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We study the distributions of the random Dirichlet series with parameters (s, β) defined by

$$S = \sum_{n=1}^{\infty} \frac{I_n}{n^s},$$

where (I_n) is an independent sequence of Bernoulli random variables taking value 1 with probability $1/n^\beta$ and 0 otherwise. Random series of this type are motivated by the record indicator sequences which have been studied in the extreme value theory in statistics. We show that the distributions have densities when $s > 0$ and $0 < \beta \leq 1$ with $s+\beta > 1$, and are purely atomic or not defined because of divergence otherwise. In particular, in the case when $s > 0$ and $\beta = 1$, we prove that the density is bounded and continuous when $0 < s < 1$, and unbounded when $s > 1$. In the case when $s > 0$ and $0 < \beta < 1$ with $s + \beta > 1$, we prove that the density is smooth. To show the absolute continuity, we obtain estimates of the Fourier transforms, employing van der Corput's method to deal with number-theoretic problems. We also give further regularity results of the densities.

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HAUSDORFF SPECTRUM OF HARMONIC MEASURE

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Abstract. This is an introduction to the paper [T] on random walks on word hyperbolic groups and their harmonic measures.

1. Introduction

Let Γ be a finitely generated group. For a probability measure μ on it, we obtain a random walk on Γ by multiplying from right independent random elements with the law μ, and the distribution of the random walk at the time n is given by the n-th convolution power μ^{*n}. There are several important quantities which capture the asymptotic behaviours of the random walks. Define the entropy h and the drift l (also called the rate of escape, or the speed) by

$$h := \lim_{n\to\infty} -\frac{1}{n}\sum_{x\in\Gamma}\mu^{*n}(x)\log\mu^{*n}(x), \quad l := \lim_{n\to\infty}\frac{1}{n}\sum_{x\in\Gamma}|x|\mu^{*n}(x),$$

where $|\cdot|$ denotes the word norm associated with a finite symmetric set of generators of Γ. It is known that the entropy, introduced by Avez [Ave], is equal to 0 if and only if all bounded μ-harmonic functions on Γ are constants ([Der], [KV]). The entropy and the drift are connected via the logarithmic volume growth v of the group which is defined by $e^v := \lim_{n\to\infty}|B_n|^{1/n}$, where $|B_n|$ denotes the cardinality of the set B_n of words of length at most n, by the inequality

$$h \le lv, \tag{1}$$

as soon as all those quantities are well-defined ([Gui], see also e.g., [BHM1] and [Ver]). The inequality (1) is also called the fundamental inequality. In [Ver], Vershik proposed to study the equality case of (1). In this paper, we focus on hyperbolic groups in the sense of Gromov and characterise the equality of (1) in terms of the boundary behaviours of the random walks. For every hyperbolic group Γ, one can define the geometric boundary $\partial\Gamma$, which is compact and admits a metric d_ε with a parameter $\varepsilon > 0$. The harmonic measure ν is defined by the hitting distribution of the random walk starting from the identity on the boundary $\partial\Gamma$, corresponding to the step distribution μ on Γ. The boundary $\partial\Gamma$ has the Hausdorff dimension $D = v/\varepsilon$ and the D-Hausdorff measure $\mathcal{H}^D$ is finite and positive on $\partial\Gamma$ [Coo]. Here the D-Hausdorff measure $\mathcal{H}^D$ is a natural measure to compare with the harmonic measure ν. We call a probability measure μ on the group Γ admissible if the support of μ generates the whole group Γ as a semigroup. In the present paper, μ is always finitely supported and admissible unless stated otherwise.

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Theorem 1.1. *For every finitely supported admissible probability measure μ on every finitely generated non-elementary hyperbolic group Γ equipped with a word metric, it holds that $h = lv$ if and only if the corresponding harmonic measure ν and the D-Hausdorff measure $\mathcal{H}^D$ are mutually absolutely continuous and their densities are uniformly bounded from above and from below.*

Blachère, Haissinsky and Mathieu established this result for every finitely supported admissible and symmetric probability measure μ [BHM2, Corollary 1.2, Theorem 1.5]. We extend it to non-symmetric measures from a completely different approach as we describe later. Recently, Gouëzel, Mathéus and Maucourant have proven that for a non-elementary hyperbolic group Γ which is not virtually free equipped with a word metric, for every finitely supported admissible probability measure μ, the equality $h = lv$ never holds [GMM2]. Together with their results, one concludes that in this setting, the harmonic measure ν and the D-Hausdorff measure $\mathcal{H}^D$ are always mutually singular. Connell and Muchnik proved that for an infinitely supported probability measure μ on Γ, the D-Hausdorff measure $\mathcal{H}^D$ (and also a Patterson-Sullivan measure) and the harmonic measure for μ can be equivalent ([CM1, Remark 0.5] and [CM2]). On the other hand, Le Prince showed that for every finitely generated non-elementary hyperbolic group Γ, there exists a finitely supported admissible and symmetric probability measure μ such that the corresponding harmonic measure ν and the D-Hausdorff measure $\mathcal{H}^D$ are mutually singular [LeP]. Ledrappier proved the corresponding result to Theorem 1.1 for non-cyclic free groups for every finitely supported admissible probability measure μ [Led, Corollary 3.15] . For free groups, it is straightforward to see that if μ depends only on the word length associated with the standard symmetric generating set, then the corresponding harmonic measure coincides with the Hausdorff measure (the uniform measure on the boundary) up to a multiplicative constant. In [GMM1], Gouëzel et al. studied a variant of the fundamental inequality (1) and also obtained some rigidity results for the equality case. Apart from Cayley graphs of groups, Lyons extensively studied the equivalence of the harmonic measure and the Patterson-Sullivan measure for universal covering trees of finite graphs [Lyo].

A novel feature of our approach is to introduce one parameter family of probability measures μ_θ, which interpolates a Patterson-Sullivan measure and the harmonic measure on the boundary $\partial\Gamma$. Let us consider for every $\theta \in \mathbb{R}$,

$$\beta(\theta) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in S_n} G(1,x)^\theta,$$

where $G(x, y)$ is the Green function associated with μ, we denote by 1 the identity of the group, and by S_n the set of words of length n. The limit exists by the Ancona inequality [Anc], and we show that β is convex, in fact, analytic except for at most finitely many points and continuously differentiable at every point. Theorem 1.1 is deduced via the following dimensional properties of the harmonic measure ν.

Theorem 1.2. *Let Γ and μ be as in Theorem 1.1, and ν be the corresponding harmonic measure on the boundary $\partial\Gamma$. It holds that*

$$\lim_{r\to 0} \frac{\log \nu\,(B(\xi, r))}{\log r} = \frac{h}{\varepsilon l}, \quad \nu\text{-a.e. } \xi. \tag{2}$$

Define the set

$$E_\alpha = \left\{ \xi \in \partial\Gamma \;\middle|\; \lim_{r \to 0} \frac{\log \nu\left(B(\xi, r)\right)}{\log r} = \alpha \right\},$$

then the Hausdorff dimension of the set E_α is given by the Legendre transform of β, i.e.,

$$\dim_H E_\alpha = \frac{\alpha\theta + \beta(\theta)}{\varepsilon},$$

for every $\alpha = -\beta'(\theta)$, where β is continuously differentiable on the whole $\mathbb{R}$, and $B(\xi, r)$ denotes the ball of radius r centred at ξ.

The measure μ_θ is constructed by the Patterson-Sullivan technique. As $\theta = 0$, the measure μ_0 is a Patterson-Sullivan measure, and thus comparable with the D-Hausdorff measure $\mathcal{H}^D$, and as $\theta = 1$, the measure μ_1 is the harmonic measure ν. The probability measure μ_θ satisfies that $\mu_\theta(E_\alpha) = 1$ for $\alpha = -\beta'(\theta)$. We call $\dim_H E_\alpha$ as a function in α the *Hausdorff spectrum* of the measure ν. To determine the Hausdorff spectrum is called multifractal analysis which has been extensively studied in fractal geometry. In fact, there are many technical similarities to analyse harmonic measures and self-conformal measures ([Fen] and [PU]). For the backgrounds on this topic, see also [Fal] and references therein. We show that the measure μ_θ satisfies a Gibbs-like property with respect to $\beta(\theta)$, where $\beta(\theta)$ is an analogue of the pressure. This measure μ_θ is also characterised by the eigenmeasures of certain transfer operator built on a symbolic dynamical system associated with an automatic structure of the group. To study the measure μ_θ, we employ the results about the Martin boundary of a hyperbolic group by Izumi, Neshveyev and Okayasu [INO] and a generalised thermodynamic formalism due to Gouëzel [Gou1]. Note that the formula (2) is proved for every non-elementary hyperbolic group and for every finitely supported symmetric probability measure μ in [BHM2, Theorem 1.3], for every non-cyclic free groups and for every probability measure μ of finite first moment in [Led, Theorem 4.15], for a general class of random walks on trees in [Kai1] and for the simple random walks on the Galton-Watson trees in [LPP].

The following result is a finitistic version of Theorem 1.2, inspired by the corresponding results for the Galton-Watson trees by Lyons, Pemantle and Peres [LPP].

Theorem 1.3. *Let Γ and μ be as in Theorem 1.1, and consider the associated random walk starting at the identity on Γ. For every $a \in (0, 1)$, there exists a subset $\Gamma_a \subset \Gamma$ such that the random walk stays in Γ_a for every time with probability at least $1 - a$, and*

$$\lim_{n \to \infty} |\Gamma_a \cap S_n|^{1/n} = e^{h/l}.$$

In particular, if $h < lv$, then the random walk is confined in an exponentially small part of the group with positive probability. This can be compared with [Ver, p.669], where a random generation of group elements which is called the Monte Carlo method is discussed. For example, the random generation of group elements according to a random walk does not produce the whole data of the group in this case; see also [GMM2].

Let us return to Theorem 1.1. For a symmetric probability measure μ, i.e., $\mu(x) = \mu(x^{-1})$ for every $x \in \Gamma$, one can define the Green metric $d_G(x, y) = -\log F(x, y)$, where $F(x, y)$ denotes the probability that the random walk starting at x ever reaches y, and show that Γ is hyperbolic with respect to d_G according to the Ancona inequality [BHM2]. The Green metric d_G is not geodesic; nevertheless, one can use approximate trees argument and most of common techniques for the geodesic case work. The harmonic measure ν is actually a quasi-conformal (in fact, conformal) measure with respect to the metric induced in the boundary $\partial\Gamma$ by the Green metric d_G, and this fact plays an essential role to deduce Theorem 1.1 and that the local dimension of the harmonic measure ν is $h/(\varepsilon l)$ for ν-a.e. in Theorem 1.2 in the symmetric case. The symmetry of μ is required to make the Green metric d_G a genuine metric in Γ; otherwise it is not clear that one can discuss about the hyperbolicity for a non-symmetric metric. Here our alternative is to introduce a measure μ_θ, whose construction is actually inspired by a recent work of Gouëzel on the local limit theorem on a hyperbolic group [Gou1], and to obtain the Hausdorff spectrum of the harmonic measure ν. In many cases, a description of the Hausdorff spectrum is a main purpose on its own right, especially, when it is motivated by a problem in statistical physics. A fundamental observation in this paper, however, is rather the converse; namely, we use the multifractal analysis to compare the harmonic measure with a natural reference measure which is the D-Hausdorff measure on the boundary of the group Γ. More precisely, we shall see that the function β is affine on $\mathbb{R}$ if and only if those two measures are mutually absolutely continuous. Furthermore, the description of the Hausdorff spectrum implies that the harmonic measure has a rich multifractal structure as soon as it is singular with respect to the D-Hasudorff measure. In particular, the range of the Hausdorff spectrum contains the interval $[h/(\varepsilon l), v/\varepsilon]$.

Let us mention about an extension to a step distribution μ of unbounded support. The arguments in the present paper work once we have the Ancona inequality and its strengthened one. Gouëzel has proven those for every admissible probability measure μ with a super-exponential tail [Gou2]. At the same time, he has also proven a failure of description of the Martin boundary in the usual sense for an admissible probability measure with an exponential tail [ibid]. Hence the results in this paper are extended to every admissible step distribution μ with a super-exponential tail, but it is obscure whether one could extend to μ with an exponential tail in the present approach. We shall also mention about an extension to a left invariant metric which is not induced by a word length in Γ. For example, one is interested in the setting where Γ acts cocompactly on the hyperbolic space $\mathbb{H}^n$ and the metric in Γ is defined by $d(x, y) := d_{\mathbb{H}^n}(xo, yo)$ for a reference point o in $\mathbb{H}^n$. In fact, in [BHM2], they proved Theorem 1.1 for symmetric μ with every metric d which is hyperbolic and quasi-isometric to a word metric in Γ (not necessarily geodesic). Some of our results still hold for such a metric d, and in fact, most of the results are expected to remain valid; but we do not proceed to this direction in the present paper for the simplification of the proofs.

Finally, we close this introduction by pointing out some related problems in a continuous setting. On the special linear groups, the regularity problem of the harmonic measures is proposed by Kaimanovich and Le Prince [KL], and they showed that there exists a finitely supported symmetric probability measure (the support can generate a given Zariski dense subgroup) on $SL(d, \mathbb{R})$ $(d \geq 2)$ such that the corresponding harmonic measure is non-atomic singular with respect to a natural smooth measure class on the Furstenberg boundary. They proved this result via the dimension inequality of the harmonic measure. Bourgain constructed a finitely supported symmetric probability measure on $SL(2,\mathbb{R})$ such that the corresponding harmonic measure is absolutely continuous with respect to Lebesgue measure on the circle [Bou]. Brieussel and the author proved analogous results in the three dimensional solvable Lie group Sol [BT], namely, they showed that random walks with finitely supported step distributions on it can produce both absolutely continuous and singular harmonic measures with respect to Lebesgue measure on the corresponding boundary. The dimension inequality is also used there to prove the existence of a finitely supported probability measure whose harmonic measure is singular. We point out, however, the dimension equality of type (2) is still missing in both cases, and in general, in the Lie group settings to the extent of our knowledge.

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