Multifractal Analysis for Complex Analogues of the Devil's Staircase and the Takagi Function in Random Complex Dynamics

Sumi, Hiroki 大阪大学

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We want to consider complex analogues of the above story. We consider the following setting.

- $\hat{\mathbb{C}} := \mathbb{C} \cup \infty \cong S^2$  (Riemann sphere).
- Let  $s \in \mathbb{N}$ .
- Let  $f_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, i = 1, \dots, s + 1$ , be rational maps with  $\deg(f_i) \ge 2$ .
- Probability parameter space of dimension s:

$$\mathcal{W} := \left\{ \vec{p} = (p_1, p_2, \dots, p_s) \in (0, 1)^s \mid \sum_{i=1}^s p_i < 1 \right\}.$$

For each p ∈ W we consider the random dynamical system on Ĉ such that at every step we choose f<sub>i</sub> with probability p<sub>i</sub>, i.e., a Markov process whose state space is Ĉ and whose transition probability is given by

$$p(x,A) := \sum_{i=1}^{s+1} p_i \mathbb{1}_A(f_i(z)), z \in \widehat{\mathbb{C}}, A \subset \widehat{\mathbb{C}}.$$

- Let  $C(\hat{\mathbb{C}}) := \{ \varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi \text{ is conti.} \}$  endowed with sup. norm.
- The transition operator  $M_{\vec{p}}: C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$  is given by

$$M_{\vec{p}}(\varphi)(z) := \sum_{i=1}^{s+1} p_i \cdot \varphi(f_i(z)), \quad \varphi \in C(\hat{\mathbb{C}}), z \in \hat{\mathbb{C}}.$$
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Let

G := {f<sub>i1</sub> ∘ · · · ∘ f<sub>in</sub> | n ∈ N, i<sub>1</sub>, . . . , i<sub>n</sub> ∈ {1, . . . , s + 1}}.

This is a semigroup whose semigroup operation is the functional composition.

This G is called the rational semigroup generated by {f<sub>1</sub>, . . . , f<sub>s+1</sub>}.

Let

F(G) := {z ∈ Ĉ | ∃ nbd U of z s.t. {h : U → Ĉ}<sub>h∈G</sub> is equiconti. on U}.
This is called the Fatou set of G.

Let J(G) := Ĉ \ F(G). This is called the Julia set of G.

## **Assumptions for** *G*:

• G is hyperbolic, i.e.,  $\overline{\bigcup_{h \in G}}$ {all critical values of  $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ }  $\subset F(G)$ .

• 
$$(f_i^{-1}(J(G))) \cap (f_j^{-1}(J(G))) = \emptyset$$
 for all  $i \neq j$ .

•  $\exists$  at least two minimal sets of G.

Here, we say that a non-empty compact set  $K\subset \hat{\mathbb{C}}$  is a **minimal set** of G if

$$K = \overline{\bigcup_{h \in G} \{h(z)\}}$$
 for each  $z \in K$ .

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**Theorem 1.1** (S, [S11-1, S13-1]). Fix  $\vec{p} \in W$  and let L be a minimal set of G. For each  $z \in \hat{\mathbb{C}}$ , let  $T_{L,\vec{p}}(z) \in [0,1]$  be the probability of tending to L starting with the initial value  $z \in \hat{\mathbb{C}}$ . Then we have the following. (1)  $\exists \alpha > 0$  s.t.  $T_{L,\vec{p}} \in C^{\alpha}(\hat{\mathbb{C}}) :=$  the space of  $\alpha$ -Hölder conti. fcns on  $\hat{\mathbb{C}}$  endowed with  $\alpha$ -Hölder norm. Moreover,  $M_{\vec{p}}(T_{L,\vec{p}}) = T_{L,\vec{p}}$ . (2)  $\exists V$  :nbd of  $\vec{p}$  in W,  $\exists \alpha > 0$  s.t.  $\vec{q} \mapsto T_{L,\vec{q}} \in C^{\alpha}(\hat{\mathbb{C}})$  is real-analytic in V. (3) The set of varying points of  $T_{L,\vec{p}}$  is equal to J(G), which is a thin fractal set (e.g.  $\dim_H(J(G)) < 2$ ).  $T_{L,\vec{p}}$  is a complex analogue of the devil's staircase or Lebesgue's singular functions. 8 Complex analogues of the Takagi function

Fix  $\vec{p} \in \mathcal{W}$  and let L be a minimal set of G.

Definition 1.2.

For  $\vec{n} = (n_1, \dots, n_s) \in (\mathbb{N} \cup \{0\})^s$  and  $z \in \hat{\mathbb{C}}$  we set

$$C_{\vec{n}}(z) := \frac{\partial^{|\vec{n}|} T_{L,(a_1,\dots,a_s,1-\sum_{i=1}^s a_i)}(z)}{\partial^{n_1} a_1 \partial^{n_2} a_2 \cdots \partial^{n_s} a_s} |_{\vec{a}=\vec{p}}$$

(note:  $C_{(1,0,...,0)}$  is a complex analogue of the Takagi function.)

Also, define the  $\mathbb{C}$ -vector space

$$\mathcal{T} := \operatorname{span}\{C_{\vec{n}} \mid \vec{n} \in (\mathbb{N} \cup \{0\})^s\} \subset C^{\alpha}(\hat{\mathbb{C}}).$$

• For  $C \in \mathcal{T}$  and  $z \in \hat{\mathbb{C}}$  consider pointwise Hölder exponent of C at z:  $H\"{o}l(C, z) := \sup\{\beta \in \mathbb{R} \mid \limsup_{y \to z, y \neq z} \frac{|C(y) - C(z)|}{d(y, z)^{\beta}} < \infty\}.$ • By the separation condition in the setting, we have  $\forall z \in J(G), \exists !i(z) \in \{1, \dots, s+1\} \text{ s.t. } f_{i(z)}(z) \in J(G).$ We define  $f : J(G) \to J(G)$  by  $f(z) = f_{i(z)}(z).$ • Define potentials  $\zeta : J(G) \to \mathbb{R}, \zeta(z) := -\log ||f'_{i(z)}(z)||$  and  $\psi : J(G) \to \mathbb{R}, \psi(z) := \log p_{i(z)}.$ Theorem 1.3 ([JS14, JS15]). Let  $C \in \mathcal{T} \setminus \{0\}, z \in J(G).$ Then  $H\"{o}l(C, z) = \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} \psi \circ f^k(z)}{\sum_{k=0}^{n-1} \zeta \circ f^k(z)}.$ 10 **Corollary 1.4.** Let  $C \in \mathcal{T} \setminus \{0\}$ . Then C is continuous on  $\hat{\mathbb{C}}$  and varies precisely on J(G) (which is a thin fractal set). In particular,  $\mathcal{T} = \bigoplus_{\vec{n} \in (\mathbb{N} \cup \{0\})^s} \mathbb{C}C_{\vec{n}}$  is a direct sum.

**Theorem 1.5.** (*Multifractal formalism*) Let  $C \in \mathcal{T} \setminus \{0\}$ . Then the level sets

 $\{z \in J(G) \mid \mathsf{H\"ol}(C, z) = \alpha\}, \ \alpha \in \mathbb{R},$ 

satisfy the multifractal formalism. That is, the Hausdorff dimension function  $\alpha \mapsto \dim_H(\{z \in J(G) \mid H\"ol(C, z) = \alpha\})$  is a real analytic strictly concave and positive function on a bounded open interval  $(\alpha_-, \alpha_+)$ , except very rare cases.

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**Example 1.6.** Let  $g_1(z) = z^2 - 1$ ,  $g_2(z) = \frac{z^2}{4}$  and  $f_1 = g_1 \circ g_1$ ,  $f_2 := g_2 \circ g_2$ . Let  $\vec{p} = (1/2, 1/2)$ . Then  $\{\infty\}$  is a minimal set of  $G = \{f_{i_1} \circ \cdots \circ f_{i_n} \mid n \in \mathbb{N}, \forall i_j \in \{1, 2\}\}$ . • The function  $T_{\infty, \vec{p}} : \hat{\mathbb{C}} \to [0, 1]$  of prob. of tending to  $\infty$ is a complex analogue of the devil' s staircase (or Lebesgue's singular functions) and it is called a **devil's coliseum**. • Also, let  $C_{(1)}(z) = \frac{\partial T_{\infty,(a,1-a)}(z)}{\partial a} \mid a = 1/2$ . Then the function  $C_{(1)} : \hat{\mathbb{C}} \to \mathbb{R}$  is a **complex analogue of the Takagi function**. • Both  $T_{\infty, \vec{p}}$  and  $C_{(1)}$  are Hölder continuous on  $\hat{\mathbb{C}}$  and vary precisely on J(G), which is a thin fractal set (e.g.  $\dim_H J(G) < 2$ ). Multifractal formalism works. 12









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