we next give a factorization of \( e^{-tL} P_0 \) which was obtained in [2, Section 5].

Let us introduce operators related to \( u_\xi \) and \( u_\xi^* \). We define the operators
\[
\mathcal{T} : L^2(\mathbb{R}) \rightarrow L^2(\Omega), \quad \mathcal{P} : L^2(\Omega) \rightarrow L^2(\mathbb{R}), \quad \Lambda : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})
\]
by
\[
\mathcal{T} \sigma = \mathcal{F}^{-1} [\hat{T}_\xi \hat{\sigma}], \quad \hat{T}_\xi \hat{\sigma} = 1_{\{|\eta| \leq \nu_0\}}(\xi) u_\xi \hat{\sigma};
\]
\[
\mathcal{P} u = \mathcal{F}^{-1} [\hat{P}_\xi \hat{u}], \quad \hat{P}_\xi \hat{u} = 1_{\{|\eta| \leq \nu_0\}}(\xi) \langle \hat{u}, u_\xi^* \rangle;
\]
\[
\Lambda \sigma = \mathcal{F}^{-1} [1_{\{|\eta| \leq \nu_0\}}(\xi) \lambda_0(\xi) \hat{\sigma}]
\]
for \( u \in L^2(\Omega) \) and \( \sigma \in L^2(\mathbb{R}) \). It follows that
\[
P_0 = \mathcal{T} \mathcal{P}, \quad 1_{\{|\eta| \leq \nu_0\}}(\xi) \hat{\Pi}(\xi) = \hat{T} \hat{P};
\]
\[
P_0 L \subset LP_0 = \Lambda P_0, \quad e^{-tL} P_0 = \mathcal{T} e^{t\Lambda} \mathcal{P}.
\]
Note that \( P_0 e^{-tL} = e^{-tL} P_0 \).

Here and in what follows we assume that
\[
\nu \geq \nu_1, \quad \frac{\gamma^2}{\nu + \nu} \geq \gamma_1^2, \quad \omega \leq \omega_1,
\]
where \( \nu_1, \gamma_1 \) and \( \omega_1 \) are the constants given in Proposition 4.1.

As for \( \mathcal{T} \), we have the following proposition.

**Proposition 4.3.** ([2, Proposition 5.1]) The operator \( \mathcal{T} \) has the following properties:

(i) \( \partial_{x_3}^l \mathcal{T} = \mathcal{T} \partial_{x_3}^l \) for \( l = 0, 1, \cdots \).

(ii) \( \| \partial_{x_3}^k \partial_{x_3}^l \mathcal{T} \sigma \|_2 \leq C \| \sigma \|_{L^2(\mathbb{R})} \) for \( k = 0, 1, \cdots k_0 \), \( l = 0, 1, \cdots \) and \( \sigma \in L^2(\mathbb{R}) \).
(iii) $T$ is decomposed as

$$T = T^{(0)} + \partial_{x_3} T^{(1)} + \partial^2_{x_3} T^{(2)},$$

where $T^{(j)} \sigma = F^{-1}[\hat{T}^{(j)}\hat{\sigma}]$ $(j = 0, 1, 2)$ with

$$\hat{T}^{(0)}\hat{\sigma} = 1_{\{|\xi| \leq \rho_0\}}(\xi) \hat{\sigma} u^{(0)},$$
$$\hat{T}^{(1)}\hat{\sigma} = 1_{\{|\xi| \leq \rho_0\}}(\xi) \hat{\sigma} u^{(1)}(\cdot),$$
$$\hat{T}^{(2)}\hat{\sigma} = -1_{\{|\xi| \leq \rho_0\}}(\xi) \hat{\sigma} u^{(2)}(\cdot, \xi).$$

Here $T^{(j)} (j = 0, 1, 2)$ satisfy estimates (i) and (ii) by replacing $T$ with $T^{(j)}$.

As for $P$, we have the following properties.

**Proposition 4.4.** ([2, Proposition 5.2]) The operator $P$ has the following properties:

(i) $\partial^l_{x_3} P = P \partial^l_{x_3}$ for $l = 0, 1, \cdots$.

(ii) $\|\partial^l_{x_3} Pu\|_{L^2(\Omega)} \leq C\|u\|_2$ for $k = 0, 1, \cdots k_0, l = 0, 1, \cdots$ and $u \in L^2(\Omega)$. Furthermore, $\|Pu\|_{L^2(\Omega)} \leq C\|u\|_1$ for $u \in L^1(\Omega)$.

(iii) $P$ is decomposed as

$$P = P^{(0)} + \partial_{x_3} P^{(1)} + \partial^2_{x_3} P^{(2)},$$

where $P^{(j)} u = F^{-1}[\hat{P}^{(j)}\hat{u}]$ $(j = 0, 1, 2)$ with

$$\hat{P}^{(0)}\hat{u} = 1_{\{||\xi| \leq \rho_0\}}(\xi) \langle \hat{u}, u^{(0)} \rangle = 1_{\{||\xi| \leq \rho_0\}}(\xi) \langle Q_0\hat{u}, \rangle,$$
$$\hat{P}^{(1)}\hat{u} = 1_{\{||\xi| \leq \rho_0\}}(\xi) \langle \hat{u}, u^{(1)} \rangle,$$
$$\hat{P}^{(2)}\hat{u} = -1_{\{||\xi| \leq \rho_0\}}(\xi) \langle \hat{u}, u^{(2)}(\xi) \rangle.$$

Here $P^{(j)} (j = 0, 1, 2)$ satisfy estimates (i) and (ii) by replacing $P$.

Concerning $A$, we have the following estimates for $e^{tA}$.

**Proposition 4.5.** ([2, Proposition 5.3]) The operator $e^{tA}$ satisfies the following estimates.

(i) $\|\partial^l_{x_3} e^{tA} Pu\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{4}{2} - \frac{l}{2}}\|u\|_1$,

(ii) $\|\partial^l_{x_3} e^{tA} P^{(j)} u\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{4}{2} - \frac{l}{2}}\|u\|_1, \quad j = 0, 1, 2$,

(iii) $\|\partial^l_{x_3} (T - T^{(0)}) e^{tA} Pu\|_2 \leq C(1 + t)^{-\frac{4}{2} - \frac{l+1}{2}}\|u\|_1$,

for $u \in L^1(\Omega)$ and $l = 0, 1, 2$.

We next consider the asymptotic behavior of $e^{tA}$. Let us define an operator $H(t)$ by

$$H(t) \sigma = F^{-1}[e^{-(i\kappa_1\xi + \kappa_0\xi^2)t}\hat{\sigma}]$$

for $\sigma \in L^2(\mathbb{R})$, where $\kappa_1 \in \mathbb{R}$ and $\kappa_0 > 0$ are given by Proposition 4.1. The asymptotic leading part of $e^{tA}$ is given by $H(t)$. In fact, we have the following estimates.
Proposition 4.6. ([2, Proposition 5.8]) For \( u \in L^2(\Omega) \), we set \( \sigma = \langle Q_0 u \rangle \). If \( u \in L^1(\Omega) \), then there holds the estimate
\[
\| \partial_{x_3}^l (e^{t \Lambda} \mathcal{P}u - H(t)\sigma) \|_{L^2(\mathbb{R})} \leq C t^{-\frac{3}{4} - \frac{3}{2l}} \| u \|_1 \quad (l = 0, 1, \ldots).
\]
Observe that, since \( e^{t \mathcal{L}} P_0 = T e^{t \Lambda} \mathcal{P} = T(0) e^{t \Lambda} \mathcal{P} + (T - T(0)) e^{t \Lambda} \mathcal{P} \), we have (4.8) by Propositions 4.3 (iii), 4.5 (iii) and 4.6.

5 Decomposition of Problem

In this section we formulate the problem. The problem (1.5)-(1.8) is written as
\[
\frac{du}{dt} + Lu = F, \quad w|_{\partial \Omega} = 0, \quad u|_{t=0} = u_0.
\]
Here \( u = T(\phi, w); \quad F = F(u) \) denotes the nonlinearity:
\[
F = T(f^0(\phi, w), f(\phi, w)).
\]

The local solvability in \( Z(T) \) for (5.1) follows from [12].

Proposition 5.1. If \( u_0 = T(\phi_0, w_0) \) satisfies the following conditions;

(i) \( u_0 \in H^2 \times (H^2 \cap H^1) \),

(ii) \(-\frac{\alpha}{\pi} \rho_1 \leq \phi_0\),

then there exists a number \( T_0 > 0 \) depending on \( \| u_0 \|_{H^2} \) and \( \rho_1 \) such that the following assertions hold. Problem (5.1) has a unique solution \( u(t) \in Z(T) \) satisfying
\[
\phi(x, t) \geq -\frac{\alpha}{\pi} \rho_1 \quad \text{for} \quad \forall (x, t) \in \Omega \times [0, T_0];
\]
and the following estimate holds
\[
\| u \|_{Z(T)}^2 \leq C_0 \{ 1 + \| u_0 \|_{H^2}^2 \}^\alpha \| u_0 \|_{H^2}^2
\]
for some positive constants \( C_0 \) and \( \alpha \).

Theorem 3.1 would follow if we would establish the a priori estimates of \( u(t) \) in \( Z(T) \) uniformly for \( T \).

To obtain the appropriate a priori estimates, we decompose the solution \( u \) into its \( P_0 \) and \( P_\infty \) parts. Let us decompose the solution \( u(t) \) of (5.1) as
\[
u(t) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t),
\]
where \( \sigma_1(t) = \mathcal{P} u(t), \quad u_1(t) = (T - T^{(0)}) \mathcal{P} u(t), \quad u_\infty(t) = P_\infty u(t). \)
Note that \( P_0 u(t) = (\sigma_1 u(0))(t) + u_1(t) \).

Since \( u_1(t) \) is written as
\[
u_1(t) = (\mathcal{T} - T^{(0)}) P u(t) = (\partial_{x_3} T^{(1)} + \partial_{x_3} T^{(2)}) \sigma_1(t),
\]
we see from Proposition 4.3 and Proposition 4.4 the following estimates for \( \sigma_1(t) \) and \( u_1(t) \).

**Proposition 5.2.** Let \( u(t) \) be a solution of (5.1) in \( Z(T) \). Then there hold the estimates
\[
\| \partial_{x_3} \sigma_1(t) \|_2 \leq C \| \partial_{x_3} \sigma_1(t) \|_2
\]
for \( 1 \leq l \leq 3 \); and
\[
\| \partial_{x_3} \partial_{x_3} \partial_t^m u_1(t) \|_2 \leq C \{ \| \partial_{x_3} \sigma_1(t) \|_2 + \| \partial_t \sigma_1(t) \|_2 \}
\]
for \( 1 \leq k + l + 2m \leq 3 \).

We derive the equations for \( \sigma_1(t) \) and \( u_1(t) \).

**Proposition 5.3.** Let \( T > 0 \) and assume that \( u(t) \) is a solution of (5.1) in \( Z(T) \). Then the following assertions hold.
\[
\sigma_1 \in \bigcap_{j=0}^{1} C^j([0, T] : H^2(\mathbb{R})), \quad u_\infty \in Z(T), \quad \phi_\infty \in C^1([0, T]; H^1).
\]
Furthermore, \( \sigma_1 \) and \( u_\infty \) satisfy
\[
\sigma_1(t) = e^{t \Lambda} P u_0 + \int_{0}^{T} e^{(t - \tau) \Lambda} P F(\tau) d\tau; \quad (5.2)
\]
and
\[
\partial_t u_\infty + L u_\infty = F_\infty, \quad u_\infty |_{\partial \Omega} = 0, \quad u_\infty |_{t=0} = u_{\infty,0}, \quad (5.3)
\]
where \( F_\infty = P_\infty F \) and \( u_{\infty,0} = P_\infty u_0 \).

Let \( u(t) \) be a solution of (5.1) in \( Z(T) \). From Proposition 5.2, we obtain
\[
\sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} \{ \| u_1(\tau) \|_2 + \| \partial_2 u_1(\tau) \|_2 \}
\]
\[
\leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} \{ \| \partial_{x_3} \sigma_1(\tau) \|_2 + \| \partial_t \sigma_1(\tau) \|_2 \},
\]
and thus, the estimates for \( u_1(t) \) follows from the ones for \( \sigma_1(t) \). Therefore, as in [3], we introduce the quantity \( M_1(t) \) defined by
\[
M_1(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} \| \sigma_1(\tau) \|_2 + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} \{ \| \partial_{x_3} \sigma_1(\tau) \|_2 + \| \partial_t \sigma_1(\tau) \|_2 \};
\]
and we define the quantity \( M(t) \geq 0 \) by
\[
M(t)^2 = M_1(t)^2 + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} E_\infty(\tau) \quad (t \in [0, T])
\]
with
\[ E_\infty(t) = \left\| u_\infty(t) \right\|_2^2. \]

We define a quantity \( D_\infty(t) \) for \( u_\infty = \mathcal{T}(\phi_\infty, w_\infty) \) by
\[ D_\infty(t) = \left\| D\phi_\infty(t) \right\|_2^2 + \left\| Dw_\infty(t) \right\|_2^2. \]

If we could show \( M(t) \leq C \) uniformly for \( t \geq 0 \), then Theorem 3.1 would follow.

The uniform estimate for \( M(t) \) is given by using the following estimates for \( M_1(t) \) and \( E_\infty(t) \).

**Proposition 5.4.** There exist positive constants \( \nu_0, \gamma_0 \) and \( \omega_0 \) such that if \( \nu \geq \nu_0, \frac{\gamma_0^2}{\nu+\nu} \geq \gamma_0^2 \) and \( \omega \leq \omega_0 \), then the following assertions hold. There is a positive number \( \epsilon_1 \) such that if a solution \( u(t) \) of (5.1) in \( Z(T) \) satisfies \( \sup_{0 \leq \tau \leq t} \left\| u(\tau) \right\|_2 \leq \epsilon_1 \) and \( M(t) \leq 1 \) for \( t \in [0, T] \), then the estimates
\[ M_1(t) \leq C\left\{ \| u_0 \|_{L^1} + M(t)^2 \right\} \] (5.4)
and
\[ E_\infty(t) + \int_0^\infty e^{-a(t-\tau)} D_\infty(\tau) d\tau \]
\[ \leq C\left\{ e^{-at} E_\infty(0) + (1+t)^{-3/2} M(t)^4 + \int_0^t e^{-a(t-\tau)} \mathcal{R}(\tau) d\tau \right\} \] (5.5)
hold uniformly for \( t \in [0, T] \) with \( C > 0 \) independent of \( T \). Here \( a = a(\nu, \bar{\nu}, \gamma) \) is a positive constant; and \( \mathcal{R}(t) \) is a function satisfying the estimate
\[ \mathcal{R}(t) \leq C\left\{ (1+t)^{-3/2} M(t)^3 + (1+t)^{-1/2} M(t) D_\infty(t) \right\} \] (5.6)
provided that \( \sup_{0 \leq \tau \leq t} \left\| u(\tau) \right\|_2 \leq \epsilon_2 \) and \( M(t) \leq 1 \).

Proposition 5.4 follows from Propositions 6.1, 7.1 and 8.1 below.

As in [3, 11], one can see from Propositions 5.1 and 5.4 that if \( \| u_0 \|_{H^2 \cap L^1} \) is sufficiently small, then
\[ M(t) \leq C\| u_0 \|_{H^2 \cap L^1} \]
uniformly for \( t \geq 0 \), which proves Theorem 3.1.

### 6 Estimates for \( P_0 \)-part of \( u(t) \)

In this section, we estimate the \( P_0 \)-part of \( u(t) \)
\[ P_0 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t), \]
where \( \sigma_1(t) = \mathcal{P} u(t) \) and \( u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)}) \mathcal{P} u(t) \). We will prove the following estimate for \( M_1(t) \).
Proposition 6.1. Let $T > 0$ and assume that $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu+\nu} \geq \gamma^2_1$ and $\omega \leq \omega_1$. Then there exists a positive constant $\epsilon$ independent of $T$ such that if a solution $u(t)$ of (5.1) in $Z(T)$ satisfies $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon$ and $M(t) \leq 1$ for all $t \in [0, T]$, then the estimate

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}$$

holds uniformly for $t \in [0, T]$, where $C$ is a positive constant independent of $T$.

Let us prove Proposition 6.1. We decompose the nonlinearity $F$ into

$$F = \sigma_1^2 F_1 + F_2,$$

where

$$F_1 = F_1(x') = -\frac{1}{2\gamma \mu(x')} \nabla' \left\{ P''(\rho_s(x')) (\phi(0)(x'))^2 \right\}, \quad 0,$$

$$F_2 = F - \sigma_1^2 F_1.$$

Here $\sigma_1^2 F_1(x')$ is the part of $F$ containing only $\sigma_1^2(t)$ but not $\partial_{x_3}\sigma_1(t)$, $u_1(t)$, $u_\infty(t)$, $\sigma_1^2(t)$ and so on.

Before going further, we introduce a notation. For a function $g$ we define $\langle g \rangle_0$ by

$$\langle g \rangle_0 = \mathcal{F}^{-1}[1_{\{|\eta| \leq r_0\}}(\xi)\hat{g}].$$

The nonlinearity $F$ has the following properties.

Lemma 6.2. There hold the following assertions.

(i) $\langle Q_0 F \rangle = -\partial_{x_3} \langle \phi w^3 \rangle$.

(ii) $\mathcal{P} F = -\partial_{x_3} \langle \phi w^3 \rangle_0 + \langle \mathcal{P}^{(1)} F + \partial_{x_3}^2 \mathcal{P}^{(2)} F \rangle$.

Proof. As for (i), we see from integration by parts that $\langle \nabla' \cdot (\phi w') \rangle = 0$. It then follows that

$$\langle Q_0 F \rangle = -\langle \text{div}(\phi w) \rangle = -\langle \partial_{x_3} (\phi w^3) \rangle = -\partial_{x_3} \langle \phi w^3 \rangle.$$

We next prove (ii). From the definition of $\mathcal{P}^{(0)}$ and (i), there holds that

$$\mathcal{P}^{(0)} F = \mathcal{F}^{-1}[1_{\{|\eta| \leq r_0\}}(\xi)\hat{Q}_0 \hat{F}]] = \langle Q_0 F \rangle_0 = -\partial_{x_3} \langle \phi w^3 \rangle_0.$$

We thus obtain (ii). This completes the proof.

Noting that $\|\sigma_1\|_{\infty} \leq C\|\sigma_1\|_2^{\frac{1}{2}}\|\partial_{x_3}\sigma_1\|_2^{\frac{1}{2}}$, one can obtain the following estimates by straightforward computations.

Lemma 6.3. There exists a positive constant $\epsilon$ such that if a solution $u(t)$ of (5.1) in $Z(T)$ satisfies $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon$ and $M(t) \leq 1$ for all $t \in [0, T]$, then the following estimates hold for $t \in [0, T]$ with a positive constant $C$ independent of $T$.
By (6.1) and (6.2) we deduce the desired estimate. This completes the proof. □

This, together with (5.2), (6.1) and Lemma 6.3, implies that

\[ \| \phi w^3(t) \|_1 \leq C(1 + t)^{-1} M(t)^2. \]

By Proposition 4.5 and Lemma 6.3 we then have

\[ \| F(t) \|_1 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2. \]

We see from Lemma 6.2 (ii) that

\[ \| \phi w^3(t) \|_1 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2. \]

We next consider

\[ \| \phi w^3(t) \|_1 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2. \]

Proof of Proposition 6.1 We see from Proposition 4.5 that

\[ \| \partial_{t}^l e^{t \Lambda} P u_0 \|_2 \leq C(1 + t)^{-\frac{l}{2} - \frac{1}{2}} \| u_0 \|_1 \quad (l = 0, 1). \]

We next consider \( \int_0^t e^{(t-\tau) \Lambda} P F(\tau) d\tau \). We write it as

\[ \int_0^t e^{(t-\tau) \Lambda} P F(\tau) d\tau = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) e^{(t-\tau) \Lambda} P F(\tau) d\tau =: I_1(t) + I_2(t). \]

We see from Lemma 6.2 (ii) that

\[ e^{(t-\tau) \Lambda} P F(\tau) = e^{(t-\tau) \Lambda} \{ -\partial_{t}^l \phi w^3_0 + \partial_{t}^l P^{(1)} F + \partial_{t}^2 P^{(2)} F \} = \partial_{t}^l e^{(t-\tau) \Lambda} \{ -\phi w^3_0 + P^{(1)} F + \partial_{t}^2 P^{(2)} F \}. \]

By Proposition 4.5 and Lemma 6.3 we then have

\[ \| \partial_{t}^l I_1(t) \|_2 \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{l}{2} - \frac{1}{2}} (\| \phi w^3_0(\tau) \|_1 + \| F(\tau) \|_1) d\tau \]

\[ \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{l}{2} - \frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} M(t)^2 d\tau \]

\[ \leq C(1 + t)^{-\frac{1}{2} - \frac{l}{2}} M(t)^2 \]

for \( l = 0, 1 \). Applying Lemma 6.3 (ii) and (v) we have

\[ \| \partial_{t}^l I_2(t) \|_2 \leq C \int_0^t (1 + t - \tau)^{-\frac{l}{2} - \frac{1}{2}} (1 + \tau)^{-1} M(t)^2 d\tau M(t)^2 \]

\[ \leq C(1 + t)^{-\frac{1}{2} - \frac{l}{2}} M(t)^2 \]

for \( l = 0, 1 \). We thus obtain

\[ \| \partial_{t}^l \sigma_1(t) \|_2 \leq C(1 + t)^{-\frac{l}{2} - \frac{1}{2}} \{ \| u_0 \|_1 + M(t)^2 \} \quad (6.1) \]

for \( l = 0, 1 \).

Let us estimate the time derivative. Since \( \lambda_0(\xi) = -(i\kappa_1 \xi + \kappa_0 \xi^2 + O(\xi^3)) = O(\xi) \), we obtain

\[ \| A \sigma_1(t) \|_2 = \| F^{-1}[1_{|\sigma_1| \leq \rho_0}](\xi) \lambda_0(\xi) \bar{\sigma}_1(t) \|_2 \leq C \| \partial_{t} x_3 \sigma_1(t) \|_2. \]

This, together with (5.2), (6.1) and Lemma 6.3, implies that

\[ \| \partial_{t} \sigma_1(t) \|_2 \leq C \{ \| \partial_{t} x_3 \sigma_1(t) \|_2 + \| F(t) \|_2 \} \leq C(1 + t)^{-\frac{3}{4}} \{ \| u_0 \|_1 + M(t)^2 \}. \quad (6.2) \]

By (6.1) and (6.2) we deduce the desired estimate. This completes the proof. □
7 Estimates for $P_\infty$-part of $u(t)$

In this section we derive the estimates for the $P_\infty$-part of $u(t)$.

Throughout this section, we assume that $u(t)$ is a solution of (5.1) in $Z(T)$ for a given $T > 0$. We show the following estimate.

**Proposition 7.1.** There exist positive constants $\nu_0 (\geq \nu_1)$, $\gamma_0 (\geq \gamma_1)$ and $\omega_0 (\leq \omega_1)$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu + \nu} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then

$$E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau$$

$$\leq C \{e^{-a t} E_\infty(0) + (1 + t)^{-\frac{3}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} \mathcal{R}(\tau) d\tau \}.$$

uniformly for $t \in [0, T]$ with $C > 0$ independent of $T$.

Proposition 7.1 is proved by the estimate (4.9) for $e^{-tL} P_\infty$ and the Matsumura-Nishida energy method.

We introduce notations. In what follows $C$ and $C_j$ ($j = 1, 2, \cdots$) denote various constants independent of $T$, $\nu$, $\nu$, and $\gamma$, whereas, $C_{\nu, \nu, \gamma, \cdots}$ denotes various constants which depends on $\nu, \nu, \gamma, \cdots$ but not on $T$.

We first establish the $H^1$-estimate for $u_\infty$ which follows from the estimate (4.9) for $e^{-tL} P_\infty$.

**Proposition 7.2.** There exist positive constants $\nu_0 (\geq \nu_1)$, $\gamma_0 (\geq \gamma_1)$ and $\omega_0 (\leq \omega_1)$ such that if

$$\nu \geq \nu_0, \quad \frac{\gamma^2}{\nu + \nu} \geq \gamma_0^2, \quad \omega \leq \omega_0; \quad (7.1)$$

then, for any $0 < a < 2a_0$,

$$\|u_\infty(t)\|_{H^1} + \int_0^t e^{-a(t-\tau)} \|u_\infty(\tau)\|_{H^1}^2 d\tau$$

$$\leq C_{\nu, \nu, \gamma} \{e^{-a t} \|u_\infty, 0\|_{H^1}^2 + \sup_{0 \leq \tau \leq t} \|F_\infty(\tau)\|_2^2 + \int_0^t e^{-a(t-\tau)} \|F_\infty(\tau)\|_{H^1}^2 d\tau \}.$$

**Proof.** We write $u_\infty(t)$ as

$$u_\infty(t) = e^{-tL} u_\infty, 0 + \int_0^t e^{-(t-\tau)L} F_\infty(\tau) d\tau.$$
Since \( u_{\infty,0} \in H^1 \times H^1_0 \), we see from (4.9) that

\[
\|u_{\infty}(t)\|_{H^1} \leq C \left\{ e^{-a_0 t} \|u_{\infty,0}\|_{H^1}^2 + \int_0^t e^{-a_0(t-\tau)} \|F_{\infty}(\tau)\|_{H^1 \times H^1} \, d\tau \right\}
+ \int_0^t e^{-a_0(t-\tau)} (t - \tau)^{-\frac{1}{2}} \|F_{\infty}(\tau)\|_2 \, d\tau \leq C \left\{ e^{-a_0 t} \|u_{\infty,0}\|_{H^1}^2 + \sup_{0 \leq \tau \leq t} \|F_{\infty}(\tau)\|_2 \right\}
+ \int_0^t e^{-a_0(t-\tau)} \|F_{\infty}(\tau)\|_{H^1 \times H^1} \, d\tau \}
\]

from which we have

\[
\|u_{\infty}(t)\|_{H^1}^2 \leq C \left\{ e^{-2a_0 t} \|u_{\infty,0}\|_{H^1}^2 + \sup_{0 \leq \tau \leq t} \|F_{\infty}(\tau)\|_2^2 + \int_0^t e^{-a(t-\tau)} \|F_{\infty}(\tau)\|_{H^1}^2 \, d\tau \right\} (7.2)
\]

for any \( 0 < a < 2a_0 \). Set \( V(t) = \int_0^t e^{-a(t-\tau)} \|F_{\infty}(\tau)\|_{H^1}^2 \, d\tau \). Then \( V(t) \) satisfies \( dV/dt + \tilde{a} V = \|F_{\infty}\|_{H^1}^2 \) and \( V(0) = 0 \). It follows that \( \int_0^t e^{-a(t-\tau)} V(t) \, d\tau \leq \int_0^t e^{-a(t-\tau)} \|F_{\infty}(\tau)\|_{H^1}^2 \, d\tau \) for any \( 0 < a < \tilde{a} \). This, together with (7.2), yields the desired inequality. This completes the proof.

We next derive the \( H^2 \) estimate for \( u_{\infty}(t) \). In what follows we set

\[
f_0 = Q_0 F_{\infty}, \quad f_{\infty} = \tilde{Q} F_{\infty}
\]

and

\[
\dot{\phi}_{\infty} = \partial_t \phi_{\infty} + v_s \partial_{x_3} \phi_{\infty} + w \cdot \nabla \phi_{\infty},
\]

where

\[
\tilde{f}_0 = f_0 - w \cdot \nabla \phi_{\infty}.
\]

Note that

\[
\|\dot{\phi}_{\infty}\|_{H^1} \leq C_{\nu \gamma} \left( \|u_{\infty}\|_{H^1 \times H^2}^2 + \|\tilde{f}_0\|_{H^1}^2 \right).
\]

The following Propositions 7.3 – 7.6 can be proved in a similar manner in [1, Section 4]. So we here give outline of proof.

We begin with the \( L^2 \) energy estimates for \( \partial_t u_{\infty} \) and \( \partial_{x_3}^2 u_{\infty} \).

**Proposition 7.3.** Under the assumption (7.1) (with \( \nu_0, \gamma_0 \) and \( \omega_0^{-1} \) replaced by suitably larger ones), the following assertions hold.

(i) There exists positive constant \( c \) such that the following inequality holds:

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma} \left\| \sqrt{\frac{\rho(t)}{\gamma \rho_s}} \partial_t \phi_{\infty} \right\|_2^2 + \left\| \sqrt{\rho_s} \partial_t w_{\infty} \right\|_2^2 \right\}
+ \frac{3}{2} \nu \left\| \nabla \partial_t w_{\infty} \right\|_2^2 + \frac{3}{2} \nu \left\| \text{div} \partial_t w_{\infty} \right\|_2^2 + c \frac{\nu \gamma}{\gamma^2} \left\| \partial_t \dot{\phi}_{\infty} \right\|_2^2 \leq C_{\nu \gamma} \|u_{\infty}\|_{H^1 \times H^2} + |A_1|.
\] (7.3)
Here
\[ A_1 = \frac{1}{2} \left( |\partial_t \phi_\infty|^2, \text{div} \left( \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} w \right) \right) + \left( [\partial_t, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \partial_t \phi_\infty \right) + \left( \partial_t \tilde{f}_\infty, \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \partial_t \phi_\infty \right) + \left( \partial_t f_\infty, \partial_\rho \partial_t w_\infty \right) + C C_\nu + \nu \|\partial_t \tilde{f}_\infty\|^2.
\]

(ii) There exists positive constant \( b \) such that the following inequality holds:
\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \left[ \sqrt{\frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty}} \partial^2_{x_3} \phi_\infty \right]^2 + \sqrt{\frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty}} \partial^2_{x_3} w_\infty \right\}^2 + \frac{1}{2} \square \|
\]
\[ \leq C \nu + \nu \|\partial^2_{x_3} \phi_\infty\|^2 + C \nu \| u_\infty \|_{H^1 \times H^2} + |A_{0,0,2}|. \]

Here
\[ A_{0,0,2} = \frac{1}{2} \left( \partial^2_{x_3} \phi_\infty, \text{div} \left( \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} w \right) \right) + \left( \partial^2_{x_3} \phi_\infty, \text{div} \left( \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \partial^2_{x_3} \phi_\infty \right) \right) + \left( \partial^2_{x_3} \tilde{f}_\infty, \partial_{x_3} \left( \partial_\rho \partial^2_{x_3} w_\infty \right) \right) + C \nu + \nu \|\partial^2_{x_3} \phi_\infty\|^2.
\]

Outline of Proof. We write \( L \) as
\[ L = A + B,
\]
where
\[ A = \begin{pmatrix}
\nu^3 \partial_{x_3} - \gamma^2 \text{div} (\rho_\infty) \\
\nabla \left( \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \right) - \frac{\nu}{\rho_\infty} \Delta I_3 - \frac{\nu + \nu'}{\rho_\infty} \nabla \text{div} + v_3^3 \partial_{x_3}
\end{pmatrix},
\]
\[ B = \begin{pmatrix}
0 \\
\nu \sqrt{\gamma_3^2 \rho_3^2} \\
0 \\
\nabla v_3^3
\end{pmatrix}.
\]

Note that
\[ \langle Au, u \rangle = \nu \|
\]
\[ \square \| w \|_2^2 + \nu \|\text{div} \ w \|_2^2. \]

In terms of \( A \) and \( B \) we rewrite problem (5.3) as
\[ \partial_t u_\infty + Au_\infty = -Bu_\infty + F_\infty. \]

Let \( j \) and \( k \) be nonnegative integers satisfying \( j + 2k = 2 \). We apply \( \partial^{j}_{x_3} \partial^{k}_{t} \) to (7.6) and obtain
\[ \partial_t \partial^{j}_{x_3} \partial^{k}_{t} u_\infty + A \partial^{j}_{x_3} \partial^{k}_{t} u_\infty = -B \partial^{j}_{x_3} \partial^{k}_{t} u_\infty + \partial^{j}_{x_3} \partial^{k}_{t} F_\infty.
\]

We take the weighted inner product \( \langle \cdot, \cdot \rangle \) of this equation with \( \partial^{j}_{x_3} \partial^{k}_{t} u_\infty \). One can then arrive at the desired estimates by using (7.5) and the relations
\[
\left( \partial^{j}_{x_3} \partial^{k}_{t} \left( w \cdot \nabla \phi_\infty \right), \partial^{j}_{x_3} \partial^{k}_{t} \phi_\infty, \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \nabla \phi_\infty \right) = \frac{1}{2} \left( w, \nabla \partial^{j}_{x_3} \partial^{k}_{t} \phi_\infty \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \right) + \left( \partial^{j}_{x_3} \partial^{k}_{t} w, \nabla \phi_\infty \right)
\]
\[ = -\frac{1}{2} \left( \partial^{j}_{x_3} \partial^{k}_{t} \phi_\infty, \nabla \phi_\infty \right) + \left( \partial^{j}_{x_3} \partial^{k}_{t} \phi_\infty \frac{P'(\rho_\infty)}{\gamma^4 \rho_\infty} \right).
\]
Under the assumption there exists positive constant suitably larger ones, the following assertions hold.

\[
\partial^2_{x_1} \partial^k_{x_3} \phi_\infty = -\gamma^2 \text{div} (\rho_s \partial^2_{x_1} \partial^k_{x_3} w_\infty) + \partial^2_{x_1} \partial^2_{x_3} f_\infty.
\]

We note that the lower order norm \(\|u_\infty\|_{H^1 \times H^2}\) arises from the lower order term \(Bw_\infty\).

We next state the interior estimate and the boundary estimates of the tangential derivatives.

**Proposition 7.4.** Under the assumption (7.1) (with \(\nu_0, \gamma_0\) and \(\omega_0^{-1}\) replaced by suitably larger ones), the following assertions hold.

(i) There exists positive constant \(b\) such that the estimate

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma} \left[ \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma \rho_s}} \partial^2_{x_1} \phi_\infty \right]^2 + \left\| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma \rho_s}} \partial^2_{x_1} w_\infty \right\|_2^2 \right\} + \frac{1}{2} \nu \chi_0 \left\| \partial^2_{x_1} w_\infty \right\|_2^2 + \frac{h \nu \overline{\partial}_{x_1} f_\infty}{\gamma} \left\| \chi_0 \partial^2_{x_1} \phi_\infty \right\|_2^2 
\]

(7.7)

holds for any \(\epsilon > 0\). Here

\[
A^{(0)} = \frac{1}{2} \left( \left[ \partial^2_{x_1} \phi_\infty \right]^2 \text{div} \left( \chi_0 \frac{P'(\rho_s)}{\gamma \rho_s} w \right) + \left( \partial^2_{x_1} w \cdot \nabla \right) \phi_\infty, \chi_0 \frac{P'(\rho_s)}{\gamma \rho_s} \partial^2_{x_1} \phi_\infty \right)
\]

\[
+ \left( \partial^2_{x_1} \overline{f}_\infty, \chi_0 \frac{P'(\rho_s)}{\gamma \rho_s} \partial^2_{x_1} \phi_\infty \right) + \left( \partial^2_{x_1} f_\infty, \partial^2_{x_1} \phi_\infty \right)
\]

\[
+ Cb \frac{h \nu \overline{\partial}_{x_1} f_\infty}{\gamma} \left\| \chi_0 \partial^2_{x_1} \phi_\infty \right\|_2^2.
\]

(ii) Let \(1 \leq m \leq N\). There exists positive constant \(b\) such that the estimate

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma} \left[ \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma \rho_s}} \partial^k_{x_3} \phi_\infty \right]^2 + \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma \rho_s}} \partial^k_{x_3} w_\infty \right\|_2^2 \right\} + \frac{1}{2} \nu \chi_m \left\| \partial^k_{x_3} w_\infty \right\|_2^2 + \frac{h \nu \overline{\partial}_{x_3} f_\infty}{\gamma} \left\| \chi_m \partial^k_{x_3} \phi_\infty \right\|_2^2 
\]

(7.8)

holds for \((k, j) = (2, 0), (1, 1)\) and any \(\epsilon > 0\). Here

\[
A^{(m)}_{0, k, j} = \frac{1}{2} \left( \left[ \partial^k_{x_3} \phi_\infty \right]^2 \text{div} \left( \chi_m \frac{P'(\rho_s)}{\gamma \rho_s} w \right) \right)
\]

\[
+ \left( \left[ \partial^k_{x_3} w \cdot \nabla \right] \phi_\infty, \chi_m \frac{P'(\rho_s)}{\gamma \rho_s} \partial^k_{x_3} \phi_\infty \right)
\]

\[
+ \left( \partial^k_{x_3} \overline{f}_\infty, \chi_m \frac{P'(\rho_s)}{\gamma \rho_s} \partial^k_{x_3} \phi_\infty \right) + \left( \partial^{k-1}_{x_3} f_\infty, \partial^2_{x_1} \phi_\infty \right)
\]

\[
+ Cb \frac{h \nu \overline{\partial}_{x_3} f_\infty}{\gamma} \left\| \chi_m \partial^k_{x_3} \phi_\infty \right\|_2^2.
\]

**Outline of Proof.** Let us consider (ii). We apply \(\partial^k_{x_3}\) to (7.6) to have

\[
\partial_t \partial^k_{x_3} u_\infty + A \partial^k_{x_3} u_\infty = -[\partial^k_{x_3}, A] \partial^k_{x_3} u_\infty - \partial^k_{x_3} B \partial^k_{x_3} u_\infty + \partial^k_{x_3} F_\infty.
\]

Here \([\partial^k, A]\) denotes the commutator of \(\partial^k\) and \(A\). We take the weighted inner product \(\langle \cdot, \cdot \rangle\) of this equation with \(\chi_m^2 \partial^k_{x_3} u_\infty\). One can then arrive at the desired
Proposition 7.5. Let $1 \leq m \leq N$. Under the assumption (7.1) (with $\nu_0$, $\gamma_0$ and $\omega_0^{-1}$ replaced by suitably larger ones), there exists positive constant $b$ such that the estimate

$$
\frac{1}{2} \left( \frac{1}{\gamma} \right) \left( \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma \rho_s}} \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right)^2 + \frac{1}{2} \left( \frac{1}{\gamma} \right) \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma \rho_s}} \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right)^2
+ b \left( \frac{\nu + \bar{\nu}}{\gamma} \right) \left( \chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right)^2
\leq C \left\{ \left( \frac{\nu + \bar{\nu}}{\gamma} \right) \left( \chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right)^2 + \frac{1}{\nu + \bar{\nu}} \left( \text{div} (\chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty) \right)^2 \right\}
+ C(b + 1) \left( \frac{\nu + \bar{\nu}}{\gamma} \right) \left( \chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right)^2 + \| f_\infty \|_{L^2}^2.
$$

(7.9)

holds for $j, k, l \geq 0$ satisfying $j + k + l = 1$. Here

$$
A_{i+1, k, j}^{(m)} = \frac{1}{2} \sum_{j+k+l=1} \left( \left| \chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right|^2 + \text{div} (\chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty) \right)^2
+ C \sum_{j+k+l=1} \left( \chi_m \partial_n^{j+1} \partial^k \partial_x \phi_\infty \right)^2 + \| f_\infty \|_{L^2}^2.
$$

Outline of Proof. We transform a scalar field $p(x')$ on $D \cap O_m$ as

$$
\tilde{p}(y') = p(x') (y' = \Psi^m(x'), x' \in D \cap O_m),
$$

where $\Psi^m(x')$ is a function given in section 2. Similarly we transform a vector field $v(x') = T(v^1(x'), v^2(x'), v^3(x'))$ into $\tilde{v}(y') = T(\tilde{v}^1(y'), \tilde{v}^2(y'), \tilde{v}^3(y'))$ as

$$
v(x') = E(y') \tilde{v}(y').
$$

where $E(y') = (e_1(y'), e_2(y'), e_3)$ with $e_1(y'), e_2(y')$ and $e_3$ given in section 2. From the proof of [1, Proposition 4.16], we have

$$
\partial_\nu \partial_{y_1} \phi_\infty + \tilde{v}_3 \partial_{y_3} \partial_{y_1} \phi_\infty + \frac{\tilde{\rho}_s \tilde{P}'(\tilde{\rho}_s)}{\nu + \bar{\nu}} \partial_{y_3} \phi_\infty = \tilde{\rho}_s h,
$$

(7.10)

where

$$
h = -\frac{\gamma^2}{\nu + \bar{\nu}} \left\{ \partial_\nu \tilde{w}^1 + \nu \left( \text{rot}_y \text{rot}_y \tilde{w}^1 \right) + \tilde{\rho}_s \partial_{y_1} \left( \frac{P'(\tilde{\rho}_s)}{\gamma \tilde{\rho}_s} \right) \tilde{\phi} + \frac{\nu}{\gamma} \tilde{\rho}_s \left( \Delta_y \tilde{v}_3 \right) \tilde{\phi} + \tilde{\rho}_s \tilde{v}_3 \partial_{y_3} \tilde{w}^1 \right\}
- \left\{ \frac{1}{\rho_s} \partial_{y_1} \tilde{v}_3 \partial_{y_3} \tilde{\phi} + \gamma^2 \frac{1}{\rho_s} \partial_{y_1} \left( \text{div}_y (\tilde{\rho}_s \tilde{w}) \right) - \gamma^2 \partial_{y_1} \text{div}_y \tilde{w} \right\}.
$$

Here $\left( \text{rot}_y \tilde{w}^1 \right)$ denotes the $e_1(y')$ component of $\text{rot}_y \tilde{w}$, and so on. We note that $\left( \text{rot}_y \text{rot}_y \tilde{w}^1 \right)$ does not contain $\partial_{y_1}^2$. See the proof of [1, Proposition 4.16]. Applying $\partial_{y_1} \partial_{y_2} \partial_{y_3}$ to (7.10) and take the $L^2$ inner product with $\chi_m \frac{P'(\rho_s)}{\gamma \rho_s} \partial_{y_1} \partial_{y_2} \partial_{y_3} \phi_\infty$, one can obtain the desired result.

Using the estimate for the Stokes system we have the following estimates.
Proposition 7.6. Under the assumption (7.1) (with \( \nu_0, \gamma_0 \) and \( \omega_0^{-1} \) replaced by suitably larger ones), the following assertions hold.

(i) There holds the estimate

\[
\frac{\nu^2}{\nu + \nu'} \| \partial^3_x w_{\infty} \|^2 + \frac{1}{\nu + \nu'} \| \partial^2_\phi \phi_{\infty} \|^2 \\
\leq C \left\{ \frac{\nu^2}{\nu + \nu'} \| \partial^2_\phi \phi_{\infty} \|^2 + \frac{1}{\nu + \nu'} \| \partial_t \partial_x w_{\infty} \|^2 + \frac{\nu^2 + \nu'}{\nu + \nu'} \| \phi_{\infty} \|^2 \right\} + C_{\nu, \nu'} \| u_{\infty} \|^2_{H^3 \times H^2}.
\]  

(ii) Let \( 1 \leq m \leq N \). There holds the estimate

\[
\frac{\nu^2}{\nu + \nu'} \| \chi_m \partial^2_\phi \partial_x w_{\infty} \|^2 + \frac{1}{\nu + \nu'} \| \chi_m \partial_x \partial_\phi \phi_{\infty} \|^2 \\
\leq C \left\{ \frac{\nu^2}{\nu + \nu'} \| \chi_m \partial_\phi \phi_{\infty} \|^2 + \frac{\nu^2 + \nu'}{\nu + \nu'} \| \chi_m \partial_x \partial_\phi \phi_{\infty} \|^2 + \frac{1}{\nu + \nu'} \| \partial_t \partial_x w_{\infty} \|^2 \right\} + C_{\nu, \nu'} \| u_{\infty} \|^2_{H^3 \times H^2}.
\]  

(iii) There holds the estimate

\[
\frac{\nu^2}{\nu + \nu'} \| \partial^2_\phi \partial_x w_{\infty} \|^2 + \frac{1}{\nu + \nu'} \| \partial_\phi \partial_x \phi_{\infty} \|^2 \\
\leq C \left\{ \frac{\nu^2 + \nu'}{\nu + \nu'} \| \partial_\phi \phi_{\infty} \|^2 + \frac{\nu^2 + \nu'}{\nu + \nu'} \| \partial_\phi \phi_{\infty} \|^2 + \frac{\nu^2 + \nu'}{\nu + \nu'} \| \partial_\phi \phi_{\infty} \|^2 \right\} + C_{\nu, \nu'} \| u_{\infty} \|^2_{H^3 \times H^2}.
\]  

**Outline of Proof.** We write problem (5.3) as

\[
\partial_t \phi_{\infty} + v_3^2 \partial_x \phi_{\infty} + \gamma^2 \text{div}(\rho_0 w_{\infty}) + w \cdot \nabla \phi_{\infty} = \tilde{f}_0, \\
\partial_t w_{\infty} - \frac{\nu}{\rho} \Delta w_{\infty} - \frac{\nu}{\rho} \nabla \text{div} w_{\infty} + \nabla \left( \frac{P(\rho)}{\gamma \rho^2} \phi_{\infty} \right) \\
+ \frac{\nu}{\gamma \rho^2} \phi_{\infty} e_3 + v_3^2 \partial_x w_{\infty} + (w' \cdot \nabla v_3^2) e_3 = f_0, \\
w_{\infty} |_{\partial \Omega} = 0,
\]

\[
(\phi_{\infty}, w_{\infty}) |_{t=0} = (\phi_{\infty,0}, w_{\infty,0}).
\]

By (7.14) and (7.15), we have

\[
\text{div} w_{\infty} = G,
\]

\[
-\Delta w_{\infty} + \nabla \left( \frac{P(\rho)}{\gamma \rho^2} \phi_{\infty} \right) = F,
\]

\[
w_{\infty} |_{\partial \Omega} = 0.
\]

Here

\[
G = \frac{1}{\gamma^2} \left( \tilde{f}_0 - \phi_{\infty} - \gamma^2 \text{div} (\rho_0 - 1) w_{\infty} \right),
\]

\[
F = \frac{\rho_0}{\nu} f_0 - \frac{\rho_0}{\nu} \left\{ \partial_t w_{\infty} - \frac{\nu}{\rho} \nabla G + \frac{\nu}{\gamma \rho^2} \phi_{\infty} e_3 + v_3^2 \partial_x w_{\infty} + (w' \cdot \nabla v_3^2) e_3 \right\}.
\]
Applying the estimate for the Stokes system, we have the estimate in (i). The estimates in (ii) and (iii) can be obtained similarly by applying $\chi_m \partial_1$ and $\partial_{x_3}$ to (7.18), respectively.

We are now in a position to prove Proposition 7.1.

**Proof of Proposition 7.1** Let $b_1$ and $b_2$ be constants satisfying $b_1, b_2 > 1$. Define $\tilde{\mathcal{E}}_2[u_\infty]$ by

$$\tilde{\mathcal{E}}_2[u_\infty] = \frac{1}{\gamma} \sum_{m=1}^{N} \left\{ b_1 \left( \| \chi_m \sqrt{\frac{P^*(\rho_m)}{\gamma \rho_m}} \partial_1 \phi_\infty \|_2^2 + \| \chi_m \sqrt{\frac{P^*(\rho_m)}{\gamma \rho_m}} \partial_{x_3} \phi_\infty \|_2^2 \right) \right\}$$

$$+ \frac{1}{\gamma} \left( \| \chi_0 \sqrt{\frac{P^*(\rho_0)}{\gamma \rho_0}} \partial_1 \phi_\infty \|_2^2 + \| \chi_0 \sqrt{\frac{P^*(\rho_0)}{\gamma \rho_0}} \partial_{x_3} \phi_\infty \|_2^2 \right)$$

$$+ b_1 \sum_{m=1}^{N} \left( \| \chi_m \sqrt{p_s \rho^2 w_\infty} \|_2^2 + \| \chi_m \sqrt{p_s \partial^2_{x_3} w_\infty} \|_2^2 \right)$$

for $u_\infty = T(\phi_\infty, w_\infty)$. We compute

$$b_2 \left[ \sum_{m=1}^{N} \left\{ b_1 \left( (7.8) \mid (k,j) = (2,0) \right) + (7.8) \mid (k,j) = (1,1) \right\} + (7.9) \mid (l,k,j) = (0,1,0) \right\} + (7.7) + b_1 (7.4) \right]$$

$$+ \sum_{m=1}^{N} (7.12) + (7.13).$$

Then

$$\frac{1}{2} \frac{d}{dt} b_2 \tilde{\mathcal{E}}_2[u_\infty] + bb_2 \nu + \frac{\nu}{\nu + \nu} \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \partial_{x_3} \phi_\infty \|_2^2 \right)$$

$$+ \frac{\nu}{\nu + \nu} \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \partial_1 w_\infty \|_2^2 + \| \partial_{x_3} \partial_{x_3} w_\infty \|_2^2 \right)$$

$$+ \frac{\nu}{\nu + \nu} \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \partial_1 w_\infty \|_2^2 + \| \partial_{x_3} \partial_{x_3} w_\infty \|_2^2 \right)$$

$$+ b_2 \left\{ b_1 \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \chi_m \partial_1 \partial_1 \phi_\infty \|_2^2 \right) \right\}$$

$$+ b_2 \left\{ b_1 \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \chi_m \partial_1 \partial_1 \phi_\infty \|_2^2 \right) \right\}$$

$$\leq C_{b_1 b_2} \left\{ \| \partial_1 \phi_\infty \|_2^2 + C_{\nu \rho_0} \| u_\infty \|_{H^1 \times H^2} + \frac{1}{\nu + \nu} \| \partial_1 \partial_1 w_\infty \|_2^2 + R_0 \right\}$$

$$+ \left\{ b_2 \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \chi_m \partial_1 \partial_1 \phi_\infty \|_2^2 \right) \right\}$$

$$+ \left\{ b_2 \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \chi_m \partial_1 \partial_1 \phi_\infty \|_2^2 \right) \right\}$$

$$+ \left\{ b_2 \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \chi_m \partial_1 \partial_1 \phi_\infty \|_2^2 \right) \right\}$$

$$+ \left\{ b_2 \sum_{m=1}^{N} \left( \| \chi_m \partial_1 \phi_\infty \|_2^2 + \| \chi_m \partial_1 \partial_1 \phi_\infty \|_2^2 \right) \right\}$$
for any $\varepsilon > 0$. Here

$$R_0 = \sum_{m=1}^{N} (|A^{(m)}_{0,0}| + |A^{(m)}_{0,1}|) + |A^{(0)}| + |A_{0,0,2}| + \sum_{m=1}^{N} (|A^{(m)}_{1,0}| + |A^{(m)}_{1,1}|).$$

Fix $b_1 > 8C_1$ and $b_2 > 8\frac{C_1}{\nu}$. Then it follows that

$$\frac{d}{dt} b_2 \mathcal{E}_2[u_\infty] + b_2 \nu + \nu
\frac{\nu}{\nu + \nu + \frac{\nu}{\nu}} \left( \sum_{m=1}^{N} \left( \|\chi_m \partial_x \partial_\phi \|_2^2 + \|\partial_x \partial_{x_3} \partial_\phi \|_2^2 \right) \right)
+ \frac{\nu}{\nu + \nu} \left( \sum_{m=1}^{N} \left( \|\chi_m \partial_x \partial_\phi \|_2^2 + \|\partial_x \partial_{x_3} \phi \|_2 \right) \right)
\leq C_1 \frac{b_2}{b_2 + 1} \left( \varepsilon + \frac{1}{2} \chi \gamma \frac{\nu}{\nu + \nu} \left( \|\chi_m \partial_x \partial_\phi \|_2^2 + \|\chi_m \partial_x \partial_{x_3} \phi \|_2 \right) \right)
+ \frac{1}{\nu + \nu} \|\partial_t \partial_x w_\infty \|_2^2 + R_0 \right)

for any $\varepsilon > 0$. Here $I_1[w_\infty]$ is given by

$$I_1[w_\infty] = \nu \left\{ \sum_{m=1}^{N} \left( \|\chi_m \nabla \partial^2 w_\infty \|_2^2 + \|\chi_m \nabla \partial_{x_3} w_\infty \|_2^2 \right) \right\}
+ \|\chi_0 \nabla \partial_x^2 w_\infty \|_2^2 + \|\nabla \partial_{x_3}^2 w_\infty \|_2^2 \right\}
+ \nu \left\{ \sum_{m=1}^{N} \left( \|\chi_m \operatorname{div} \partial^2 w_\infty \|_2^2 + \|\chi_m \operatorname{div} \partial_{x_3} w_\infty \|_2^2 \right) \right\}
+ \|\chi_0 \operatorname{div} \partial_x^2 w_\infty \|_2^2 + \|\operatorname{div} \partial_{x_3}^2 w_\infty \|_2^2 \right\}.$$

Let $b_3$ and $b_4$ be constants satisfying $b_3, b_4 > 1$. Define $E_2[u_\infty]$ by

$$E_2[u_\infty] = b_3 b_4 b_4 \mathcal{E}_2[u_\infty] + b_4 \frac{1}{2} \sum_{m=1}^{N} \left( \|\chi_m \sqrt{\frac{P}{\gamma'} \partial_x^2 \phi_\infty} \|_2^2 \right)
+ \left( \frac{1}{2} \|\sqrt{\frac{P}{\gamma'} \partial_\phi \|_2^2 + \|\sqrt{P} \partial_t w_\infty \|_2^2 \right)$$

for $u_\infty = T(\phi_\infty, w_\infty)$. We compute

$$b_4 \left\{ b_3 (7.19) + \sum_{m=1}^{N} (7.9) \right\} + (7.11) + b_5 (7.3).$$
It follows that
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_2[w_\infty] + b_4 \frac{\nu + \bar{\nu}}{\gamma} \| \partial_x^3 \phi_\infty \|_2^2 + \frac{\nu^2}{\nu + \bar{\nu}} \| \partial_x^3 w_\infty \|_2^2 + \frac{1}{2} \frac{\nu^2}{\nu + \bar{\nu}} \| \partial^2_x \phi_\infty \|_2^2 \\
+ b_3 b_4 \left\{ \frac{\nu^2}{\nu + \bar{\nu}} \left( \sum_{m=1}^N \| \chi_m \partial_x \partial^2 w_\infty \|_2 \right) + \frac{1}{\nu + \bar{\nu}} \left( \sum_{m=1}^N \| \chi_m \partial_x \partial \phi_\infty \|_2 \right) \right\} + \frac{1}{2} b_2 b_3 b_4 I_1[w_\infty] \\
+ \frac{b_4}{2} \frac{\nu + \bar{\nu}}{\gamma} \sum_{m=1}^N \frac{\nu}{\gamma} \| \partial_x \phi_\infty \|_2^2 + \frac{1}{2} \left\{ \nu \| \partial_t \phi_\infty \|_2^2 + \bar{\nu} \| \partial_t w_\infty \|_2^2 + e^{\frac{\nu + \bar{\nu}}{\gamma}} \| \partial_t \phi_\infty \|_2^2 \right\} \\
\leq C_{b_1 \ldots b_4} \left\{ \left( \varepsilon + \frac{1}{\gamma} + \frac{\nu + \bar{\nu}}{\gamma} \right) \| \partial^2_t \phi_\infty \|_2^2 + \frac{1}{\nu + \bar{\nu}} \| \partial_t \partial_x w_\infty \|_2^2 + C_{\nu \bar{\nu}, \omega} \| u_\infty \|_{H^1 \times H^2}^2 \\
+ R \right\} + C_2 \left\{ \frac{b_4 \nu^2}{\nu + \bar{\nu}} \sum_{m=1}^N \left( \| \chi_m \partial \partial_x^2 w_\infty \|_2^2 + \| \chi_m \nabla \partial \partial_x \phi_\infty \|_2^2 \right) \\
+ \| \chi_m \nabla \partial \partial_x \phi_\infty \|_2^2 + \frac{\nu + \bar{\nu}}{\gamma} \| \partial^2_t \phi_\infty \|_2^2 \right\}
\]
for any \( \varepsilon > 0 \). Here
\[
R = R_0 + \sum_{m=1}^N |A_{2,0,0}^{(m)}| + |A_1|.
\]

Fix \( b_3 \) and \( b_4 \) so large that \( b_3 > 8C_2 \) and \( b_4 > 2\frac{C_2}{\nu} \). We assume that \( \nu, \bar{\nu} \) and \( \gamma \) also satisfy \( \gamma^2 > 8C_{b_1 \ldots b_4} \) and \( \gamma^2 > 8C_{b_1 \ldots b_4}(\nu + \bar{\nu}) \). We take \( \varepsilon > 0 \) sufficiently small such that \( \varepsilon < \frac{1}{8C_{b_1 \ldots b_4} - \frac{1}{\nu + \bar{\nu}}} \). It then follows that
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_2[w_\infty] + b_4 \frac{\nu + \bar{\nu}}{\gamma} \| \partial_x^3 \phi_\infty \|_2^2 + \frac{\nu^2}{\nu + \bar{\nu}} \| \partial_x^3 w_\infty \|_2^2 + \frac{1}{2} \frac{\nu^2}{\nu + \bar{\nu}} \| \partial^2_x \phi_\infty \|_2^2 \\
+ \frac{\nu^2}{\nu + \bar{\nu}} \left( \sum_{m=1}^N \| \chi_m \partial_x \partial w_\infty \|_2 \right) + \frac{1}{\nu + \bar{\nu}} \left( \sum_{m=1}^N \| \chi_m \partial_x \partial \phi_\infty \|_2 \right) \\
+ \frac{1}{\nu + \bar{\nu}} \left( \sum_{m=1}^N \| \chi_m \partial_x \partial \phi_\infty \|_2 \right) + \| \partial_x \phi_\infty \|_2^2 \\
+ I_1[w_\infty] + \frac{1}{\nu + \bar{\nu}} \sum_{m=1}^N \| \chi_m \frac{P(\rho)}{\gamma} \partial \nabla \phi_\infty \|_2^2 \\
+ \frac{1}{2} \left\{ \nu \| \partial_t \phi_\infty \|_2^2 + \bar{\nu} \| \partial_t w_\infty \|_2^2 + e^{\frac{\nu + \bar{\nu}}{\gamma}} \| \partial_t \phi_\infty \|_2^2 \right\} \\
\leq \{ C_{\nu \bar{\nu}, \omega} \| u_\infty \|_{H^1 \times H^2}^2 + R \}.
\]

We thus see that there are positive constants \( c_1, c_2 \) and \( C_3 \) such that
\[
\frac{d}{dt} \mathcal{E}_2[w_\infty] + c_1 \mathcal{E}_2[w_\infty] \\
+ c_2 \left( \| \partial_x^3 w_\infty \|_2^2 + \| \partial^2_x \phi_\infty \|_2^2 + \| \phi_\infty \|_{H^2} + \| \partial_t w_\infty \|_{H^1}^2 + \| \partial_t \phi_\infty \|_2^2 \right) \\
\leq C_{\nu \bar{\nu}, \omega} \| u_\infty \|_{H^1 \times H^2}^2 + R.
\]

Since
\[
\| \partial_x^2 w_\infty \|_2^2 \leq \eta \| \partial_x^3 w_\infty \|_2^2 + C_\eta \| w_\infty \|_2^2
\]
holds for any \( \eta > 0 \), taking \( \eta \) so small that \( \eta < \frac{1}{2} \min\{ \frac{c_2}{c_2 + c_{v\nu v}}, 1 \} \), we obtain

\[
\frac{d}{dt} E_2[u_{\infty}] + c_1 E_2[u_{\infty}] \\
+ \frac{c_2}{2} \left( ||\partial^2_{x} w_{\infty}||^2_2 + ||\partial^2_{x} \phi_{\infty}||^2_2 + ||\phi_{\infty}||^2_{H^1} + ||\partial_{t} w_{\infty}||^2_{H^1} + ||\partial_{t} \phi_{\infty}||^2_2 \right) \\
\leq 2C_{v\nu v}(u_{\infty})^2_{H^1,L^2} + \mathcal{R}).
\] (7.21)

We see from (7.21) and Proposition 7.2 that there exist positive constants \( \bar{c}_1, \bar{c}_2 \) and \( C_{v\nu v} \) such that

\[
\mathcal{E}_2[u_{\infty}(t)] + ||u_{\infty}(t)||^2_{H^1} + \bar{c}_2 \int_0^t e^{-c_1(t-\tau)} \left( ||\partial^3_{x} w_{\infty}||^2_2 + ||\partial^2_{x} \phi_{\infty}||^2_2 + ||u_{\infty}||^2_{H^1} \\
+ ||\phi_{\infty}||^2_{H^1} + ||\partial_{t} u_{\infty}||^2_{H^1,L^2} \right) d\tau \\
\leq C_{v\nu v} \left\{ e^{-c_1(t-\tau)} \mathcal{E}_2[u_{\infty, 0}] + ||u_{\infty, 0}||^2_{H^1} \right\} + \sup_{0 \leq \tau \leq t} \| F_\infty(\tau) \|_{\frac{3}{2}} + \int_0^t e^{-c_1(t-\tau)} \mathcal{R} d\tau \}.
\] (7.22)

It remains to estimate the term \( ||\partial^2_{x} w_{\infty}(t)||_2 \). We write the second equation of (5.1) as

\[-\nu \Delta w_{\infty} - \nu \nabla \text{div} w_{\infty} = J, \quad w_{\infty} |_{\partial \Omega} = 0,\]

where

\[ J = -\rho_s \left\{ \partial_{t} w_{\infty} + \nabla \left( \frac{P'(\rho_s)}{\tau_p^\rho_s} \phi_{\infty} \right) + \nu \Delta v_3^2 \phi_{\infty} e_3 + v_3^2 \partial_{x_3} w_{\infty} + (w_{\infty} \cdot \nabla) v_3^2 e_3 - f_{\infty} \right\}. \]

Since \( J \in L^2(\Omega) \), we obtain, by elliptic estimate,

\[ ||\partial^2_{x} w_{\infty}||^2_2 \leq C( ||w_{\infty}||^2_2 + ||J||^2_2) \leq C_{v\nu v}(\mathcal{E}_2[u_{\infty}] + ||u_{\infty}||^2_{H^1} + ||f_{\infty}||^2_2). \]

From this, with (7.22), we see that

\[
\mathcal{E}_2[u_{\infty}(t)] + ||u_{\infty}(t)||^2_{H^1} + \mathcal{E}_2 \left\{ e^{-c_1(t-\tau)} \mathcal{E}_2[u_{\infty, 0}] + ||u_{\infty, 0}||^2_{H^1} \right\} + \sup_{0 \leq \tau \leq t} \| F_\infty(\tau) \|_{\frac{3}{2}} + \int_0^t e^{-c_1(t-\tau)} \mathcal{R} d\tau \}
\] (7.23)

As we will see in section 8 below, it holds that

\[ \sup_{0 \leq \tau \leq t} || F_\infty(\tau) ||_{\frac{3}{2}} \leq C(1 + t)^{-\frac{3}{2}} M(t)^4. \] (7.24)

Proposition 7.1 follows from (7.23) and (7.24). This completes the proof. \( \square \)

8 Estimates of nonlinear terms

In this section we prove the estimate (7.24) and (5.6).
We first make an observation. By the Sobolev inequality we have
\[ \| \phi(t) \|_\infty \leq C \phi(t)_{H^2} \leq C_1 [u(t)]_2 \]
for a positive constant \( C_1 \). It then follows that
\[ \rho(x, t) = \rho_s(x') + \gamma^{-2} \phi(x, t) \geq \rho_1 - \gamma^{-2} \| \phi(t) \|_\infty \geq \rho_1 - C_1 \gamma^{-2} [u(t)]_2. \]

Fix a positive constant \( \epsilon_s \) satisfying \( \epsilon_s \leq \frac{1}{4} \frac{\gamma^2 \rho_1}{C_1} \). If \( [u(t)]_2 \leq \epsilon_s \), then it holds that
\[ \| \phi(t) \|_\infty \leq \frac{1}{4} \gamma^2 \rho_1, \quad \rho(x, t) \geq \frac{3}{4} \rho_1 > 0. \]
This implies that \( \bar{Q}F(t) \) is smooth whenever \( [u(t)]_2 \leq \epsilon_s \).

We will show the following

**Proposition 8.1.** If \( [u(t)]_2 \leq \epsilon_s \) and \( M(t) \leq 1 \), then
\[ \sup_{0 \leq \tau \leq t} \| F_{\infty}(\tau) \|_2^2 \leq C(1 + t)^{-\frac{3}{2}} M(t)^4, \quad (8.1) \]
\[ R(t) \leq C \{ (1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{4}} M(t) D_\infty(t) \} \quad (8.2) \]

To prove Proposition 8.1, we prepare several lemmas.

**Lemma 8.2.**

(i) For \( 2 \leq p \leq p \leq \infty \). If \( j \) and \( k \) are nonnegative integers satisfying
\[ 0 \leq j < k, \quad k > j + n(\frac{1}{2} - \frac{1}{p}), \]
then there exists a positive constant \( C \) such that
\[ \| \partial^{j} f \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^{\frac{2a}{a-n}}(\mathbb{R}^2)} \| f \|_{H^k(\mathbb{R}^n)}, \]
Here \( a = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p}). \)

(ii) For \( 2 \leq p \leq \infty \). If \( j \) and \( k \) are nonnegative integers satisfying
\[ 0 \leq j < k, \quad k > j + 3(\frac{1}{2} - \frac{1}{p}), \]
then there exists a positive constant \( C \) such that
\[ \| \partial^{j} f \|_{L^p(\Omega)} \leq C \| f \|_{H^k(\Omega)}. \]

(iii) If \( f \in H^1(\Omega) \) and \( f = f(x') \) is independent of \( x' \in D \), then it holds that
\[ \| f \|_{L^\infty(\Omega)} \leq C \| f \|_{L^2(\Omega)} \| \partial_{x_3} f \|_{L^2(\Omega)}^{\frac{1}{2}} \]
for a positive constant \( C \).

The proof of Lemma 8.2 can be found, e.g., in [11, 16].
Lemma 8.3. (i) For nonnegative integers \( m \) and \( m_k \) (\( k = 1, \ldots, l \)) and multiindex \( \alpha_k \) (\( k = 1, \ldots, l \)), we assume that

\[
m \geq \left[ \frac{n}{2} \right], \quad 0 \leq |\alpha_k| \leq m_k \leq 2 + |\alpha_k| \quad (k = 1, \ldots, l)
\]

and

\[
\sum_{k=1}^{l} m_k \geq 2(l - 1) + \sum_{k=1}^{l} |\alpha_k|.
\]

then the estimate holds

\[
\| \prod_{k=1}^{l} \partial_{x_k}^{\alpha_k} f_k \|_2 \leq C \prod_{k=1}^{l} \|f_k\|_{H^{m_k}}
\]

for a positive constant \( C \).

(ii) For \( 1 \leq k \leq m \). We assume that \( F(x, t, y) \) is a smooth function on \( \Omega \times [0, \infty) \times I \) with a compact interval \( I \subset \mathbb{R} \). For \( |\alpha| + 2j = k \) the estimates hold

\[
\| \left[ \partial_{x}^{\alpha} \partial_{t}^{j} F(x, t, f_1) \right] f_2 \|_2 \leq \begin{cases} C_0(t, f_1(t)) [f_2]_{k-1} + C_1(t, f_1(t)) \{1 + \|Df_1\|_{m-1}[f_2]_k\} \|Df_1\|_{m-1}[f_2]_k, \\
C_0(t, f_1(t)) [f_2]_{k-1} + C_1(t, f_1(t)) \{1 + \|Df_1\|_{m-1}[f_2]_k\} \|Df_1\|_{m-1}[f_2]_{k-1}, 
\end{cases}
\]

where

\[
C_0(t, f_1(t)) = \sum_{(\beta, l) \leq (\alpha, j), (\beta, l) \neq (0, 0)} \sup_x \left| \partial_{x}^{\beta} \partial_{t}^{j} F(x, t, f_1(x, t)) \right|,
\]

\[
C_1(t, f_1(t)) = \sum_{(\beta, l) \leq (\alpha, j), 1 \leq p \leq |\alpha|} \sup_x \left| \partial_{x}^{\beta} \partial_{t}^{p} \partial_{y}^{j} F(x, t, f_1(x, t)) \right|.
\]

(iii) For \( m \geq 2 \) the estimates hold

\[
\|f_1 \cdot f_2\|_{H^m} \leq C_1\|f_1\|_{H^m}\|f_2\|_{H^m}, \quad [f_1 \cdot f_2]_m \leq C_2[f_1]_m[f_2]_m
\]

for a positive constants \( C_1 \) and \( C_2 \).

See, e.g., [13, 16] for the proof of Lemma 8.3.

We begin with the following preliminary estimates for \( \sigma_1 \) and \( u_j \) (\( j = 1, \infty \)).

Lemma 8.4. We assume that \( u(t) = T(\phi(t), w(t)) = (\sigma_1 u(0))(t) + u_1(t) + u_\infty(t) \) be a solution of (5.1) in \( Z(T) \). The following estimates hold for all \( t \in [0, T] \) with a positive constant \( C \) independent of \( T \).

(i) \( \|\sigma_1(t)\|_2 \leq C(1 + t)^{-\frac{1}{4}} M(t) \),

(ii) \( \|u(t)\|_2 \leq C(1 + t)^{-\frac{1}{4}} M(t) \),

(iii) \( \|D\sigma_1(t)\| \leq C(1 + t)^{-\frac{3}{2}} M(t) \).
Proposition 8.5. Let $u(t)$ be a solution of (5.1) with a positive constant $C$.

Proof. The estimates (i)-(iii) and (vi) can be proved by applying Lemmas 8.2 and 8.3 similarly to the proof of [11, Proposition 8.5]. As for (iv), we have

\[
\left| \left( \partial^2_{x_2} \phi_\infty \right|^2, \text{div} \left( \frac{P'(\rho)}{\gamma \rho_s} \phi \right) \right| + \left| \left( \partial^2_{x_2} \phi_\infty \right|^2, \text{div} \left( \lambda_0^2 \frac{P'(\rho)}{\gamma^4 \rho_s} \phi \right) \right| + \left| \left( \partial_t \phi_\infty \right|^2, \text{div} \left( \frac{P'(\rho)}{\gamma \rho_s} \phi \right) \right| \\
+ \frac{N}{m=1} \sum_{j+k+l=1} \left| \left( \partial^{k+1} \partial^{l+1}_{x_2} \phi_\infty \right|^2, \text{div} \left( \lambda_m^2 \frac{P'(\rho)}{\gamma^4 \rho_s} \phi \right) \right| \\
\leq C(1+t)^{-1} M(t) \sqrt{D_\infty(t)}.
\]
We next consider (v). We observe that $[\partial^{k+1}\partial^i_{x3}, w \cdot \nabla] \phi_\infty$ and $[\partial^{k+1}\partial^i_{x3}, w \cdot \nabla] \phi_\infty$ are written as a linear combination of terms of the forms $a[\partial^i_{x}, w] \nabla \phi_\infty$ and $(w \cdot \nabla a) \partial^i_{x3} \phi_\infty$ with smooth function $a = a(x')$ and integer $q$ satisfying $1 \leq q \leq 2$. Therefore, applying Lemma 8.3, we obtain the desired estimate. This completes the proof.

Let us consider the nonlinearity $\tilde{Q}F = T(0, f)$. We write $\tilde{Q}F = T(0, f)$ in the form

$$\tilde{Q}F = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3,$$

where $\tilde{F}_l = T(0, h_l)$ ($l = 0, 1, 2, 3$). Here

$$h_0 = -w \cdot \nabla w + f_1(\rho_s, \phi)\left(-\partial^2_{x3} \sigma_1 w^{(0)} + \frac{\partial^2_{x3} \nu_{s}}{\gamma^2 \rho_s} (\phi_1 + \phi_\infty)\right) + f_2(\rho_s, \phi)(-\partial^2_{x3} \sigma_1 w^{(0)} - \partial_{x3} \sigma_1 \partial_{x} w^{(0)}) + f_{0, 1}(x', \phi) \phi_1 + f_{0, 2}(x', \phi) \partial_{x3} \sigma_1 + f_{0, 3}(x', \phi) \phi(\phi_1 + \phi_\infty),$$

$$h_1 = -\text{div}(f_1(\rho_s, \phi) \nabla (w_1 + w_\infty)),$$

$$h_2 = -\nabla (f_2(\rho_s, \phi) \text{div}(w_1 + w_\infty)) + (\text{div}(w_1 + w_\infty)) \nabla'(f_2(\rho_s, \phi)),$$

$$h_3 = \nabla (f_3(x', \phi) \phi(\phi_1 + \phi_\infty)) - (\phi_1 + \phi_\infty) \nabla (f_3(x', \phi) \phi).$$

Here $f_{0,l} = f_{0,l}(x', \phi)$ ($l = 1, 2, 3$) and $f_3(x', \phi)$ are smooth functions of $x'$ and $\phi$.

**Proposition 8.6.** We assume that $u(t)$ is a solution of (5.1) in $Z(T)$. If $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq 0$ and $M(t) \leq 1$ for all $t \in [0, T]$, then the following estimates hold with a positive constant $C$ independent of $T$.

(i) $\|\tilde{Q}F(t)\|_2 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2$.

(ii) $\|h_0(t)\|_2 \leq C\{(1 + t)^{-\frac{1}{2}} M(t)^2 + (1 + t)^{-\frac{3}{4}} M(t) \|Dw_\infty(t)\|_2\}$.

(iii) $\|h_l(t)\|_{H^1} \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{3}{4}} M(t) \|Dw_\infty(t)\|_2\}, \quad (l = 1, 2, 3)$.

(iv) $\|\partial_t h_l(t)\|_2 \leq C\{(1 + t)^{-1} M(t)^2 + (1 + t)^{-\frac{3}{4}} M(t) \|Dw_\infty(t)\|_2\}, \quad (l = 1, 2, 3)$.

Proposition 8.6 can be proved in a similar manner to the proof of [11, Proposition 8.6] and [3, Proposition 8.6].

**Proof of Proposition 8.1** We first prove (8.1). We see from Proposition 8.5 and Proposition 8.6 that

$$\|Q_0 F\|_2 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2,$$

$$\|\tilde{Q} F\|_2 \leq C(1 + t)^{-\frac{1}{2}} M(t)^2,$$

and hence,

$$\|\tilde{Q} F\|_2^2 \leq C \|F\|_2^2 \leq C(1 + t)^{-\frac{3}{2}} M(t)^4.$$
This shows (8.1).
We next prove (8.2). We write
\[ \mathcal{R} = \sum_{i=1}^{4} I_i, \]
where
\[ I_1 = \left| \left( \frac{\partial^2 \phi}{\partial x_3^2} \phi \right)^2, \div \left( \frac{\gamma_0}{\gamma_p} \phi \right) w \right| + \left| \left( \frac{\partial^2 \phi}{\partial x_1^2} \phi \right)^2, \div \left( \chi^2 \frac{P'(\rho_0)}{\gamma_p} \phi \right) w \right| \]
\[ + \left| \left| \frac{\partial \phi}{\partial x_3}, \div \left( \frac{\gamma_0}{\gamma_p} \phi \right) w \right|^2 \right| + \sum_{j+k=1}^{N} \left| \left( \frac{\partial^j+1 \partial^k \phi}{\partial x_3}, \div \left( \chi^2 \frac{P'(\rho_0)}{\gamma_p} \phi \right) w \right) \right| \]
\[ I_2 = \left| \left( \partial_1, w \cdot \nabla \phi \right) \phi, \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi \right| + \left| \left( \frac{\partial^2 \phi}{\partial x_1^2}, w \cdot \nabla \phi \right) \phi, \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi \right| \]
\[ + \sum_{m=1}^{N} \left\{ \sum_{j+k=1}^{N} \left| \left( \chi^2 \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi \right) \right|^2 \right\}, \]
\[ I_3 = \left| \left( \frac{\partial^2 \phi}{\partial x_3^2} \phi, \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi \right) \frac{\partial^0 \phi}{\partial x_3} \phi \right| + \left| \left( \frac{\partial^2 \phi}{\partial x_3^2} \phi, \frac{\gamma_0}{\gamma_p} \partial_3 \phi \right) \right| + \left| \left( \partial_1 f^0, \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi \right) \right| \]
\[ + \sum_{m=1}^{N} \sum_{j+k=1}^{N} \left| \left( \frac{\partial^j+1 \partial^k \phi}{\partial x_3}, \div \left( \frac{\gamma_0}{\gamma_p} \phi \right) w \right) \right|^2 \]
\[ I_4 = \left| \left( \partial_1 f \phi, \partial_3 (\rho_3 \partial_3^2 w) \right) \phi \right| + \left| \left( \partial_1 f \phi, \partial_3 (\chi^2 \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi) \right) \phi \right| \]
\[ + \sum_{m=1}^{N} \sum_{j+k=1}^{N} \left| \left( \frac{\partial^j \partial^k \phi}{\partial x_3}, \div \left( \chi^2 \frac{P'(\rho_0)}{\gamma_p} \partial_3 \phi \right) \right) \phi \right| + \left| f \phi \right| + \left| \partial_1 f \phi \right| \]
\[ \leq C \left( (1+t)^{-\frac{1}{2}} M(t) D_\infty (t), \right) \]
\[ \leq C \left( (1+t)^{-\frac{1}{2}} M(t) \sqrt{D_\infty (t)[\phi_\infty]} \right) \]
\[ \leq C \left( (1+t)^{-1} M(t)^2 \sqrt{D_\infty (t)} \right) \]
\[ \leq C \left\{ (1+t)^{-\frac{3}{4}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty (t) \right\}. \]
As for $I_3$ and $I_4$, we have
\[ I_3 + I_4 \leq C \left\{ \left\| \partial_1 f^0 \right\|_{H^2} \left\| \phi_\infty \right\|_{H^2} + \left\| f_\infty \right\|_{H^1} \left\| w_\infty \right\|_{H^2} \right\} \]
\[ + \left\| \partial_1 f^0 \right\|_{H^2} \left\| \partial_1 \phi_\infty \right\|_{H^2} + \left\| \partial_1 f_\infty \right\|_{H^2} \left\| \partial_1 w_\infty \right\|_{H^2} \right\}. \]
Since $\left\| Q_0 P_\infty \Phi \right\|_{1} + \left\| \tilde{Q} P_\infty \Phi \right\|_{2} \leq C \left\| \Phi \right\|_{1}$, we find from Proposition 8.5 and Proposition 8.6 that
\[ \left\| \partial_1 f^0 \right\|_{H^2} + \left\| f_\infty \right\|_{H^1} + \left\| \partial_1 f_\infty \right\|_{2} \leq C \left\{ (1+t)^{-\frac{3}{4}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) \left\| D_\infty (t) \right\|_{2} \right\}. \]
It then follows from Lemma 8.4 that
\[
\| f_0^0 \|_{H^2} \| \phi_{\infty} \|_{H^2} + \| f_\infty \|_{H^1} \| w_\infty \|_{H^2} + \| \partial_t f_\infty \|_2 \| \partial_t w_\infty \|_2 \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^3 + (1 + t)^{-\frac{1}{4}} M(t)^2 \sqrt{D_\infty(t)}\}.
\]

It remains to estimate \( \| \partial_t f_\infty \|_2 \| \partial_t \phi_{\infty} \|_2 \). Since
\[
\partial_t \phi_{\infty} = -Q_0 L P u + Q_0 P F,
\]
we see from Lemma 8.3 and Proposition 8.5 (i) – (iii) that
\[
\| \partial_t \phi_{\infty} \|_{H^1} \leq C \{ \| v_0^0 \partial_x \phi_{\infty} \|_{H^1} + \| \partial_x w_\infty \|_{H^1} + \| Q_0 F_\infty \|_{H^1} \} \leq C (1 + t)^{-\frac{3}{2}} M(t).
\]

This, together with Lemma 8.3 and Proposition 8.5 (i) – (iii), then yields
\[
\| \partial_t f_\infty \|_2 \| \partial_t \phi_{\infty} \|_2 \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^2 + (1 + t)^{-\frac{1}{4}} M(t)^2 \sqrt{D_\infty(t)}\},
\]
and therefore, we have
\[
I_3 + I_4 \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^2 + (1 + t)^{-\frac{1}{4}} M(t) D_\infty(t)\}.
\]

We thus conclude that
\[
R(t) \leq C \{(1 + t)^{-\frac{3}{2}} M(t)^2 + (1 + t)^{-\frac{1}{4}} M(t) D_\infty(t)\}.
\]

This completes the proof.

\[\square\]

9 Asymptotic behavior

In this section we prove the asymptotic behavior (3.2).

Since \( M(t) \leq C \| u_0 \|_{H^2 \cap L^1} \) for all \( t \geq 0 \), we see that
\[
\| u(t) - (\sigma_1 u(0))(t) \|_2 \leq C (1 + t)^{-\frac{3}{2}} \| u_0 \|_{H^2 \cap L^1}.
\]

Therefore, to prove (3.2), it suffices to show the following

**Proposition 9.1.** Let \( \sigma = \sigma(x_3, t) \) be the solution of (3.3) with initial value \( \sigma|_{t=0} = \langle \phi_0 \rangle \). Assume that \( \nu \geq \nu_0, \frac{\gamma_0^2}{\nu + \omega} \geq \gamma_0^2 \) and \( \omega \leq \omega_0 \). Then there exists \( \epsilon > 0 \) such that if \( \| u_0 \|_{H^2 \cap L^1} \leq \epsilon \), then
\[
\| \sigma_1(t) - \sigma(t) \|_2 \leq C (1 + t)^{-\frac{3}{2} + \delta} \| u_0 \|_{H^2 \cap L^1} \quad (\delta > 0).
\]

To prove Proposition (9.1), we prepare two lemmas.

**In what follows we denote by \( \sigma = \sigma(x_3, t) \) the solution of (3.3) with initial value \( \sigma|_{t=0} = \sigma_0 \).**

It is well-known that \( \sigma(t) \) has the following decay properties.
Lemma 9.2. Assume that \( \sigma(t) \) is a solution of (3.3) with \( \sigma|_{t=0} = \sigma_0 \in H^1 \cap L^1 \). Then

\[
\begin{align*}
\| \partial_x \sigma(t) \|_2 & \leq C(1 + t)^{-\frac{1}{2} - \frac{x}{2}} \| \sigma_0 \|_{H^1 \cap L^1} \quad (l = 0, 1), \\
\| \sigma(t) \|_\infty & \leq C(1 + t)^{-\frac{1}{2}} \| \sigma_0 \|_{H^1 \cap L^1}.
\end{align*}
\]

We decompose \( \mathcal{H}(t) \) into two parts. We define \( \mathcal{H}_0(t) \) and \( \mathcal{H}_\infty(t) \) by

\[
\mathcal{H}_0(t) = \mathcal{F}^{-1}\{ \phi_0 \}(\xi)e^{-(\imath \xi \cdot x + \rho_0 t)} f, \quad \mathcal{H}_\infty(t) = \mathcal{H}(t) - \mathcal{H}_0(t).
\]

Then \( \mathcal{H}(t) = \mathcal{H}_0(t) + \mathcal{H}_\infty(t) \) and \( \mathcal{H}_0(t) \) and \( \mathcal{H}_\infty(t) \) have the following properties.

Lemma 9.3. There hold the following estimates.

\[
\begin{align*}
\| \partial_x \mathcal{H}_0(t) \sigma_0 \|_2 & \leq C(1 + t)^{-\frac{1}{2} - \frac{x}{2}} \| \sigma_0 \|_1, \\
\| \partial_x \mathcal{H}_\infty(t) \sigma_0 \|_2 & \leq C1^{\frac{1}{2}} e^{-\frac{\rho_0}{2} t} \| \sigma_0 \|_2, \\
\| \partial_x (e^{\lambda_3} \sigma_0 - \mathcal{H}_0(t) \sigma_0) \|_2 & \leq C(1 + t)^{-\frac{1}{2} - \frac{x}{2}} \| \sigma_0 \|_1.
\end{align*}
\]

Lemma 9.3 can be proved in a similar manner to the proof of [2, Proposition 5.8]; and we omit the proof.

We now prove Proposition 9.1.

Proof of Proposition 9.1. Let \( \sigma_0 = \langle \phi_0 \rangle \). We define \( N(t) \) by

\[
N(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2} - \delta} \| \sigma_1(t) - \sigma(t) \|_{H^1}.
\]

We write \( \sigma \) as

\[
\sigma(t) = \mathcal{H}(t) \sigma_0 - \kappa_2 \int_0^t \mathcal{H}(t - \tau) \partial_{x_3} (\sigma^2) (\tau) d\tau. \tag{9.1}
\]

As for \( \sigma_1(t) \), by Lemma 6.2 (ii), we have

\[
\mathcal{F}[[P] F] = -\imath \xi 1_{\{|\eta| \leq \rho_0\}}(\xi) \langle \phi w^3 \rangle + \partial_{x_3} \mathcal{F}[[P] (1) F] + \partial_{x_3} \mathcal{F}[[P] (2) F]
\]

\[
= -\imath \xi \kappa_{21} 1_{\{|\eta| \leq \rho_0\}}(\xi) (\sigma^2) - i \xi 1_{\{|\eta| \leq \rho_0\}}(\xi) (\langle \phi w^3 \rangle - \langle \phi(0) w^{(0),3} \rangle \langle \sigma^3 \rangle)
\]

\[
+ \partial_{x_3} \mathcal{F}[[P] (1) (\sigma^2 F_1 + F_2)] + \partial_{x_3} \mathcal{F}[[P] (2) F],
\]

where \( \kappa_{21} = \langle \phi(0) w^{(0),3} \rangle \). Furthermore,

\[
\mathcal{F}[[P] (1) (\sigma^2 F_1)] = 1_{\{|\eta| \leq \rho_0\}}(\xi) \langle \sigma^2 F_1, u^{(1)} \rangle = 1_{\{|\eta| \leq \rho_0\}}(\xi) [\langle F_1, u^{(1)} \rangle (\sigma^2)]
\]

\[
= -\kappa_{22} 1_{\{|\eta| \leq \rho_0\}}(\xi) (\sigma^2),
\]

where \( \kappa_{22} = -\langle F_1, u^{(1)} \rangle \). We thus obtain

\[
e^{(t-\tau) \lambda} P F = -\kappa_2 e^{(t-\tau) \lambda} \partial_{x_3} (\sigma^2) - e^{(t-\tau) \lambda} \partial_{x_3} \{ \langle \phi w^3 \rangle - \langle \phi(0) w^{(0),3} \rangle \sigma^2 \}
\]

\[
+ e^{(t-\tau) \lambda} J_4 + e^{(t-\tau) \lambda} J_5.
\]

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Here we set $\kappa_2 = \kappa_{21} + \kappa_{22}$,

$$J_4 = \partial_{x_3} P^{(1)} F_2 + \partial^2_{x_3} P^{(2)} F_2,$$

$$J_5 = \partial^2_{x_3} P^{(2)} (\sigma^2_1 F_1).$$

It then follows from (5.2) and (9.1) that $\sigma_1(t) - \sigma(t)$ is written as

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^5 I_j(t),$$

where

$$I_0(t) = e^{tA} P u_0 - H(t) \sigma_0 + \kappa_2 \int_0^t H_\infty(t - \tau) \partial_{x_3} (\sigma^2)(\tau) \, d\tau,$$

$$I_1(t) = -\kappa_2 \int_0^t H_0(t - \tau) \partial_{x_3} (\sigma^2_1 - \sigma^2) \, d\tau,$$

$$I_2(t) = -\kappa_2 \int_0^t (e^{(t-\tau)A} - H_0(t - \tau)) \partial_{x_3} (\sigma^2_1)(\tau) \, d\tau,$$

$$I_3(t) = -\int_0^t \partial_{x_3} e^{(t-\tau)A} (\langle \phi w^3 \rangle - \langle \phi^{(0)} w^{(0),3} \rangle) \, d\tau,$$

$$I_j(t) = \int_0^t e^{(t-\tau)A} J_j(\tau) \, d\tau, \quad (j = 4, 5).$$

We see from Proposition 4.6 and Lemmas 9.2, 9.3 that

$$\|I_0(t)\|_{H^1} \leq C \left\{ (1 + t)^{-\frac{3}{4}} \|u_0\|_{H^1 \cap L^1} + \int_0^t (t - \tau)^{-\frac{1}{2}} e^{-\frac{\theta}{2} \tau} (t-\tau) \|\sigma\|_{\infty} \|\partial_{x_3} \sigma\|_2(\tau) \, d\tau \right\}$$

$$\leq C \left\{ (1 + t)^{-\frac{3}{4}} \|u_0\|_{H^1 \cap L^1} + \int_0^t (t - \tau)^{-\frac{1}{2}} e^{-\frac{\theta}{2} \tau} (t-\tau)^{\frac{3}{4}} (1 + \tau)^{-\frac{1}{4}} \, d\tau \|u_0\|_{H^1 \cap L^1}^2 \right\}$$

$$\leq C (1 + t)^{-\frac{3}{4}} \|u_0\|_{H^1 \cap L^1} \{1 + \|u_0\|_{H^1 \cap L^1}\}.$$

As for $I_1(t)$, we first observe

$$\|(\sigma^2_1 - \sigma^2)(t)\|_1 \leq \|(\sigma + \sigma)(t)\|_2 \|(\sigma_1 - \sigma)(t)\|_2 \leq C (1 + t)^{-1+\delta} N(t) \|u_0\|_{H^2 \cap L^1}.$$
for \( k = 0, 1 \).

As for \( I_3(t) \), since
\[
\| \langle \delta w^3 \rangle (\tau) - \langle \delta(0) w^{(0)}.3(0) \rangle \sigma^2_1(\tau) \|_1 \leq C \{ \| \sigma_1(\tau) \|_2 \| u(\tau) - \sigma_1(\tau) u^{(0)} \|_2 + \| u(t) \|_2 \| u(\tau) - \sigma_1(\tau) u^{(0)} \|_2 \} \leq C (1 + \tau)^{-1} M(t)^2,
\]
we have
\[
\| \partial_{x_3}^k I_3(t) \|_2 \leq C M(t)^2 \int_0^t (1 + \tau)^{-\frac{3}{2}} - \frac{1}{2} (1 + \tau)^{-1} d\tau \leq C (1 + t)^{-\frac{1}{2}} \log(1 + t) \| u_0 \|_{H^2 L^1}^2
\]
for \( k = 0, 1 \).

By Proposition 4.5 and Lemma 6.3, \( I_4(t) \) is estimated as
\[
\| \partial_{x_3}^k I_4(t) \|_2 = \| \int_0^t e^{(t-\tau)\Lambda} \partial_{x_3} (P^{(1)} F_2(\tau) + \partial_{x_3} P^{(2)} F_2(\tau)) d\tau \|_2 M(t)^2 \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{2}} - \frac{1}{2} (1 + \tau)^{-1} d\tau \| u_0 \|_{H^2 L^1}^2 \leq C (1 + t)^{-\frac{1}{2}} \log(1 + t) \| u_0 \|_{H^2 L^1}^2
\]
for \( k = 0, 1 \).

As for \( I_5(t) \), since \( \partial_{x_3} P^{(2)}(\tau) = P^{(2)} \partial_{x_3} \), we see from Lemma 6.3 that
\[
\| \partial_{x_3}^k I_5(t) \|_2 \leq \| \int_0^t e^{(t-\tau)\Lambda} \partial_{x_3}^{k+1} P^{(2)} (\partial_{x_3} (\sigma^2_1) F_1)(\tau) d\tau \|_2 \leq C \left\{ \int_0^t (1 + t - \tau)^{-\frac{3}{2}} - \frac{1}{2} (1 + \tau)^{-1} d\tau M(t)^2 \right\} \leq C (1 + t)^{-\frac{1}{2}} \log(1 + t) \| u_0 \|_{H^2 L^1}^2
\]
for \( k = 0, 1 \).

Therefore, we obtain
\[
\| (\sigma_1 - \sigma)(t) \|_{H^1} \leq C (1 + t)^{-\frac{1}{2}} \log(1 + t) \| u_0 \|_{H^2 L^1}^2 \{ 1 + \| u_0 \|_{H^2 L^1}^2 + \| u_0 \|_{H^2 L^1}^2 + N(t) \}.
\]
It then follows that if \( \| u_0 \|_{H^2 L^1} \) is sufficiently small, then
\[
N(t) \leq C \| u_0 \|_{H^2 L^1}.
\]
We thus see that if \( \| u_0 \|_{H^2 L^1} \ll 1 \), then
\[
\| \sigma_1(t) - \sigma(t) \|_2 \leq C (1 + t)^{-\frac{1}{2}} \log(1 + t) \| u_0 \|_{H^2 L^1}^2 \]
This completes the proof.  

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