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# Spectrum of the quantum Rabi model and representation theory

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## Non-commutative harmonic oscillators I

The story has begun by the study of non-commutative harmonic oscillators (NcHO):
The normal form the Hamiltonian $Q_{(\alpha,\beta)}(x, D)$ of NcHO is given by

$$Q_{(\alpha,\beta)}(x, D) = A\Big( -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\Big) + J\Big(x\frac{d}{dx} + \frac{1}{2}\Big),$$

where $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

A. Parmeggiani and M. Wakayama:
*Oscillator representations and systems of ordinary differential equations*, Proc. Natl. Acad. Sci. USA **98** (2001), 26–30.

*Non-commutative harmonic oscillators-I, II, Corrigenda and remarks to I*, Forum. Math. **14** (2002), 539–604, 669–690, ibid **15** (2003), 955–963.

## Non-commutative harmonic oscillators II

- Development of the study of NcHO including number theoretic investigations can be found in the book and its references:
  A. Parmeggiani, Spectral Theory of Non-commutative Harmonic Oscillators: An Introduction. LNM. **1992**, Springer, 2010.
- There is a second degree element $\mathcal{R}$ of $U(\mathfrak{sl}_2)$ such that the image of $\mathcal{R}$ under the oscillator representation of the Lie algebra $\mathfrak{sl}_2$ gives the NcHO:

  H. Ochiai, *Non-commutative harmonic oscillators and Fuchsian ordinary differential operators*, Comm. Math. Phys. **217** (2001), 357–373.



## Purposes of the talk (I)

- Draw the following pictures:

  1) the image of $\mathcal{R}$ under the non-unitary principal series representation of $\mathfrak{sl}_2$ gives a Heun ODE. Moreover, this Heun ODE provides the Heun picture of the quantum Rabi model under suitable (including a parameter of the representation) confluent procedure [2].

  2) there exists another second degree element $\mathcal{K}$ of $U(\mathfrak{sl}_2)$ such that the image of $\mathcal{K}$ under non-unitary principal series representation of $\mathfrak{sl}_2$ gives the quantum Rabi model [1].

## Purposes of the talk (II)

- Describe the representation theoretic explanation (finite dimensional representations of $\mathfrak{sl}_2$) of the degenerate spectrum of the quantum Rabi model, which was described by Küs:
  M. Kus: *On the spectrum of a two-level system*, J. Math. Phys., **26** (1985) 2792–2795.
- Provide a conjectural statement about the non-degenerate exceptional spectrum (relating the discrete series representation of $\mathfrak{sl}_2$). Numerical evidence is found in
  A.J. Maciejewski, M. Przybylska and T. Stachowiak: *Full spectrum of the Rabi model*, Phys. Letter A **378**, (2014), 16-20.



## the quantum Rabi model

The quantum Rabi model is defined by the Hamiltonian

$$H_{\mathrm{Rabi}}/\hbar = \omega\psi^\dagger\psi + \Delta\sigma_z + g\sigma_x(\psi^\dagger + \psi).$$

Here $\psi = (x+\partial_x)/\sqrt{2}$ (resp. $\psi^\dagger = (x-\partial_x)/\sqrt{2}$) is the annihilation (resp. creation) operator for a bosonic mode of frequency $\omega$, $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices for the two-level system, $2\Delta$ is the energy difference between the two levels, and $g$ denotes the coupling strength between the two-level system and the bosonic mode. For simplicity and without loss of generality we may set $\hbar = 1$ and $\omega = 1$.

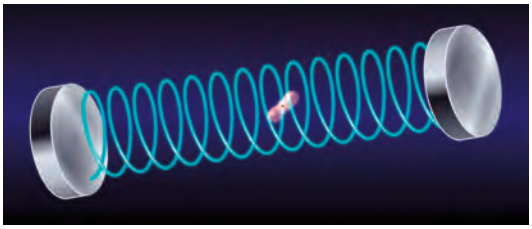

Figure: Courtesy of APS/Alan Stonebraker in E. Solano, Viewpoint: The dialogue between quantum light and matter, Physics 4, 68 (2011).

The Rabi model describes the simplest interaction between quantum light and matter. The model considers a two-level atom coupled to a quantized, single-mode harmonic oscillator.



## Spectrum of the quantum Rabi model

Spectrum of $H_{\mathrm{Rabi}}$ is classified as

$$\mathrm{Spec}(H_{\mathrm{Rabi}}) = \{\text{Regular eigen.}\} \sqcup \{\text{Exceptional eigen.}\}$$

Exceptional eigenvalues $\lambda$ are of the form $\lambda = N - g^2$ ($N \in \mathbb{Z}$). Regular eigenvalues are the ones not of the form. Moreover,

$$\begin{aligned}&\{\text{Exceptional eigen.}\}\\ =&\{\text{non} - \text{degenerate Exceptional eigen.}\} \sqcup \{\text{degenerate Exceptional eigen.}\}\end{aligned}$$

Degenerate exceptional eigenvalues ($\lambda = N - g^2$ for some $N \in \mathbb{N}$) are described by Küs (1985).
The regular spectrum was described by D. Braak for the first time in about 70 years after the proposition of the quantum Rabi model:
D. Braak, *On the Integrability of the Rabi Model*, Phys. Rev. Lett. **107** (2011), 100401–100404.

## The element $\mathcal{R} \in U(\mathfrak{sl}_2)$

For the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$, define a second order element $\mathcal{R}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ by

$$\mathcal{R} := \frac{2}{\sinh 2\kappa}\Big\{\Big[(\sinh 2\kappa)(E-F) - (\cosh 2\kappa)H + \nu\Big](H-\nu) + (\varepsilon\nu)^2\Big\}.$$

Here $H, E$ and $F$ be the standard generators of the Lie algebra $\mathfrak{sl}_2$ defined by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

They satisfy the commutation relations

$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = H.$$



## NcHO and $\mathcal{R}$

Suppose that $\alpha \neq \beta$. Determine the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$ by the formulas

$$\cosh\kappa = \sqrt{\frac{\alpha\beta}{\alpha\beta - 1}}, \ \sinh\kappa = \frac{1}{\sqrt{\alpha\beta - 1}}, \ \varepsilon = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|, \ \nu = \frac{\alpha + \beta}{2\sqrt{\alpha\beta(\alpha\beta - 1)}}\lambda.$$

Then the eigenvalue problem $Q\varphi = \lambda\varphi$ ($\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$) is equivalent to the equation $\pi'(\mathcal{R})u = 0$ ($u \in \overline{\mathbb{C}[y]}$). Here $\pi'$ is the oscillator representation of $\mathfrak{sl}_2$ defined on the space $\mathbb{C}[y]$ by

$$\pi'(H) = y\partial_y + 1/2, \ \pi'(E) = y^2/2, \ \pi'(F) = -\partial_y^2/2.$$

## The element $\mathcal{K} \in U(\mathfrak{sl}_2)$

$$\mathcal{K} := \Big[\frac{1}{2}H - E + 1 - \frac{\lambda + g^2}{2}\Big][F + 4g^2] + (\frac{1}{2} - \frac{\lambda + g^2}{2})[H - \frac{1}{2}].$$

$$\Lambda_a := 4g^2\Big(\frac{1}{2}a + 1 - \frac{\lambda + g^2}{2}\Big) + (\frac{1}{2} - \frac{\lambda + g^2}{2})(a - \frac{1}{2}).$$

Here $a$ is a parameter of non-unitary principal series $\varpi_a$ of the $\mathfrak{sl}_2$. (See [1].) Under the representation $\varpi_a$ of $\mathfrak{sl}_2$ we have the confluent Heun picture of the quantum Rabi model as ([1])

$$\mathcal{H}_1^{\mathrm{Rabi}}(\lambda) = \{x(x-1)\}^{-1} x^{-\frac{1}{2}(a-\frac{1}{2})}(\varpi_a(\mathcal{K}) - \Lambda_a) x^{\frac{1}{2}(a-\frac{1}{2})} \quad (a := -(\lambda + g^2)),$$

where

$$\mathcal{H}_1^{\mathrm{Rabi}}(\lambda) := \frac{d^2}{dx^2} + \Big\{ -4g^2 + \frac{1 - (\lambda + g^2)}{x} + \frac{1 - (\lambda + g^2 + 1)}{x - 1} \Big\} \frac{d}{dx} + \frac{4g^2(\lambda + g^2)x + \mu}{x(x-1)}$$

with the accessory parameter

$$\mu := (\lambda + g^2)^2 - 4g^2(\lambda + g^2) - \Delta^2.$$



## References

The talk presented in the workshop is based on the results in the following works:

[1] M. Wakayama and T. Yamasaki, *The quantum Rabi model and Lie algebra representations of* $\mathfrak{sl}_2$, J. Phys. A: Math. Theor. **47** (2014), 335203 (17pp).

[2] M. Wakayama, *Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun differential equations, eigenstates degeneration and the Rabi model*, Int. Math. Res. Not., doi:10.1093/imrn/RNV145 (May 25, 2015) (36pp).