Volatility Modelling in Finance : A Survey

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https://doi.org/10.15017/1522106

出版情報：経済論究. 152, pp.27-63, 2015-07-27. 九州大学大学院経済学会

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Keywords: Volatility Models, GARCH, MGARCH, SV, Realised Volatility, VIRF, Finance

1 Introduction

Volatility modelling in empirical finance has played a crucial role in modern asset pricing, portfolio allocation and risk management. Bauwens *et al*. (2012) argue that volatility modelling is still and will remain for long, one of the most active research topics in financial econometrics. The recent surge in popularity of volatility models in finance is explained by the fact that they can be used for forecasting volatility of financial assets. As the variance of returns is used as a measure of risk, it is important for investors to know how the volatility of their portfolios can be expected to change in the future. That is why, the development of volatility models has gone along with their application and progressive use in the financial markets. The financial crisis in the period 2008 to 2009 and its dramatic economic consequences have made it apparent that academics, regulators and policymakers still have a lot of progress to make in their understanding of financial risks. These risks have been compounded by development of sophisticated financial products and the strong linkages between financial institutions due to increasing globalisation. This paper surveys the literature on volatility models; compares and contrasts several models with emphasis on recent contributions and provides an overview of the most recent advances in the field. Further, our survey shall consider both univariate and multivariate

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classes of volatility models with financial applications.

The main approaches to modelling volatility in the literature so far have been: conditional volatility, stochastic volatility (SV) and realised volatility (RV). An alternative method, known as implied volatility is based on the Black-Scholes option pricing model. The estimation methods and analyses have been within both the classical and Bayesian frameworks. Harvey (2013) states that the generalised autoregressive conditional heteroscedasticity (GARCH) and SV models have provided the principal means of analysing, modelling and monitoring volatility changes in the last three decades. Interestingly, the applications of GARCH models to financial time series have been quite successful. Shephard (2005) stressed the fact that "the development of this subject has been highly multidisciplinary, with results drawn from financial economics, probability theory and econometrics, blending to produce methods and models which have aided our understanding of the realistic pricing of options, efficient asset allocation and accurate risk assessment". Asset pricing theory in particular is underscored by the idea that higher rewards may be expected when we face higher risks, but these risks change through time in complex ways. That is why a theory of dynamic volatilities is required to help analyse such changes.

Since the development of ARCH models in the 1980s, several generalisations and extensions have been proposed with the aim of increasing their flexibility. Some of the most widely applied volatility models include GARCH, *exponential* GARCH (EGARCH), *threshold* GARCH (TGARCH), *power* GARCH (PGARCH), *Markov-switching* GARCH (MS-GARCH), etc. Engle (2003) note that these extensions recognise the presence of nonlinearities, asymmetry and long memory properties associated with volatility and the non-normal nature of returns distribution. Several versions of these models have been applied to analyse inflation (e.g. Engle, 1982; Evans and Wachtel, 1993), the stock returns (e.g. Babikir *et al*., 2012 and King and Botha, 2015), interest rates (see Gospodinov, 2005; Song, 2014) and the exchange rates (see, e.g. Rapach and Strauss, 2008 and Ozer-Imer and Ozkan, 2014). Teräsvirta *et al*. (2010) stress that "the changing conditional variance is typically modelled using past values of the error process and past values of the conditional variance itself". A related but slightly distinct class of volatility models are the SV models (e.g. Shephard and Andersen, 2009), autoregressive conditional duration (ACD) models (Engle and Russel, 1998; Engle, 2000) and dynamic conditional correlation (DCC) models (see Engle, 2002 and Tse and Tsui, 2002). The SV methods are also employed to model series particularly those with nonstationarity in their variances and are now among the main tools time-varying volatility is modelled in financial markets.

Several surveys of these models exists in the literature see e.g., Bollerslev *et al*. (1992, 1994), Bera and Higgins (1993) and Tsay (2005). Andersen *et al.'s* (2009) handbook of financial time series contains several chapters on volatility models while Shephard (2005) contains selected articles on SV models. Bauwens *et al.'s* (2012) handbook of volatility models and their applications is a volume devoted to the theory and practice of volatility models in financial engineering which provides overview of the most recent advances in the field. Many initial surveys focus on univariate models, but the recent reviews are beginning to emphasise on multivariate models and according to Teräsvirta *et al*. (2010) "they are at the moment still less frequently applied than their univariate counterparts". Recently, Harvey (2013) proposes a generalised dynamic conditional score (DCS) model with emphasis on robust modelling of outliers in the levels of time series and to the treatment of time-varying relationships.

Another class of nonlinear volatility models that have been widely used in empirical finance and that allows for jumps in volatility are the (MS)-ARCH and MS-GARCH models. These models often consider discrete time, with the volatility stochastically switching between a finite number of fixed regimes. A key feature of MS models is their ability to capture endogenous regime shifts and other nonlinear dynamics. A wide range of potential nonlinearities can be entertained when modelling variances but the *switching*-ARCH (SWARCH) and MS-GARCH models are the most widely used because switching in these models is stochastic. Leading contributions in this area include Hamilton and Susmel (1994), Cai (1994), Dueker (1997), Gray (1996), and Bauwens *et al*. (2010). In the same period, researchers began developing simulation based inference methods through the application of Markov Chain Monte Carlo (MCMC) and Bayesian techniques to address the same issues. Influential papers in this field range from Kim *et al*. (1998) to Song (2014).

Although much of the contributions in the late 1980s and the 1990s were aimed at capturing the key properties of ARCH processes such as volatility clustering, asymmetry and long memory. Recent studies are beginning to focus on incorporating into the modelling process, third (skewness) and higher moments (e.g. kurtosis) (see, e.g. Back, 2014; Kim *et al*., 2014). For instance, Bauwens and Laurent (2005) propose a flexible method to introduce skewness in multivariate symmetric distributions and applied their procedure to the multivariate student t density leading to a multivariate skew-student density. Massacci (2014) proposes a two-regime threshold model for the conditional distribution of stock returns whereby returns follow a distinct skewed student t distribution within each regime, i.e. the model enables the capturing of time variation in the conditional distribution of returns and its higher order moments. He finds that the model estimates conditional volatility more accurately and produces useful risk assessment as measured by the term structure of value-at-risk (VaR).

In this paper, models used for estimating the volatility of financial assets are surveyed with particular emphasis on recent developments related to empirical finance. We begin with an overview of some of the most significant theoretical contributions in the parameterisation and development of volatility models and a survey of the extensive empirical applications using financial data. Specifically, this paper is structured as follows: Section two outlines univariate volatility models, while Section three presents nonlinear ARCH/GARCH volatility models ranging from MS-type models to time-varying volatility models. Section four reviews the development of SV models, while multivariate volatility models are discussed in Section five. The RV models and forecasting techniques for volatility models are surveyed in Sections six and seven respectively. In Section eight, we present and survey several applications of volatility models to financial returns and the volatility impulse response function (VIRF) methodology is discussed in Section nine. Section ten concludes.

2 Univariate Volatility Models

2.1 Exponentially Weighted Moving Average (EWMA) Model

The exponentially weighted moving average (EWMA) model is essentially a simple extension of the historical average volatility measure, which allows more recent observations to have a stronger impact on the forecast of volatility than older data points (EWMA smoothly downweights the more distant data) (Brooks, 2008). In an EWMA specification, the latest observations decline exponentially over time. The EWMA model [which in effect is a restricted version of Engle's (1982) ARCH (∞) model] can be expressed in several ways, for example

$$h_t=(1-\lambda)\sum_{j=1}^{\infty}\lambda^j(r_{t-j}-\bar{r})^2, \tag{1}$$

where h_t is the variance estimate for period t (which also becomes the forecast of future volatility for all period), $\bar{r}$ is the average return estimated over the observations and λ is the decay factor which determines how much weight is given to recent versus older observations. As recommended by Riskmetrics, λ should be set at 0.94 in the event of analysing daily data. There are several methods that have been proposed in the literature in computing the EWMA, but the main point is that when the infinite sum in eqn. (1) is replaced with a finite sum of observable data, the weights from the given expression will now sum to less than 1. The infinite sum can be eliminated by substitution to give $h_t=(1-\lambda)h_{t-1}+\lambda r(t)^2$ which is a weighted average of last period's volatility and this period's squared return.

2.2 Autoregressive Conditional Heteroscedasticity (ARCH) Models

Engle's (1982) ARCH specification, is considered in many studies as the first formal volatility model with application to finance where $y_t=g(x_t\,;\,\Theta)+\varepsilon_t$ with $g(x_t\,;\,\Theta)$ as the conditional mean and ε_t as the error term. The conditional variance is $h_t=var(\varepsilon_t|\mathcal{F}_{t-1})$ where $\mathcal{F}_{t-1}$ is the information set. The ARCH model's variance equation is given by

$$h_t=\alpha_0+\sum_{j=1}^{q}\alpha_j\varepsilon_{t-j}^2,\quad \varepsilon_t=z_t\sigma_t,\quad z_t\sim i.i.d.(0,\ 1), \tag{2}$$

where $\alpha_0>0$, $\alpha_j\geq 0$, $j=1, 2, \ldots, q-1$ and $\alpha_q>0$. The mean is given by α_0, while α_j is the ARCH coefficient. The parameter restrictions form a necessary and sufficient condition for a positive conditional variance. The GARCH (p, q) model proposed independently by both Bollerslev (1986) and Taylor (1986) has the conditional variance of the form

$$h_t=\alpha_0+\sum_{j=1}^{q}\alpha_j\varepsilon_{t-j}^2+\sum_{j=1}^{p}\beta_j h_{t-j}, \tag{3}$$

A sufficient condition for the conditional variance to be positive with probability 1 is $\alpha_0>0$, $\alpha_j\geq 0$, $j=1, 2, \ldots, q-1$; $\alpha_q>0$; $\beta_j\geq 0$, $j=1, \ldots, p-1$; $\beta_p>0$. The model is specified as a function of three terms: α_0, ε_{t-j}^2 (ARCH term) and h_{t-j} (GARCH term). The persistence of h_t is captured by $\alpha+\beta$ and covariance stationarity requires that $\alpha+\beta<1$, while the unconditional variance is equal to $\alpha/(1-\sum_{j=1}^{q}\alpha_j+\sum_{j=1}^{p}\beta_j)$. Several generalisations and extensions of GARCH models were proposed to accommodate asymmetric effects. Some of the influential models include the GJR-GARCH model of Glosten *et al*. (1993) and Sentana's (1995) *quadratic* GARCH. The GJR-GARCH model is expressed as

$$h_t=\alpha_0+\sum_{j=1}^{q}[\alpha_j+\delta_j I(\varepsilon_{t-j}<0)]\varepsilon_{t-j}^2+\sum_{j=1}^{p}\beta_j h_{t-j}, \tag{4}$$

The GJR-GARCH model is also referred to as *threshold* GARCH or TGARCH model in the literature. The asymmetric GARCH (AGARCH) model of Ding *et al*. (1993) is specified as

$$h_t=\alpha_0+\sum_{j=1}^{q}\alpha_j(|\varepsilon_{t-j}|-\delta_j\varepsilon_{t-j})^2+\sum_{j=1}^{p}\beta_j h_{t-j}, \tag{5}$$

Studies such as Schwert (1990) have suggested modelling the conditional standard deviation instead of the conditional variance, while Zakoian (1994) considers the asymmetric version of these specifications with further generalisations by Ding *et al*. (1993). Taylor (1986) and Schwert (1990) propose a power GARCH (PGARCH) model given by

$$h_t^k=\alpha_0+\sum_{j=1}^{q}\alpha_j|\varepsilon_{t-j}|^{2k}+\sum_{j=1}^{p}\beta_j h_{t-j}^k,\quad k>0, \tag{6}$$

where k is the parameter to be estimated. The EGARCH model introduced by Nelson (1991) allows for asymmetric effects between positive and negative asset returns. The specification for conditional variance is expressed as

$$\ln h_t=\alpha_0+\sum_{i=1}^{q}\frac{\alpha_i|\varepsilon_{t-i}|+\gamma_i\varepsilon_{t-i}}{\sigma_{t-i}}+\sum_{j=1}^{p}\beta_j\ln h_{t-j}, \tag{7}$$

where the γ captures asymmetry, if $\gamma\neq 0$. The EGARCH is covariance stationary provided $\sum_{j=1}^{p}\beta_j<1$ (Zivot, 2009). If parameters of GARCH models are restricted to sum to 1 and the constant term is dropped, it gives the integrated GARCH (IGARCH) model given by

$$h_t=\sum_{j=1}^{q}\alpha_j\varepsilon_{t-j}^2+\sum_{j=1}^{p}\beta_j h_{t-j}. \tag{8}$$

Diebold (2004) states that the IGARCH is to GARCH what ARIMA is to ARMA although with other interesting twists. Engle *et al*. (1987) extend the basic ARCH model so that the conditional volatility can generate a risk premium which is part of the expected returns. The model is known as ARCH-in-the-mean (ARCH-M) model. The GARCH-M model extends the conditional mean equation $[y_t = g(x_t;\Theta) + \varepsilon_t]$ to include additional regressor $g(\sigma_t)$ which can be an arbitrary function of conditional volatility σ_t. The most common specifications are $g(\sigma_t) = \sigma_t^2$, σ_t or $\ln(\sigma_t^2)$ (Zivot, 2009). Exogenous independent variables may also be added to the conditional variance specification in eqn. (3) just as exogenous variables may be added to the conditional mean equation as in the case of the GARCH-M model. This may be represented by the specification in eqn. (9) below

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} + \sum_{k=1}^{K} \delta_k' z_{t-k}, \tag{9}$$

where z_t is a $m \times 1$ vector of variables and δ is a $m \times 1$ vector of positive coefficients. In the mid-1990s, a new class of fractionally integrated (FIGARCH) model was proposed by Baillie *et al.* (1996) aimed at explaining volatility and long memory in financial market volatility. The empirical evidence from the model estimations by Bollerslev and Mikkelsen (1996) suggest that the long-run dependence in US stock market volatility is best described by a mean-reverting fractionally integrated process, so that a shock to the optimal forecast of the future conditional variance dissipates at a slow hyperbolic rate. The starting point for the FIGARCH model is from eqn. (10) below whereby

$$(1-L)\varepsilon_{t-j}^2 = \alpha_0 + \nu_t - \beta_1 \nu_{t-1}. \tag{10}$$

The FIGARCH (1, d, 0) model is obtained from eqn. (10) by replacing the difference operator by a fractional difference operator yielding eqn. (11) below

$$(1-L)^d \varepsilon_{t-j}^2 = \alpha_0 + \nu_t - \beta_1 \nu_{t-1}, \tag{11}$$

where typically $0 < d < 1$. The FIGARCH model can be written as an infinite-order ARCH model by applying the definition $\nu_t = \varepsilon_t^2 - h_t$ to it. This yields

$$h_t = \alpha_0 (1-\beta_1)^{-1} + \lambda(L)\varepsilon_t^2, \tag{12}$$

where $\lambda(L) = [1-(1-L)^d(1-\beta_1 L)^{-1}]\varepsilon_t^2 = \sum_{j=1}^{\infty} \lambda_j L^j \varepsilon_t^2$ and $\lambda_j \geq 0$ for all j. Expanding the fractional difference operator into an infinite sum yields the result that for long lags j,

$$\lambda_j = [(1-\beta_1)\Gamma(d)^{-1}] j^{-(1-d)} = c j^{-(1-d)}, \quad c > 0, \tag{13}$$

where $\Gamma(d)$ is the gamma function. From eqn. (13), it is seen that the effect of ε_{t-j}^2 on the conditional variance h_t decays hyperbolically as a function of the lag length j. This is why the FIGARCH model was introduced as it would conveniently explain the apparent long memory in autocorrelation functions of squared observations of many daily return series.

3 Nonlinear GARCH Volatility Models

3.1 Markov-switching ARCH/GARCH Volatility Models

Hamilton and Susmel (H&S) (1994) propose an MS model whereby instead of the lagged variance term providing strong connection for volatility from one period to the next, an MS model govern switches between several variance regimes. The H&S mean model which is assumed to be fixed among regimes is given by $y_t = \alpha + \beta y_{t-1} + \varepsilon_t$, where $E\varepsilon_t^2 \equiv h_t = a_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$. In the H&S model, what switches is a variance inflation factor, which gives the formula $E\varepsilon_t^2 = g(S_t)\left(a_0 + \sum_{i=1}^{q} \frac{a_i(\varepsilon_{t-i}^2)}{g(S_{t-1})}\right)$. H&S also include an asymmetry term given by $E\varepsilon_t^2 = g(S_t)\left(a_0 + \xi \frac{(\varepsilon_{t-i}^2)}{g(S_{t-1})}(\varepsilon_{t-1} < 0) + \sum_{i=1}^{q} \frac{a_i(\varepsilon_{t-i}^2)}{g(S_{t-1})}\right)$ and use conditional student's t errors. Subsequently, Cai (1994) proposes a SWARCH model by replacing a_0 with $a(S_t)$ given by

$$h_t = a(S_t) + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \quad \alpha_i \geq 0, \quad \varepsilon_t = u_t \sqrt{h_t}, \quad u_t \sim N(0, 1), \tag{14}$$

where $a(S_t) = a_0 + a_1 S_t$, $a_0 > 0$, $a_1 > 0$. Dueker (1997) extends their approach to GARCH models whereby given $y_t = \mu_t + \varepsilon_t$, which assumes a student's t distribution with n_t degrees-of-freedom (df) in y, $\varepsilon_t \sim (0, n_t, h_t)$ $n_t > 2$. The conditional mean, μ_t is allowed to switch according to a two-state Markov process governed by a state variable, $S_t: \mu_l S_t + \mu_h(1 - S_t)$, $S_t \in (0, 1)$ for all t. Let $\mathbf{y}_t$ be a vector of observed variables and let S_t denote an unobserved random variable that can take on the values 1, 2, ..., or K. Suppose that S_t can be described by a Markov process transition probability given by

$$\begin{aligned} &\text{Prob}(S_t = j \mid S_{t-1} = i, \ S_{t-2} = k, \ldots, \ \mathbf{y}_{t-1}, \ \mathbf{y}_{t-2}, \ldots), \\ &= \text{Prob}(S_t = j \mid S_{t-1} = i) = p_{ij}, \end{aligned} \tag{15}$$

for $i, j = 1, 2, \ldots, K$ (see, Hamilton and Susmel, 1994). The transition probabilities can be collected in a $(K \times K)$ matrix expressed as

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{K1} \\ p_{12} & p_{22} & \cdots & p_{K2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{1K} & p_{2K} & \cdots & p_{KK} \end{bmatrix}.$$

Note that empirically, each column of $\mathbf{P}$ sums to 1. Dueker's (1997) MS-GARCH model which enhances connection between the state variable and the mean return, variance and kurtosis takes the form in eqn. (16) below

$$h_t^{(j)} = a_0 + \alpha(S_{t-1} = j)(\varepsilon_{t-1}^{(j)})^2 + \beta(S_{t-1} = j)\hat{h}_{t-1}. \tag{16}$$

The GARCH analog to Cai's model which assumes Markov switching in intercept is given by

$$h_t^{(i,j)} = a_0(S_t = i) + \alpha(\varepsilon_{t-1}^{(j)})^2 + \beta h_{t-1}^{(j)} \tag{17}$$

with constants α and β. In practice, $a_0(S_t)$ is parameterised as $g(S_t)a_0$ where $g(S=1)$ is normalised to unity. An MS analog to Hansen's (1994) model in which the variance follows a GARCH process and the student's t df parameter (n_t) is allowed to switch is given in eqn. (18). The n_t follows a Markov process governed by $S_t : n_l S_t + n_h(1-S_t)$. The GARCH process is still a function of the state variable although n_t does not enter eqn. (16) as switching in the mean implies that ε_t is a function of the state variable. Since the kurtosis of a student's t random variable equals $3(n_t-2)/(n_t-4)$ the model below is referred to as GARCH with switching in the conditional kurtosis (GARCH-K) given by

$$h_t^{(j)} = a_0 + \alpha(\varepsilon_{t-1}^{(j)})^2 + \beta \hat{h}_{t-1}. \tag{18}$$

Another specification similar to GARCH-K is the GARCH-DF(GARCH with switching in the df parameter) whereby the variance is assumed to be $\sigma_t^2 = h_t n_t/(n_t-2)$ rather than $\sigma_t^2 = h_t$. It is expressed in eqn. (19) by

$$h_t^{(j)} = a_0 + \alpha(1-2v_{t-1}^{(j)})(\varepsilon_{t-1}^{(j)})^2 + \beta \hat{h}_{t-1}. \tag{19}$$

From eqn. (19) the GARCH process scales the variance of ε_t for a given value of the shape parameter when we define $v_t = 1/n_t$ so that $(1-2v_t) = [(n_t-2)/n_t]$ yields the GARCH-DF equation. With the GARCH-K model, the GARCH-DF model shares the features of time-varying conditional kurtosis so that the kurtosis is not assumed to be constant. The SWARCH model with a leverage effect (SWARCH-L) as in H&S (1994) is given in eqn. (20). The model has a switching in a normalisation factor in variance i.e. $\sigma_t^2 = g_t h_t$ where h_t follows an ARCH (2) process with a leverage effect expressed as

$$h_t^{(j,k)} = a_0 + \frac{(\alpha_1 + \xi D_{t-1}^{(j)})}{g(S_{t-1}=j)}(\varepsilon_{t-1}^{(j)})^2 + \frac{\alpha_2}{g(S_{t-2}=k)}(\varepsilon_{t-2}^{(k)})^2, \tag{20}$$

where $D_{t-1}^{(j)}$ is a dummy variable that equals 1 when $\varepsilon(S_{t-1}=j)_{t-1} < 0$. The leverage effect parameter ξ is expected to have a positive sign. Another influential MS-GARCH model is Gray's (1996) specification which has been used to analyse stock returns volatility by e.g. Babikir *et al*. (2010). Based on empirical evidence, it has been shown that the SWARCH model of HS produces more robust and stable parameter estimates than Gray's (1996) MS-GARCH model (see, King and Botha, 2015). A nonlinear version of the GJR-GARCH model can be obtained by making the transition between regimes smooth. A smooth transition GARCH (STGARCH) model may be defined as

$$h_t = \alpha_{10} + \sum_{j=1}^{q} \alpha_{1j}\varepsilon_{t-j}^2 + (\alpha_{20} + \sum_{j=1}^{q} \alpha_{2j}\varepsilon_{t-j}^2) G(\gamma, \mathbf{c}; \varepsilon_{t-j}) + \sum_{j=1}^{p} \beta_j h_{t-j}^f, \tag{21}$$

where the transition function

$$G(\gamma, \mathbf{c}; \varepsilon_{t-j}) = [1 + \exp(-\gamma \Pi_{k=1}^{K}(\varepsilon_{t-j} - c_k))]^{-1}, \ \gamma > 0. \tag{22}$$

When $K=1$, eqn. (22) becomes a simple logistic function that controls the change of the intercept from α_{10} to $\alpha_{10}+\alpha_{20}$ and the coefficient of ε_{t-j}^2 from α_{1j} to $\alpha_{1j}+\alpha_{2j}$ as a function of ε_{t-j}. Zivot

(2009) state that often, a GARCH model with non-normal error distribution is required to fully capture the observed fat-tails in returns. Accordingly, it is expected that with time-varying volatility, there will be changing distributions and as a result MS models should perform better than simple data partitions based on thresholds. Studies have argued that what appears to be fat tails in the full sample may in fact be an artefact of the attempt to model two or more distinct regimes with a single distribution (Hamilton and Susmel, 1994).

3.2 Time-Varying GARCH Volatility Models

One common argument in the finance literature is that in empirical applications, the assumptions of constant parameter GARCH models may not be appropriate when the series to be modelled are quite long. One possibility is to assume that the parameters change at specific points of time, divide the series into subseries according to the location of the break-points and fit seperate GARCH models to the subseries. The main statistical challenge is then finding the number of break-points and their location because they are often not known in advance. It is equally possible to model the switching standard deviation regimes using the TGARCH model. This is done by assuming that the threshold variable is the time. Another possibility is to modify the STGARCH model to fit the situation. This has been implemented by defining the transition function as a function of time as follows

$$G(\gamma, \mathbf{c}; t^*)=[1+\exp(-\gamma\Pi_{k=1}^{K}(t^*-c_k))]^{-1}, \quad \gamma>0, \tag{23}$$

where $t^*=t/T$. Standardising the time variable between 0 and 1 makes interpretation of the parameters c_k, $k=1, \ldots, K$, easy as they indicate where in relative terms the changes in the process occur. The time-varying GARCH (TV-GARCH) model then takes the form

$$h_t=\alpha_0(t)+\sum_{j=1}^{q}\alpha_j(t)\varepsilon_{t-j}^2+\sum_{j=1}^{p}\beta_j(t)h_{t-j}, \tag{24}$$

where $\alpha_0(t)=\alpha_{01}+\alpha_{02}G(\gamma, \mathbf{c}; t^*)$, $\alpha_j(t)=\alpha_{j1}+\alpha_{j2}G(\gamma, \mathbf{c}; t^*)$, $j=1, \ldots, q$ and $\beta_j(t)=\beta_{j2}G(\gamma, \mathbf{c}; t^*)$, $j=1, \ldots, p$ [i.e. $\alpha_0(t)$, $\alpha_j(t)$ and $\beta_j(t)$ are all functions of time]. Some of the time-varying parameters in eqn. (24) may be restricted to constants *a priori*. For example, it may be assumed that only intercept $\alpha_0(t)$ is time-varying, which implies that the unconditional variance is changing over time. Baillie and Morana (2009) recently generalised the FIGARCH model in this way. Č îžek and Spokoiny (2009) offer a new method of estimating and forecasting TV-G(ARCH) models including global parametric, smooth transition and change point models as special cases.

4 Stochastic Volatility (SV) Models

SV models have been extensively used within the fields of financial engineering and mathematical finance to capture the impact of time-varying volatility on financial markets. Shephard and

Andersen (2009) note the highly multidisciplinary nature of the development of SV models. Unlike the ARCH processes which explicitly model the conditional variance of returns given observed past returns with the likelihood function easily delivered, in the SV approach the predictive distribution of returns is specified indirectly through the structure of the model. Some of the earlier contributions to the SV literature began in the continuous-time framework through the It'o stochastic integral equation given by

$$M_t=\int_0^t \sigma_s dW_s, \tag{25}$$

where the non-negative spot volatility σ is assumed to have cǎdlǎg sample paths. This however, allows for jumps in the volatility process and the SV model in eqn. (25) has continuous sample paths even if σ does not. A necessary and sufficient condition for M to constitute a martingale is that $E\sqrt{\int_0^t \sigma_s^2 ds}<\infty$ (see, Shephard and Andersen, 2009). The squared volatility process is often termed the spot variance. When σ and W are independent, we obtain the crucial simplification that $M_t\left|\int_0^t \sigma_s^2 ds \sim N\left(0,\ \int_0^t \sigma_s^2 ds\right)\right.$. The directing process is known as integrated variance defined as $IV_t=\int_0^t \sigma_s^2 ds$ and arises naturally as a quantity of major interest in empirical applications. Hull and White (1987) allow the spot volatility process to follow a general diffusion. In their method, the spot variation process is given as a solution to a univariate stochastic differential equation expressed as

$$d\sigma^2=\alpha(\sigma^2)dt+\omega(\sigma^2)dB, \tag{26}$$

where B is a second Brownian motion and $\alpha(.)$ and $\omega(.)$ are deterministic functions which can be specified quite generally but must ensure that σ^2 remains strictly positive. Wiggins (1987) starts from eqn. (26) but then focused on the special case where log volatility follows a Gaussian Ornstein-Uhlenbeck process,

$$d\log\sigma^2=\alpha(\mu-\log\sigma^2)dt+\omega dB,\quad \alpha>0. \tag{27}$$

The SV model often used in estimating unobserved volatility and with a different data generating process compared to the observation-driven GARCH models can be expressed as

$$\begin{cases} y_t=\mu+\sqrt{h_t}\varepsilon_t, & \varepsilon_t\sim N(0,\ 1),\ t=1,\ldots,\ T, \\ \log h_t=\alpha+\phi\log h_{t-1}+\eta_t, & \eta_t\sim N(0,\ \sigma_\eta^2),\ |\phi|\leq 1, \end{cases} \tag{28}$$

where ϕ measures persistence in volatility. The disturbances ε_t and η_t are mutually and serially uncorrelated, where $E(\log\varepsilon_t^2)=-1.27$, $var(\log\varepsilon_t^2)=\pi^2/2\cong 4.93$ and the observation error $(\log\varepsilon_t^2)$ is not standard normal but a $\log\chi^2$ distributed error. From eqn. (28) which is a nonlinear model as both h_t and ε_t are stochastic, the unconditional mean of the process is defined as $\alpha_h=(1-\phi)^{-1}\alpha$, while the unconditional variance is $\sigma_h{}^2=(1-\phi^2)^{-1}\sigma_\eta{}^2$. The measurement equation describes the relationship between the observations and the latent factors, and the state equation describes the dynamic properties of the latent factors while the relative variance is given by $\psi=\log(\sigma_\eta^2/\sigma_\varepsilon^2)$. The leverage effect can be incorporated into the SV model by allowing correlation between the innovations of the state and the observation equation. The SV model with leverage and based on an AR(1) process for log volatility can be represented by

$$y_t=\sigma\exp\left(\frac{1}{2}h_t\right)\varepsilon_t,\quad h_{t+1}=\phi h_t+\eta_t,\quad \begin{pmatrix}\varepsilon_t\\ \eta_t\end{pmatrix}\sim N\left((0,\ \begin{bmatrix}1 & \sigma_\eta\rho\\ \sigma_\eta\rho & \sigma_\eta^2\end{bmatrix}\right) \tag{29}$$

for $t=1,\ldots,T$. The correlation coefficient ρ is typically negative, implying that negative shocks in the return are accompanied by positive shocks in the volatility and vice versa. Koopman and Hol-Uspensky (2002) propose capturing the volatility feedback effect by including volatility as a regression effect in the mean, known as SV-in-mean (SVM) model given in eqn. (30) below

$$\mu_t=a+by_{t-1}+d\sigma\quad\exp\left(\frac{1}{2}h_t\right), \tag{30}$$

where a, b, d and σ^2 are parameters. The volatility feedback effect coefficient d is typically negative, if not zero. The observation density for eqn. (30) is given by

$$\log p(y_t|h_t)=-\frac{1}{2}\log 2\pi-\frac{1}{2}\log\sigma^2-\frac{1}{2}h_t-\frac{(y_t-a-by_{t-1}-d\sigma_t^2)^2}{2\sigma_t^2}. \tag{31}$$

The conditional variances for ARCH models are specified as a function of past squared innovations and lagged conditional variances, while SV models' variances are modelled to follow some unobserved stochastic process. Although the likelihood functions of ARCH models could be readily derived, the estimation techniques proposed for SV models are often more computationally involved ranging from quasi-maximum likelihood methods to a variety of simulation techniques. For an overview of SV techniques and their connection to continuous-time option pricing models, see Tsay (2005), Shephard and Andersen (2009), Teräsvirta *et al*. (2010), etc.

The main distinction between GARCH and SV models is that SV models have separate disturbance terms in the mean and variance equations, precluding direct observation of the variance process (Koopman and Uspensky, 2002). One of the difficulties in applications of SV based models is that compared with their ARCH cousins, they are hard to estimate efficiently due to the unobservable nature of the volatility state variable. More recent papers resort to Bayesian methods in order to efficiently estimate SV model parameters. Kim *et al*. (1998) used MCMC methods to provide a likelihood-based framework for SV models and find that the simple SV model fits the data as well as more parameterised GARCH models.

5 Multivariate GARCH Volatility Models

As highlighted in the previous sections, the pioneering articles on ARCH by Engle (1982) and

Bollerslev (1986) introduced univariate models while subsequent extensions aimed at greater flexibility were proposed by Nelson (1991), Glosten *et al*. (1993) and Baillie *et al*. (1996) among others. The "second generation" studies extend the models to a multivariate setting starting with Bollerslev *et al.'s* (1988) diagonal VECH (DVECH), leading to further extensions such as Bollerslev's (1990) constant conditional correlation-GARCH (CCC-GARCH), Engle and Kroner's (1995) Baba-Engle-Kraft-Kroner-GARCH (BEKK-GARCH), Engle's (2002) DCC, Tse and Tsui's (2002) TVCC-GARCH and McAleer *et al.'s* (2009) VARMA-AGARCH models, etc. For the multivariate case, suppose $\varepsilon_t=(\varepsilon_{1t}, \ldots, \varepsilon_{mt})'$ such that $E\varepsilon_t=\mathbf{0}$ and $E(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1})=\mathbf{H}_t=[h_{ij,t}]$ where $\mathbf{H}_t$ is positive definite and $\varepsilon_t=\mathbf{H}_t^{1/2}z_t$ with $\mathbf{z}_t\sim i.i.d.(\mathbf{0}, \mathbf{I}_m)$. The conditional variance equation in the spirit of Bollerslev *et al*. (1988) takes the form

$$vech(\mathbf{H}_t)=\mathbf{C}+\sum_{k=1}^{q}\mathbf{A}_k vech(\varepsilon_{t-k}\varepsilon'_{t-k})+\sum_{k=1}^{p}\mathbf{B}_k vech(\mathbf{H}_{t-k}), \tag{32}$$

where vech denotes the half-vectorisation operator which stacks the columns of a square matrix from the diagonal downwards in a vector while the vectorisation operator vec stacks the whole columns. $\mathbf{A}_k=[\alpha_{kij}]$ and $\mathbf{B}_k=[\beta_{kij}]$ are coefficient matrices with $m(m+1)/2$ rows and columns, and $\mathbf{C}$ is an $[m(m+1)/2]\times 1$ intercept vector with positive elements. The diagonal vech uses only the diagonal elements of $\mathbf{A}$ and $\mathbf{B}$, and sets all values of $\alpha_{ij}=\beta_{ij}=0$, for $i\neq j$. The CCC-GARCH model has the following structure: $\mathbf{H}_t=\mathbf{D}_t\mathbf{R}\mathbf{D}_t$ where $\mathbf{D}_t=diag(h_{11t}^{1/2}, \ldots, h_{mmt}^{1/2})$ and $\mathbf{R}=[\rho_{ij}]$ is the correlation matrix, so that $h_{ijt}=\rho_{ij}(h_{iit}h_{jjt})^{1/2}$ for $i, j=1, \ldots, m$. Each conditional variance h_{jjt}, $j=1, \ldots, m$, is assumed to follow a basic univariate GARCH model. The CCC-GARCH model assumes that the conditional variances of each return, h_{it}, $i=1, \ldots, m$, follows a GARCH process given by

$$h_{it}=\alpha_0+\sum_{j=1}^{q}\alpha_{ij}\varepsilon_{i,t-j}^2+\sum_{j=1}^{p}\beta_{ij}h_{i,t-j}. \tag{33}$$

Due to limitations of the CCC-GARCH models which presumes that the conditional variances are independent across returns and their neglect of asymmetric behaviour, Ling and McAleer (2003) propose a VARMA model of the conditional mean defined by

$$\Phi(L)(Y_t-\mu)=\Psi(L)\varepsilon_t, \quad \varepsilon_t=D_t\eta_t, \tag{34}$$

where $\Phi(L)=I_m-\Phi_1L-\ldots-\Phi_pL^p$ and $\Psi(L)=I_m-\Psi_1L-\ldots-\Psi_qL^q$ are polynomials in L, I_K is the $k\times k$ identity matrix with conditional variance of the VARMA-GARCH defined as

$$H_t=C+\sum_{i=1}^{q}A_i vec(\varepsilon)_{t-i}+\sum_{i=1}^{p}B_jH_{t-j}, \tag{35}$$

where $H_t=(h_{1t}, \ldots, h_{mt})'$, $vec(\varepsilon)_t=(\varepsilon_{1t}^2, \ldots, \varepsilon_{mt}^2)'$ and C, A_i for $i=1, \ldots, q$ and B_j for $j=1, \ldots, p$ are $m\times m$ matrices. The VARMA-GARCH model assumes that shocks (positive or negative), have identical impacts on conditional variance. To address this shortcoming, McAleer *et al*. (2009) propose a VARMA-AGARCH specification for the conditional variance expressed as

$$H_t = C + \sum_{i=1}^{q} A_i vec(\varepsilon)_{t-i} + \sum_{i=1}^{K} D_i I_{t-i} vec(\varepsilon)_{t-i} + \sum_{j=1}^{p} B_j H_{t-j}, \tag{36}$$

where D_i are $m \times m$ matrices for $i=1, \ldots, K$ and $I_t = diag(I_{1t}, \ldots, I_{mt})$, where

$$I_t = \begin{cases} 0, & \varepsilon_{it} > 0 \\ 1, & \varepsilon_{it} \leq 0, \end{cases} \tag{37}$$

I_t is an indicator function. VARMA-AGARCH reduces to VARMA-GARCH when $D_i=0$ for all i. If $D_i=0$, A_i and B_j are diagonal matrices for all i and j, then VARMA-GARCH reduces to CCC-GARCH model (Allen *et al*., 2011). For the BEKK-GARCH model, its main advantage is that its conditional variance matrices are always positive definite. It is given by

$$\mathbf{H}_t = (\mathbf{C}_0^*)'\mathbf{C}_0^* + \sum_{k=1}^{K}\sum_{j=1}^{q} (\mathbf{A}_{jk}^*)'(\varepsilon_{t-j}\varepsilon_{t-j}')\mathbf{A}_{jk}^* + \sum_{k=1}^{K}\sum_{j=1}^{p} (\mathbf{B}_{jk}^*)'\mathbf{H}_{t-j}\mathbf{B}_{jk}^*, \tag{38}$$

where $\mathbf{C}_0^* = (\mathbf{c}_{01}', \ldots, \mathbf{c}_{0m}')'$ is an $m \times m$ triangular matrix, $\mathbf{A}_{jk}^* = [\alpha_{jk,il}^*]$ and $\mathbf{B}_{jk}^* = [\beta_{jk,il}^*]$ are $m \times m$ coefficient matrices. $(\mathbf{C}_0^*)'\mathbf{C}_0^*$ is positive definite when $\mathbf{C}_0^*$ is of full rank (Teräsvirta *et al*., 2010). If $K=p=q=1$, eqn. (38) becomes

$$\mathbf{H}_t = (\mathbf{C}_0^*)'\mathbf{C}_0^* + (\mathbf{A}^*)'(\varepsilon_{t-1}\varepsilon_{t-1}')\mathbf{A}^* + (\mathbf{B}^*)'\mathbf{H}_{t-1}\mathbf{B}^*. \tag{39}$$

The elements of $\mathbf{A}^*$ capture the effects of shocks on volatility while the elements of $\mathbf{B}^*$ capture the effects of past conditional variances measuring the diagonal parameters of the effects of past own shocks and past volatility in both cases (see, Miralles-Marcelo *et al*., 2013). The number of parameters in the BEKK (1, 1, 1) is $N(5N+1)/2$. Considering a bivariate case where $m=2$, the elements of $\mathbf{H}_t$ in eqn. (39) are

$$\begin{aligned} h_{11,t} &= \mathbf{c}_{01}'\mathbf{c}_{01} + (\alpha_{11}^{*2}\varepsilon_{1,t-1}^2 + \alpha_{21}^{*2}\varepsilon_{2,t-1}^2 + 2\alpha_{11}^*\alpha_{21}^*\varepsilon_{1,t-1}\varepsilon_{2,t-1}) + \beta_{11}^{*2}h_{11,t-1} + \beta_{21}^{*2}h_{22,t-1} + 2\beta_{11}^*\beta_{21}^*h_{12,t-1}, \\ h_{12,t} &= \mathbf{c}_{01}'\mathbf{c}_{02} + \alpha_{11}^*\alpha_{12}^*\varepsilon_{1,t-1}^2 + (\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*)\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{12}^*\alpha_{22}^*\varepsilon_{2,t-1}^2 + \beta_{11}^*\beta_{12}^*h_{11,t-1} \\ &+ (\beta_{11}^*\beta_{22}^* + \beta_{12}^*\beta_{21}^*)h_{12,t-1} + \beta_{21}^*\beta_{22}^*h_{12,t-1}, \\ h_{22,t} &= \mathbf{c}_{02}'\mathbf{c}_{02} + (\alpha_{12}^{*2}\varepsilon_{1,t-1}^2 + \alpha_{22}^{*2}\varepsilon_{2,t-1}^2 + 2\alpha_{22}^*\alpha_{12}^*\varepsilon_{1,t-1}\varepsilon_{2,t-1}) + \beta_{12}^{*2}h_{1,t-1} + \beta_{22}^{*2}h_{22,t-1} + 2\beta_{22}^*\beta_{12}^*h_{12,t-1}. \end{aligned} \tag{40}$$

Findings from recent studies revealed that asymmetry is a common feature in the analysis of equity markets owing to the leverage effect property of stock market return series (Teräsvirta *et al*., 2010). In the analyis of DCC, Engle's (2002) DCC model which is a generalisation of Bollerslev's (1990) CCC estimator has recently been extended to incorporate asymmetry. In Bollerslev's CCC model, $\mathbf{H}_t = \mathbf{D}_t\mathbf{R}\mathbf{D}_t$, where $D_t = diag\{\sqrt{h}_{i,t}\}$ (the diagonal matrix) and $\mathbf{R}$ is the correlation matrix containing the conditional correlations that does not depend on t. Tse and Tsui (2002) proposed a model similar to Engle's (2002) DCC model known as the varying CC-GARCH (VCC-GARCH) model which is an extension of the CCC-MGARCH model by introducing time-variation in the correlation matrix. Engle's (2002) DCC model assumes $\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$. The estimated conditional correlations are obtained using $\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{iit}q_{jjt}}}$. The DCC model can be represented by the following specifications:

$$
\begin{aligned}
& r_t|\mathcal{F}_{t-1} \sim N(0,\ \mathbf{H}_t), \\
& \mathbf{H}_t = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t,\ \ \varepsilon_t = D_t^{-1} r_t, \\
& \mathbf{D}_t = diag(\sigma_{11,t}^{1/2},\ \ldots,\ \sigma_{kk,t}^{1/2}), \\
& \mathbf{Q}_t = (1-\alpha-\beta)\mathbf{R}_t + \alpha\varepsilon_{t-1}\varepsilon'_{t-1} + \beta\mathbf{Q}_{t-1}, \\
& \mathbf{R}_t = (diag\mathbf{Q}_t)^{-1/2}\mathbf{Q}_t(diag\mathbf{Q}_t)^{-1/2},
\end{aligned} \tag{41}
$$

where $\mathbf{H}_t$ is the conditional variance matrix and $\sigma_{ii,t}$ is specified as a univariate GARCH-type equation. The $\mathbf{Q}_t$ is the covariance matrix with typical elements $q_{ij,t}$ as a weighted average of a positive definite and a positive semidefinite matrix. The $\mathbf{R}_t$ is the $k\times k$ unconditional correlation matrix of ε_t, and α and β are positive scalar parameters satisfying $\alpha+\beta<1$ (Bauwens and Laurent, 2005).

5.1 Factor GARCH Volatility Models

Diebold (2004) argues that factor GARCH models are effectively models of cointegration in variance [i.e. they model common feature (persistence) in variance]. The key merit of the factor GARCH model is that it has an economic interpretation. Teräsvirta *et al*. (2010) state that theories of asset pricing suggests that the risk premium of an asset depends on its covariance with other assets as well as its own variance and the factor GARCH model represents those covariances parsimoniously making it possible to consider larger number of assets at the time than many other vector GARCH models would in practice. Consider the BEKK-GARCH model in eqn. (38) and assume that for each k, the matrices $\mathbf{A}^*_{jk}$ and $\mathbf{B}^*_{jk}$ in this model rank 1 and the same $m\times 1$ left and right eigenvectors $\mathbf{A}^*_{jk}=\alpha_k g_k \mathbf{f}'_k$ and $\mathbf{B}^*_{jk}=\beta_k g_k \mathbf{f}'_k$, with $g'_k\mathbf{f}_i=0$, $i\neq k$, and $g'_k\mathbf{f}_i=1$. Thus, the BEKK-GARCH model in eqn. (38) becomes

$$
\mathbf{H}_t = (\mathbf{C}_0^*)'\mathbf{C}_0^* + \sum_{k=1}^{K}\mathbf{g}_k\mathbf{g}'_k\left[\sum_{j=1}^{q}\alpha_{kj}^2(\mathbf{f}'_k\varepsilon_{t-j})^2 + \sum_{j=1}^{p}\beta_{kj}^2\mathbf{f}'_k\mathbf{H}_{t-j}\mathbf{f}_k\right], \tag{42}
$$

where $\mathbf{f}_k$ are the factor weights while c_k, α_{kj}^2 and β_{kj}^2 are the common conditional variances. This is referred to as the K-factor GARCH model. Each factor $\gamma_{kt}=\mathbf{f}'_k\varepsilon_t$ has a univariate GARCH (p, q) structure: $\gamma_{kt}|\mathcal{F}_{t-1}\sim\mathcal{D}(0,\ \mathbf{f}'_k\mathbf{H}_t\mathbf{f}_k)$ where D denotes a distribution, such that the conditional variance is given by

$$
\mathbf{f}'_k\mathbf{H}_t\mathbf{f}_k = c_k + \sum_{j=1}^{q}\alpha_{kj}^2(\mathbf{f}'_k\varepsilon_{t-j})^2 + \sum_{j=1}^{p}\beta_{kj}^2\mathbf{f}'_k\mathbf{H}_{t-j}\mathbf{f}_k, \tag{43}
$$

where $c_k=\mathbf{f}'_k(\mathbf{C}_0^*)'\mathbf{C}_0^*\mathbf{f}_k$. Specifying $\gamma_{kt}=z_{kt}(\mathbf{f}'_k\mathbf{H}_t\mathbf{f}_k)^{1/2}$, $k=1,\ \ldots,\ K$, where $z_{kt}\sim i.i.d.(0,\ 1)$ such that $Ez_{kt}z_{jt}=\omega_{kj}$, while γ_{kt} and γ_{jt} have a CCC ω_{kj}. The model (43) characterising ε_t is a linear combination of the K conditional variances of the univariate common factors γ_{kt}. The model is known as an observable factor model because the factors γ_{kt}, $k=1,\ \ldots,\ K$, are not unobservable variables and most often $\mathbf{f}_k$, $k=1,\ \ldots,\ K$, are assumed to be known. A key feature of the factor GARCH model is that the factors are generally correlated. On the contrary, uncorrelated factors would describe genuinely different common components driving the returns and for this reason be more economic than correlated ones. Another distinguishing feature of these models is related to whether the number of heteroscedastic factors is less than the number of assets or not. Other observable factor models are the generalised orthogonal (GO-) GARCH model of van der Weide (2002) which is an extension of the orthogonal GARCH model proposed by Alexander and Chibumba (1996). The generalised orthogonal factor (GOF-) GARCH model of Lanne and Saikkonen (2007) (whereby some factors can be conditionally homoscedastic) contains not only systematic but also idiosyncratic components of risk.

5.2 Estimation Methods and Approaches

5.2.1 Maximum Likelihood Estimation (MLE) Approach

Several studies have shown that financial return series exhibit excess kurtosis and fat tails to the extent that the assumption of normality becomes unrealistic. Since this is important in empirical finance, using a more appropriate distribution would help to account for excess kurtosis (see Bollerslev *et al*. 1992 and Engle, 2003). A number of studies consider both the student's t and normal distributions for the standardised residuals of returns innovations. For an MGARCH (1, 1) model which can be constructed by allowing all the volatility terms to interact with each other, the log likelihood function can be expressed as

$$\ln L = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln|H_t| - \frac{1}{2}\sum_{t=1}^{T}\varepsilon_t' H_t^{-1}\varepsilon_t, \tag{44}$$

where ε_t is a $k\times 1$ column vector, and (2π) a constant. To account for leptokurtosis, Bollerslev (1987) advocates the use of t-distribution. The density for random variable x is given by

$$f_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi(\nu-2)}}\left(1+\frac{x^2}{\nu-2}\right)^{-(\nu+1)/2}. \tag{45}$$

The joint density is $g(x) = \Pi_{i=1}^{N} f_{\nu_i}(x_i)$ which enables different df (ν_i) for each component x_i. Under this assumption, the conditional distribution of ε_t is $\varepsilon_t|\mathcal{F}_{t-1} \sim g(H_t^{-1/2}\varepsilon_t)\,|H_t^{-1/2}|$ and the contribution of ε_t to the log-likelihood is given by $l_t = \ln g(H_t^{-1/2}\varepsilon_t) + \ln|H_t^{-1/2}|$. For the SV model in eqn. (29), estimates can be obtained by treating ε_t as though it were Gaussian and maximising the resulting likelihood function (Harvey *et al*., 1994; Jungbacker and Koopman, 2009). The conditional log density $p(y_t|h_t)$ is then given by

$$\log p(y_t|h_t) = -\frac{1}{2}\log 2\pi - \frac{1}{2}h_t - \frac{1}{2}\exp(-h_t)(y_t-\mu)^2,\quad t=1,\ldots,\ T. \tag{46}$$

The unknown coefficients that need to be estimated are α, ϕ, and σ_η. In relation to a new class of GARCH model with multivariate skew t densities that allows for skewness in multivariate symmetric distributions. The standardised skew-student's t density proposed by Bauwens and Laurent (2005) is expressed as

$$f(\varepsilon|\xi,\ \nu)=\left(\frac{2}{\sqrt{\pi}}\right)^k\left(\Pi_{i=1}^k\frac{\xi_i s_i}{1+\xi_i^2}\right)\frac{\Gamma((\nu+k)/2)}{\Gamma(\nu/2)(\nu-2)^{k/2}}\left(1+\frac{\varepsilon^{*\prime}\varepsilon^*}{\nu-2}\right)^{-(k+\nu)/2}, \tag{47}$$

where $\varepsilon^*=(\varepsilon_1^*,\ldots,\ \varepsilon_k^*)'$, $\varepsilon_i^*=(s_i z_i+m_i)\xi_i^{I_i}$ and $m_i=\frac{\Gamma((\nu-1)/2)\sqrt{\nu-2}}{\sqrt{\pi}\Gamma(\nu/2)}\left(\xi_i-\frac{1}{\xi_i}\right)$. Furthermore, $s_i^2=\left(\xi_i^2+\frac{1}{\xi_i{}^2-1}\right)-m_i^2$ and $I_i=\begin{cases}-1 & \text{if } z_i\geq-\frac{m_i}{s_i}\\ 1 & \text{if } z_i<-\frac{m_i}{s_i}\end{cases}$. Bauwens and Laurent (2005) suggests that using a more appropriate distribution may lead to improved empirical modelling and financial decision making. The main drawback of normal or student's t distributions is that the density is symmetric whereas the distribution of financial returns is almost always skewed. Other multivariate asymmetric densities have been proposed in the literature see e.g. Vlaar and Palm (1993) for mixtures of multivariate normal densities, the multivariate skew-normal density of Azzalini and Dalla Valle (1996) and Barndoff-Nielsen and Shephard (BNS)(2001) in the case of the generalised hyperbolic distributions. For Engle's DCC model, the log likelihood for the estimators in eqn. (41) can be expressed as

$$\begin{aligned}
&r_t|\mathcal{F}_{t-1}\sim N(0,\ H_t),\\
&L=-\frac{1}{2}\Sigma_{t=1}^T(k\log(2\pi)+\log|H_t|+r_t'H_t^{-1}r_t),\\
&=-\frac{1}{2}\Sigma_{t=1}^T(k\log(2\pi)+\log|D_tR_tD_t|+r_t'D_t^{-1}R_t^{-1}D_t^{-1}r_t),\\
&=-\frac{1}{2}\Sigma_{t=1}^T(k\log(2\pi)+2\log|D_t|+\log|R_t|+\varepsilon_t'R_t^{-1}\varepsilon_t),\\
&=-\frac{1}{2}\Sigma_{t=1}^T(k\log(2\pi)+2\log|D_t|+r_t'D_t^{-1}D_t^{-1}r_t-\varepsilon_t'\varepsilon_t+\log|R_t|+\varepsilon_t'R_t^{-1}\varepsilon_t,
\end{aligned} \tag{48}$$

which according to Engle (2002) can simply be maximised over the parameters of the model. During periods of financial turmoil asset price volatilities often exhibit jumps and breaks leading to extreme values and distributions with fatter tails than that of a normal distribution. Given that the fourth order moment exists, Bollerslev (1986) shows that the kurtosis implied by a GARCH (1, 1) model with normal errors is greater than 3. Zivot (2009) observes that most often, a GARCH model with non-normal error distribution is required to fully capture the observed fat-tails in returns.

5.2.2 Bayesian and Markov Chain Monte Carlo Approach

As earlier stated recent studies now consider the analysis of returns volatility from a Bayesian perspective by means of simulation based on importance sampling. The likelihood function used in the simulation is defined as

$$L(\psi)=p(y|\psi)=\int p(y,\ \theta|\psi)d\theta=\int p(y|\theta,\ \psi)p(\theta|\psi)d\theta, \tag{49}$$

This requires a simulation device for sampling from an importance density $\tilde{p}(\theta|y, \psi)$ which relates to the true density $p(y|\theta, \psi)$. For the model with the parameter vector ψ, the Kalman filter and smoother provide posterior means, variances, and covariances of the state vector given the data. The likelihood function can be rewritten as

$$L(\psi)=\int p(y, \theta|\psi)\frac{p(\theta|\psi)}{g(\theta|y, \psi)}g(\theta|y, \psi)d\theta=\tilde{E}\left\{p(y, \theta|\psi)\frac{p(\theta|\psi)}{g(\theta|y, \psi)}\right\}, \tag{50}$$

where $\tilde{E}$ is expectation with respect to the importance density $g(\theta|y, \psi)$. The likelihood function associated with the importance density is given by

$$L_g(\psi)=g(\theta|\psi)=\frac{g(y, \theta|\psi)}{g(\theta|y, \psi)}=\frac{g(y|\theta, \psi)p(\theta|\psi)}{g(\theta|y, \psi)} \tag{51}$$

and it follows that $p(\theta|\psi)/g(\theta|y, \psi)=L_g(\psi)/g(\theta|y, \psi)$ and substitution results in $L(\psi)=L_g(\psi)\tilde{E}\left\{\frac{p(y|\theta, \psi)}{g(y|\theta, \psi)}\right\}$ which is the expression to be used in the calculation. For the SV model in eqn. (28), the autoregressive parameter ϕ is restricted to have a value between zero and one, and we estimate $\hat{\psi}_1$ where $\phi=\psi_1=\exp(\hat{\psi}_1)/1+\exp(\hat{\psi}_1)$, $\hat{\psi}_1=\ln(\phi/1-\phi)$. The linear approximating model is obtained with $\tilde{H}_t=2\sigma^2\exp(\tilde{\theta}_t)/y_t^2$, $y=\tilde{\theta}_t-1/2\tilde{H}_t+1$. In the Bayesian approach to SV modelling of volatility, several authors often consider the Gaussian log-density given by

$$\log L(y|\theta)=-\frac{T}{2}\log(2\pi)-\frac{1}{2}\sum_{t=1}^{T}\log F_t-\frac{1}{2}\sum_{t=1}^{T}\frac{\nu_t^2}{F_t}, \tag{52}$$

where $y=(y_1, \ldots, y_T)$, ν_t is the one-step-ahead prediction error for the best linear estimator of $\log y_t^2$ and F_t is the mean squared error.

6 Realised Volatility (RV) Modelling

The advent of commonly available high-frequency data has moved RV to the forefront of volatility research recently. RV offers an alternative way of measuring volatility. A related development is the rapidly accumulating theoretical and applied research on how to exploit this high-frequency data to estimate the increments of the quadratic variation (QV) process and then to use this estimate to project QV into the future in order to predict future levels of volatility. Influential studies in this field within the context of finance include BNS (2002a) and BNS (2002b).

A number of studies have argued that the squared daily return of an asset is a noisy measure of daily volatilty of the asset. This is because the daily return is a difference between two (log) prices, most often the closing prices of two consecutive days. That is why it has been suggested by some authors to be replaced by RV which is defined as the sum of squared intra-daily returns. Another intraday volatility measure is the intraday price range, i.e. the difference between the highest and lowest observed price within the day. Formally, let p_t be the (log) price of a financial asset at time t, and assuming we normalise the daily time interval to unity. The discrete price process of continuosly compunded returns per day is defined by

$$r_t = p_t - p_{t-1}, \quad t = 1, \ldots. \tag{53}$$

The estimator of daily RV is given by

$$RV_t = \sum_{j=1}^{M} r_{t,j}^2, \tag{54}$$

The instantaneous return is given by $dp_t = \sigma_t dw_t$. The continuos-time counterpart of the RV eqn. is defined as the integral over squared instantaneous return as follows

$$QV_t = \int_0^1 \sigma_{t+\tau}^2 d\tau, \tag{55}$$

This integral is called the daily QV. If the discrete returns are uncorrelated, it can be shown that $plim_{m\to\infty}(RV_t - QV_t) = 0$. This means that the daily RV_t is a consistent estimator of the daily QV (Teräsvirta *et al.*, 2010). Recently, studies such as Hansen *et al.* (2011) have introduced a new framework known as the realised GARCH for the joint modelling of returns and realised measures of volatility. The realised GARCH model which integrates the GARCH model with RV relates the realised measure to the conditional variance of returns.

7　Volatility Models Forecasting

There are many surveys in the literature that focus on the forecast performance of GARCH models ranging from Hansen and Lunde (2005), Liu and Hung (2010) to King and Botha (2015). Some of these surveys cover the mathematical and statistical properties of volatility models, while others focus on the empirical issues associated with estimating GARCH models and forecasting volatility. One of the main objectives of modelling conditional volatility is to generate accurate forecasts for both the future value of a financial time series and its conditional volatility. As already highlighted, volatility forecasts are used for option pricing, risk management, trading strategies and model evaluation. A number of studies have highlighted that by allowing for a time-varying conditional variance, GARCH models can generate accurate forecasts of future volatility, especially over the short horizons. Results from several empirical analyses have shown mixed results in this regard. In this subsection, we compare and contrast several volatility models with particular emphasis on their forecast performance and provides an overview of the most recent advances that are aimed at improving their forecast accuracy.

When we consider the basic GARCH (1, 1) model where $\varepsilon_t = z_t \sigma_t$ such that $z_t \sim i.i.d.(0, 1)$ and has a symmetric distribution. Assume the model is to be fitted over the period $t = 1, 2, \ldots, T$. The optimal, in terms of mean-squared error, forecast of h_{T+k} given information at time T is $E_T[h_{T+k}]$ and can be computed using a simple recursion. For $k = 1$,

$$E_T[h_{T+1}] = \alpha_0 + \alpha_1 E_T[\varepsilon_T^2] + \beta_1 E_T[h_T], \\ = \alpha_0 + \alpha_1 {\varepsilon_T}^2 + \beta_1 h_T, \tag{56}$$

where it is assumed that ε_T^2 and h_T are known. Similarly, for $k=2$

$$E_T[h_{T+2}] = \alpha_0 + \alpha_1 E_T[\varepsilon_{T+1}^2] + \beta_1 E_T[h_{T+1}], \\ = \alpha_0 + (\alpha_1 + \beta_1) E_T[h_{T+1}], \tag{57}$$

Since $E_T[\varepsilon_{T+1}^2] = E_T[z_{T+1}^2 h_{T+1}] = E_T[h_{T+1}]$. Generally, for $k \geq 2$

$$E_T[h_{T+k}] = \alpha_0 + (\alpha_1 + \beta_1) E_T[h_{T+k-1}], \\ = \alpha_0 \sum_{i=0}^{k-1} (\alpha_1 + \beta_1)^i + (\alpha_1 + \beta_1)^{k-1} (\alpha_1 \varepsilon_T^2 + \beta_1 h_T). \tag{58}$$

The forecasting algorithm above produces forecasts for the conditional variance h_{T+k}. The GARCH (1, 1) forecasting algorithm above is related closely to an EWMA of past values of ${\varepsilon_t}^2$ mainly used by RiskMetrics. GARCH-type models are often assessed by their out-of-sample forecasting ability. This forecast ability is often measured using traditional forecast error metrics and other criteria such as VaR violations, option pricing accuracy, or portfolio performance. In the literature, out-of-sample forecasts used in model comparison are typically computed using either of the two prominent methods. The first produces recursive forecasts while the second method produces rolling forecasts.

In the empirical context, Hansen and Lunde (2004) provide compelling evidence that it is difficult to find a volatility model that outperforms the simple GARCH (1, 1) model. West and Cho (1995) compare the out-of-sample forecasting performance of univariate GARCH models for conditional variances using 5 bilateral exchange rates for the dollar from 1973 to 1989. They find that for a one-week horizon, GARCH models tend to make slightly more accurate forecasts, while for longer horizons, it is difficult to find grounds for choosing between the various models. Recently, King and Botha (2015) investigate whether accounting for structural changes in the conditional variance process using MS models improve estimates and forecast of stock return volatility over those of single-state GARCH models within and across selected African markets. They find that univariate single-state GARCH models perform poorly in terms of forecasting compared with MS models while the MS models were shown to be unable to fully capture hetersocedasticity in the data.

Liu and Hung (2010) examine volatility forecasting for S&P-100 stock index from 1997 to 2003 and identify the essential source of performance improvements between distributional assumption and volatility specification using distribution and asymmetry-type models. Their results indicate that asymmetry-type models achieve more accurate volatility forecasts. Koopman *et al*. (2005) compare the predictive abilities of realized volatility models with those of SV and GARCH models for daily returns series. From the surveyed literature, the overall conclusion is that GARCH volatility models forecast results have been mixed and often do not forecast very well.

8 Applications of Volatility Models to Financial Returns

The global financial crisis (GFC) in the period 2007 to 2009 and its impact across financial markets has led to renewed interest in the analysis of volatility in asset prices. Most econometric models that attempt to describe the evolution of financial assets over time use return, instead of prices of assets due to its attractive statistical properties such as stationarity and ergodicity (Campbell *et al*., 1997; Tsay, 2005). There are several definitions of returns that range from simple return, log-return, absolute return to squared return and the actual time interval is vital in the analysis and comparison of returns. However, the basic patterns of simple and log returns tend to be similar. The most widely used tool in the analysis of volatility has been the GARCH models. The analysis of asset prices such as exchange rate, stock prices, bonds, house prices, commodities and derivative products prices as well as those of other markets such as insurance premiums are particularly significant in financial econometrics. In this section, we shall review some significant contributions in financial econometrics with emphasis on recent studies and survey the applications of volatility models in the analysis of financial returns with particular focus on stock returns, inflation and interest rate and exchange rate returns respectively.

8.1 Applications of Volatility Models to Stock Market Returns

As already highlighted, the application of volatility models to the analysis of financial returns has been quite successful and the analysis of stock return volatility has witnessed extensive applications of this models. It is established in finance literature that the variance of financial time series tends to change over time. That is why an assessment of dynamic volatilities is necessary to help analyse such changes. A common feature associated with stock returns is volatility clustering, asymmetry and fat tails and all these have important theoretical and empirical implications. In majority of the studies, very significant ARCH effects have been found in stock market returns (comprising both individual and market stock returns) (see, e.g. Engle, 2003; Babikir *et al*. 2012; King and Botha, 2015 etc.).

A common characteristic exhibited by financial time series are heavy tails, non-stationarity and time-varying volatility. Andersen *et al*. (2009) define non-stationarity as "the lack of reversion to a common value such as the mean of the series", while time-varying volatility refers "to a tendency of small values being followed by small values and large values being followed by large values". According to Bollerslev *et al*. (1992) most empirical implementations of GARCH (p, q) models adopt low orders for the lag lengths p and q which seems sufficient in modelling the variance dynamics of most stock market returns. On nonlinearities and GARCH-type models, when applying the BDS test, the literature tends to find that once ARCH effects are removed the BDS test on standardised residuals exhibit little evidence of nonlinear dependence, indicating that most of the seeming nonlinearities work through the conditional variance [see, Bollerslev *et al*. (1992)]. Recent studies such as Caraiani (2014) have focused on examining factors that drive nonlinearity of time series by investigating whether dynamics at different frequencies present different degree of nonlinearity and how much they may influence any nonlinearity in the aggregate original series. Caraiani finds strong evidence in support of the idea that nonlinearities are mostly found at high frequencies.

Another common feature of stock return investigated in the literature is that they tend to exhibit nonnormal unconditional densities in the form of skewness and excess kurtosis. To account for conditional leptokurtosis, Bollerslev (1987) advocates the use of t distribution. More recent studies such as Brooks *et al*. (2005), Back (2014) and Kim *et al*. (2014) now attempt to incorporate higher moments into the modelling of financial returns. For example, Brooks *et al*. (2005) introduced a new model for ARCH and kurtosis via a time-varying df parameter that allows the kurtosis to evolve over time. With the need for the computation of VaR with fatter tails and portfolio choice with higher moments, the model seems to have a lot of potential for application in empirical finance in the future. Massacci (2014) proposes a two-regime threshold model for the conditional distribution of stock returns whereby returns follow a skewed student t distribution within each regime. The model allows capturing time variation in the conditional distribution of returns and higher order moments. He applies the model to daily US stock returns to illustrate its advantages compared with other alternatives. He finds that the model performs well in terms of in-sample fit, produces useful risk assessment based on the term structure of VaR and accurately estimates the conditional volatility.

In addition to the above features of stock return data, Black (1976) uncovers a negative correlation between current returns and future volatility known as the leverage effect. Bollerslev *et al*. (1992) define the leverage effect as when a reduction in the equity value raises the debt-to-equity ratio, hence raising the riskiness of the firm as manifested by an increase in future volatility. As a result, making the future volatility to be negatively related to the current return on the stock. Empirical evidence on the incidence of leverage effect especially in emerging markets has so far been mixed. Morimune (2007) clarifies the distinction between asymmetry and leverage. He states that their exist asymmetric effect in volatility when the effects of positive return on volatility are different from those of negative returns of similar magnitude while leverage refers to the negative correlation between the current return and future volatility.

However, the linear GARCH model is unable to capture this type of dynamic pattern and that is why the EGARCH, TGARCH and other asymmetric models were proposed so as to incorporate this effects which were found to be present in some financial markets [see, Nelson (1991), GJR

(1993) and Baillie *et al*. (1996)]. Black (1976), French *et al*. (1987), and Schwert (1989) have all argued that leverage alone cannot account for the magnitude of the negative relationship. Other studies however claim that the leverage effect can only partially explain the strong negative correlation between current return and volatility in the stock market. In contrast to this, the risk-return relation predicts a positive correlation between future return and future volatility. Another alternative explanation suggested in the literature is the volatility feedback effect. Recently, King and Botha (2015) find that a unique feature of the African stock markets is the lack of a clear leverage effect.

A significant property of stock market volatility relates to the phenomenon of persistence of shocks to the variance. The degree of persistence in volatility shocks was first formally investigated by Lamoreux and Lastrapes (1990) who find that by introducing dummy variables, the level of persistence reduces and later by Engle and Mustafa (1992) using the Black-Scholes option pricing formula with a stochastic variance process modelled by an ARCH equation. The volatility of financial time series has been found to be affected significantly by sudden changes, regime shifts, and variance breaks corresponding to economic, financial and political events. Such sudden changes tend to affect the degree of persistence of the volatility of returns. Babikir *et al*. (2012) highlighted how persistence –a crucial component of risk management, investment portfolios, and derivatives pricing is affected by breaks; as its presence is closely connected to the predictability of volatility. Other advances in modeling volatility structural breaks and long memory models include the spline-GARCH model of Engle and Rangel, 2008; the adaptive FIGARCH of Baillie and Morana (2009), and the time-varying parameter (TVP) model of Amado and Teräsvirta (2009), among others.

The importance of ARCH volatility models in finance is underscored partly from the direct link between variance and risk and the tradeoff relations between risk and return. This is connected to the popular finance theory of capital asset pricing model (CAPM) and its subsequent extensions. The ARCH model of Engle *et al*. (1987) provide a platform for estimating this relationship. The parameter that measures the impact of the conditional variances on the excess returns corresponds to the coefficient of relative risk aversion. Applications of this model to different stock market returns have been reported by several authors, e.g. Bollerslev *et al*. (1988). Koopman and Uspensky (2002) regard SV and SVM models as alternatives to ARCH models and present evidence of a negative but weak link between returns and volatility. The use of the ARCH-M model as an implementation of the CAPM is not without criticism. Bollerslev *et al*. (1992) note that in the ARCH-M model, the parameter estimates in the conditional mean equation are not asymptotically independent of the parameter estimates in the conditional variance and hence any mispecification in the variance equation generally leads to biased and inconsistent parameter estimates in the mean equation.

Bauwens *et al*. (2010) using Bayesian estimation with Gibbs sampling algorithm develop an MS-GARCH model where both the mean and variance switch in time from one GARCH process to another with the switching governed by a hidden Markov chain. Recent dynamic MS-GARCH models have been shown to have the ability to capture time-varying skewness and kurtosis not captured by GARCH models with the common single component asymmetric fat-tailed distributions as innovations. Wang and Theobald (2008) investigate regime-switching behaviour in the returns of 6 East-Asian emerging stock markets from 1970 to 2004 and examine the features of each regime using an MS model. Their result show evidence of more than one regime in each of these markets and the conditional probabilities in each regime provide mixed evidence of the impact of financial liberalisation on return volatility.

Earlier, Engle and Russel (1998) and Engle (2000) proposed the autoregressive conditional duration (ACD) models for the analysis of data that arrive at irregularly spaced interval. In Engle (2000) an ACD(1,1) model for IBM transactions is fitted based on an exponential density. He finds that both returns and variances are negatively influenced by long durations, a result consistent with asymmetric information models of market microstructure. Other more recent contributions include Bauwens and Hautsch (2006) who also propose a technique for modelling irregularly spaced data. They combine an observation-based multivariate intensity model with a dynamic latent factor component that they term the stochastic conditional intensity model. The empirical section of the paper analyses the price intensities of several shocks and tests for the presence of a common underlying factor.

Although our preceding survey was largely in the realm of univariate volatility models, many of the interesting questions in empirical finance can only be effectively handled within multivariate context. It is typical to examine a large number of assets, e.g. bonds with different maturities, multiple currencies, diverse equities, and so on. Doan (2013) argues that "not only is the volatility of each asset likely to be described fairly by a GARCH process, but within a group of assets, the movements are likely to be highly correlated". Thus, this is expected to result in efficiency gains when modelling them jointly. This is one of the main motivations of resorting to MGARCH models in empirical analysis. In Bollerslev *et al*. (1988) an MGARCH model was used to fit a CAPM model for a market portfolio consisting of stocks, bonds and bills. The results indicate a significant positive mean variance tradeoff among the three assets. Further extensions of the initial MGARCH model include CCC-GARCH, BEKK-GARCH, DCC-GARCH, time-varying conditional correlation-GARCH (TVCC-GARCH) and VARMA-GARCH models among others. In the process of generalising these models, one of the key issues of concern is the computational difficulties in applications, especially in finance.

Canopius (2006) stresses the fact that a fundamental issue in international portfolio diversification relates to the evolution over time of the correlations and variances of assets. Since the GFC, there has been extensive investigation of this issue in the literature. If correlations were to increase in bad times or when markets were highly volatile, the diversification benefits would certainly be compromised when most needed (Canopius, 2006). To effectively capture asymmetric effects in DCCs, Cappiello *et al*. (2006) propose the asymmetric generalised-DCC (AG-DCC) model which extends previous specifications by allowing for series-specific news impact/smoothing parameters and conditional asymmetries in correlation dynamics among different asset classes. In addition, they investigate the presence of asymmetric responses in conditional variances and correlations to negative returns. This is a generalisation of Engle's (2002) DCC and both models solve the curse of dimensionality that bedevilled earlier MGARCH models with changing covariances. They apply the model to 21 equity and 13 bond indices and confirm the intuition that conditional equity correlation increases among regional groups when bad news hits financial markets.

In a related study, Berben and Jansen (2005) employ TVCC-GARCH model to examine whether stock markets have become more integrated in the sense that correlations between volatilities in these markets have become stronger over time. Bauwens *et al*. (2013) propose a new multiplicative multivariate DCC (mDCC) model that allows for both smooth changes in the unconditional volatilities and correlations and for conditional volatility and correlations clustering around the smoothly changing level. They find that the mDCC model achieves higher forecasting performance compared to Engle's (2002) DCC model. In sum, due to the dynamic nature of stock markets, most of the proposed models and their extensions have so far attempted to incorporate well known behaviours in stock markets and asset returns into their models such as asymmetry, time variation, fat tails and to some extent they have succeded in doing that with the possible exception of the ability to anticipate and predict financial crisis. Nevertheless, they have contributed in deepening understanding about the behaviour of asset returns and stock markets which can help improve monetary policy and the functioning of financial markets.

8.2 Applications of Volatility Models to Interest Rates and Inflation

Engle's (1982) path breaking paper on the theory and application of dynamic volatility models is perhaps the most widely cited paper in the history of financial econometrics. Engle's (1982) model along with Bollerslev's (1986) generalisations of the model were all applied quite succesfully to the study of inflation dynamics. Furthermore, the relationship between short and long-term interest rates and concept of a risk premium in explaining the term structure have received increased attention recently. Accordingly, several studies have attempted to provide insights into the nature of common factors needed to describe the dynamic evolution of the term structure of interest rates. In this subsection, we will review relevant studies, that use econometric techniques to model time-varying conditional variances and risk premia in the term structure of interest rates and the dynamics of inflation volatility.

The ARCH model although nonlinear because it involves variances which are expectations of squares, are effectively linear models of squared variables (Diebold, 2004). Engle *et al*. (1987) used ARCH-M model to analyse quarterly excess holding yield of 6 months over 3 months treasury bills. Recently, Engle (2003) argues that financial price volatility is caused by arrival of new information and that news intensity is high during economic distress or major economic news announcements. This has been corroborated by studies such as Koopman *et al*. (2005) and Zivot (2009). Shields *et al*. (2005) examine the response of uncertainty about inflation and output growth to shocks and find a statistically significant size and sign bias and spillover effects. Their result further reveals that both inflation and growth volatility respond asymmetrically to positive and negative shocks. They argue that uncertainty about inflation is a determinant of output uncertainty and that higher growth volatility tends to raise inflation volatility. In sum, their finding suggests that negative growth and inflation shocks lead to higher and more persistent uncertainty than shocks of equal magnitude but opposite sign.

Kasman *et al*. (2011) find that interest and exchange-rate volatility are key determinants of conditional bank stock return volatility in Turkey. Their result indicates that interest and exchange-rate volatility are the major determinants of bank stock return volatility. Earlier, Marotta (2009) investigates whether size and speed of pass-through of market rates into short-term business lending rates have increased with the introduction of the Euro. His results were contrary to the intuition that a reduced volatility in money market rates is bound to mitigate uncertainty and to ease the transfer of policy rate changes to retail rates. Gospodinov (2005) proposes an LM test for linearity in the conditional mean and variance functions in a TAR model with GARCH errors. His finding indicates the presence of threshold nonlinearities in the AR and GARCH equations of the conditional moments of interest rate and that allowing for nonlinearities in the mean and variance leads to significant forecast improvements. He adopts a two-regime threshold process for the autoregressive mean and the GARCH conditional variance.

MS-G(ARCH) models have recently been applied to analyse inflation uncertainty and interest-rate dynamics. These models often consider discrete time, with the volatility stochastically switching between a finite number of fixed regimes. The main feature of MS models is their ability to capture endogenous regime shifts. For instance, Evans and Wachtel (1993) examine inflation regimes and the sources of inflation uncertainty within this context. A wide range of potential nonlinearities can be entertained when modelling variances but the SWARCH and MS-GARCH models are the most widely used. Leading contributions in this area include Dueker (1997), Gray (1996) and Song (2014). Cai (1994) incorporates the main features of both Hamilton's (1994) MS model and Engle's ARCH model to examine volatility persistence in the monthly excess returns of the 3-months treasury bill. He concludes that regime shifts have a greater impact on the properties of the data and was unable to reject the null hypothesis of "no ARCH" effects within the regimes.

Recent studies have began developing simulation based inference methods through the application of MCMC and Bayesian techniques to the study of interest rate and inflation volatility. For instarce Song (2014) proposes an infinite hidden Markov model (iHMM) to integrate regime switching and structural break dynamics in a unified Bayesian framework. The iHMM has the ability to model and incorporate infinite number of states as existing MS models assume a finite number of states (usually 2 or at most 4). Conventional MS models assume that the future is like the past. Applying his model to US real interest rates, this flexible approach allows for regime persistence and estimates the number of states automatically. His results show that iHMM provides the best out-of-sample density forecast and both regime switching and structural break dynamics are important for analysing real interest rates.

8.3 Applications of Volatility Models to Exchange Rate Returns

The analyses of exchange-rate movements and volatility have always attracted considerable attention in finance literature. This is due to its implications for many issues in international finance ranging from portfolio management, foreign investment to trade. The exchange rate and its volatility are main factors that influence economic activities and sustained volatility have led to currency crises, distortion of production patterns and sharp fluctuations in countries' external reserves recently. Since the adoption of floating exchange rate system in 1973, literature on exchange-rate volatility has grown tremendously. New set of theories evolved explaining exchange-rate behaviour and how exchange-rate dynamics affect macroeconomic/financial variables. This has led to several attempts to examine volatility of asset prices. Recently, exchange rate debates have taken centre-stage with the euro-zone currency and debt crises, concerns about China's exchange rates value and yen depreciation. Most of these issues have been investigated using GARCH-type models, often well suited to modelling such behaviour.

Ozer-Imer and Ozkan (2014) emphasise the vital role of exchange rate in decision making of investors, portfolio managers, international financial institutions and central banks. Due to recent financial crises and monetary policies in both emerging and developed economies, the major exchange rates have experienced significant fluctuations. This has in some cases led to higher currency volatility with implications on business cycles, trade and capital flows. Several studies have examined the dynamic and distributional properties of exchange rates and their returns in major FX markets. These studies mostly within the univariate framework have been able to establish stylised facts ranging from volatility clustering, fat tails (corresponding to a kurtosis larger than 3), volatility mean reversion, skewness to asymmetry. However, not many attempt to analyse the transmission of moments higher than the first and second moments until quite recently. Lastrapes (1989) who analyses exchange-rate volatility using the ARCH model, finds that there is a significant reduction in estimated volatility persistence if controls for monetary regime shifts are incorporated into the model.

On the sources of intermarket and intramarket volatility, Bollerslev *et al*. (1992) note that the ARCH effects associated with high-frequency data could be due to the amount of information or the quality of the information reaching the market in clusters, or from the time it takes market participants to fully process the information. Engle *et al*. (1990) examine the impact of news in one market on the time path of per-hour volatility in other markets using a volatility type of VAR analysis. Their objective was to explain the causes of volatility clustering in exchange rates by the testing of two hypotheses: heat waves (i.e. when the volatility has only country-specific autocorrelation) and meteor showers (phenomenon of intra-daily volatility spillovers from one market to the next). They find that the empirical evidence is generally against the null hypothesis of the heat wave. Andersen and Bollerslev (1998) examine the DM/US dollar intraday volatility based on a one-year sample of 5 minutes returns with emphasis on activity patterns, macroeconomic announcement and calendar effects. They find that market activity is correlated with price variability and that scheduled releases occasionally induce large price changes, but the associated volatility shocks appear short lived.

Bollerslev *et al*. (1992) state that several policy-oriented questions relating to the impact of exchange rate on different macroeconomic variables also require an understanding of the exchange rate volatility. They stress that "whereas stock returns have been found to exhibit some degree of asymmetry in their conditional variances, the two-sided nature of the FX market makes such asymmetries less likely". In the context of regime switching analysis, Beine *et al*. (2003) examine whether direct purchases and sales made by certain central banks in the FX market can explain the observed volatility regime switches associated with weekly yen and DM returns against the US dollar. They find that depending on the prevailing volatility level, coordinated central bank intervention (CBI) can lead to either a stabilising or destabilising effect. Frenkel *et al*. (2005) find a positive link between Bank of Japan's (BOJ's) interventions in the FX market and the volatility of the yen/US dollar return during the period 1993 to 2000. Walid *et al*. (2011) examine the link between stock price volatility and exchange-rate changes using an MS-EGARCH model for selected emerging markets. They find that the relation between stock and FX markets is regime dependent and stock volatility responds asymmetrically to events in the FX markets.

In line with findings for both stock returns and interest rates, the level of volatility persistence in the FX market is equally quite high. Bollerslev *et al*. (1992) claim that even though many different currencies may exhibit IGARCH-type behaviour, it is certainly possible that this persistence is common across different rates. They further stresss that "the presence of such co-persistence among the variances has many important practical implications (e.g., in optimal portfolio allocation decisions involving a trade-off between future expected returns and the associated risk)". Engle (2002) proposes the DCC model which is not linear but can be fitted with univariate or two-step based methods on the likelihood function (with a series of univariate GARCH and correlation estimates). He finds that the bivariate DCC model provides a very good approximation to a variety of time-varying correlation processes. Ozer-Imer and Ozkan (2014) investigate the impact of the recent GFC on the co-movement of 16 currencies using Engle's (2002) DCC model. They find that volatilities increase atleast 2-fold with the outbreak of the crisis and that there is an inverse relationship between volatility and duration of the crisis.

In the multivariate context, time-varying correlations are often estimated using MGARCH models that are linear in squares and cross products of the series [see, Bollerslev (1990) and Cappiello *et al*. (2006)]. MGARCH models have also been useful in addressing policy issues related to the FX market. Bollerslev (1990) using an MGARCH (1, 1) model with CCC to examine the effect on short-run exchange rate volatility due to the creation of the European monetary system (EMS) argue that the coherence increased over the EMS period. The empirical evidence on the impact of CBI on exchange-rate volatility is mixed. Recent studies have argued that interventions tend to influence both the level and variance of exchange rates. Dominguez (1998) observes that since CBI and currency volatility are often correlated, it may be that volatility causes intervention rather than the other way round. His study explores the impact of CBI on the behaviour of exchange rates. He finds that CBI generally increase exchange-rate volatility. Similar finding is by Chang and Taylor (1998) who using intraday data shows that intervention by the BOJ had positive and significant impact on the yen/US dollar volatility. An impressive survey of the previous literature on intervention is by Edison (1993).

Studies such as Baumohl and Lyocsa (2014) have identified increasing co-movement among markets during the GFC period as implying that correlations tend to be higher in bullish than in bearish markets. Cappiello *et al*. (2006) note that the relationship between volatilities and correlations are important in taking vital financial decisions especially as it concerns risk management and pricing derivatives, which makes them important components of risk. They further observe that "if correlations and volatilities move together, then risks in the long run are greater than they seem in the short run". Ozer-Imer and Ozkan (2014) find that co-movements between major currencies became stronger during financial crises and panics, and gets closer to positive. This has implications for foreign and domestic investors as investments become riskier due to increased volatilities arising from financial crises. Recent studies however note that it is increased financial linkages that drive co-movements. Accordingly, variations in currencies have been found to be influenced by other factors including central bank decisions such as interventions in the FX market, and structural economic challenges such as in the case of the

Euro-zone debt crises, among others. Some studies have highlighted that currency volatility is a central policy variable since exchange rate plays a crucial role in the decision making processes of economic agents both in the real and financial sectors (Ozer-Imer and Ozkan, 2014). Based on a majority of extant literature examined, most previous studies have been inconclusive as to the credibility of some of the identified transmission mechanisms.

9 Volatility Impulse Response Functions (VIRF) Modelling

Since the influential paper on vector autoregression (VAR) by Sims in 1980 which introduces the impulse response functions (IRF), the technique has evolved into an important tool for analysing the dynamics of macroeconomic and financial systems designed mainly for the conditional mean of linear models. Prior to few years ago, there was no unified methodology to uncover volatility dynamics operating between multiple variables in nonlinear models. Although, GARCH models have been popular, the analysis of the IRF that traces the dynamics of the conditional variance from past squared innovations was only formally considered and extended to MGARCH models by Lin in 1997. Gallant *et al*. (1993), Koop *et al*. (1996), Hafner and Herwartz (hereafter, H&H) (2006) and several others have attempted to trace the impact of shocks over time through concepts such as IRF, volatility IRF (VIRF) and generalised IRF (GIRF). Lin (1997) states that "determining how new information will affect future expected volatility plays a prominent role in pricing primary and derivative assets, in designing strategies for portfolio diversification and in analysing the dynamic effect of government policy on major economic variables".

Before the advent of VIRFs, previous studies often use two-stage procedure in carying out similar analysis by fitting univariate GARCH models prior to using the estimated conditional variances to construct a VAR system which allows for the construction of IRF and variance decompositions to determine how shocks in one market influences dynamic adjustment of volaility in other markets. Omrane and Hafner (2009) use an IRF method with data consisting of 5 minute returns to analyse the effects of US macroeconomic news announcements on exchange rate volatility. They consider and allow for two types of news: positive and negative for the US economy. Using a MGARCH model with exogenous news effects, they find that the initial impact of positive news on the volatility of the Pound is higher than that of the Euro while the persistence of shocks is highest for the yen. For negative news, they find that a significant part of the impact on the yen and Pound is induced by volatility spillover from the Euro.

The main properties of VIRF are: (i) is an even function while IRF are odd functions, (ii) not homogenous of any degree, and (iii) depends on history through the volatility state while IRF does not depend on history [see, H&H (2006)]. The VIRF technique makes it possible to solve the entire system in order to obtain the variance and covariance forecast for every variable in the model and that the difference in the volatility forecasts for any two sets of the initial values comprise the VIRF (Enders, 2015). Theoretical and empirical models on IRFs have been extensively investigated since the 1980s. Sims examine a VAR whereby each of the 6 variables in his model is predicted as a linear combination of past values of all 6 variables in the system (Sims, 1980). However, Sims' approach is sensitive to the order in which the variables enter the model (See, H&H, 2006 and Hatemi-J, 2014). Gallant *et al*. (GRT) (1993) develop an approach for analysing dynamics of a nonlinear time series represented by a nonparametric estimate of its one-step ahead conditional density. This entails examining the conditional moment profiles corresponding to certain shocks which they defined as the conditional expectation evaluated at time t of a time-invariant function evaluated at time $t+j$ regarded as a function of j.

GRT (1993) state that "the key idea in IRF analysis is to trace through the system the effects of small movements in the innovations, or linear combinations of the innovations". Recent studies have outlined strategies for examining IRF analysis of nonlinear time series models including H&H (2006) and Hatemi-J (2014). Earlier, Koop *et al*. (1996) and Pesaran and Shin (1998) introduced the GIRF. They describe the IRF as the measure of the time profile of the effects of a shock on the behaviour of a time series. Lin (1997) extends GRT's (1993) method to MGARCH models and employs a Monte Carlo technique to assess the finite-sample properties of the standard errors and provides an empirical example of the dependence of exchange-rate volatility. H&H (2006) develop an alternative VIRF approach by adapting Sims' IRF to volatility setting in the spirit of Koop *et al*. (1996) with particular focus on the effects of market news and central bank decisions. Comparing the VIRF with GRT's conditional moment profile, they show that for shocks affecting FX rates asymmetrically, the difference between the two methodologies and their interpretation can be substantial. Another main difference is that H&H (2006) focus on the conditional variance rather than the conditional mean.

Compared to IRFs in linear framework, VIRFs are much more complicated. Gallant *et al*. (1993), and Koop *et al*. (1996) offer competing definitions of IRF in nonlinear models. The key difference between the two lies in how they define a realistic shock to the system and the choice of a benchmark against which to measure the impact of the shock. An important feature of VIRF is that the impact of a shock depends on the current level of volatility and thus a given shock will not always increase expected volatility. The VIRF has three major advantages, (1) it enables the determination of precisely how a shock to one market influences the dynamic adjustment of volatility to another market and the persistence of these spillover effects, (2) VIRFs avoid typical orthogonolisation and ordering problems common in IRFs, and (3) VIRFs depend on both the volatility state and the unexpected returns vector when the shock occurs. Pesaran and Shin (1998) building on Koop *et al*. (1996) propose a GIRF analysis for VAR and cointegrated VAR models. The main merit of their approach is that it does not require orthogonolisation of shocks or ordering of the variables in the VAR and it can be used in the construction of order-invariant forecast error variance decomposition. Potter (2000) also extends the linear IRF analysis to the nonlinear case within the framework of the GIRF.

Koop *et al*. (1996) present a unified approach to IRF analysis applicable to both linear and nonlinear multivariate models. They develop the GIRF defined as the difference between the mean response of volatility, conditional on both history and a shock and the mean response conditioned on history only. Their approach provides a way of dealing with the problems of history, shock and compositional dependence of IRFs for multiple linear and nonlinear time series. Panopoulou and Pantelidis (2009) employ VIRFs to examine international volatility transmission between the stock markets of the US and other G-7 countries using daily stock returns data. They find using split-sample analysis, that the linkages between the markets have substantially changed recently, suggesting increased volatility interdependence among the G-7 markets especially after 1995. Recently, Hatemi-J (2014) suggests a new approach for allowing asymmetry in the IRFs which was hitherto neglected in the literature. He shows how variables can be transformed into cumulative positive and negative changes in order to estimate the impulses to an asymmetric innovation. From the survey of literature on VIRFs, most studies use the technique to unearth empirical evidence on past economic, financial and monetary policy decisions and events in order to assess the impact of shocks on conditional volatility.

10 Conclusion

In this paper, volatility models used for estimating the volatility of financial assets are surveyed with particular emphasis on recent developments. The recent surge in popularity of volatility models is explained by the fact that they can be used for forecasting volatility of financial assets. As the variance of returns is used as a measure of risk, it is important for investors to know how the volatility of their portfolios can be expected to change in the near future. The development of volatility models has gone along with their application in the financial markets. The challenges of the global economy since the recent GFC and its dramatic economic consequences, have made it clear that academics, financial analysts and policymakers have still a lot of progress to make in their understanding of financial risks. These risks are compounded by the development of sophisticated financial products and the strong linkages between financial institutions because of the globalisation of the world economy. In sum, due to the dynamic nature of modern financial markets, most of the proposed models and their extensions have so far attempted to incorporate well known behaviours in these markets into their models such as asymmetry, time variation, fat tails and to some extent they have succeded in that with the possible exception of the ability to anticipate and predict financial crisis. Nevertheless, they have contributed in deepening understanding about the dynamics of asset returns which can help improve monetary policy and the functioning of financial markets.

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