

Subsequential scaling limits of simple random walk on the two-dimensional uniform spanning tree

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UNIFORM SPANNING TREE IN TWO DIMENSIONS

Let $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$.

A subgraph of the lattice is a **spanning tree** of Λ_n
if it connects all vertices, no cycles.

Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected **uniformly
at random from all possibilities**.

The UST on \mathbb{Z}^2 , \mathcal{U} , is then the local limit of $\mathcal{U}^{(n)}$.
NB. Wired/free boundary conditions unimportant.

Almost-surely, \mathcal{U} is a **spanning tree of \mathbb{Z}^2** .

[Aldous, Benjamini, Broder, Häggström, Kirchoff, Lyons,
Pemantle, Peres, Schramm. . .]

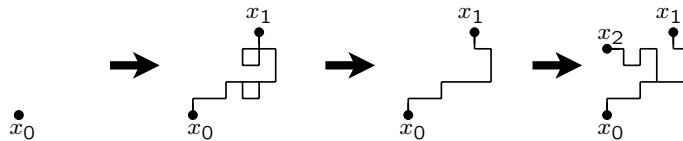
WILSON'S ALGORITHM ON \mathbb{Z}^2

Let $x_0 = 0, x_1, x_2, \dots$ be an enumeration of \mathbb{Z}^2 .

Let $\mathcal{U}(0)$ be the graph tree consisting of the single vertex x_0 .

Given $\mathcal{U}(k-1)$ for some $k \geq 1$, define $\mathcal{U}(k)$ to be the union of $\mathcal{U}(k-1)$ and the **loop-erased random walk (LERW)** path run from x_k to $\mathcal{U}(k-1)$.

The UST \mathcal{U} is then the local limit of $\mathcal{U}(k)$.



LERW SCALING IN \mathbb{Z}^d

Consider LERW as a process $(L_n)_{n \geq 0}$.

In \mathbb{Z}^d , $d \geq 5$, L rescales diffusively to BM [Lawler 1980].

In \mathbb{Z}^4 , with logarithmic corrections rescales to BM [Lawler].

In \mathbb{Z}^3 , $\{L_n : n \in [0, \tau]\}$ has a scaling limit [Kozma 2007].

In \mathbb{Z}^2 , $\{L_n : n \in [0, \tau]\}$ has **SLE(2)** scaling limit, **UST peano curve** has **SLE(8)** scaling limit [Lawler/Schramm/Werner 2004]. Growth exponent is **5/4** [Kenyon, Masson, Lawler].

Let $M_n = |\text{LERW}(0, B_E(0, n))|$ be the length of a LERW run from 0 to $B_E(0, n)^c$.

Theorem.(d=2)

[Kenyon 2000] $\lim_{n \rightarrow \infty} \frac{\log E^0 M_n}{\log n} = 5/4$

[Lawler 2014] $c_1 n^{5/4} \leq E^0 M_n \leq c_2 n^{5/4}$

Now consider random walk on the UST.

RW on random graphs: General theory.

Let $\mathcal{G}(\omega)$ be a random graph on (Ω, \mathbb{P}) . Assume $\exists 0 \in \mathcal{G}(\omega)$.

Let $D \geq 1$. For $\lambda \geq 1$, we say that $B(0, R)$ in $\mathcal{G}(\omega)$ is λ -good if

$$\begin{aligned} \lambda^{-1} R^D &\leq |B(0, R)| \leq \lambda R^D, \\ \lambda^{-1} R &\leq R_{\text{eff}}(0, B(0, R)^c) \leq R + 1. \end{aligned}$$

λ -good is a nice control of the volume and resistance for $B(0, R)$.

Theorem. [Barlow/Jarai/K/Slade 2008, K/Misumi 2008]

Suppose $\exists p > 0$ such that

$$\mathbb{P}(\{\omega : B(0, R) \text{ is } \lambda\text{-good.}\}) \geq 1 - \lambda^{-p} \quad \forall R \geq R_0, \forall \lambda \geq \lambda_0.$$

Then $\exists \alpha_1, \alpha_2 > 0$ and $N(\omega), R(\omega) \in \mathbb{N}$ s.t. the following holds for \mathbb{P} -a.e. ω :

$$\begin{aligned} (\log n)^{-\alpha_1} n^{-\frac{D}{D+1}} &\leq p_{2n}^\omega(0,0) \leq (\log n)^{\alpha_1} n^{-\frac{D}{D+1}}, & \forall n \geq N(\omega), \\ (\log R)^{-\alpha_2} R^{D+1} &\leq E_\omega^0 \tau_{B(0,R)} \leq (\log R)^{\alpha_2} R^{D+1}, & \forall R \geq R(\omega). \end{aligned}$$

In particular,

$$d_s(G) := \lim_{n \rightarrow \infty} \frac{\log p_{2n}^\omega(0,0)}{\log n} = \frac{2D}{D+1}$$

Examples. (See K 2014: LNM (St. Flour Lect. Notes))

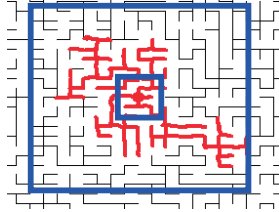
$D = 2$ and $d_s = 4/3$

- Critical percolation on regular trees conditioned to survive forever. (Barlow/K '06)
- Infinite incipient cluster (IIC) for spread out oriented percolation for $d \geq 6$ (Barlow/Jarai/K/Slade '08)
- Invasion percolation on a regular tree. (Angel/Goodman/den Hollander/Slade '08)
- IIC for percolation on \mathbb{Z}^d , $d \geq 19$ (Kozma/Nachmias '09)

More general

- α -stable Galton-Watson trees conditioned to survive forever (Croydon/K '08) $d_s = 2\alpha/(2\alpha - 1)$

VOLUME AND RESISTANCE ESTIMATES
[BARLOW/MASSON 2010,2011]



With high probability,

$$B_E(x, \lambda^{-1}R) \subseteq B_U(x, R^{5/4}) \subseteq B_E(x, \lambda R),$$

as $R \rightarrow \infty$ then $\lambda \rightarrow \infty$.

It follows that with high probability,

$$\mu_U(B_U(x, R)) \asymp R^{8/5}.$$

Also with high probability,

$$\text{Resistance}(x, B_U(x, R)^c) \asymp R.$$

\Rightarrow Exit time for intrinsic ball radius R is $R^{13/5}$,

HK bounds $p_{2n}^U(0,0) \asymp n^{-8/13}$. ($D = 8/5, d_s = 16/13$)

(Q) How about scaling limit for UST?

Barlow/Masson obtained further detailed properties.

Theorem.[Barlow/Masson 2010]

$$\begin{aligned} \mathbb{P}(M_n > \lambda EM_n) &\leq 2e^{-c_1 \lambda}, \\ \mathbb{P}(M_n < \lambda^{-1} EM_n) &\leq 2e^{-c_2 \lambda^{c_3}} \end{aligned}$$

Theorem.[Barlow/Masson 2011]

$$\begin{aligned} \mathbb{P}(B_U(0, R^{5/4}/\lambda) \not\subseteq B_E(0, R)) &\leq c_4 e^{-\lambda^{2/3}}, \\ \mathbb{P}(B_E(0, R) \not\subseteq B_U(0, \lambda R^{5/4})) &\leq c_\epsilon \lambda^{-4/15-\epsilon}. \end{aligned}$$

While for most points $x \in \mathbb{Z}^2$, the balls $B_E(0, R)$ and $B_{\mathcal{U}}(0, R^{5/4})$ will be comparable, there are neighboring points in \mathbb{Z}^2 which are far in \mathcal{U} .

Lemma. [Benjamini et. al. 2001]

The box $[-n, n]^2$ contains with probability 1 neighbouring points $x, y \in \mathbb{Z}^2$ with $d_{\mathcal{U}}(x, y) \geq n$.

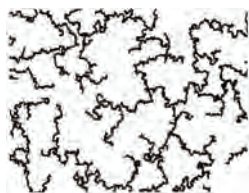
Proof. Consider the path (in \mathbb{Z}^2) of length $8n$ around the box $[-n, n]^2$: If each neighboring pair were connected by a path in \mathcal{U} of length less than n , then this path would not contain 0. So we would obtain a loop around 0 — which is impossible since \mathcal{U} is a tree.

UST SCALING [SCHRAMM 2000]

Consider \mathcal{U} as an ensemble of paths:

$$\mathfrak{U} = \{(a, b, \pi_{ab}) : a, b \in \mathbb{Z}^2\},$$

where π_{ab} is the unique arc connecting a and b in \mathcal{U} .
cf. [Aizenman/Burchard/Newman/Wilson 1999].



Picture: Oded Schramm

Scaling limit \mathfrak{T} a.s. satisfies:

- each pair $a, b \in \mathbb{R}^2$ connected by a path;
- if $a \neq b$, then this path is simple;
- if $a = b$, then this path is a point or a simple loop;
- the trunk, $\cup_{\mathfrak{T}} \pi_{ab} \setminus \{a, b\}$, is a dense topological tree with degree at most 3.

ISSUE : This topology does not carry information about intrinsic distance, volume, or resistance.

GENERALISED GROMOV-HAUSDORFF TOPOLOGY (cf. [GROMOV, LE GALL/DUQUESNE])

Define \mathbb{T} to be the collection of measured, rooted, spatial trees, i.e.

$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}),$$

where:

- $(\mathcal{T}, d_{\mathcal{T}})$ is a locally compact real tree;
- $\mu_{\mathcal{T}}$ is a Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$;
- $\phi_{\mathcal{T}}$ is a cont. map from $(\mathcal{T}, d_{\mathcal{T}})$ into \mathbb{R}^2 ;
- $\rho_{\mathcal{T}}$ is a distinguished vertex in \mathcal{T} .

On \mathbb{T}_c (compact trees only), define a distance Δ_c by

$$\inf_{\substack{Z, \psi, \psi', C: \\ (\rho_{\mathcal{T}}, \rho'_{\mathcal{T}}) \in C}} \left\{ d_P^Z(\mu_{\mathcal{T}} \circ \psi^{-1}, \mu'_{\mathcal{T}} \circ \psi'^{-1}) + \sup_{(x, x') \in C} (d_Z(\psi(x), \psi'(x')) + |\phi_{\mathcal{T}}(x) - \phi'_{\mathcal{T}}(x')|) \right\}.$$

Can be extended to locally compact case.

TIGHTNESS OF UST

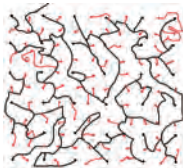
Theorem. If \mathbf{P}_{δ} is the law of the measured, rooted spatial tree

$$(\mathcal{U}, \delta^{5/4} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}(\cdot), \delta \phi_{\mathcal{U}}, 0)$$

under \mathbf{P} , then the collection $(\mathbf{P}_{\delta})_{\delta \in (0,1)}$ is tight in $\mathcal{M}_1(\mathbb{T})$.

Proof involves:

- strengthening estimates of [Barlow/Masson],
- comparison of Euclidean and intrinsic distance along paths.



UST LIMIT PROPERTIES

If $\tilde{\mathbf{P}}$ is a **subsequential limit** of $(\mathbf{P}_\delta)_{\delta \in (0,1)}$, then for $\tilde{\mathbf{P}}$ -a.e. $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ it holds that:

- (i) $\mu_{\mathcal{T}}$ is non-atomic, supported on the leaves of \mathcal{T} ,
i.e. $\mu_{\mathcal{T}}(T^o) = 0$, where $T^o := \mathcal{T} \setminus \{x \in \mathcal{T} : \deg_{\mathcal{T}}(x) = 1\}$;
- (ii) for any $R > 0$,

$$\liminf_{r \rightarrow 0} \frac{\inf_{x \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)} \mu_{\mathcal{T}}(B_{\mathcal{T}}(x, r))}{r^{8/5} (\log r^{-1})^{-c}} > 0,$$

- (iii) $\phi_{\mathcal{T}}$ is a homeo. between T^o and $\phi_{\mathcal{T}}(T^o)$ (dense in \mathbb{R}^2);
- (iv) $\max_{x \in \mathcal{T}} \deg_{\mathcal{T}}(x) = 3$;
- (v) $\mu_{\mathcal{T}} = \mathcal{L} \circ \phi_{\mathcal{T}}$.

To prove this, we need the following '**uniform control**':

$$\lim_{\eta \rightarrow 0} \liminf_{\delta \rightarrow 0} \mathbf{P} \left(\sup_{\substack{x, y \in B_{\mathcal{U}}(0, c_1 \delta^{-5/4} r): \\ d_{\mathcal{U}}(x, y) \leq c_2 \delta^{-5/4} \eta}} d_{\mathcal{U}}^S(x, y) > \delta^{-1} \varepsilon \right) = 0,$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathbf{P} \left(\inf_{\substack{x, y \in B_{\mathcal{U}}(0, \delta^{-5/4} r): \\ d_{\mathcal{U}}(x, y) \geq \delta^{-5/4} \eta}} d_{\mathcal{U}}^S(x, y) < \delta^{-1} \varepsilon \right) = 0,$$

where $d_{\mathcal{U}}^S = \text{diam}(\gamma(x, y))$ (Euclidean diameter of the LERW between x and y ; Schramm's distance).

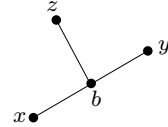
\Rightarrow This involves uniform control and requires more detailed estimates than those of Barlow/Masson.

Given such generalized G-H convergence of trees, we can prove **convergence of the process on the trees** (generalization of the theory due to Crodon (2008)).

On the (limiting) real tree $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ s.t. $\mu^{\mathcal{T}}$ has full support, one can define a 'Brownian motion' $X^{\mathcal{T}} = (X_t^{\mathcal{T}})_{t \geq 0}$.

- For $x, y, z \in \mathcal{T}$,

$$P_z^{\mathcal{T}}(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b(x, y, z), y)}{d_{\mathcal{T}}(x, y)}.$$



- Mean **occupation density** when started at x and killed at y ,

$$2d_{\mathcal{T}}(b(x, y, z), y)\mu^{\mathcal{T}}(dz).$$

Requirement :

$$\liminf_{r \rightarrow 0} \frac{\inf_{x \in \mathcal{T}} \mu_{\mathcal{T}}(B_{\mathcal{T}}(x, r))}{r^{\kappa}} > 0. \quad \exists \kappa > 0.$$

LIMITING PROCESS FOR SRW ON UST

Suppose $(\mathbb{P}_{\delta_i})_{i \geq 1}$, the laws of

$$\left(\mathcal{U}, \delta_i^{5/4} d_{\mathcal{U}}, \delta_i^2 \mu_{\mathcal{U}}, \delta_i \phi_{\mathcal{U}}, 0 \right),$$

form a convergent sequence with limit $\tilde{\mathbb{P}}$.

Let $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \sim \tilde{\mathbb{P}}$.

It is then the case that \mathbb{P}_{δ_i} , the annealed laws of

$$\left(\delta_i X_{\delta_i^{-13/4}t}^{\mathcal{U}} \right)_{t \geq 0},$$

converge to $\tilde{\mathbb{P}}$, the annealed law of

$$\left(\phi_{\mathcal{T}}(X_t^{\mathcal{T}}) \right)_{t \geq 0},$$

as probability measures on $C(\mathbb{R}_+, \mathbb{R}^2)$.

HEAT KERNEL ESTIMATES FOR SRW LIMIT

Let $R > 0$. For $\tilde{\mathbf{P}}$ -a.e. realisation of $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$, there exist random constants $c_1, c_2, c_3, c_4, t_0 \in (0, \infty)$ and deterministic constants $\theta_1, \theta_2, \theta_3, \theta_4 \in (0, \infty)$ such that the heat kernel associated with the process $X^{\mathcal{T}}$ satisfies:

$$p_t^{\mathcal{T}}(x, y) \leq c_1 t^{-8/13} \ell(t^{-1})^{\theta_1} \exp \left\{ -c_2 \left(\frac{d_{\mathcal{T}}(x, y)^{13/5}}{t} \right)^{5/8} \ell(d_{\mathcal{T}}(x, y)/t)^{-\theta_2} \right\},$$

$$p_t^{\mathcal{T}}(x, y) \geq c_3 t^{-8/13} \ell(t^{-1})^{-\theta_3} \exp \left\{ -c_4 \left(\frac{d_{\mathcal{T}}(x, y)^{13/5}}{t} \right)^{5/8} \ell(d_{\mathcal{T}}(x, y)/t)^{\theta_4} \right\},$$

for all $x, y \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)$, $t \in (0, t_0)$, where $\ell(x) := 1 \vee \log x$.