Ramification of local fields and Fontaine’s property (Pm)

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RAMIFICATION OF LOCAL FIELDS AND
FONTAINE’S PROPERTY \((P_m)\)

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Abstract. We prove that the ramification filtration of the absolute Galois group of a complete discrete valuation field with perfect residue field is characterized in terms of Fontaine’s property \((P_m)\).

1. Introduction

Let \(K\) be a complete discrete valuation field with perfect residue field \(k\) of characteristic \(p > 0\), \(\mathcal{O}_K\) its valuation ring, \(v_K\) its valuation normalized by \(v_K(K^\times) = \mathbb{Z}\), \(K^{\text{alg}}\) a fixed algebraic closure of \(K\) and \(\bar{K}\) the separable closure of \(K\) in \(K^{\text{alg}}\). In this paper, we construct a certain decreasing filtration of the absolute Galois group \(G_K := \text{Gal}(\bar{K}/K)\) to measure the ramification of extensions of \(K\).

If \(E\) is an algebraic extension of \(K\), we denote by \(\mathcal{O}_E\) the integral closure of \(\mathcal{O}_K\) in \(E\). The valuation \(v_K\) can be extended to \(E\) uniquely and the extension is also denoted by \(v_K\). For an algebraic extension \(E\) of \(K\) and a real number \(m\), we put

\[ a^m_{E/K} = \{ x \in \mathcal{O}_E \mid v_K(x) \geq m \} \]

which is an ideal of \(\mathcal{O}_E\). For a finite Galois extension \(L/K\) and a real number \(m\), we consider the following property studied in [Fo]:

\((P_m)\) For any algebraic extension \(E/K\), if there exists an \(\mathcal{O}_K\)-algebra homomorphism \(\mathcal{O}_L \to \mathcal{O}_E/a^m_{E/K}\), then there exists a \(K\)-embedding \(L \hookrightarrow E\).

For a finite Galois extension \(L\) of \(K\), we put

\[ m_{L/K} := \inf \{ m \in \mathbb{R} \mid (P_m) \text{ is true for } L/K \} \]

The property \((P_m)\) is stable under composition of extensions of \(K\) (Prop. 2.3). Hence we can define two filtrations of \(G_K\) as follows: For a real number \(m\), we denote by \(\bar{K} < m\) (resp. \(\bar{K} \leq m\)) the composite field of all finite Galois extensions \(L\) of \(K\) in \(\bar{K}\) such that \(m_{L/K} < m\) (resp. \(m_{L/K} \leq m\)). We define two closed normal subgroups \(G^{\geq m}_K\) and \(G^{> m}_K\) of \(G_K\) by

\[ G^{\geq m}_K := \text{Gal}(\bar{K}/\bar{K} < m), \quad G^{> m}_K := \text{Gal}(\bar{K}/\bar{K} \leq m) \]

The filtration \((G^{\geq m}_K)_{m \in \mathbb{R}}\) satisfies \(\bigcap_m G^{\geq m}_K = 1\) and \(G^{> 0}_K = G_K\) (Thm. 2.6 (i)). Moreover, \(G^{> 1}_K\) is the inertia subgroup of \(G_K\) and \(G^{> 1}_K\) is the wild inertia subgroup of \(G_K\) (Thm. 2.6 (iii), Rem. 2.7).

On the other hand, we denote by \(G^{(m)}_K\) the \(m\)th upper numbering ramification group in the sense of [Fo]. Namely, we put \(G^{(m)}_K := G^{m-1}_K\), where the latter is the \(m\)th upper numbering ramification group defined in [Se]. In addition, we put

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$G^{(m+)}_K := \bigcup_{m' > m} G^{(m')}_K$, where the overline means the closure with respect to the Krull topology. These define two decreasing filtrations of $G_K$ and they are well-known in the classical ramification theory.

We denote by $\bar{K}(m)$ (resp. $\bar{K}(m+)$) the fixed field of $\bar{K}$ by $G^{(m)}_K$ (resp. $G^{(m+)}_K$). Our main result in this paper is:

**Theorem 1.1.** For a real number $m$, we have $ar{K}<m = \bar{K}(m)$ and $\bar{K}\geq m = \bar{K}(m+)$, so that $G^{\geq m}_K = G^{(m)}_K$, $G^{\geq m}_K = G^{(m+)}_K$.

We prove this theorem by showing the equality $m_{L/K} = u_{L/K}$ for a finite Galois extension $L$ of $K$, where $u_{L/K}$ is the greatest upper ramification break of $L/K$ in the sense of [Fo].

The property (P$_m$) is useful for obtaining ramification bounds for certain Galois representations ([CL], [Fo], [Ha]). Indeed, Fontaine proved the following: in the case where the characteristic of $K$ is 0, for an integer $n \geq 1$, if we denote by $\mathfrak{S}$ a finite flat group scheme over $O_K$ killed by $p^n$, then the ramification of $\mathfrak{S}(\bar{K})$ is bounded by $m$ (meaning that $G^{(m)}_K$ acts trivially on $\mathfrak{S}(\bar{K})$) if $m > e(n+1/(p-1))$, where $e$ is the absolute ramification index of $K$ ([Fo], Thm. A). He obtained the ramification bound by showing that if (P$_m$) is true for a finite Galois extension $L/K$ and a real number $m$ then $m > u_{L/K} - e_{L/K}^{-1}$, where $e_{L/K}$ is the ramification index of $L/K$ (Prop. 3.2 (ii)).

In Section 2, we study some properties of (P$_m$) and the number $m_{L/K}$. By using these results, we define our filtrations of $G_K$ and deduce its properties. In Section 3, after recalling the classical ramification theory for Galois extensions of $K$ ([Fo], [Se]), we show the equality $m_{L/K} = u_{L/K}$ to prove Theorem 1.1. In Section 4, we begin with a review of the ramification theory of Abbes and Saito ([AS1], [AS2]). Their theory does not require the assumption that the residue field $k$ is perfect. Then we consider the property (P$_m$) in the imperfect residue field case, and translate our results in Section 3 into the language of their theory. In the Appendix, we prove a Galois theoretic property on filtrations of the absolute Galois group of an arbitrary field. Theorem 1.1 is proved by the equality $m_{L/K} = u_{L/K}$ and the property checked in the Appendix.

**Convention and Notation.** Fix an algebraic closure $K^{alg}$ of $K$ and denote by $\bar{K}$ the separable closure of $K$ in $K^{alg}$. We assume throughout that all algebraic extensions of $K$ under discussion are contained in $K^{alg}$. If $E$ is an algebraic extension of $K$, then we denote by $e_{E/K}$ the ramification index of $E/K$ and by $O_E$ the integral closure of $O_K$ in $E$. The valuation $v_K$ of $K$ extends to $K^{alg}$ uniquely and the extension is also denote by $v_K$.

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2. Ramification theory via (P\(m\))

In this section, we study the property (P\(m\)). For a finite Galois extension L of K, we put

\[ m_{L/K} := \inf \{ m \in \mathbb{R} \mid (P_m) \text{ is true for } L/K \}. \]

If \( L = K \), the property (P\(m\)) holds for all real numbers \( m \), so that we have \( m_{L/K} = -\infty \). The following proposition is a basic property of the number \( m_{L/K} \):

**Proposition 2.1.** Let L be a finite Galois extension of K such that \( L \neq K \). Then (P\(m\)) is not true for \( L/K \) and any real number \( m \leq 0 \), and is true for sufficiently large real number m. In particular, the number \( m_{L/K} \) is non-negative and finite.

**Proof.** For any real number \( m \leq 0 \), \( \mathcal{O}_K/a_{K/K}^m \) is a zero ring. Then the zero map \( \mathcal{O}_L \to \mathcal{O}_K/a_{K/K}^m \) is an \( \mathcal{O}_K \)-algebra homomorphism. However, there is no \( K \)-embedding \( L \hookrightarrow K \) by assumption. Hence (P\(m\)) is not true for \( L/K \) and any real number \( m \leq 0 \). Thus we have \( m_{L/K} \geq 0 \). Next, we show that (P\(m\)) is true for sufficiently large real number m. Choose an element \( \alpha \) of \( \mathcal{O}_L \) such that \( \mathcal{O}_L = \mathcal{O}_K[\alpha] \).

Let \( P \) be the minimal polynomial of \( \alpha \) over K and \( \alpha = \alpha_1, \ldots, \alpha_n \) the zeros of \( P \) in \( \bar{K} \). Suppose there exists an \( \mathcal{O}_K \)-algebra homomorphism \( \eta : \mathcal{O}_L \to \mathcal{O}_E/a_{E/K}^m \) for an algebraic extension \( E \) of K and \( m > n \sup_{i \neq 1} v_K(\alpha - \alpha_i) \). Then we have \( v_K(P(\beta)) \geq m \), where \( \beta \) is a lift of \( \eta(\alpha) \) in \( \mathcal{O}_E \). By the inequalities

\[ n \sup_i v_K(\beta - \alpha_i) \geq v_K(P(\beta)) \geq m > n \sup_{i \neq 1} v_K(\alpha - \alpha_i), \]

we have \( v_K(\beta - \alpha_{i_0}) > \sup_{i \neq 1} v_K(\alpha - \alpha_i) \) for some \( i_0 \). By Krasner’s lemma, we have \( K(\alpha_{i_0}) \subset K(\beta) \). Thus we obtain a \( K \)-embedding \( L = K(\alpha) \supseteq K(\alpha_{i_0}) \subset K(\beta) \subset E \). Hence (P\(m\)) is true for \( m > n \sup_{i \neq 1} v_K(\alpha - \alpha_i) \). Therefore, we have

\[ m_{L/K} \leq n \sup_{i \neq 1} v_K(\alpha - \alpha_i) < \infty \]
exists a $K'$-embedding $L' = LK' \hookrightarrow E$. Hence $(P_{e,m})$ is true for $L'/K'$. Next, we assume $K'/K$ is an unramified subextension of $L/K$ such that $L \neq K'$ and $(P_m)$ is true for $L/K'$ and $m$. Note that $m > 0$ by Proposition 2.1. Then we want to show that $(P_m)$ is also true for $L/K$ and $m$. Suppose there exists an $O_K$-algebra homomorphism $\eta : O_L \rightarrow O_E/a_{E/K}^m$ for an algebraic extension $E$ of $K$. The composite map

$$\eta' : O_{K'} \rightarrow O_L \xrightarrow{\eta} O_E/a_{E/K}^m$$

is an $O_K$-algebra homomorphism. Since $K'/K$ is unramified, $\eta'$ lifts to an $O_K$-algebra homomorphism $O_{K'} \rightarrow O_E$. Hence $\eta$ is an $O_K$-algebra homomorphism. By the property $(P_m)$, there exists a $K'$-embedding $L \hookrightarrow E$ corresponding to $\eta$. This is also a $K$-embedding. Therefore, $(P_m)$ is true for $L/K$ and $m$. \hfill \Box

To define filtrations of $G_K$, we show that the property $(P_m)$ is stable under composition of finite Galois extensions of $K$ as follows:

**Proposition 2.3.** Let $L$ and $K'$ be finite Galois extensions of $K$. For a real number $m$, if $(P_m)$ is true for both $L/K$ and $K'/K$, then $(P_m)$ is also true for the composite extension $LK'/K$. In particular, we have $m_{LK'/K} \leq \max\{m_{L/K}, m_{K'/K}\}$.

**Proof.** Put $L' := LK'$. Assume $(P_m)$ is true for $L/K$ and $K'/K$. Suppose there exists an $O_K$-algebra homomorphism $\eta : O_L \rightarrow O_E/a_{E/K}^m$ for an algebraic extension $E$ of $K$. Then the composite maps defined by

$$\eta' : O_{L'} \xrightarrow{\eta} O_E/a_{E/K}^m, \quad \eta'' : O_{K'} \xrightarrow{\eta} O_{L'} \xrightarrow{\eta'} O_E/a_{E/K}^m$$

are also $O_K$-algebra homomorphisms. Since $(P_m)$ is true for both $L/K$ and $K'/K$, there exist $K$-embeddings $L \hookrightarrow E$ and $K' \hookrightarrow E$ corresponding to $\eta'$ and $\eta''$ respectively. Since $L/K$ and $K'/K$ are Galois extensions, we obtain a $K$-embedding $L' \hookrightarrow E$. Therefore, $(P_m)$ is true for $L'/K$. \hfill \Box

By this proposition, we can define two increasing filtrations $(\tilde{K}_{<m})_{m \in \mathbb{R}}$ and $(\bar{K}_{\leq m})_{m \in \mathbb{R}}$ of $\bar{K}$ as follows: For any real number $m$, $\tilde{K}_{<m}$ (resp. $\bar{K}_{\leq m}$) is defined by the composite field of all finite Galois extensions $L$ of $K$ in $\bar{K}$ such that $m_{L/K} < m$ (resp. $m_{L/K} \leq m$). Then we put

$$G_{\tilde{K}}^{m} := \text{Gal}(\bar{K}/\tilde{K}_{<m}), \quad G_{\bar{K}}^{m} := \text{Gal}(\bar{K}/\bar{K}_{\leq m}),$$

which are closed normal subgroups of $G_K$. Clearly, these subgroups form decreasing filtrations of $G_K$.

**Remark 2.4.** In fact, Proposition 2.1, 2.2 and 2.3 remain true in the case where the residue field $k$ may be imperfect, though we have to show the finiteness of $m_{L/K}$ in Proposition 2.1 by a different way via Proposition 4.3. Hence the filtrations $G_{\tilde{K}}^{m}$ and $G_{\bar{K}}^{m}$ can be defined even when the residue field of $K$ is imperfect.

The property $(P_m)$ has the following property for unramified extensions of $K$:

**Proposition 2.5.** Let $L$ be a finite Galois extension of $K$. Then the following conditions are equivalent:

(i) $L/K$ is unramified.
(ii) $m_{L/K} \leq 0$.
(iii) $m_{L/K} < 1$. 


Proof. First, assume $L/K$ is unramified. Then we show that $(P_m)$ is true for $L/K$ and $m > 0$. Suppose there exists an $O_K$-algebra homomorphism $\eta : O_L \to O_E/a_E^m$ for an algebraic extension $E$ of $K$. Since $L/K$ is unramified, $\eta$ lifts to an $O_K$-algebra homomorphism $O_L \to O_E$. Thus (i) implies (ii). Since it is clear that (ii) implies (iii), it is enough to verify that (iii) implies (i). To prove this, we show that if $L/K$ is not unramified, then $m_{L/K} \geq 1$. Let $K'$ be the maximal unramified subextension of $L/K$ and $\pi_K$ (resp. $\pi_L$) a uniformizer of $O_K$ (resp. $O_L$). Then there is an $O_K$-algebra homomorphism $O_L \to O_L/\pi_L O_L \cong O_K/\pi_K O_K = O_K/a_{K/K}$. However, there is no $K$-embedding $L \hookrightarrow K'$. Hence $(P_1)$ is not true for $L/K$, so that $m_{L/K} \geq 1$.

By the properties of the number $m_{L/K}$, our filtration $(G_K^{\geq m})_{m \in \mathbb{R}}$ has the following properties:

**Theorem 2.6.** (i) For a real number $m \leq 0$, we have $G_K^{\geq m} = G_K$. Moreover, we have $\bigcap_{m \geq 1} G_K^{\geq m} = 1$ and $\bigcup_{m \geq 1} G_K^{\geq m} = G_K$.

(ii) Let $K'$ be a finite separable extension of $K$, of ramification index $e'$. We identify the Galois group $G_{K'} := \text{Gal}(K'/K)$ with a subgroup of $G_K$. Then, for a real number $m > 0$, we have $G_{K'}^{\leq m} \subset G_K^{\geq m}$, with equality if $K'/K$ is unramified.

(iii) For a real number $0 < m \leq 1$, $G_K^{\geq m}$ is the inertia subgroup of $G_K$.

Proof. The assertion (i) follows from Proposition 2.1. (iii) follows from Proposition 2.5. The first assertion of (ii) follows from Proposition 2.2. Hence we prove the second assertion of (ii). Assume $K'/K$ is unramified. It suffices to show $K'_{< m} \subset K_{< m}$. By the definition of $K'_{< m}$, it is enough to show that if a finite Galois extension $L'$ of $K'$ contained in $K$ satisfies $m_{L'/K'} < m$, then $L' \subset K_{< m}$. Since the case $L' = K'$ is true by Proposition 2.5, we may assume $L' \neq K'$. Take the Galois closure $K''$ of $K'$ over $K$ in $K$ and put $L'' := L'K''$. Note that $K''/K$ is an unramified Galois extension and $L''/K''$ is a Galois extension. Then we have $m_{L''/K''} \leq m_{L'/K'} < m$ by Proposition 2.2. Let $L''$ be the Galois closure of $L''$ over $K$ in $K$. If $L'' = K''$, then Proposition 2.5 shows $m_{L''/K} \leq 0 < m$, so that $L' \subset L'' \subset K_{< m}$. Thus we may assume $L'' \neq K''$. Any $\sigma \in \text{Gal}(L''/K)$ satisfies $\sigma(K'') = K''$ since $K''/K$ is a Galois extension, so that

$$m_{\sigma(L'')}/K'' = m_{\sigma(L'')} = m_{L''/K''} < m.$$ 

By this inequality and Proposition 2.3, we have

$$m_{L''/K''} \leq \max\{m_{\sigma(L'')} | \sigma \in \text{Gal}(L''/K)\} < m$$

since $L''/K$ is the composite field of all the conjugate fields $\sigma(L'')$ ($\sigma \in \text{Gal}(L''/K)$). Thus we have $m_{L''/K} = m_{L''/K''} < m$ by Proposition 2.2 since $K''/K$ is unramified. Therefore, we have $L' \subset L'' \subset K_{< m}$. \qed

**Remark 2.7.** We can prove that $G_K^{1}$ is the wild inertia subgroup of $G_K$ by using the property $(P_m)$ together with the classical theory of Herbrand functions in a similar way to the proof of Proposition 1.5, (ii), of [Fo]. However, we restricted ourselves here to showing what can be derived rather directly from $(P_m)$.
3. Ramification breaks

In this section, we compare our ramification filtration with the classical one. First, we recall the classical ramification theory for Galois extensions of $K$. Let $L$ be a finite Galois extension of $K$ with Galois group $G$. The order function $i_{L/K}$ is defined on $G$ by

$$i_{L/K}(\sigma) := \inf_{a \in \mathcal{O}_L} v_{K}(\sigma(a) - a), \ \sigma \in G.$$ 

Then the $i$th lower numbering ramification group $G_{(i)}$ are defined for any real number $i$ by

$$G_{(i)} := \{ \sigma \in G \mid i_{L/K}(\sigma) \geq i \}.$$ 

The transition function $e_{\varphi_{L/K}} : \mathbb{R} \to \mathbb{R}$ of $L/K$ is defined by

$$e_{\varphi_{L/K}}(u) := \int_0^u \#G(\tau) d\tau,$$

where $\#G(\tau)$ is the cardinality of $G(\tau)$. Then $\varphi_{L/K} : \mathbb{R} \to \mathbb{R}$ is piecewise linear, strictly increasing and bijective ([Se], Chap. IV, Sect. 3, Prop. 12). Denote by $u_{L/K}$ its inverse function. We also define another function $u_{L/K}$ on $G$ by

$$u_{L/K}(\sigma) := \varphi_{L/K}(i_{L/K}(\sigma)), \ \sigma \in G.$$ 

Then the $u$th upper numbering ramification group $G^{(u)}$ are defined for any real number $u$ by

$$G^{(u)} := \{ \sigma \in G \mid u_{L/K}(\sigma) \geq u \}.$$ 

For any non-negative real number $u$, we have $G^{(u)} = G^{u-1}$, where the latter is the $u$th upper numbering ramification group defined in [Se] (cf. [Fo], Rem. 1.2). We denote by $u_{L/K}$ (resp. $i_{L/K}$) the greatest upper (resp. lower) ramification break of $L/K$ defined by

$$u_{L/K} := \inf\{u \in \mathbb{R} \mid G^{(u)} = 1\}, \ \ i_{L/K} := \inf\{u \in \mathbb{R} \mid G_{(i)} = 1\}.$$ 

We put $u_{K/K} = -\infty$ by convention. The next lemma is a basic property of the number $u_{L/K}$:

**Lemma 3.1.** For finite Galois extensions $M \subset L$ of $K$, we have $u_{M/K} \leq u_{L/K}$.

**Proof.** By the compatibility with the quotient (cf. [Se], Chap. IV, Sect. 3, Prop. 14), $\text{Gal}(L/K)^{(u)} = 1$ implies $\text{Gal}(M/K)^{(u)} = 1$ for any real number $u$. Thus we obtain the inequality. \qed

Fontaine proved the following proposition:

**Proposition 3.2** ([Fo], Prop. 1.5). Let $L$ be a finite Galois extension of $K$ and $m$ a real number. Then there are the following relations:

(i) If we have $m > u_{L/K}$, then $(P_m)$ is true.

(ii) If $(P_m)$ is true, then we have $m > u_{L/K} - e_{L/K}^{-1}$.

By this proposition, we have the inequalities

$$u_{L/K} - e_{L/K}^{-1} \leq m_{L/K} \leq u_{L/K},$$

for a finite Galois extension $L$ of $K$. More precisely, we have the following equality:

**Proposition 3.3.** For a finite Galois extension $L$ of $K$, we have $m_{L/K} = u_{L/K}$.
Proof. It is enough to show that (P\textsubscript{m}) is not true for L/K and m < u\textsubscript{L/K}. Suppose L/K is unramified. Then we have u\textsubscript{L/K} = m\textsubscript{L/K} = 0 if L \neq K, and u\textsubscript{L/K} = m\textsubscript{L/K} = \infty if L = K, so that the proposition follows. Therefore, we may assume L/K is not unramified. The number u\textsubscript{L/K} is stable under unramified base change. Thus we may assume L/K is a totally ramified extension by Proposition 2.2. If L/K is a tamely ramified extension, (P\textsubscript{m}) is not true even for m = u\textsubscript{L/K} = 1 because we can find a counter-example to (P\textsubscript{m}) for m = u\textsubscript{L/K} as follows: Let \pi\textsubscript{L} (resp. \pi\textsubscript{K}) be a uniformizer of O\textsubscript{L} (resp. O\textsubscript{K}). Then there is an O\textsubscript{K}-algebra homomorphism O\textsubscript{L} \rightarrow O\textsubscript{L}/\pi\textsubscript{L}O\textsubscript{L} \cong O\textsubscript{K}/\pi\textsubscript{K}O\textsubscript{K} = O\textsubscript{K}/a\textsubscript{K/K}. However, there is no K-embedding L \hookrightarrow K. Therefore, we may assume L/K is a wildly ramified extension. To prove this proposition, we shall find a counter-example to (P\textsubscript{m}) for L/K and m = u\textsubscript{L/K} = e\textsuperscript{-1}, where e can be taken to be an arbitrarily large number. Take a finite tame ramified Galois extension K' of K. Put L' := LK' and e' := e\textsubscript{L'/K}. If we apply (ii) of Proposition 3.2 to L'/K, then there exists an algebraic extension E of K such that there exists an O\textsubscript{K}-algebra homomorphism \eta : O\textsubscript{L'} \rightarrow O\textsubscript{E}/a\textsubscript{E/K}, but there is no K-embedding L' \hookrightarrow E, where m\textsubscript{0} := u\textsubscript{L'/K} = e\textsuperscript{-1}. By Lemma 3.1, we have m\textsubscript{0} \geq m\textsubscript{1}, where m\textsubscript{1} := u\textsubscript{L/K} = e\textsuperscript{-1}. Consider the two O\textsubscript{K}-algebra homomorphisms defined by composite maps:

\[ \eta' : O\textsubscript{K} \hookrightarrow O\textsubscript{L} \xrightarrow{\eta} O\textsubscript{E}/a\textsubscript{E/K} \rightarrow O\textsubscript{E}/a\textsubscript{E/K}, \quad \eta'' : O\textsubscript{K} \hookrightarrow O\textsubscript{L} \xrightarrow{\eta} O\textsubscript{E}/a\textsubscript{E/K} \rightarrow a\textsubscript{E/K}. \]

Since K'/K is a tamely ramified extension, we have u\textsubscript{K'/K} \leq 1. On the other hand, since L'/K is a wildly ramified extension, we have e'm\textsubscript{0} > e' as shown in the proof of [Fo], Proposition 1.5, (ii), hence we deduce m\textsubscript{0} > 1. Thus we have m\textsubscript{0} > u\textsubscript{K'/K}. According to (i) of Proposition 3.2 for K'/K, there exists a K-embedding K' \hookrightarrow E corresponding to \eta''. If we suppose there exists a K-embedding L \hookrightarrow E, then there exists a K-embedding L' = LK' \hookrightarrow E since L/K and K'/K are Galois extensions. This is a contradiction. Therefore, (P\textsubscript{m}) is not true for L/K and m = m\textsubscript{1}. Hence the result follows.

Remark 3.4. By Proposition 2.3, Lemma 3.1 and Proposition 3.3, we deduce the equality u\textsubscript{L/K} = \max\{u\textsubscript{L/K}, u\textsubscript{K'/K}\} for any finite Galois extensions L and K' of K.

Remark 3.5. In the above proposition, we proved the equality m\textsubscript{L/K} = u\textsubscript{L/K} with the assumption that the residue field k is perfect. We are also interested in the case where k may be imperfect. In Chapter IV of [Se], the ramification filtration is defined in the case where L/K is unfierociously\textsuperscript{1} ramified. Our proof of Proposition 3.3 remains true in this case since so does Fontaine’s proof of Proposition 3.2 and the composite field of L/K and any tamely ramified extension K'/K is still unfierociously ramified.

Theorem 1.1 follows from Propositions 3.3 and 5.4.

4. The ramification theory of Abbes and Saito

First, we recall the ramification theory of Abbes and Saito ([AS1], [AS2]). In Subsection 4.1, we generalize the property (P\textsubscript{m}) to the imperfect residue field case. In Subsection 4.2, we translate our results in Section 3 into the language of the

\textsuperscript{1}We mean by an unfierociously ramified extension L/K a finite algebraic extension whose residue field extension is separable.
ramification theory of Abbes and Saito. Let $K$ be a complete discrete valuation field whose residue field $k$ may not be perfect. Let $K^{alg}$ be a fixed algebraic closure of $K$, $\bar{K}$ the separable closure of $K$ in $K^{alg}$ and $G_K := \text{Gal}(\bar{K}/K)$ the absolute Galois group. Abbes and Saito defined a decreasing filtration $(G^m_K)_{m \geq 0}$ by closed normal subgroups $G^m_K$ of $G_K$ indexed by rational numbers $m \geq 0$, in such a way that $\cap_{m \geq 0} G^m_K = 1$, $G^0_K = G_K$ and $G^1_K$ is the inertia subgroup of $G_K$. It is defined by using certain functors $F$ and $F^m$ from the category $\mathcal{FE}_K$ of finite étale $K$-algebras to the category $\mathcal{S}_K$ of finite $G_K$-sets. We recall the definition of $F$ and its quotients $F^m$ for positive rational numbers $m$. Let $L$ be a finite étale $K$-algebra and $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in $L$. We define $F(L) := \text{Hom}_K(L, \bar{K}) = \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K)$. The functor $F$ gives an anti-equivalence of $\mathcal{FE}_K$ with $\mathcal{S}_K$, thereby making $\mathcal{FE}_K$ a Galois category. To define $F^m$, we proceed as follows: An embedding of $\mathcal{O}_L$ is a pair $(\mathcal{B}, \mathfrak{b} \to \mathcal{O}_L)$ consisting of an $\mathcal{O}_K$-algebra $\mathcal{B}$ which is formally of finite type and formally smooth over $\mathcal{O}_K$, and a surjection $\mathcal{B} \to \mathcal{O}_L$ of $\mathcal{O}_K$-algebras which induces an isomorphism $\mathcal{B} / \mathfrak{m} \mathcal{B} \to \mathcal{O}_L / \mathfrak{m}_L$, where $\mathfrak{m}_B$ and $\mathfrak{m}_L$ are respectively the radicals of $\mathcal{B}$ and $\mathcal{O}_L$ (cf. [AS2], Def. 1.1). Let $I$ be the kernel of the surjection $\mathcal{B} \to \mathcal{O}_L$. Write $m = m_2/m_1$ for some positive integers $m_1$ and $m_2$. Then the affinoid algebra $\mathcal{B}[(f^m_1/x^m_1)] \otimes_{\mathcal{O}_K} K$ does not depend on the presentation of $m$ ([AS2], Lem. 1.4, 4), where $\pi_K$ is a uniformizer of $K$ and $\Lambda$ means the $\pi_K$-adic completion. Hence we denote this ring by $\mathcal{B}^m$. Let $X^m(\mathfrak{b} \to \mathcal{O}_L)$ be the affinoid variety $\text{Sp}(\mathcal{B}^m)$ associated with $\mathcal{B}^m$. For any affinoid variety $X$ over $K$, let $\pi_0(X_K)$ denote the set limit $\pi_0(X \otimes_K K')$ of geometric connected components, where $K'$ runs through the finite separable extensions of $K$ contained in $\bar{K}$. Then we define the functor $F^m$ by

$$F^m(L) := \lim_{(\mathcal{B} \to \mathcal{O}_L)} \pi_0(X^m(\mathfrak{b} \to \mathcal{O}_L)_K),$$

where $(\mathcal{B} \to \mathcal{O}_L)$ runs through the category of embeddings of $\mathcal{O}_L$ (cf. [AS2], Def. 1.1). The projective system in the right-hand side is constant ([AS2], Lem. 1.9). The finite set $F(L)$ can be identified with a subset of $X^m(\mathfrak{b} \to \mathcal{O}_L)(\bar{K})$, and this induces a natural surjective map $F(L) \to F^m(L)$. The $m$-th ramification subgroup $G^m_K$ is characterized by the property that $F(L)/G^m_K = F^m(L)$ for all $L$. If the residue field of $K$ is perfect, this filtration $(G^m_K)_m$ defined as above coincides with the classical one $(G^{(m)}_K)_m$ defined in Section 3 (cf. [AS1], Subsect. 6.1).

4.1. Generalization of Fontaine’s proposition. In this subsection, we generalize Fontaine’s proposition to the imperfect residue field case. Let $L$ be a finite Galois extension of $K$ and $m$ a positive rational number. We define the property $(P_m)$ and the number $m_{L/K}$ in the same way as those in the Introduction. For an affinoid variety $X$ over $K$ and a point $x \in X(K^{alg})$, we denote by $K(x)$ and $X_x$, respectively, the definition field of $x$, and the geometric connected component of $X$ which contains $x$. The ring $\mathcal{O}_L$ is a complete intersection over $\mathcal{O}_K$. Namely, we have $\mathcal{O}_L \cong \mathcal{O}_K[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$ ([AS1], Lem. 7.1). We denote by $z_1, \ldots, z_d$ the common zeros of $f_1, \ldots, f_n$ in $\bar{K}$. Let $I := (f_1, \ldots, f_n)$ be the ideal of $\mathcal{O}_K[T_1, \ldots, T_n]$ generated by $f_1, \ldots, f_n$. Consider the surjection $\varphi : \mathcal{O}_K[T_1, \ldots, T_n] \to \mathcal{O}_K[T_1, \ldots, T_n]/(f_1, \ldots, f_n) \cong \mathcal{O}_L$. Put $w_i := \varphi(T_i)$ for

---

2We say that an $\mathcal{O}_K$-algebra $A$ is formally of finite type over $\mathcal{O}_K$ if $A$ is semi-local, $\mathfrak{m}_A$-adically complete Noetherian and the quotient $A/\mathfrak{m}_A$ is finite over $k$, where $\mathfrak{m}_A$ is the radical of $A$ (cf. [AS2], Sect. 1).
If

Let

For any

Obvious.

Let

\( x \) and \( z \)

and \( L/K \) the affinoid variety over \( K \)

\( B := \lim_{\rightarrow} \) and \( x \)

of \( L \) is due to Hiranouchi and Taguchi.

Proof

Proposition 4.3.

If the residue field \( K \)

By definition, the property (Q′)

Proof

Lemma 4.2.

Remark 4.1.

Let

\( c \)

\( K \)

Now we can prove Proposition 4.3 as follows: Let

\( \sigma \)

\( 1 \)

\( f \)

\( n \)

\( x \)

\( y \)

\( K \)

\( x \)

Indeed, if \( z_i \in X^{m}_x \), then \( X^{m}_x = X^{m}_{y} \), which contradicts \( x \in X^{m}_x \) and \( x \not\in X^{m}_{z_i} \) for any \( z_i \neq z \). Thus \( K(z) \subset K(x) \) by Lemma 4.4. Hence (P′_m) is true. \( \square \)
Remark 4.5. The author does not know whether the equality $m_{L/K} = e_{L/K}$ remains true in the case where the residue field of $K$ is imperfect. However, we can show at least the following:

**Proposition 4.6.** Let $L$ be a finite Galois extension of $K$. Let $K'$ be a weakly unramified$^3$ extension of $K$ such that $L' := LK'/K'$ is unferociously ramified (the existence of such an extension is proved in [AS1], Append. Cor. A.2). Then we have $c_{L'/K'} \leq m_{L/K}$.

*Proof.* We have $m_{L'/K'} \leq m_{L/K}$ by Proposition 2.2 (cf. Rem. 2.4). Since $L'/K'$ is unferociously ramified, we can apply Proposition 3.3 to $L'/K'$ (cf. Rem. 3.5). Then we have $c_{L'/K'} = m_{L'/K'}$. Thus the desired inequality $c_{L'/K'} \leq m_{L/K}$ holds. □

4.2. Comparison with the ramification theory of Abbes and Saito. In this subsection, we translate our results in Section 3 into the language of the ramification theory of Abbes and Saito. Let $K$ be a complete discrete valuation field with perfect residue field and $L$ a finite Galois extension of $K$. We define a non-Archimedean valuation on $K^{alg}$ by $|y| = \theta^{v(y)}$, where $0 < \theta < 1$ is a real number. Fix a generator $z$ of $O_L$ as an $O_K$-algebra. Let $P$ be the minimal polynomial of $z$ over $K$, and $z = z_1, \ldots, z_d$ the zeros of $P$ in $K$. Let $X^m$ be the affine variety over $K$ as defined in the previous subsection, so that $X^m(K^{alg}) = \{x \in O_K^{alg} \mid v_K(P(x)) \geq m\}$. If the residue field of $K$ is perfect, we can rewrite ($P'_m$) for $L/K$ and a positive rational number $m$ as follows:

($P'_m$) For any $x \in X^m(K^{alg})$, there exists a zero $x$ of $P$ in $K$ which is $K(x)$-rational.

On the other hand, we consider the following property:

($Q'_m$) For any $x \in X^m(K^{alg})$, there exists a zero $x$ of $P$ in $K$ such that $|z - x| = \min_i |x - z_i|$ and $|z - x| < \min_{i \neq 1} |z - z_i|$.

**Proposition 4.7.** The properties ($Q'_m$) and ($Q''_m$) are equivalent.

*Proof.* Put $D(z_i, \theta^m) := \{x \in O_K^{alg} \mid |x - z_i| \leq \theta^m\}$ for $i = 1, \ldots, d$. The disc $D(z_i, \theta^m)$ is connected and contains $z_i$. Denote $\psi := \psi_{L/K}$ for simplicity. Then we have the following by Lemma 4.8 below:

$X^m(K^{alg}) = \{x \in O_K^{alg} \mid |P(x)| \leq \theta^m\}$

$= \{x \in O_K^{alg} \mid \min_i |x - z_i| \leq \theta^\bar{\psi}(m)\}$

$= \bigcup_i D(z_i, \theta^\bar{\psi}(m))$.

The property ($Q'_m$) is true if and only if $X^m_{z_i}$ ($1 \leq i \leq d$) are disjoint. The assertion of the proposition follows from the following equivalences:

$X^m_{z_i} \cap X^m_{z_j} = \emptyset \ (i \neq j) \iff \min_{i \neq j} |z_i - z_j| > \theta^\bar{\psi}(m)$

$\iff \min_i |z_i - x| < \min_{i \neq j} |z_i - z_j|$ for all $x \in X^m(K^{alg})$.

The first equivalence is proved as follows: Let $i \neq j$. Assume $X^m_{z_i}$ ($1 \leq i \leq d$) are disjoint. Then we have $z_j \not\in X^m_{z_i}$. On the other hand, we have $D(z_i, \theta^\bar{\psi}(m)) \subset X^m_{z_i}$.

$^3$We mean by a weakly unramified extension $K'/K$ a finite algebraic extension such that $e_{K'/K} = 1$. 
since $D(z_i, \theta \tilde{\psi}(m))$ is connected and contained in $X^m(K^{alg})$. Hence the zero $z_j$

is not contained in $D(z_i, \theta \tilde{\psi}(m))$, so that $|z_i - z_j| > \theta \tilde{\psi}(m)$. Conversely, suppose

$|z_i - z_j| > \theta \tilde{\psi}(m)$. Then we have $D(z_i, \theta \tilde{\psi}(m)) \cap D(z_j, \theta \tilde{\psi}(m)) = \emptyset$. Hence we

obtain the decomposition $X^m(K^{alg}) = \bigsqcup_i D(z_i, \theta \tilde{\psi}(m))$. Thus we deduce $X^m_{x_1} =

D(z_i, \theta \tilde{\psi}(m))$. In particular, the connected components $X^m_{x_i}$ ($0 \leq i \leq d$) are disjoint.

Finally, we prove the second equivalence. Assume $\min_i |z_i - z_j| > \theta \tilde{\psi}(m)$. For a point

$x \in X^m(K^{alg})$, take $i_1$ such that $x \in D(z_{i_1}, \theta \tilde{\psi}(m))$. Then we have

$$\min_i |z_i - x| \leq |z_{i_1} - x| \leq \theta \tilde{\psi}(m) < \min_{i \neq j} |z_i - z_j|.$$ 

Conversely, assume $\min_i |z_i - x| < \min_{i \neq j} |z_i - z_j|$ for any $x \in X^m(K^{alg})$. Take $y \in K^{alg}$

such that $|y| = \theta^m$ ($< 1$) and take $x \in K^{alg}$ such that $P(x) = y$. Then we have $x \in O_{K^{alg}}$

and $|P(x)| = \theta^m$. In particular, this shows $x \in X^m(K^{alg})$. By assumption and Lemma 4.8 below, we have the inequality

$$\theta \tilde{\psi}(m) = \min_i |z_i - x| < \min_{i \neq j} |z_i - z_j|.$$ 

□

Lemma 4.8 ([Fo]. Prop. 1.4). Let $x$ be an element of $K^{alg}$. Put $i := \sup_i v_K(z_i - x)$

and $u := v_K(P(x))$. Then we have

$$u = \tilde{\psi}_{L/K}(i), \quad \tilde{\psi}_{L/K}(u) = i.$$ 

We obtain the following consequences:

Proposition 4.9. We have the following relations:

(i) If $(Q^m_{\epsilon})$ is true, then $(P^m_{\epsilon})$ is true.

(ii) If $(P^m_{\epsilon})$ is true, then $(Q^m_{\epsilon+\epsilon})$ is true for any $\epsilon > 0$. 

In particular, we have the equality $m_{L/K} = c_{L/K}$.

Proof. The above (i) is the special case of Proposition 4.3. (ii) follows from Propositions 3.3, Proposition 4.7 and the equality $u_{L/K} = c_{L/K}$. □

5. Appendix

In this section, we prove a Galois theoretic property of a filtration of the absolute

Galois group of an arbitrary field. This section is independent of the other sections. Let $K$ be a field, $K$ a fixed separable closure of $K$, $G_K := \text{Gal}(\bar{K}/K)$ the absolute

Galois group of $K$ and $\mathfrak{S}$ the set of all finite Galois extensions of $K$ contained in $\bar{K}$. Throughout this Appendix, all separable extensions of $K$ are assumed to be subfields of $\bar{K}$. Let $R$ be a totally ordered set.

Definition 5.1. Assume we are given a system of decreasing filtrations $(\text{Gal}(L/K)^u)_{L/K \in \mathfrak{S}, u \in R}$. Then we say that the system of filtrations $(\text{Gal}(L/K)^u)_{L/K \in \mathfrak{S}, u \in R}$ is quotient-compatible if, for any $L, L' \in \mathfrak{S}$ such that $L' \subset L$, the image of $\text{Gal}(L/K)^u$ under the natural projection $\text{Gal}(L/K) \to \text{Gal}(L'/K)$ coincides with $\text{Gal}(L'/K)^u$.

Proposition 5.2. There is a natural one-to-one correspondence between the set of decreasing filtrations $(G^u_K)_{u \in R}$ on $G_K$ consisting of closed subgroups of $G_K$ and the set of quotient-compatible systems of decreasing filtrations $(\text{Gal}(L/K)^u)_{L/K \in \mathfrak{S}, u \in R}$. 

Proof. Assume we are given a decreasing filtration \((G^u_K)_{u \in R}\) on \(G_K\) consisting of closed subgroups of \(G_K\). Let \(L\) be a finite Galois extension of \(K\) with Galois group \(G\). Then a decreasing filtration \(G^u\) can be defined by the image of \(G^u_K\) by the restriction map \(G_K \to G\). Conversely, suppose we are given a quotient-compatible system of decreasing filtrations \((\text{Gal}(L/K)^u)_{L/K \in \mathcal{O}, u \in R}\). For any finite Galois extensions \(L' \subset L\) of \(K\), the compatibility with the quotient induces a natural projection \(\text{Gal}(L/K)^u \to \text{Gal}(L'/K)^u\) by the restriction map. Hence we can define a decreasing filtration \(G^u_K\) on \(G_K\) by

\[
G^u_K := \lim_{\longleftarrow} \text{Gal}(L/K)^u,
\]

where \(L\) runs through the set of all finite Galois extensions of \(K\) contained in \(\bar{K}\). This correspondence induces the desired bijection. \(\Box\)

Definition 5.3. Let \(G\) be a set and \((G^u)_{u \in R}\) a decreasing filtration on \(G\). Then we say that \((G^u)_{u \in R}\) is separated if \(\bigcap_u G^u = 1\) and \((G^u)_{u \in R}\) is left continuous if \(G^u = \bigcap_{m < u} G^m\).

Let \((G^u_K)_{u \in R}\) be a decreasing filtration on \(G_K\) which is separated and left continuous, and \(L\) a finite Galois extension of \(K\) with Galois group \(G\). Put \(G^u_K := \bigcup_{u \sim u} G^u_K\), where the overline means the closure with respect to Krull topology. Then we denote by \(K_{(u)}\) (resp. \(K_{(u+)}\)) the fixed field of \(\bar{K}\) by \(G^u_K\) (resp. \(G^{u+}_K\)). Define \(G^u\) (resp. \(G^{u+}\)) as the image of \(G^u_K\) (resp. \(G^{u+}_K\)) by the restriction map \(\pi : G_K \to G\). Put

\[
u_u := \inf \{u \in R \mid G^u = 1\},
\]

assuming that the infimum exists in \(R\). We denote by \(K_{\leq u}\) (resp. \(K_{\leq u}\)) the union of all finite Galois extension \(L\) of \(K\) in \(\bar{K}\) such that \(\nu_u < u\) (resp. \(\nu_u \leq u\)).

Proposition 5.4. We have \(K_{\leq u} = K_{(u)}\) and \(K_{\leq u} = K_{(u+)}\) for any \(u \in R\).

Proof. If \(L\) is a finite Galois extension of \(K\) with Galois group \(G\), then the left continuousness makes \(G^u_{L/K} \neq 1\). Hence \(\nu_u < u\) (resp. \(\nu_u \leq u\)) is equivalent to \(G^u = 1\) (resp. \(G^{u+} = 1\)). This is equivalent to \(G^u_K \subset \text{Ker}(\pi) = G_L\) (resp. \(G^{u+}_K \subset G_L\)). The result follows it. \(\Box\)

References

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