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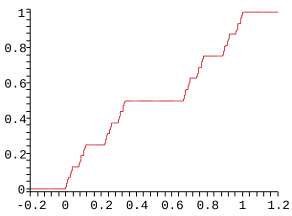
1 Introduction

(1) The devil's staircase is equal to the restriction of function of probability of tending to $+\infty$ w.r.t. the random dynamics on $\mathbb{R}$ s.t. we take $h_1(x) = 3x$ with prob. $1/2$ and we take $h_2(x) = 3x - 2$ with prob. $1/2$.



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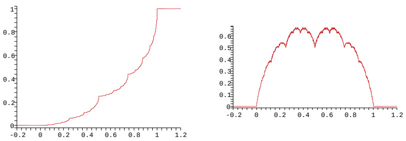
(2) Lebesgue's singular function L_p with parameter $p \in (0,1)$ is equal to the restriction of function of probability of tending to $+\infty$ w.r.t. the random dynamics on $\mathbb{R}$ s.t.
we take $h_1(x) = 2x$ with prob. p and
we take $h_2(x) = 2x - 1$ with prob. $1 - p$.

(3) The Takagi function (on $[0,1]$) is equal to the function $x \mapsto \frac{1}{2} \cdot \frac{\partial L_p(x)}{\partial p}|_{p=1/2}, \ x \in [0,1]$.



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We want to consider complex analogues of the above story. We consider the following setting.

- $\hat{\mathbb{C}} := \mathbb{C} \cup \infty \cong S^2$ (Riemann sphere).
- Let $s \in \mathbb{N}$.
- Let $f_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, i = 1, \ldots, s+1$, be rational maps with $\deg(f_i) \geq 2$.
- Probability parameter space of dimension s:

$$\mathcal{W} := \left\{ \vec{p} = (p_1, p_2, \ldots, p_s) \in (0,1)^s \mid \sum_{i=1}^{s} p_i < 1 \right\}.$$

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- For each $\vec{p} \in \mathcal{W}$ we consider the **random dynamical system** on $\hat{\mathbb{C}}$ such that at every step we choose f_i with probability p_i, i.e., a Markov process whose state space is $\hat{\mathbb{C}}$ and whose transition probability is given by

$$p(x, A) := \sum_{i=1}^{s+1} p_i 1_A(f_i(z)), z \in \hat{\mathbb{C}}, A \subset \hat{\mathbb{C}}.$$

- Let $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi$ is conti.$\}$ endowed with sup. norm.
- The transition operator $M_{\vec{p}} : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ is given by

$$M_{\vec{p}}(\varphi)(z) := \sum_{i=1}^{s+1} p_i \cdot \varphi(f_i(z)), \quad \varphi \in C(\hat{\mathbb{C}}), z \in \hat{\mathbb{C}}.$$

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- Let
$$G := \{f_{i_1} \circ \cdots \circ f_{i_n} \mid n \in \mathbb{N}, i_1, \ldots, i_n \in \{1, \ldots, s+1\}\}.$$

  This is a semigroup whose semigroup operation is the functional composition.
  This G is called the **rational semigroup** generated by $\{f_1, \ldots, f_{s+1}\}$.

- Let
$$F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } \{h : U \to \hat{\mathbb{C}}\}_{h \in G} \text{ is equiconti. on } U\}.$$
  This is called the **Fatou set** of G.

- Let $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This is called the **Julia set** of G.

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Assumptions for G:

- G is hyperbolic, i.e., $\overline{\cup_{h\in G}\{\text{all critical values of } h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}} \subset F(G)$.

- $(f_i^{-1}(J(G))) \cap (f_j^{-1}(J(G))) = \emptyset$ for all $i \neq j$.

- $\exists$ at least two minimal sets of G.

  Here, we say that a non-empty compact set $K \subset \hat{\mathbb{C}}$ is a **minimal set** of G if

$$K = \overline{\cup_{h\in G}\{h(z)\}} \text{ for each } z \in K.$$

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Theorem 1.1 (S, [S11-1, S13-1]). *Fix $\vec{p} \in \mathcal{W}$ and let L be a minimal set of G. For each $z \in \hat{\mathbb{C}}$, let $T_{L,\vec{p}}(z) \in [0,1]$ be the probability of tending to L starting with the initial value $z \in \hat{\mathbb{C}}$. Then we have the following.*

(1) *$\exists \alpha > 0$ s.t. $T_{L,\vec{p}} \in C^{\alpha}(\hat{\mathbb{C}})$:= the space of α-Hölder conti. fcns on $\hat{\mathbb{C}}$ endowed with α-Hölder norm. Moreover, $M_{\vec{p}}(T_{L,\vec{p}}) = T_{L,\vec{p}}$.*

(2) *$\exists V$:nbd of $\vec{p}$ in $\mathcal{W}$, $\exists \alpha > 0$ s.t. $\vec{q} \mapsto T_{L,\vec{q}} \in C^{\alpha}(\hat{\mathbb{C}})$ is real-analytic in V.*

(3) *The set of varying points of $T_{L,\vec{p}}$ is equal to $J(G)$, which is a thin fractal set (e.g. $\dim_H(J(G)) < 2$).*

$T_{L,\vec{p}}$ is a complex analogue of the devil's staircase or Lebesgue's singular functions.

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Complex analogues of the Takagi function

Fix $\vec{p} \in \mathcal{W}$ and let L be a minimal set of G.

Definition 1.2.
For $\vec{n} = (n_1, \ldots, n_s) \in (\mathbb{N} \cup \{0\})^s$ and $z \in \hat{\mathbb{C}}$ we set

$$C_{\vec{n}}(z) := \frac{\partial^{|\vec{n}|} T_{L,(a_1,\ldots,a_s,1-\sum_{i=1}^{s} a_i)}(z)}{\partial^{n_1} a_1 \partial^{n_2} a_2 \cdots \partial^{n_s} a_s}|_{\vec{a}=\vec{p}}.$$

(note: $C_{(1,0,\ldots,0)}$ is a complex analogue of the Takagi function.)
Also, define the $\mathbb{C}$-vector space

$$\mathcal{T} := \text{span}\{C_{\vec{n}} \mid \vec{n} \in (\mathbb{N} \cup \{0\})^s\} \subset C^{\alpha}(\hat{\mathbb{C}}).$$

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- For $C \in \mathcal{T}$ and $z \in \hat{\mathbb{C}}$ consider **pointwise Hölder exponent of C at z:**
$$\text{Höl}(C, z) := \sup\{\beta \in \mathbb{R} \mid \limsup_{y \to z, y \neq z} \frac{|C(y) - C(z)|}{d(y, z)^{\beta}} < \infty\}.$$
- By the separation condition in the setting, we have $\forall z \in J(G)$, $\exists! i(z) \in \{1, \ldots, s+1\}$ s.t. $f_{i(z)}(z) \in J(G)$. We define $f : J(G) \to J(G)$ by $f(z) = f_{i(z)}(z)$.
- Define potentials
$\zeta : J(G) \to \mathbb{R}, \zeta(z) := -\log \|f'_{i(z)}(z)\|$ and
$\psi : J(G) \to \mathbb{R}, \psi(z) := \log p_{i(z)}$.

Theorem 1.3 ([JS14, JS15]). *Let $C \in \mathcal{T} \setminus \{0\}$, $z \in J(G)$. Then*
$$\text{Höl}(C, z) = \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} \psi \circ f^k(z)}{\sum_{k=0}^{n-1} \zeta \circ f^k(z)}.$$

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Corollary 1.4. *Let $C \in \mathcal{T} \setminus \{0\}$. Then C is continuous on $\hat{\mathbb{C}}$ and varies precisely on $J(G)$ (which is a thin fractal set). In particular, $\mathcal{T} = \oplus_{\vec{n} \in (\mathbb{N} \cup \{0\})^s} \mathbb{C} C_{\vec{n}}$ is a direct sum.*

Theorem 1.5. (*Multifractal formalism*) *Let $C \in \mathcal{T} \setminus \{0\}$. Then the level sets*

$$\{z \in J(G) \mid \text{Höl}(C, z) = \alpha\}, \ \alpha \in \mathbb{R},$$

satisfy the multifractal formalism.
That is, the Hausdorff dimension function $\alpha \mapsto \dim_H(\{z \in J(G) \mid \text{Höl}(C, z) = \alpha\})$ is a real analytic strictly concave and positive function on a bounded open interval (α_-, α_+), except very rare cases.

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Example 1.6. Let $g_1(z) = z^2 - 1, g_2(z) = \frac{z^2}{4}$ and $f_1 = g_1 \circ g_1, f_2 := g_2 \circ g_2$. Let $\vec{p} = (1/2, 1/2)$. Then $\{\infty\}$ is a minimal set of $G = \{f_{i_1} \circ \cdots \circ f_{i_n} \mid n \in \mathbb{N}, \forall i_j \in \{1, 2\}\}$.

- The function $T_{\infty,\vec{p}} : \hat{\mathbb{C}} \to [0, 1]$ of prob. of tending to ∞ is a complex analogue of the devil' s staircase (or Lebesgue's singular functions) and it is called a **devil's coliseum**.
- Also, let $C_{(1)}(z) = \frac{\partial T_{\infty,(a,1-a)}(z)}{\partial a} \mid a = 1/2$. Then the function $C_{(1)} : \hat{\mathbb{C}} \to \mathbb{R}$ is a **complex analogue of the Takagi function**.
- Both $T_{\infty,\vec{p}}$ and $C_{(1)}$ are Hölder continuous on $\hat{\mathbb{C}}$ and vary precisely on $J(G)$, which is a thin fractal set (e.g. $\dim_H J(G) < 2$). **Multifractal formalism works**.

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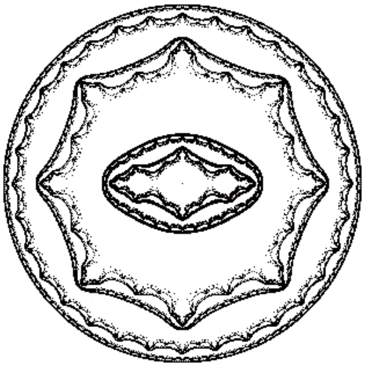
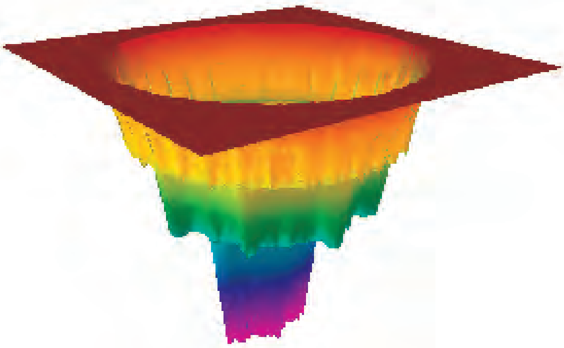
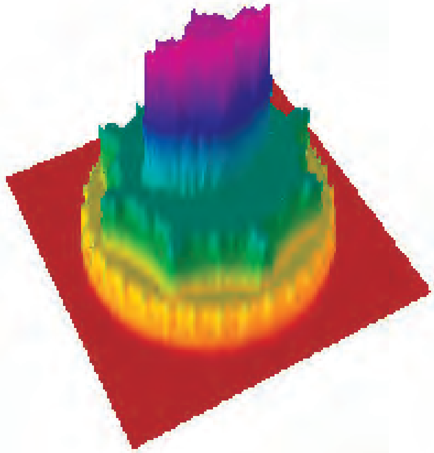
$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2.\ G := \langle h_1, h_2 \rangle.\ G \in \mathcal{G}_{dis}.$
The figure of $J(G)$. $\sharp \mathrm{Con}(J(G)) > \aleph_0$.



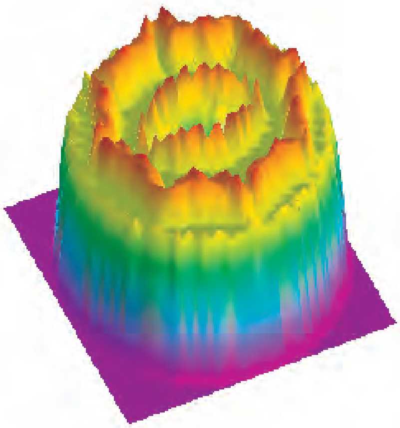
$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2,\ \tau := \frac{1}{2}\delta_{h_1} + \frac{1}{2}\delta_{h_2}.$
The graph of $z \mapsto T_{\tau,\infty}(z)$.
(Devil's Coliseum (Complex analogue of devil's staircase).)



The graph of $z \mapsto 1 - T_{7,\infty}(z)$.



$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2.$
The graph of $z \mapsto \frac{\partial T}{\partial p}(h_1, h_2, \frac{1}{2}, z)$.
(A complex analogue of the Takagi function.)



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