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BETA EXPANSION WITH ROTATION

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A JOINT WORK WITH JONATHAN CAALIM.

The beta transform

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor$$

gives a generalization of binary and decimal expansions to a real base $\beta > 1$. Its ergodic property is well known:

- There is a unique absolutely continuous invariant probability measure (ACIM) equivalent to 1-dim Lebesgue measure [17]
- The invariant measure is made explicit ([16, 7])
- The system is exact, consequently it is mixing of any degree.
- Its natural extension is Bernoulli.

When β is not an integer, the digits $\{0, 1, \dots, \lfloor \beta \rfloor\}$ are not independent. There are many studies on the associated symbolic dynamics, in particular, when it becomes SFT, sofic, specification, etc. They are described by the forward orbit of the discontinuity $1 - 0$, but not so easy to give algebraic criteria of them. For example, if β is a Pisot number, then the system is sofic, but it is not easy to characterize SFT cases among them. Here Pisot number is an algebraic integer greater than one whose all other conjugates have modulus less than one.

Number theoretical generalizations had been studied by means of numeration system in complex bases, e.g., [10, 4, 2, 13]. In this talk, we wish to generalize beta expansion in a dynamical way to the complex plane introducing rotation action. Let $1 < \beta \in \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$. Fix $\xi, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1/\eta_2 \notin \mathbb{R}$. Then $\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0, 1), y \in [0, 1)\}$ is a fundamental domain of the lattice $\mathcal{L}$ generated by η_1 and η_2 in $\mathbb{C}$, i.e.,

$$\mathbb{C} = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d)$$

is a disjoint partition of $\mathbb{C}$. Define a map $T : \mathcal{X} \to \mathcal{X}$ by $T(z) = \beta\zeta z - d$ where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta\zeta z \in \mathcal{X} + d$. Given a point z in $\mathcal{X}$, we obtain an expansion

$$\begin{aligned} z &= \frac{d_1}{\beta\zeta} + \frac{T(z)}{\beta\zeta} \\ &= \frac{d_1}{\beta\zeta} + \frac{d_2}{(\beta\zeta)^2} + \frac{T^2(z)}{(\beta\zeta)^2} \\ &= \sum_{i=1}^{\infty} \frac{d_i}{(\beta\zeta)^i} \end{aligned}$$

with $d_i = d(T^{i-1}(z))$. In this case, we write $d_T(z) = d_1 d_2 \ldots$. We call T the *rotational beta transformation* and $d_T(z)$ the *expansion* of z with respect to T. We note that the map T generalizes the notions of beta expansion [17, 16, 7] and negative beta expansion [6, 15, 8] in a natural dynamical manner to the complex plane $\mathbb{C}$.

Since T is a piecewise expanding map, by a general theory developed in [11, 12, 5, 18, 19, 3, 20], there exists an invariant probability measure μ which is absolutely continuous to the two-dimensional Lebesgue measure. The number of ergodic components is known to be finite [11, 5, 18]. An explicit upper bound in terms of the constants in Lasota-Yorke type inequality was given by Saussol [18]. However this bound may be large. We shall give two explicit constants B_1 and B_2 depending only on η_1 and η_2 that T has a unique ACIM if $\beta > B_1$. Further if $\beta > B_2$ then the ACIM is equivalent to the 2-dimensional Lebesgue measure on $\mathcal{X}$. Note that if β is small, then we can give examples of T's with at least two ergodic ACIM's. An interesting remaining problem is to improve B_1 and B_2. We feel that they are still far from best possible.

For general cases, it is difficult to make explicit the Radon-Nikodym density of the ACIM. It is of interest to study when the symbolic system associated to the rotational beta expansion is sofic, where we can compute the density explicitly.

Restricting to a rotation generated by q-th root of unity ζ with all parameters in $\mathbb{Q}(\zeta, \beta)$, it gives a sofic system when $\cos(2\pi/q) \in \mathbb{Q}(\beta)$ and β is a Pisot number. It is interesting to point out that this result gives examples of sofic rotational expansion with any finite order rotation, like 7-fold or 11-fold.

We will also also show that the condition $\cos(2\pi/q) \in \mathbb{Q}(\beta)$ is necessary by giving a family of non-sofic systems for $q = 5$. Anyway this gives a sufficient condition of soficness but it is not necessary. It is of interest to characterize sofic cases.

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