β-Encoder: Symbolic Dynamics and Electronic Implementation for AD/DA Converters

Kohda, Tohru
Kyushu University : Professor Emeritus

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β-Encoders: Symbolic Dynamics and Electronic implementation for AD/DA converters

Tohru Kohda[1]

Extended Abstract: Almost all signal processing systems need analog signals to be discretized. Discretization in time and in amplitude are called *sampling* and *quantization*, respectively. These two operations constitute *analog-to-digital (A/D) conversion*. The A/D conversion includes *pulse-code modulation (PCM)* [1, 2, 3] and $\Sigma - \Delta$ modulation [4, 5, 6, 7, 8]. PCM has a precision of $O(2^{-L})$ for L iterations but has a serious problem when it is implemented in an electronic circuit, *e.g.,* if PCM has a threshold shift, then the quantization errors do not decay. In contrast, $\Sigma - \Delta$ modulation achieves a precision that decays like an inverse polynomial in L but has the practical advantage for analog circuit implementation.

In 2002, Daubechies *et al.*[15] introduced a new A/D converter using an amplifier with a factor β and a flaky quantizer with a threshold ν, known as a β-*encoder*, and showed that it has exponential accuracy even if it is iterated at each step in the successive approximation of each sample by using an imprecision quantizer with a quantization error and offset parameter, Furthermore, in a subsequent paper, Daubechies *et al.*[16] introduced a "flaky" version of an imperfect quantizer derfined as

$$Q^{\text{flaky}}_{[\nu_0,\nu_1]}(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad z < \nu_0, \\ 1, & \text{if} \quad z \geq \nu_1, \\ 0 \text{ or } 1, & \text{if} \quad z \in [\nu_0, \nu_1], \nu_0 < \nu_1 \end{cases} \tag{1}$$

which is a model of a quantizer $Q_\nu(z)$ with a varying threshold $\nu \in [\nu_0, \nu_1], \nu_0 < \nu_1$, defined as

$$Q_\nu(x) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad x < \nu, \\ 1, & \text{if} \quad x \geq \nu \end{cases} \tag{2}$$

They made the remarkable observation that "greedy"($\nu = \nu_{\rm G} = 1$) and "lazy"($\nu = \nu_{\rm L} = (\beta - 1)^{-1}$) expansions, as well as "cautious"($\nu_{\rm G} < \nu < \nu_{\rm L}$) expansions[2] in the β-encoder with such a flaky quantizer exhibit exponential accuracy in the bit rate L, and they gave the decoded values as

$$\hat{x}^{\rm DDGV}_{\rm L} = \sum_{i=1}^{L} b_i \gamma^i,\ b_i \in \{0, 1\},\ \gamma = \beta^{-1}. \tag{3}$$

[1]Professor Emeritus, Kyushu University, E-mail:torukoda81@wind.ocn.ne.jp

[2]*Intermediate expansions*[30, 32] between the greedy and lazy expansions[28, 29] are called "*cautious*" by Daubechies[16].

Furthermore, Daubechies and Yilmätz[17] proposed a β-encoder that is not only robust to quantizer imperfections but also robust with to the amplification factor β, and gave the β-recovery method that relies upon embedding the value of β in the encoded bit stream for each sample value separately without measureing its value. This β-encoder is a signinificant achievement in Nyquist-rate A/D and D/A conversions in the sense that it may become a good alternative for PCM [9, 10, 11].

In our recent paper[18], we gave comprehensive reviews for A/D conversions including PCM, $\Sigma - \Delta$ modulation, and β-encoder (see Fig.1 for its single-loop feedback form) as well as symbolic dynamics. [3] Furthermore, we gave the fact that β-encoders using a flaky quantizer with the threshold ν are characterized by the symbolic dynamics of the multi-valued Rényi-Parry map, defined as[22, 23]

$$T_\beta(x) = \beta x \bmod 1 \tag{4}$$

or Parry's (β, α)-map, defined as[24]

$$T_{\beta,\alpha}(x) = \beta x + \alpha \bmod 1 \tag{5}$$

in the middle interval (see Fig.2). Dynamical systems theory[37, 38] tells us that a sample x is always confined to a subinterval of a contracted interval, as shown in Fig.3 and so its decoded sample can be defined as [18, 19, 20],

$$\widehat{x}_L^{\rm KHA} = \sum_{i=1}^{L} b_i \gamma^i + \frac{\gamma^L}{2(\beta-1)},\ b_i \in \{0, 1\}, \tag{6}$$

because the decoded sample is equal to the midpoint of the subinterval. The decoded sample $\widehat{x}_L^{\rm KHA}$ also yields the characteristic equation for recovering β, which improves the quantization error by more than 3dB over the bound given by Daubechies *et al.*[16] and Daubechies and Yilmätz[17].[4]

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[3] Several tutorial papers and textbooks are available (see e.g.[9, 10, 11] for digital communication, [12, 13, 14] for the basics of dynamical systems theory, and [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] for β-transformation. See a review paper[18] and the detailed references cited therein for fundamental of quantization for digital communications and various AD/DA conversions and β-encoder fundamentals.

[4] Ward[21] has recently proposed new AD/DA algorithms for generating a binary sequence $\{b_i^{\rm Ward}\}_{i=1}^{\infty}, b_i^{\rm Ward} \in \{-1, 1\}$ for a real-valued $y \in (-1, 1)$ using a flaky version of an imperfect quantizer and gave its decoded value as $\widehat{y}_L^{\rm Ward} = \sum_{i=1}^{L} b_i^{\rm Ward} \gamma^i$. Ward's flaky quantizer is also realized exactly by the multi-valued Rényi-Parry map because it is topological conjugate to Parry's map:$T_{\beta,\alpha}(x)$ via the conjugacy $y = h(x) = 2x - (\beta - 1)^{-1}$.

In order to show the self-correction property of the amplification factor β in β-encoder, Daubechies and Yilmätz[17] presented an equation governed by the sample data bit sequences as follows. Using the β-expansion sequences $\{b_i\}_{i=1}^{L}$ for $x \in [0,1)$ and $\{c_i\}_{i=1}^{L}$ for $y = 1-x$, $1 \leq i \leq L$ yields a root of the *algebraic equation of* β, defined by

$$P_L^{\rm DY}(\gamma) = 1 - \sum_{i=1}^{L}(b_i + c_i)\gamma^i. \tag{7}$$

On the contray, our β-recovering equation with index p_L [18, 19, 20] is

$$P_L^{\rm KHA}(\gamma, p_L) = 1 - \sum_{i=1}^{L}(b_i + c_i)\gamma^i - p_L\frac{\gamma^{L+1}}{1-\gamma},\ p_L \in \{0,1,2\}, \tag{8}$$

which is based on an L-bit truncated expansion with index p_L, defined as

$$\widehat{x}_L^{\rm KHA}(\gamma, p_L) = \sum_{i=1}^{L} b_i\gamma^i + p_L \cdot \frac{\gamma^L}{2(\beta-1)} \tag{9}$$

The associated quantization error is bounded by

$$|x - \widehat{x}_L^{\rm KHA}(\gamma, p_L)| \leq \left(\frac{1+|p_L - 1|}{2}\right)\cdot(\beta-1)^{-1}\gamma^L \tag{10}$$

so that the cases where $p_L = 0, 1, 2$ correspond to the leftmost, intermediate, and rightmost points of the Lth subinterval, respectively; the case where $p_L = 0$ is equal to Dabechies et al.'s decoded value.

As thoroughly discussed in our recent paper[35], the probabilistic behavior of this flaky quantizer is explained by the deterministic dynamics of a *multi-valued Rényi-Parry map* on the middle interval[18, 19, 20] (see Fig.2). This map is an eventually locally onto map of $[\nu-1, \nu)$, which is topologically conjugate to Parry's (β, α)-map $T_{\beta,\alpha}(x)$ with $\alpha = (\beta-1)(\nu-1)$. β-encoders have a closed subinterval $[\nu-1, \nu)$, which includes an *attractor*[36, 37, 38]. This *β-expansion attractor*[35] seems to be irregularly oscillatory but performs the β-expansion of each sample stably and precisely (see Fig.3). This viewpoint allows us to obtain a decoded sample(eq.6 or eq.9), which is equal

The homeomorphism $\widehat{y}_L^{\rm Ward} = h(\widehat{x}_L^{\rm KHA})$, however, does not necessarily imply equivalence in terms of the quatization errors; in fact, Ward's algorirthm doubles the maximum quantization error and quadruples its mean square error.

to the midpoint of the subinteval, and its associated characteristic equation for recovering β (eq.8), and shows that ν should be set to around the midpoint of its associated greedy and lazy values. This leads us to design β-encoders realizing *ordinary* (see Fig.4) and *negative scaled β-maps*[20] (see Fig.5) and observe β-expansion attractors embedded in these β-encoders[35].

Finally, we note that parts of this article draw on our previous work in [18, 19, 20, 35], which were supported by the Aihara Innovation Mathematical Modelling Project (Aihara Project), the Japan Society for the Promotion of Science (JSPS) through the "Fundamental Program for World-Leading Innovation R&D on Science and Technology(FIRST Program)", initiated by the Counicil for Science and technology Policy (CSTP). The FIRST Program also supported the β-encoder group to implement these β-encoders in an LSI (Large-Scale Integrated) circuit and evaluate quantization errors and their performance in practically realized LSI circuits based on a simple β-recovery method suited to operation of AD/DA conversions in LSI cicuits, .[42, 43, 44, 45].

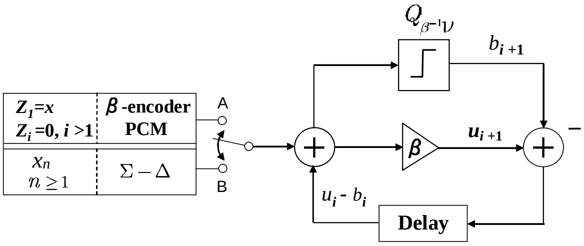

Fig.1. A discrete-time, single-loop feedback system using an amplifier with an ampflication factor β and a 1-bit quantizer $Q_{\beta^{-1}\nu}$ with a threshold ν that realizes PCM when $\beta = 2$ and $\nu = 1$; a β-encoder when $1 < \beta < 2$ and $\nu \in [1, (\beta-1)^{-1}]$, proposed by Daubechies et al.[15]; and $\Sigma-\Delta$ modulation when $\beta = 1$ and $\nu = 0$. The input is $z_1 = x \in [0, 1), z_i = 0, i > 1$ for the PCM and β-encoder, and the input is $x_n, n \geq 1$ for the $\Sigma - \Delta$ modulation. The initial conditions are given by $u_0 = b_0 = 0$. The output sequence $\{b_i\}_{i=1}^{L}, b_i \in \{0, 1\}$ gives the L-bit β-expansion for x, and the averaging sequence $\{b_i\}_{i=1}^{M}, b_i \in \{0, 1\}$ over i is the output in response to the input sequence $\{x_n\}_{n\geq1}$ for the $\Sigma - \Delta$ modulation. The β-encoder provides the greedy, lazy, and cautious schemes for $\nu = \nu_{\mathrm{G}} = 1, \nu = \nu_{\mathrm{L}} = (\beta - 1)^{-1}$, and $\nu_{\mathrm{G}} < \nu < \nu_{\mathrm{L}}$, respectively.

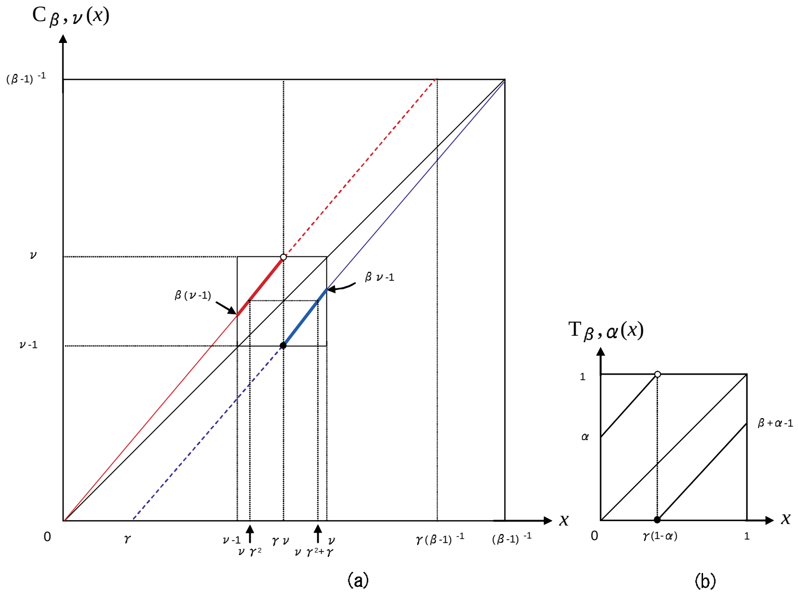

Fig.2. The expansion map $C_{\beta,\nu}(x) \stackrel{\text{def}}{=} \beta x - Q_{\beta^{-1}\nu}(x)$ realizing the Daubechies et al.'s flaky quantizer[16] $Q^{\text{flaky}}_{(\gamma\nu_{\text{G}},\gamma\nu_{\text{L}})}(z)$, $1 = \nu_{\text{G}} < \nu < \nu_{\text{L}} = (\beta-1)^{-1}$ [5] (b) renormalizing the interval $[\nu-1,\nu]$ into the unit interval $[0,1]$, which shows that such an eventually locally onto map is equivalent to the Parry (β,α)-transformation:$T_{\beta,\alpha}(x) = \beta x + \alpha \bmod 1$. The transformation $T_{\beta,\alpha}(x)$ has a finite (signed) invariant measure $\nu(E) = \int_{\text{E}} h(x)dx$, where $h(x)$ is given by $h(x) = \sum\limits_{x<T^n_{\beta,\alpha}(1)} \beta^{-n} - \sum\limits_{x<T^n_{\beta,\alpha}(0)} \beta^{-n}$.[24, 39]

[5] $C_{\beta,\nu}(x)$ has its *eventually locally onto* map with the *strongly invariant* subinterval $C^{-1}_{\beta,\nu}([0,\gamma\nu]) \cap C^{-1}_{\beta,\nu}([\gamma\nu,(\beta-1)^{-1}]) = [\nu-1,\nu]$. (Let $\tau : E \to E$ be a continuous map. Let $F \subset E$. If $\tau(F) \subset F$, then F is called *invariant*. If $\tau(F) = F$, then F is called *strongly invariant*. [14]).

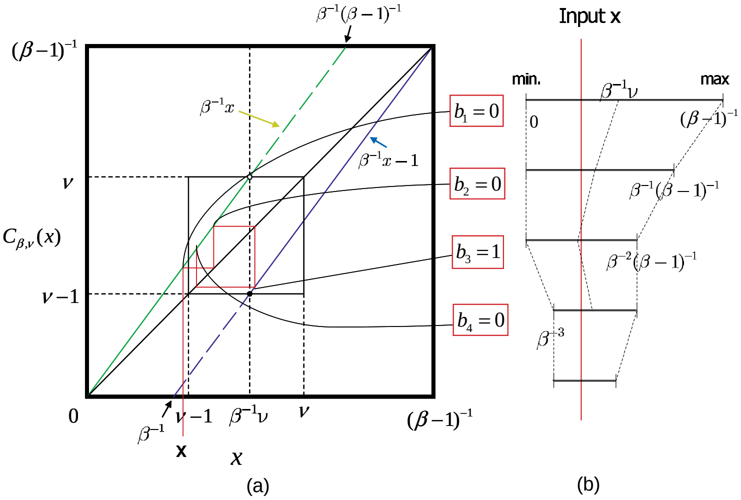

Fig.3. (a) The *multi-valued Rényi-Parry map* $C_{\beta,\nu}(x) \stackrel{\text{def}}{=} \beta x - Q_{\beta^{-1}\nu}(x)$ on the middle interval $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$ with its discontinuity $x = \beta^{-1}\nu$, which is *eventually locally onto* $[\nu-1, \nu)$, where $1 \leq \nu \leq (\beta-1)^{-1}$. An eventually locally onto map of $[\nu-1, \nu)$ with $\nu = 1 + \alpha/(\beta-1)$ is topologically conjugate to Parry's (β, α)-transformation $T_{\beta,\alpha}(x)$ via the conjugacy $\varphi^{-1}(x) = x + \alpha/(\beta-1)$, i.e., $\varphi(C_{\beta,\nu}(\varphi^{-1}(x)) = T_{\beta,\alpha}(x)$ when $\alpha = (\beta-1)(\nu-1)$. The map $C_{\beta,\nu}(x)$ realizes Daubechies et al.'s flaky quantizer[16] $Q^{\text{flaky}}_{[\beta^{-1},\beta^{-1}(\beta-1)^{-1}]}(z)$. (b) The contraction process by the first 4 binary β-expansions of the input x using $C_{\beta,\nu}(x)$ while the binary digits are obtained. The associated subintervals with a contraction ratio β^{-1} are given as $[0, (\beta-1)^{-1}), [0, \beta^{-1}(\beta-1)^{-1}), [0, \beta^{-2}(\beta-1)^{-1}), [\beta^{-3}, \beta^{-2}(\beta-1)^{-1})$. The input x is always confined to the ith subinterval.

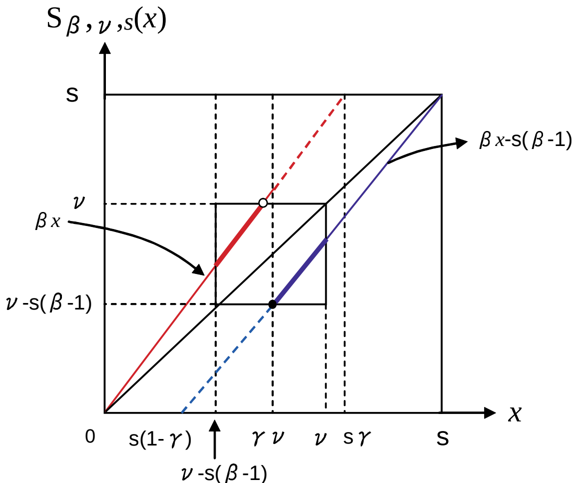

Fig.4. The *scale-adjusted ordinary β-map* $S_{\beta,\nu,s}(x) \stackrel{\text{def}}{=} \beta x - s(\beta-1)Q_{\gamma\nu}(x) = \begin{cases} \beta x & \text{when} \quad x \in [0, \gamma\nu), \\ \beta x - s(\beta-1), & \text{when} \quad x \in \gamma\nu, s), \end{cases}$ with its eventually locally onto map

$[\nu - s(\beta-1), \nu) \to [\nu - s(\beta-1), \nu)$, $\nu \in [s(\beta-1), s)$. Such an eventually locally onto map with $\nu = s(\alpha+\beta-1)$ is topologically conjugate to $T_{\beta,\alpha}(x)$ via the conjugacy $\varphi_{\mathrm{S}}^{-1}(x) = s(\beta-1)x + s\alpha$, i.e., $\varphi_S(S_{\beta,\nu,s}(x)) = T_{\beta,\alpha}(\varphi_{\mathrm{S}}(x))$.

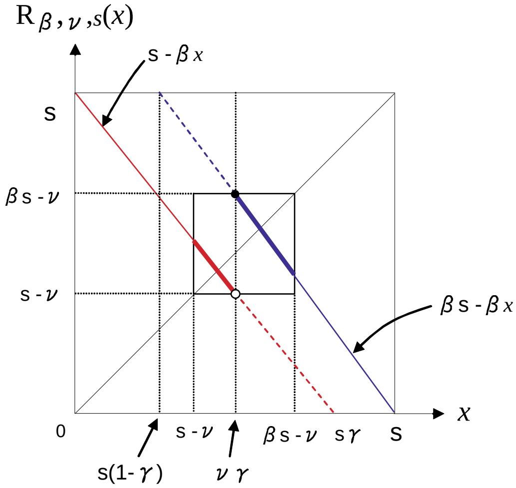

Fig.5. The *scale-adjusted negative β-map*

$$R_{\beta,\nu,s}(x) \stackrel{\text{def}}{=} -\beta x + s[1 + (\beta-1)Q_{\gamma\nu}(x)] = \begin{cases} s - \beta x & \text{when} \quad x \in [0, \gamma\nu), \\ \beta s - \beta x, & \text{when} \quad x \in \gamma\nu, s), \end{cases}$$

with its eventually locally onto map $[s-\nu, \beta s - \nu) \to [s - \nu, \beta s - \nu)$ when $(\beta^2 - \beta + 1)/(\beta+1)s \leq \nu < (2\beta-1)/(\beta+1)s$. [6] Such an eventually locally onto map with $\nu = s[(\beta-1)\alpha + \beta]/(\beta+1)$ is topologically conjugate to *Parry's transformation with negative slope*[20, 40] $T_{-\beta,\alpha}(x) \stackrel{\text{def}}{=} -\beta x + \alpha \bmod 1$, $\beta \geq 1, 0 \leq \alpha < 1$ via the conjugacy $\varphi_{\mathrm{R}}^{-1}(x) = s(\beta-1)x + s - \nu$, i.e., $\varphi_{\mathrm{R}}(R_{\beta,\nu,s}(x)) = T_{-\beta,\alpha}(\varphi_{\mathrm{R}}(x))$. The transformation $T_{-\beta,\alpha}(x)$ has a finite (signed) invariant measure $\nu(E) = \int_{\mathrm{E}} h(x)dx$, where $h(x)$ is given by $h(x) = \sum_{x < T^n_{-\beta,\alpha}(1)} (-\beta)^{-n} - \sum_{x < T^n_{-\beta,\alpha}(0)} (-\beta)^{-n}$.[41, 18]

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Workshop on β-transformation and related topics
At IMI, Kyushu University 3/10,2k15

β-encoders: Symbolic dynamics and Electronic Implementation for AD/DA converters

Tohru Kohda (Emeritus of Kyushu Univ.),
Collaborated with
Y.Horio (Tokyo Denki Univ.)
Y.Takahashi (Emeritus, the Univ. of Tokyo),
K.Aihara (the Univ. of Tokyo)

Contents

1) **β- encoder** for A/D, D/A conversion:
β-transformations and (β, α)-transformations realize flaky version of quantiser of AD-converter

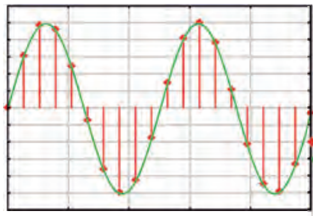
2) Renyi-Parry map with threshold has its eventually locally onto-map

i) **β-encoder** using Daubechies et.al.'s quantizer
ii) Scaled **β-encoder,** iii) Scaled negative **β-encoder**

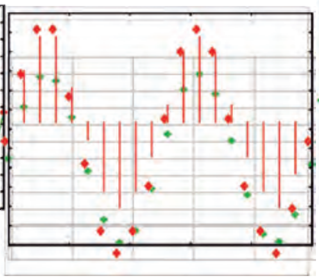
3) **β**-expansion and negative **β-expansion attractors** observed in A/D converters

Analog-to-Digital (A/D), Digital-to-Analog (D/A) conversion

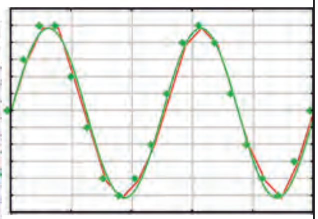
- **A/D D/A conversion are foundation for a variety of applications,**
  e.g., audio, image and communication etc.
- **Accuracy and stability**



sampling



quantization

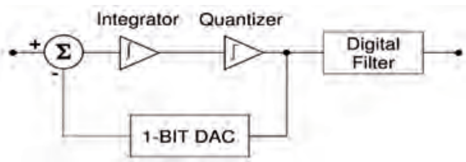


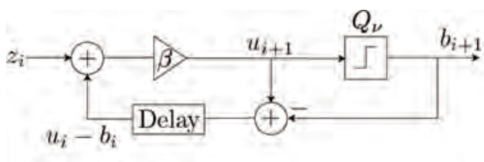
D/A conversion

quantize error
robust to fluctuation

Conventional methods of A/D, D/A conversion

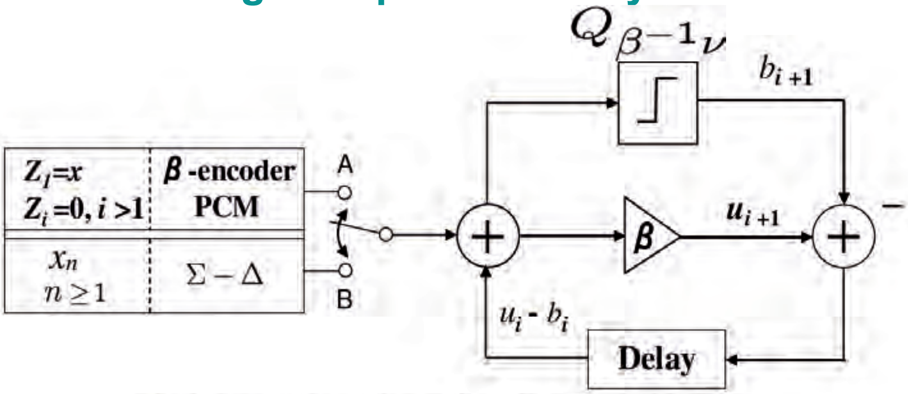
- **PCM has high precision, but doesn't possess stability.**
- **ΣΔ modulation**
  has stability,
  but its precision is lower than PCM .





β-encoder
high precision, stability

Several kinds of AD/DA-converters **in a single-loop feedback system**



"flaky" quantizer: $Q_{\beta^{-1}\nu}(\cdot)$ with threshold $\beta^{-1}\nu$

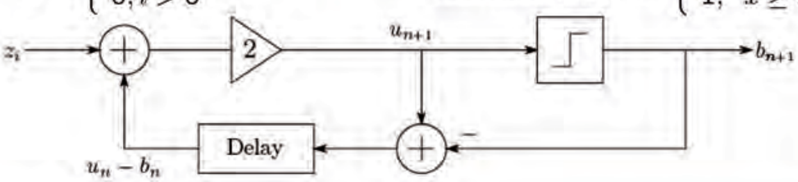
PCM when $\beta = 2, \nu = 1; L$ bits

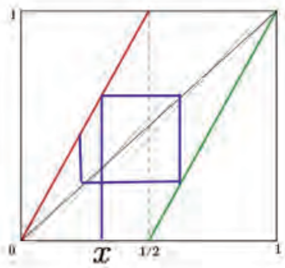
β-**encoder** when $1 < \beta < 2, 1 < \nu < (\beta - 1)^{-1}; L$ bits

$\Sigma - \Delta$ **modulation** when $\beta = 1, \nu = 0$; 1 bit

Pulse Code Modulation (PCM)

$$z_i = \begin{cases} x \text{ (input)}, & i = 0 \\ 0, & i > 0 \end{cases} \qquad \text{quantiser } Q_1(x) := \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$





Bernoulli shift (binary expansion):

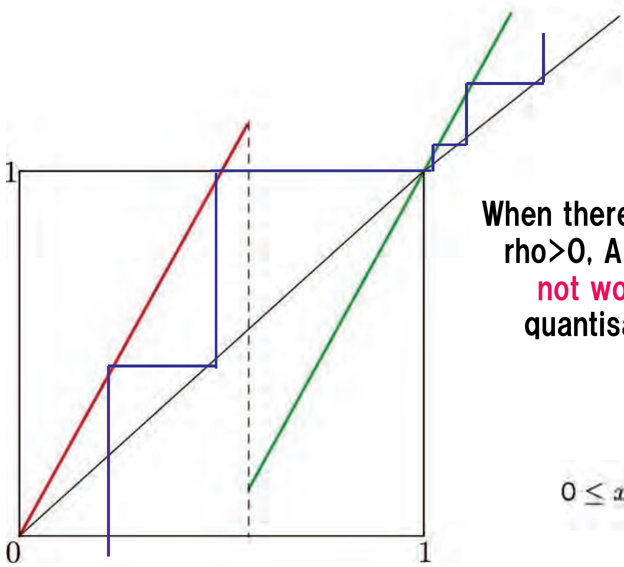
$$B(x) = \begin{cases} 2x, & x < 1/2 \\ 2x - 1, & x \geq 1/2 \end{cases}$$

$$b_i = \begin{cases} 0, & B^{i-1}(x) < 1/2 \\ 1, & B^{i-1}(x) \geq 1/2 \end{cases}$$

$$x = \sum_{i=1}^{\infty} b_i 2^{-i}$$

6

Divergence of a value x in PCM



When there is a threshold shift rho>0, A/D converter does not work well because quantisation errors don't decay

$$0 \le x - \sum_{i=1}^{\infty} b_i 2^{-i} < \frac{\rho}{2}$$

7

Background

Hardware implementation

- **Inose and Yasuda '64:** ΣΔ modulation
- **Gray '87 :** Oversampled ΣΔ modulation
- **Karanicolas '93:** A 15-b 1-Msamples/s Digitally self-Calibrated Pipelined ADC=pipelined PCM with self-correction

Ergodic theory

Renyi '57:f-expansion

- Parry '60,'64: β-expansion, (β, α) expansion

Takahashi '73, Ito & Takahashi '74, and Takahashi '83
Markov automorphism, Markov shift, orbit basis

Erdoes and Joo '90:
greedy and lazy expansions

Dajani and Kraikaamp '02, and Sidorov'02:
intermediate expansion
= (β, α) expansion

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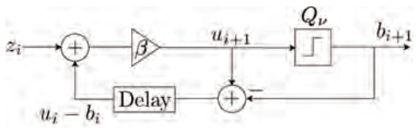
8

Hardware implementation
• Unsolved problems from mathematical standpoint

Ergodic theory

Daubechies et.al.,'06 Ⅰ: analyse



stability of β-encoder:

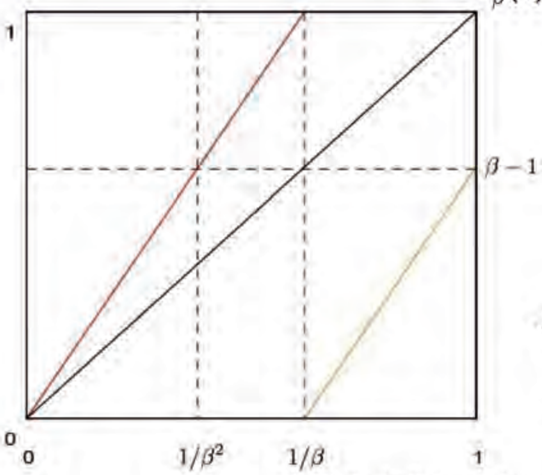
robustness to fluctuations to β of AMP and threshold v of quantiser Qv(･),

exponential accuracy in bit rate L

- Daubechies et.al., '06 Ⅱ: DA conversion using estimated β without knowing exact β Milestone !

β-transform (Renyi '57,Parry'60)

$$T_\beta(x) = \beta x \bmod 1, \beta > 1, \beta \notin \mathbf{Z}$$



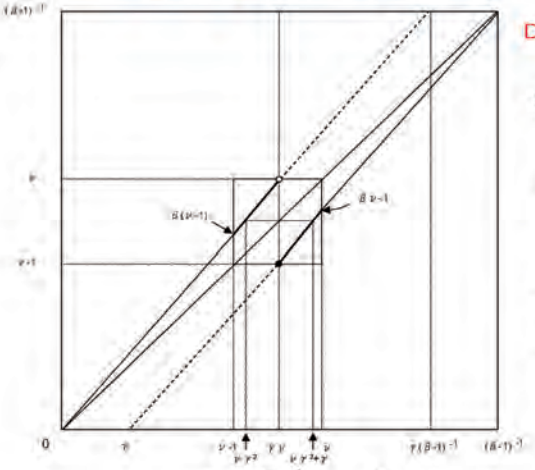
$$\beta - 1 = 1/\beta \Rightarrow$$

$$\beta = G = \frac{1+\sqrt{5}}{2}$$

$$P(\beta) = \begin{bmatrix} 1/\beta & 1-1/\beta \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \lambda(P) = \frac{1-\sqrt{5}}{2} < 0$$

Multi-valued Renyi-**Parry's** map realizing
flaky version of an imperfect quantizer



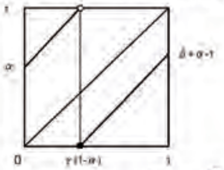
Daubechies *et.al.*'s quantiser(2k6):

$$Q^f{}_{[\nu_0,\nu_1]}(z) = \begin{cases} 0, & \text{if } z \le \nu_0 \\ 1, & \text{if } z \ge \nu_1 \\ 0 \text{ or } 1 & \text{if } z \in (\nu_0,\nu_1) \end{cases}$$

Parry's (β,α)-map

$$T_{\beta,\alpha}(x) = \beta x + \alpha \bmod 1,$$

$$\beta > 1, 0 \le \alpha < 1$$

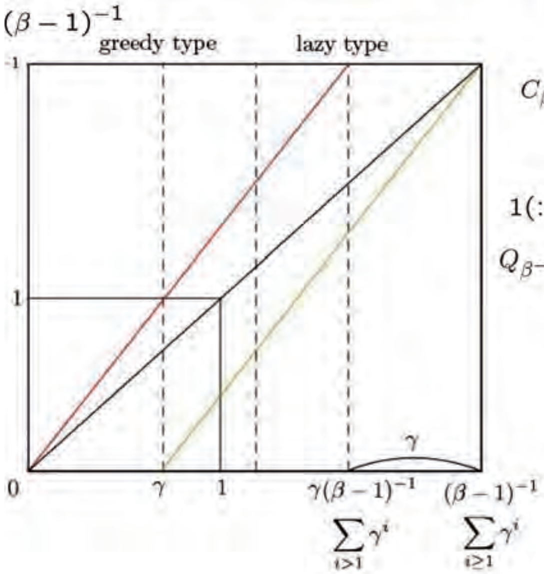


$$\alpha = (\beta - 1)(\nu - 1)$$

multi-valued Renyi—Parry Map and
its eventually locally onto-map

11

Cautious map $C_{\beta,\nu}(x)$



$$C_{\beta,\nu}(x) = \beta x - Q_{\beta^{-1}\nu}(x)$$

$$= \begin{cases} \beta x, & x < \beta^{-1}\nu \\ \beta x - 1, & x \ge \beta^{-1}\nu \end{cases}$$

$1(:\text{gr}) \le \nu \le (\beta-1)^{-1}(:\text{la})$

$$Q_{\beta^{-1}\nu}(x) = \begin{cases} 0, & x < \beta^{-1}\nu \\ 1, & x \ge \beta^{-1}\nu \end{cases}$$

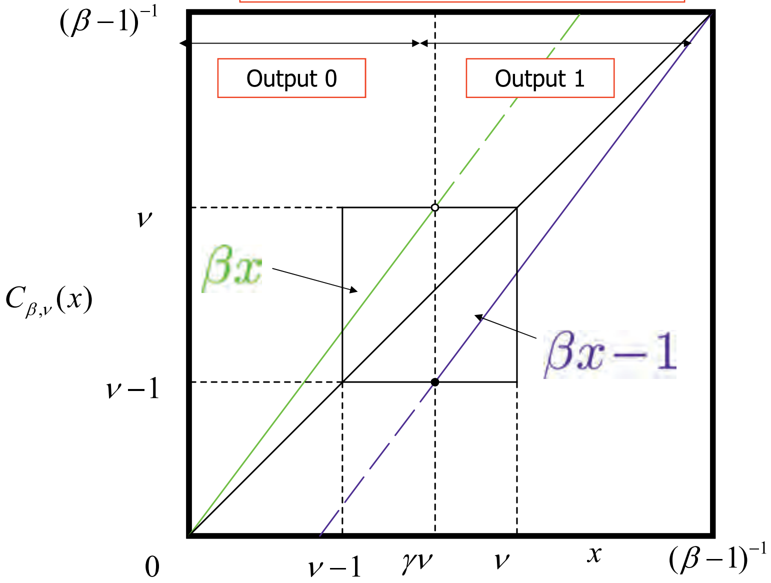
$$b_i = \begin{cases} 0, & C^{i-1}_{\beta,\nu}(x) < \beta^{-1}\nu \\ 1, & C^{i-1}_{\beta,\nu}(x) \ge \beta^{-1}\nu \end{cases}$$

$$x = \sum_{i=1}^{L} b_i\gamma^i + \gamma^L C^L_{\beta,\nu}(x)$$

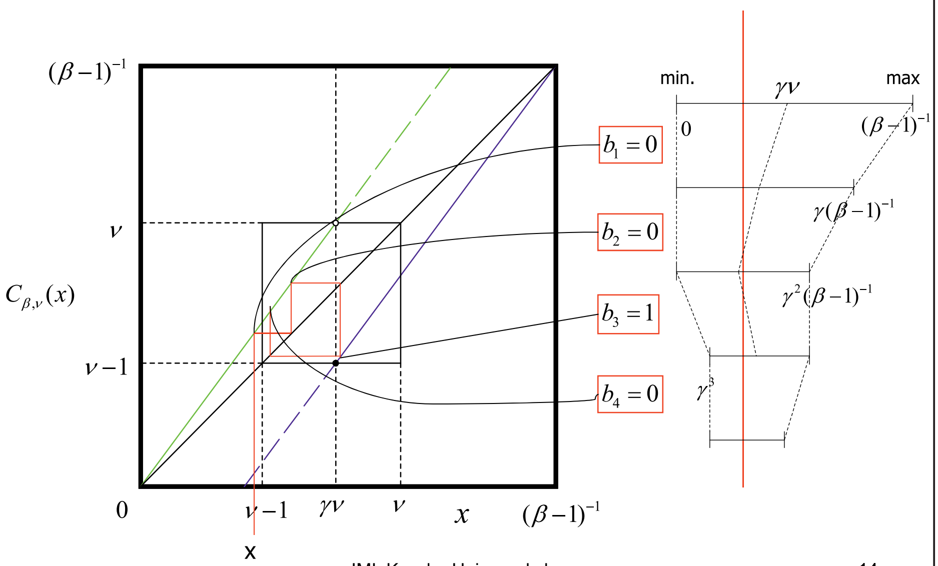
$$\gamma = \beta^{-1}$$

12

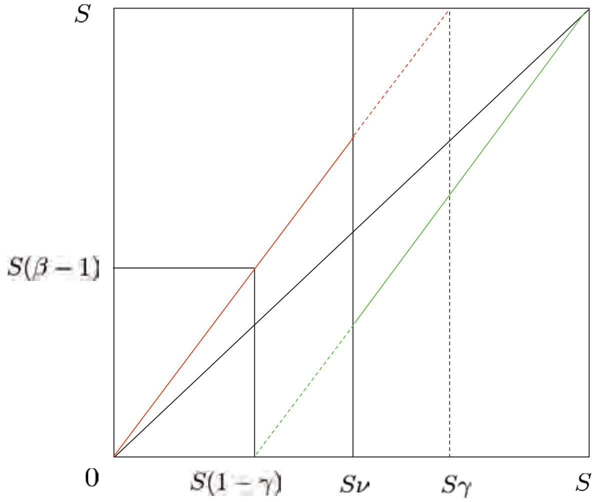
Eventually locally onto-map



Contraction process of subintervals



scaling map of β-expansion

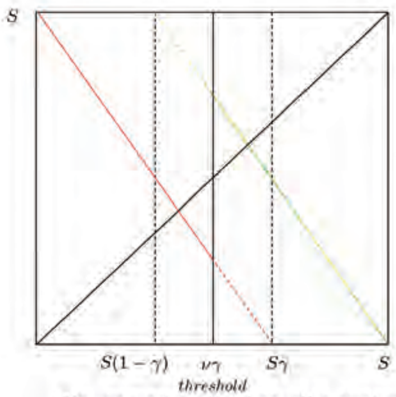


bit budget L,
threshold's tolerance:
$\sigma_{\beta,s} = s(2-\beta)$:

$\beta_{\text{opt.}} = \frac{2L}{L+1}$
$s = \sigma_{\beta,s}(L+1)/2$

Eventually locally onto-map is topologically conjugate to $T_{\beta,\alpha}(x)$

Negative beta-expansion (2k9)



$$R_{\beta,\nu,s}(x) := \begin{cases} s-\beta x, & x \in [0,\nu\gamma) \\ \beta s-\beta x, & x \in [\nu\gamma, s) \end{cases}$$

$s > 0$, $\nu \in [s(\beta-1), s]$

$$x = (-\gamma)^L R^L(x) - s\sum_{i=1}^{L} (b_i\beta + \overline{b_i})(-\gamma)^i$$

$$\hat{x} = s\{\frac{(-\gamma)^L}{2} - \sum_{i=1}^{L} (b_i\beta + \overline{b_i})(-\gamma)^i\}$$

Decoded value

Its eventually locally onto map is topologically conjugate to
Parry $(-\beta,\alpha)$-map: $T_{-\beta,\alpha}(x) = -\beta x + \alpha \bmod 1, \beta > 1, 0 \le \alpha < 1$

Main result Ⅰ :β-decoding using interval analysis（2k7）

Theorem 1: The decoded value $\tilde{x}$ given by the interval analysis is defined by

$$\tilde{x} = \sum_{i=1}^{L} b_i \gamma^i + \boxed{\frac{\gamma^{L+1}}{2(1-\gamma)}},$$

Dust (but essential)

which gives

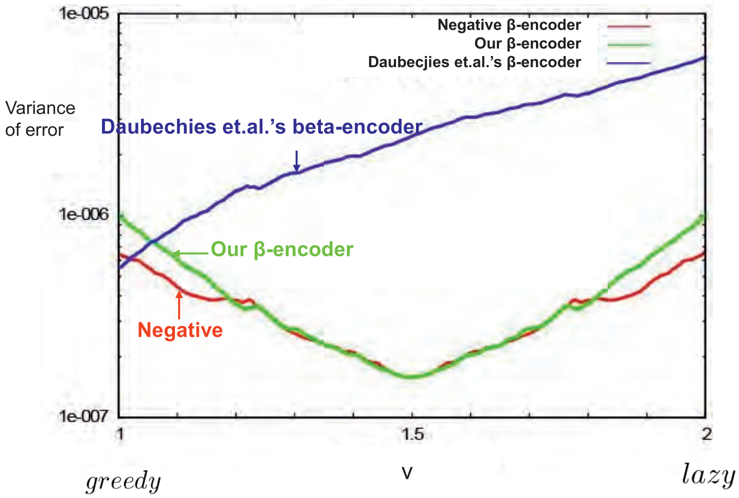
$$0 \le |x - \tilde{x}| \le \frac{(\beta-1)^{-1}\gamma^L}{2} < \gamma^L \le \underline{\nu\gamma^L} \quad \text{when } \beta > 3/2.$$

3dB improved when β> 3/2

cf.)Daubechies et.al.,: $0 \le |x - \tilde{x}_{Dau}| \le \nu\gamma^L$. $\tilde{x}_{Dau} = \sum_{i=1}^{L} b_i \gamma^i$.

Quantization Error

$(\beta = 1.5, s = (\beta - 1)^{-1}, L = 16)$



Main result 2: Characteristic equation for β reconstruction（2k7）

sequence $\{b_j\}_{j=1}^{L}$ for x and sequence $\{c_j\}_{j=1}^{L}$ for $y = 1 - x$

$$\tilde{x} = \sum_{j=1}^{L} b_j \gamma^j + \frac{\gamma^{L+1}}{2(1-\gamma)}, \quad \tilde{y} = \sum_{j=1}^{L} c_j \gamma^j + \frac{\gamma^{L+1}}{2(1-\gamma)},$$

Dust term Dust term

Daubechies et.al.'s idea

Since $\tilde{x} + \tilde{y} = 1$, the estimated value of γ is a root of $P(\gamma)$, referred to as characteristic equation of γ, defined by

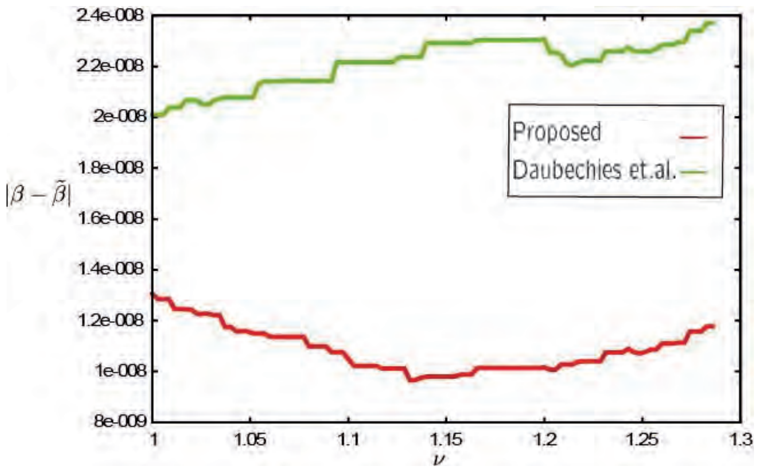
$$P(\gamma) = 1 - \sum_{i=1}^{L} (b_i + c_i)\gamma^i - \frac{\gamma^{L+1}}{1-\gamma} = 0.$$

Dust term

cf.) $P_{Dau}(\gamma) = 1 - \sum_{i=1}^{L} (b_i + c_i)\gamma^i = 0.$

19

The precision of beta estimation



For $N = 32$ and $\beta = 1.77777$, the worst precision of the estimation for β when varing x and ν.

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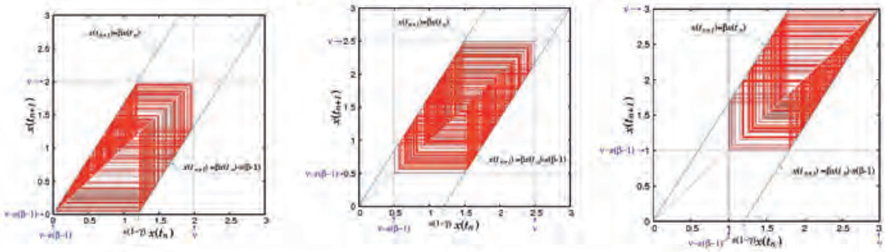
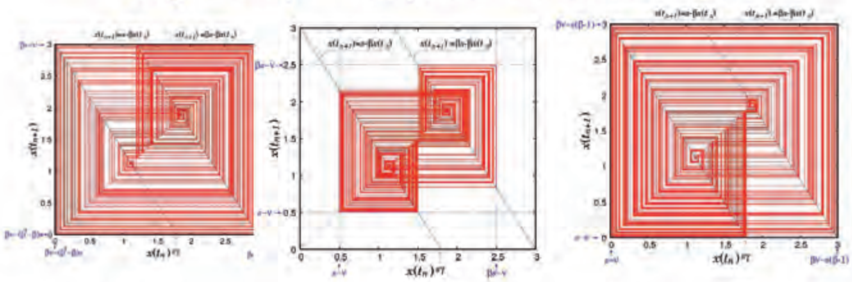
β − expansion attractors



(a) (b) (C)

Examples of the chaotic attractors obtained from the scale-adjusted β-encoder circuit.
(a) The greedy expansion attractor,
(b) the cautious expansion attractor, and
(c) the lazy expansion attractor.

Negative β − expansion attractors



(a) (b) (C)

Examples of the chaotic attractors obtained from the negative β-encoder circuit.
(a) The greedy expansion attractor,
(b) the cautious expansion attractor, and
(c) the lazy expansion attractor.

Implementation of β-encoder in an LSI (Large-Scale Integrated) circuit

- We note that parts of this article draw on our previous work:

1) T. Kohda ,Y. Horio, Y. Takahashi, and K. Aihara,
"Beta Encoders: Symbolic Dynamics and Electronic Implementation ",
Int. Journal of Bifurcation and Chaos, **22, no9**, 2012,

2) T. Kohda ,Y.Horio, and K. Aihara,"**β-** Expansion Attractors observed in A/D Converters", AIP Chaos,**22**,no.4,2012,

supported by the Aihara Project, JSPS through FIRST Program.

The FIRST Program also supported the **β-** encoder group to implement **β-**encoders in an LSI circuit and evaluate quantization errors and their performance in practically realized LSI circuits using a simple **β-**recovery method suited to operation of AD/DA converters in LSI circuits,

T.Makino, Y.Iwata, K.Shinohara, Y.Jitsumatsu, M.Hotta, H.San, and K.Aihara,
"Rigorous Estimates of Quantization error and Adaptive Decoding Scheme for an A/D Converter Based on a Beta-Map," NOLTA (Nonlinear Theory and Its Applications), IEICE, 6-1,99-111,2015.