On Parameter Estimation of Autoregressive Process in the Presence of Noise

Jia, Li-Juan
Department of Electrical and Electronic Systems Engineering, Graduate School of Information Science and Electrical Engineering, Kyushu University : Doctoral Student

Jin, Chun-Zhi
Department of Electrical and Electronic Systems Engineering, Faculty of Information Science and Electrical Engineering, Kyushu University

Yang, Zi-Jiang
Department of Electrical and Electronic Systems Engineering, Faculty of Information Science and Electrical Engineering, Kyushu University

Wada, Kiyoshi
Department of Electrical and Electronic Systems Engineering, Faculty of Information Science and Electrical Engineering, Kyushu University

https://doi.org/10.15017/1515734

出版情報：九州大学大学院システム情報科学紀要. 6 (2), pp.185-190, 2001-09-26. 九州大学大学院システム情報科学研究院
バージョン：

九州大学大学院
システム情報科学紀要
第6巻 第2号 平成13年9月

Research Reports on Information Science and
Electrical Engineering of Kyushu University
Vol.6, No.2, September 2001

On Parameter Estimation of Autoregressive Process in the Presence of Noise

Li-Juan JIA*, Chun-Zhi JIN**, Zi-Jiang YANG** and Kiyoshi WADA **

(Received June 15, 2001)

Abstract: The paper studies the problem of parameter estimation for autoregressive (AR) process in the presence of white observation noise. A new type of bias compensated least-square (BCLS) algorithm is proposed to obtain consistent parameter estimate for AR models. The main feature of the proposed algorithm is that an auxiliary backward output parameter estimator is introduced in order to estimate the variance of observation noise. The proposed algorithm compensates the bias via the estimated variance of observation noise and hence yields a consistent parameter estimate. Some comments are given to illustrate that the proposed algorithm is less computational burden and more numerically reliable. Numerical results are provided to support these comments.

Keywords: Autoregressive process, Parameter estimation, Bias compensation, Asymptotic bias

1. Introduction

The problem of parameter estimation of autoregressive (AR) process with output data corrupted by white noise is studied. In this paper a new type of bias compensated least squares (BCLS) algorithm is proposed based on the bias compensation principle which is proposed by Sagara and Wada [1].

It is known that LS estimate is biased for AR parameter estimation in the presence of white observation noise [2),3),4),5]. Theoretical analysis shows that the bias of the LS estimate is caused by the observation noise. If observation noise variance is known, the BCLS method can be directly used to simply produce consistent estimate. However in the practical case, the observation noise variance is unknown. So if the observation noise variance can be estimated, the BCLS algorithm can compensate the bias via the estimate of the observation noise variance and hence yield consistent parameter estimates. Therefore the key of the BCLS algorithm is how to estimate the observation noise variance.

Over the last decades, many efforts have been devoted to this problem and several methods have been proposed to estimate the unknown noise variance [2),4),5]. In Zheng's method which is named the ILS method [5], prefiltering the noisy data or increasing the order of the underlying AR model is used to obtain the estimate of noise variance. On the other hand, Sakai and Arase' algorithm is proposed based on the BCLS method, in which an output noise variance estimation algorithm is derived [2]. Wada and Eguchi proposed a modified version of Sakai and Arase's algorithm. Several simulation results are given to indicate that Wada and Eguchi's algorithm is more effective than Sakai and Arase's from statistical and computational points of view [4].

In this paper, a straightforward variance estimation algorithm is developed to obtain a new type of BCLS algorithm to give the consistent parameter estimate. We introduce a backward output parameter estimator to derive the algorithm for observation noise variance. Computer simulations are carried out under various noise levels. The results illustrate that the proposed algorithm is accurate even though in high noise level. It is shown that the proposed algorithm is more effective and simpler than the other existing algorithms from a computational point of view.

The paper is organized as follows. In the next section, the problem statement is presented and the bias of LS estimate resulting from observation noise is described. In section 3, Wada and Eguchi's algorithm is reviewed briefly and the new type of BCLS algorithm is derived. Several comments on Wada and Eguchi's algorithm and the new proposed algorithm are also given. Section 4 summarizes the implementation procedure of the new proposed BCLS algorithm. Simulation results are presented in section 5 and section 6 summarizes the conclusions.

2. Problem Statement

Consider a noisy AR process $\{x_t\}$ with pth order described as

* Department of Electrical and Electronic Systems Engineering, Doctoral Student

** Department of Electrical and Electronic Systems Engineering

$$x_t = \sum_{i=1}^{p} a_i x_{t-i} + e_t \tag{1}$$
$$y_t = x_t + n_t \tag{2}$$

where $\{e_t\}$ is a zero-mean driving white noise sequence with finite variance σ_e^2, and $\{n_t\}$, which represents observation noise, is a white noise sequence with finite variance σ_n^2.

Assume that $\{e_t\}$ and $\{n_t\}$ are mutually uncorrelated , then we have

$$\left.\begin{aligned} E[e_k] = 0, E[e_k^2] = \sigma_e^2; \\ E[n_k] = 0, E[n_k^2] = \sigma_n^2; \\ E[e_k n_k] = 0. \end{aligned}\right\} \tag{3}$$

The problem under study is, using noisy observations y_t, to make a consistent estimation of the AR model parameters.
Let

$$\boldsymbol{a}^T = [a_1\ a_2 \dots\ a_p] \tag{4}$$
$$\boldsymbol{x}_t^T = [x_{t-1}\ x_{t-2}\ \dots x_{t-p}] \tag{5}$$
$$\boldsymbol{y}_t^T = [y_{t-1}\ y_{t-2} \cdots\ y_{t-p}] \tag{6}$$

we have

$$y_t = \boldsymbol{x}_t^T \boldsymbol{a} + e_t + n_t \tag{7}$$
$$= \boldsymbol{y}_t^T \boldsymbol{a} + v_t \tag{8}$$

where

$$v_t = e_t + n_t - \boldsymbol{n}_t^T \boldsymbol{a} \tag{9}$$
$$\boldsymbol{n}_t^T = [n_{t-1}\ n_{t-2} \cdots\ n_{t-p}] \tag{10}$$

The LS estimate is given by

$$\widehat{\boldsymbol{\alpha}}_N = \Big[\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_t^T\Big]^{-1} \sum_{t=1}^{N} \boldsymbol{y}_t y_t. \tag{11}$$

Substituting (8) into (11) yields

$$\widehat{\boldsymbol{\alpha}}_N = \boldsymbol{a} + \Big[\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_t^T\Big]^{-1} \sum_{t=1}^{N} \boldsymbol{y}_t v_t. \tag{12}$$

Taking probability limit, we have

$$\operatorname*{plim}_{N\to\infty} \widehat{\boldsymbol{\alpha}}_N = \boldsymbol{a} + E[\boldsymbol{y}_t \boldsymbol{y}_t^T]^{-1} E[\boldsymbol{y}_t v_t]. \tag{13}$$

This equation indicates that in the case of output data corrupted by white noise, the LS estimate is biased for AR parameter estimation.

The bias of the LS estimate is described by

$$\boldsymbol{b} = E[\boldsymbol{y}_t \boldsymbol{y}_t^T]^{-1} E[\boldsymbol{y}_t v_t]. \tag{14}$$

From (3), we can obtain the following expression:

$$E[\boldsymbol{y}_t v_t] = -\sigma_n^2 \boldsymbol{a}. \tag{15}$$

Thus the estimate of the asymptotic bias can be obtained as

$$\widehat{\boldsymbol{b}}_N = -N P_N \sigma_n^2 \widehat{\boldsymbol{a}}_{N-1} \tag{16}$$

where

$$P_N = \Big[\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_t^T\Big]^{-1}. \tag{17}$$

3. BCLS Algorithm

According to bias compensation principle which is proposed by Sagara and Wada[1)] , the BCLS estimate is given by

$$\widehat{\boldsymbol{a}}_N = \widehat{\boldsymbol{\alpha}}_N + N \widehat{\sigma}_n^2 P_N \widehat{\boldsymbol{a}}_{N-1}. \tag{18}$$

It is clear that if we obtain the estimate of observation noise variance $\widehat{\sigma}_n^2$, we can give the consistent BCLS eatimate. In the following we will review the BCLS algorithm proposed by Wada and Eguchi and describe the new type of BCLS algorithm.

3.1 Wada and Eguchi's Algorithm

Introduce an estimator $\widehat{\varphi}_N$ defined by the following equation:

$$\widehat{\varphi}_N = \Big(\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_{t-1}^T\Big)^{-1} \sum_{t=1}^{N} \boldsymbol{y}_t y_{t-1}. \tag{19}$$

The recursive algorithm for $\widehat{\varphi}_N$ is given by

$$\widehat{\varphi}_N = \widehat{\varphi}_{N-1} + \frac{Q_{N-1} \boldsymbol{y}_N (y_{N-1} - \boldsymbol{y}_{N-1}^T \widehat{\varphi}_{N-1})}{1 + \boldsymbol{y}_{N-1}^T Q_{N-1} \boldsymbol{y}_N} \tag{20}$$

$$Q_N = Q_{N-1} - \frac{Q_{N-1} \boldsymbol{y}_N \boldsymbol{y}_{N-1}^T Q_{N-1}}{1 + \boldsymbol{y}_{N-1}^T Q_{N-1} \boldsymbol{y}_N} \tag{21}$$

where

$$Q_N = \left(\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_{t-1}^T \right)^{-1}.$$

From (19), we have

$$\sum_{t=1}^{N} \boldsymbol{y}_t (y_{t-1} - \boldsymbol{y}_{t-1}^T \widehat{\boldsymbol{\varphi}}_N) = \mathbf{o}. \tag{22}$$

Define an error fuction g_N as

$$g_N = \sum_{t=1}^{N} (y_t - \boldsymbol{y}_t^T \widehat{\boldsymbol{\alpha}}_N)(y_{t-1} - \boldsymbol{y}_{t-1}^T \widehat{\boldsymbol{\varphi}}_N). \tag{23}$$

Since

$$y_t - \boldsymbol{y}_t^T \widehat{\boldsymbol{\alpha}}_N = \boldsymbol{y}_t^T (\boldsymbol{a} - \widehat{\boldsymbol{\alpha}}_N) + v_t, \tag{24}$$

g_N can be rewritten as

$$g_N = \sum_{t=1}^{N} v_t (y_{t-1} - \boldsymbol{y}_{t-1}^T \widehat{\boldsymbol{\varphi}}_N). \tag{25}$$

Taking probability limits of (25), we have

$$\operatorname*{plim}_{N\to\infty} \frac{g_N}{N} = \sigma_n^2 (a_1 + \boldsymbol{a}^T H \operatorname*{plim}_{N\to\infty} \widehat{\varphi}_N) \tag{26}$$

where H is a $p \times p$ shift matrix defined by

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & \ddots & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then the observation noise variance estimate can be given as

$$\widehat{\sigma}_n^2 = \frac{g_N}{N(\widehat{a}_{1,N-1} + \widehat{\boldsymbol{a}}_{N-1}^T H \widehat{\boldsymbol{\varphi}}_N)}. \tag{27}$$

The BCLS estimate becomes

$$\widehat{\boldsymbol{a}}_N = \widehat{\boldsymbol{\alpha}}_N + \frac{g_N}{\widehat{a}_{1,N-1} + \widehat{\boldsymbol{a}}_{N-1}^T H \widehat{\boldsymbol{\varphi}}_N} P_N \widehat{\boldsymbol{a}}_{N-1}. \tag{28}$$

3.2 The new type of BCLS algorithm

Now we describe the new type of BCLS algorithm.

Introduce an auxiliary backward output parameter estimate $\widehat{\boldsymbol{\beta}}_N$ defined as

$$\widehat{\boldsymbol{\beta}}_N = \left[\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_t^T \right]^{-1} \sum_{t=1}^{N} \boldsymbol{y}_t y_{t-p-1}, \tag{29}$$

then

$$\sum_{t=1}^{N} \boldsymbol{y}_t (y_{t-p-1} - \boldsymbol{y}_t^T \widehat{\boldsymbol{\beta}}_N) = \mathbf{o}. \tag{30}$$

Denote an error function f_N as

$$f_N = \sum_{t=1}^{N} (y_t - \boldsymbol{y}_t^T \widehat{\boldsymbol{\alpha}}_N)(y_{t-p-1} - \boldsymbol{y}_t^T \widehat{\boldsymbol{\beta}}_N). \tag{31}$$

Since

$$y_t - \boldsymbol{y}_t^T \widehat{\boldsymbol{\alpha}}_N = \boldsymbol{y}_t^T (\boldsymbol{a} - \widehat{\boldsymbol{\alpha}}_N) + v_t, \tag{32}$$

f_N can be rewritten as

$$f_N = \sum_{t=1}^{N} v_t (y_{t-p-1} - \boldsymbol{y}_t^T \widehat{\boldsymbol{\beta}}_N). \tag{33}$$

Taking probability limits of (33), we have

$$\begin{aligned} \operatorname*{plim}_{N\to\infty} \frac{f_N}{N} &= E[v_t n_{t-p-1}] - E[v_t \boldsymbol{n}_t^T] \operatorname*{plim}_{N\to\infty} \widehat{\boldsymbol{\beta}}_N \\ &= \sigma_n^2 \boldsymbol{a}^T \operatorname*{plim}_{N\to\infty} \widehat{\boldsymbol{\beta}}_N. \end{aligned} \tag{34}$$

Then the estimate of the observation noise variance $\widehat{\sigma}_n^2$ can be given as

$$\widehat{\sigma}_n^2 = \frac{f_N}{N \widehat{\boldsymbol{a}}_{N-1}^T \widehat{\boldsymbol{\beta}}_N}. \tag{35}$$

The BCLS-B estimate becomes

$$\widehat{\boldsymbol{a}}_N = \widehat{\boldsymbol{\alpha}}_N + \frac{f_N}{\widehat{\boldsymbol{a}}_{N-1}^T \widehat{\boldsymbol{\beta}}_N} P_N \widehat{\boldsymbol{a}}_{N-1}. \tag{36}$$

3.3 Comments on the two BCLS based algorithms

We now make some comments on the two BCLS based algorithms. In order to distinguish between these algorithms, we call the proposed algorithm BCLS-B (backward) algorithm.

Like Wada and Eguchi's algorithm, in the BCLS-

B algorithm the auxiliary estimator is also introduced to obtain the output noise variance. But in Wada and Eguci's approach, a product moment matrix in the introduced auxiliary estimate

$$\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_{t-1}^T \tag{37}$$

lacks the symmetry, so that the matrix might sometimes become ill-conditioned and results in numerical instability. Our new approach guarantees the symmetry because of using system product moment matrix $\sum_{t=1}^{N} \boldsymbol{y}_t \boldsymbol{y}_t^T$.

Moreover, because of introducing the product moment matrix (37), it is clear from (20) and (21) that the computational load of Wada and Eguchi's algorithm will increase. On the contrary, BCLS-B algorithm does not increase the computational load.

The above discussion indicates that the proposed BCLS-B algorithm is simpler, more stable and effective in the computational aspect.

3.4 Calculation of f_N

After finishing the comments on Wada and Eguchi's algorithm and the proposed algorithm, we will continue to decsribe the derivation of the proposed algorithm following subsection 3.2.

Equation (36) shows that it is necessary to calculate the value of f_N to obtain the consistent parameter estimates. We will give the recursive form of f_N in the following.

From (30) and (31), f_N can be described as

$$f_N = \sum_{t=1}^{N} y_t \left(y_{t-p-1} - \boldsymbol{y}_t^T \widehat{\boldsymbol{\beta}}_N \right), \tag{38}$$

then we have

$$f_N = \sum_{t=1}^{N} y_t y_{t-p-1} - \sum_{t=1}^{N} y_t \boldsymbol{y}_t^T \widehat{\boldsymbol{\beta}}_N. \tag{39}$$

So we can give the recursive form for computing f_N as follows:

$$f_N = c_{yy,N} - \boldsymbol{c}_{xy,N}^T \widehat{\boldsymbol{\beta}}_N \tag{40}$$

where

$$c_{yy,N} = c_{yy,N-1} + y_N y_{N-p-1}, \tag{41}$$

$$\boldsymbol{c}_{xy,N} = \boldsymbol{c}_{xy,N-1} + y_N \boldsymbol{y}_N. \tag{42}$$

4. Implementation of the BCLS-B Algorithm

Based on the above discussions, the recursive BCLS-B algorithm is summarized as follows:

***Step* 1**:
Calculate the LS estimate $\widehat{\boldsymbol{\alpha}}_N$ and the auxiliary estimate $\widehat{\boldsymbol{\beta}}_N$ using noisy data by the conventional RLS algorithm:

$$\widehat{\boldsymbol{\alpha}}_N = \widehat{\boldsymbol{\alpha}}_{N-1} + \frac{P_{N-1}\boldsymbol{y}_N \left(y_N - \boldsymbol{y}_N^T \widehat{\boldsymbol{\alpha}}_{N-1} \right)}{1 + \boldsymbol{y}_N^T P_{N-1} \boldsymbol{y}_N},$$

$$\widehat{\boldsymbol{\beta}}_N = \widehat{\boldsymbol{\beta}}_{N-1} + \frac{P_{N-1}\boldsymbol{y}_N \left(y_{N-p-1} - \boldsymbol{y}_N^T \widehat{\boldsymbol{\beta}}_{N-1} \right)}{1 + \boldsymbol{y}_N^T P_{N-1} \boldsymbol{y}_N},$$

$$P_N = P_{N-1} - \frac{P_{N-1}\boldsymbol{y}_N \boldsymbol{y}_N^T P_{N-1}}{1 + \boldsymbol{y}_N^T P_{N-1} \boldsymbol{y}_N}.$$

***Step* 2**:
Calculate f_N:

$$\begin{aligned} c_{yy,N} &= c_{yy,N-1} + y_N y_{N-p-1}, \\ \boldsymbol{c}_{xy,N} &= \boldsymbol{c}_{xy,N-1} + y_N \boldsymbol{y}_N, \\ f_N &= c_{yy,N} - \boldsymbol{c}_{xy,N}^T \widehat{\boldsymbol{\beta}}_N. \end{aligned}$$

***Step* 3**:
Calculate the observation noise variance estimate:

$$\widehat{\sigma}_n^2 = \frac{f_N}{N \widehat{\boldsymbol{a}}_{n-1}^T \widehat{\boldsymbol{\beta}}_N}.$$

***Step* 4**:
Calculate the consistent parameter estimate via

$$\widehat{\boldsymbol{a}}_N = \widehat{\boldsymbol{\alpha}}_N + N \widehat{\sigma}_n^2 P_N \widehat{\boldsymbol{a}}_{N-1}.$$

***Step* 5**:
N=N+1 and return to 1 until convergence.

Initial values at $N = 0$ are given as follows:

$$P_0 = \alpha \boldsymbol{I}, \widehat{\boldsymbol{\alpha}}_0 = \mathbf{o}, \widehat{\boldsymbol{\beta}}_0 = \mathbf{o},$$

$$f_0 = 0, c_{yy,0} = 0, \boldsymbol{c}_{xy,0} = \mathbf{o}.$$

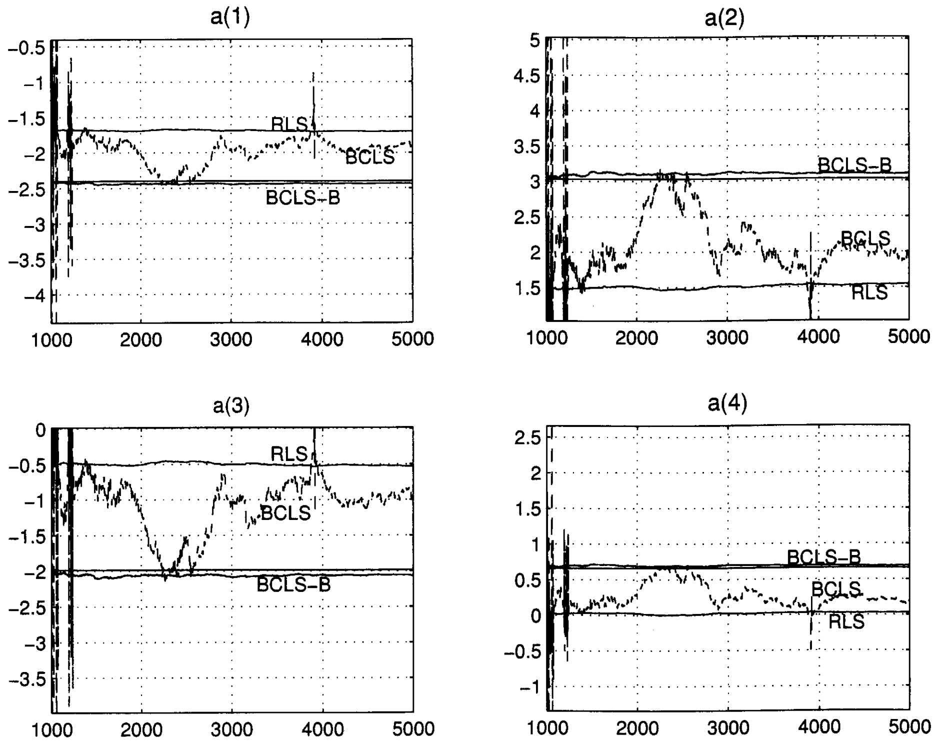

Fig.1 Parameter estimates in $\sigma_n^2 = 0.2$

Table-1 Mean values of parameter estimates

σ_n^2		0.2	0.4	0.6	0.8
LS	$\widehat{\alpha}_1$	1.7286	1.4946	1.3691	1.2991
	$\widehat{\alpha}_2$	-1.5993	-0.1346	-0.9259	-0.8079
	$\widehat{\alpha}_3$	0.5798	0.1415	-0.0411	-0.1377
	$\widehat{\alpha}_4$	0.0485	0.1243	0.1801	0.2037
BCLS	$\widehat{a}_1$	-1.2264	2.1315	2.0828	2.1295
(WE)	$\widehat{a}_2$	4.9816	-2.4419	-2.3494	-2.4450
	$\widehat{a}_3$	-6.0126	1.4018	1.3148	1.4088
	$\widehat{a}_4$	2.9389	-03985	-0.3666	-0.4073
BCLS	$\widehat{a}_1$	2.3937	2.4287	2.3731	2.4004
-B	$\widehat{a}_2$	-3.0200	-3.0851	-2.9742	-3.0309
(new)	$\widehat{a}_3$	1.9771	2.0378	1.9316	1.9874
	$\widehat{a}_4$	-0.6571	-0.6779	-0.6377	-0.6610

5. Simulation Results

Consider a 4th-order AR model described as

$$x_t - 2.4x_{t-1} + 3.03x_{t-2} - 1.986x_{t-3} + 0.6586x_{t-4} = e_t$$

$$\boldsymbol{a}^T = [2.4 \quad -3.03 \quad 1.986 \quad -0.6586]$$

where $\{e_t\}$ is a white noise with unit variance.

Data length and runs are chosen 5000 and 10, respectively.

The mean values of parameter estimates obtained by the LS method, Wada and Eguchi's algorithm and the new type of BCLS algorithm in four kinds of noise level are listed in Table 1. It seems that the LS estimates are usually biased, Wada and Eguchi's algorithm may not give accurate results. However the BCLS-B algorithm can estimate parameters consistently in variours noise levels. It is also shown that the estimation accuracy of the BCLS-B algorithm is very satisfactory.

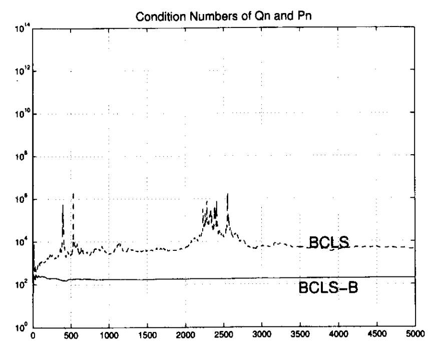

Fig.3 Condition numbers of P_N and Q_N in $\sigma_n^2 = 0.2$

Figure 1 and Figure 2 display the results of AR parameter estimates obtained by the RLS algorithm and two BCLS based algorithms in one trial under the two noise levels ($\sigma_n^2 = 0.2$, $\sigma_n^2 = 0.8$). In the figures, "RLS" indicates the parameter estimates obtained by the RLS method, "BCLS" in-

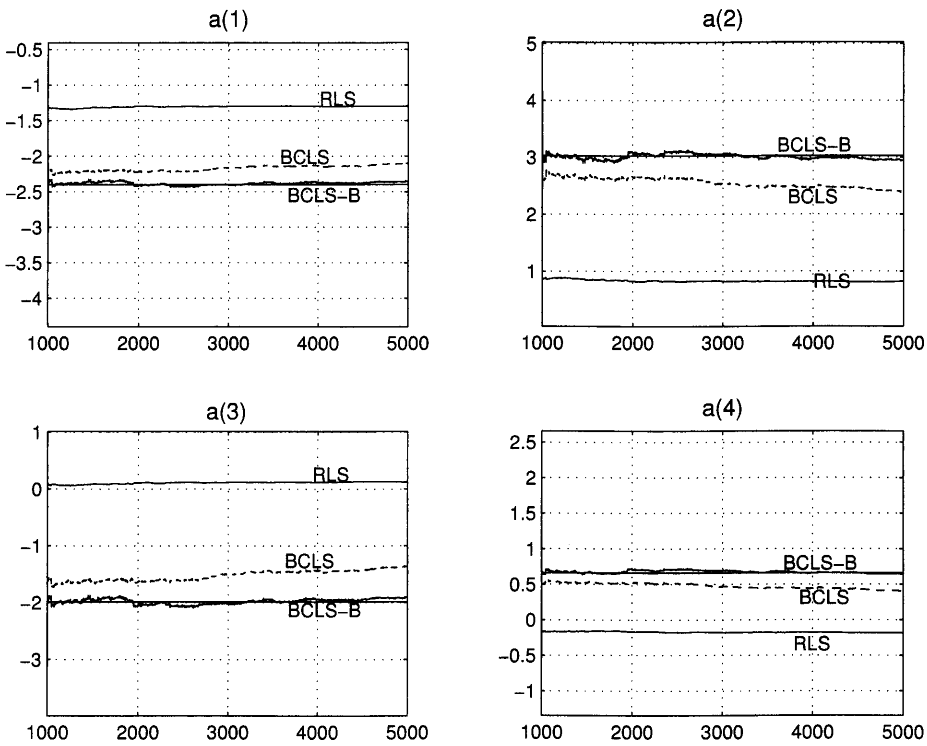

Fig.2 Parameter estimates in $\sigma_n^2 = 0.8$

dicates the parameter estimates obtained by Wada and Eguchi's BCLS algorithm and "BCLS-B" indicates the parameter estimates obtained by the new proposed BCLS algorithm. From Figure 2 we can know the BCLS estimates sometimes are unstable because the product moment matrix Q_N in the BCLS algorithm sometimes becomes ill-conditioned (shown in Figure 3). On the contrary, the proposed BCLS-B algorithm is stable and reliable. Figure 2 shows that the proposed BCLS-B algorithm is accurate even though in high noise level.

Numerical computation also indicates that the proposed BCLS-B algorithm can decrease more 30 percent computational burden than the former algorithm.

6. Conclusions

In this paper, we discuss the problem of parameter estimation of AR process in the presence of white noise with unknown variance. The new type of BCLS algorithm is proposed to produce consistent parameter estimates of AR model. In our algorithm, the auxiliary backward output estimator is utilized for the estimation of the noise variance. Owing to this, the proposed algorithm is simpler, more stable and effective in computation than the existing algorithms. The satisfactory quality of the proposed BCLS algorithm is proved by computer simulation. The future work is how to explain some simulation results from therotical points of view.

References

1) S. Sagara and K. Wada : *On-line modified least-squares parameter estimation on linear discrete dynamic systems.*, Int.J.Control.Vol.25,No.3 pp.329-343, 1977
2) H.Sakai and M.Arase: *Recursive parameter estimation of an autoregressive process disturbed by white noise.*, Int. J. Control, Vol. 30, No.6, pp.946-966, 1979
3) P.Stoica, A.Nehorai and S.M.Kay: *Statistical analysis of the least squares autoregressive estimator in the presence of noise*, IEEE Trans. Acoust., Speech, Signal Processing, Vol.ASSP-35, pp.1273-1281, 1987
4) K.Wada, M.Eguchi, T.Matsumoto: *Parameter estimation of AR process via bias compensated least squares method (in Japanese)*, ISCIE, Vol.2, N0.6, pp. 175-182, 1989
5) W. X. Zheng: *A least squares based method for autoregressive signals in the presence of noise.* IEEE Trans. Circuits and systems-II. Vol.46, pp. 81-85, 1999