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Kinoshita, Takehiko
Faculty of Mathematics, Kyushu University

Nakao, Mitsuhiro T.
Faculty of Mathematics, Kyushu University

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T. Kinoshita & M. T. Nakao

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2-projection

Takehiko Kinoshita*, Mitsuhiro T. Nakao

Faculty of Mathematics, Kyushu University, Hakozaki, Higashiku, Fukuoka 812-8581, Japan

Abstract

We present constructive a priori error estimates for H_0^2-projection into a space of polynomials on a one-dimensional interval. Here, "constructive" indicates that we can obtain the error bounds in which all constants are explicitly given or are represented in a numerically computable form. Using the properties of Legendre polynomials, we consider a method by which to determine these constants to be as small as possible. Using the proposed technique, the optimal constant could be enclosed in a very narrow interval with results verification. Furthermore, constructive error estimates for finite element H_0^2-projection in one dimension are presented. This type of estimates will play an important role in the numerical verification of solutions for nonlinear fourth-order elliptic problems as well as in the guaranteed a posteriori error analysis for the finite element method or the spectral method.

Key words: Constructive a priori error estimates, Legendre polynomials, Fourth-order elliptic problem
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*Corresponding author
Email addresses: t-kinoshita@math.kyushu-u.ac.jp (Takehiko Kinoshita), mtnakao@math.kyushu-u.ac.jp (Mitsuhiro T. Nakao)

1. Introduction

In the present paper, we consider the smallest constant C in a priori error estimates of the form

$$\left\|u - P^2 u\right\|_{H_0^2(\Lambda)} \leq C\, |u|_{H^4(\Lambda)}\,, \quad \forall u \in H_0^2(\Lambda) \cap H^4(\Lambda), \tag{1}$$

where P^2 is an H_0^2-projection on a one-dimensional interval Λ, and $\|\cdot\|_{H_0^2(\Lambda)}$ and $|\cdot|_{H^4(\Lambda)}$ are the norm in H_0^2 and the seminorm in H^4, respectively. The purpose of the present study is to find the upper and lower bounds of optimal constants in the above estimates. These constants not only play an important role in theoretically verifying the solutions of differential equations (e.g. [8, 2, 6]), but also contribute to highly reliable computing in numerical simulation using the finite element method or the spectral method. In general, C should be made as small as possible.

In the case of the H_0^1-projection, for approximation spaces with linear and quadratic polynomials, the optimal constants can be theoretically determined as $\frac{1}{\pi}$ and $\frac{1}{2\pi}$, respectively (see [7, 5]). Such a constant can also be computed for higher-order polynomials (see [4]), although it is not optimal.

For the H_0^2-projection, Schultz obtained constructive a priori error estimates based on piecewise cubic interpolation (see [7]), which is not optimal.

In the present paper, we propose a method that is an extension and improvement of the technique presented in [4] to obtain a constant very close to the optimal constant with guaranteed accuracy. Note that the proposed technique improves Schultz's result and can also be applied to obtain the optimal constants in the case of higher-order polynomials. Furthermore, using the present results, it will be possible to realize more efficient computations in the numerical verification of solutions related to fourth-order elliptic problems, such as those described in [2, 6, 9].

2. Legendre polynomials

Let $\Lambda = (a, b)$, $(a < b \in \mathbb{R})$ be a one-dimensional interval. The Legendre polynomials on Λ are defined as a complete orthogonal system in $L^2(\Lambda)$, for an arbitrary non-negative integer n,

$$P_n(x) := \frac{(-1)^n}{n!\,|\Lambda|^n} \left(\frac{d}{dx}\right)^n (b-x)^n (x-a)^n, \tag{2}$$

where $|\Lambda| := b-a$. Furthermore, P_n has the following properties ([1]):

$$\frac{d}{dx}\left((b-x)(x-a)P_n'(x)\right)+n(n+1)P_n(x)=0, \quad \forall n\geq 0, \tag{3}$$

$$(P_m,P_n)_{L^2(\Lambda)}=\frac{|\Lambda|}{2n+1}\delta_{m,n}, \quad \forall m,n\geq 0, \tag{4}$$

$$(2n+1)P_n=\frac{|\Lambda|}{2}\left(P_{n+1}'-P_{n-1}'\right), \quad \forall n\geq 1, \tag{5}$$

$$P_n(a)=(-1)^n, \quad P_n(b)=1, \quad \forall n\geq 0, \tag{6}$$

where $(P_m,P_n)_{L^2(\Lambda)}$ denotes the L^2 inner product on Λ and $\delta_{m,n}$ denotes Kronecker's δ.

Lemma 2.1. *For any $u\in H^1(\Lambda)$ and integer $n\geq 1$, we have*

$$(u,P_n)_{L^2(\Lambda)}=\frac{|\Lambda|}{2(2n+1)}\Big((u',P_{n-1})_{L^2(\Lambda)}-(u',P_{n+1})_{L^2(\Lambda)}\Big). \tag{7}$$

Proof: From (5) and (6)

$$\begin{aligned}
(u,P_n)_{L^2(\Lambda)} &= \left(u,\frac{|\Lambda|}{2(2n+1)}\left(P_{n+1}'-P_{n-1}'\right)\right)_{L^2(\Lambda)} \\
&= \frac{|\Lambda|}{2(2n+1)}u(b)\big(P_{n+1}(b)-P_{n-1}(b)\big)-\frac{|\Lambda|}{2(2n+1)}u(a)\big(P_{n+1}(a)-P_{n-1}(a)\big) \\
&\quad -\frac{|\Lambda|}{2(2n+1)}\Big((u',P_{n+1}-P_{n-1})_{L^2(\Lambda)}\Big) \\
&= \frac{|\Lambda|}{2(2n+1)}\Big((u',P_{n-1})_{L^2(\Lambda)}-(u',P_{n+1})_{L^2(\Lambda)}\Big),
\end{aligned}$$

which implies (7). □

3. Error estimates for H_0^2-projection on a one-dimensional interval

Let $H_0^2(\Lambda)$ be a function space on Λ defined as

$$H_0^2(\Lambda)\equiv\left\{u\in H^2(\Lambda)\ ;\ u(a)=u(b)=u'(a)=u'(b)=0\right\}$$

with associated inner product

$$(u,v)_{H_0^2(\Lambda)} := (u'',v'')_{L^2(\Lambda)}.$$

First, we define the following set of functions

Definition 3.1. *For any integer $n \geq 4$, an n-th order polynomial ϕ_n on Λ is defined as*

$$\phi_n(x) = \frac{(-1)^n\sqrt{2n-3}}{(n-2)!\,|\Lambda|^{n-3/2}} \left(\frac{d}{dx}\right)^{n-4} (b-x)^{n-2}(x-a)^{n-2}. \tag{8}$$

Then, we have

Theorem 3.2. *The set of functions $\{\phi_n\}_{n\geq 4} \subset H_0^2(\Lambda)$ is a complete orthonormal system in $H_0^2(\Lambda)$.*

Proof: First, we show the orthogonality. From (8) we have, for arbitrary $n \geq 2$,

$$\begin{aligned}\phi''_{n+2}(x) &= \sqrt{\frac{2n+1}{|\Lambda|}}\frac{(-1)^n}{n!\,|\Lambda|^n}\left(\frac{d}{dx}\right)^n (b-x)^n(x-a)^n \\ &= \sqrt{\frac{2n+1}{|\Lambda|}}P_n(x).\end{aligned}$$

Hence, for any $m,\ n \geq 2$, by using the property given in (4), it holds that

$$\begin{aligned}(\phi_{m+2},\phi_{n+2})_{H_0^2(\Lambda)} &= \left(\phi''_{m+2},\phi''_{n+2}\right)_{L^2(\Lambda)} \\ &= \left(\sqrt{\frac{2m+1}{|\Lambda|}}P_m, \sqrt{\frac{2n+1}{|\Lambda|}}P_n\right)_{L^2(\Lambda)} \\ &= \frac{\sqrt{(2m+1)(2n+1)}}{|\Lambda|}\left(P_m,P_n\right)_{L^2(\Lambda)} \\ &= \delta_{m,n},\end{aligned}$$

which implies that $\{\phi_n\}_{n\geq 4}$ is an orthonormal system in $H_0^2(\Lambda)$.

Next, we prove the completeness. For an element $u \in H_0^2(\Lambda)$, suppose that $(u,\phi_{n+2})_{H_0^2(\Lambda)} = 0$ for all $n \geq 2$. Then, we have

$$(u,\phi_{n+2})_{H_0^2(\Lambda)} = \sqrt{\frac{2n+1}{|\Lambda|}}\left(u'',P_n\right)_{L^2(\Lambda)}, \quad \forall n \geq 2.$$

Namely,

$$\left(u'',P_n\right)_{L^2(\Lambda)} = 0, \quad \forall n \geq 2.$$

Moreover, by $u \in H_0^2(\Lambda)$ we have the following equalities:

$$(u'', P_0)_{L^2(\Lambda)} = u'(b) - u'(a) = 0,$$
$$(u'', P_1)_{L^2(\Lambda)} = -\frac{2}{|\Lambda|}(u', P_0)_{L^2(\Lambda)} = -\frac{2}{|\Lambda|}\big(u(b) - u(a)\big) = 0.$$

Since $\{P_n\}_{n\geq 0}$ is a complete orthogonal system in $L^2(\Lambda)$, it holds that $u'' = 0$ in $L^2(\Lambda)$. Thus, we have $u = 0$ in $H_0^2(\Lambda)$, which proves the completeness of $\{\phi_n\}_{n\geq 4}$. □

Definition 3.3 (H_0^2-projection). *For an integer $N \geq 4$, we define the finite-dimensional subspace S_N of $H_0^2(\Lambda)$ by $S_N \equiv \mathrm{span}_{4\leq n\leq N}\, \phi_n$. Then, we define the H_0^2-projection P_N^2 from $H_0^2(\Lambda)$ into S_N by*

$$\big(u - P_N^2 u, v_N\big)_{H_0^2(\Lambda)} = 0, \quad \forall v_N \in S_N. \tag{9}$$

We also set $S_3 = \{0\}$ and $P_3^2 \equiv 0$.

Now, we have the following basic constructive error estimates for the H_0^2-projection of a function u with H^4-regularity.

Theorem 3.4. *For an arbitrary integer $N \geq 3$, there exists a constant $\tilde{C}(|\Lambda|, N) > 0$ such that*

$$\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)} \leq \tilde{C}(|\Lambda|, N)\, |u|_{H^4(\Lambda)}, \quad \forall u \in H_0^2(\Lambda) \cap H^4(\Lambda), \tag{10}$$

where

$$\tilde{C}(|\Lambda|, N) = \begin{cases} \sqrt{3}\left(\frac{|\Lambda|}{2}\right)^2 \frac{1}{\sqrt{2N-5}(2N-3)\sqrt{2N-1}}, & \text{if } N = 3, \\ \sqrt{3}\left(\frac{|\Lambda|}{2}\right)^2 \frac{\sqrt{10N-3}}{(2N-3)\sqrt{(2N-1)(2N+1)(2N+3)}}, & \text{if } 4 \leq N \leq 38, \\ 3\sqrt{2}\left(\frac{|\Lambda|}{2}\right)^2 \frac{1}{\sqrt{(2N-1)(2N+1)(2N+5)(2N+7)}}, & \text{if } 39 \leq N. \end{cases}$$

Here, the H^4 seminorm is defined as $|u|_{H^4(\Lambda)} \equiv ||u''''||_{L^2(\Lambda)}$.

Proof: From Theorem 3.2, any $u \in H_0^2(\Lambda) \cap H^4(\Lambda)$ can be expanded by $\{\phi_n\}$. That is,

$$u = \sum_{n=4}^{\infty} a_n \phi_n \tag{11}$$
$$\text{with} \quad a_n = (u, \phi_n)_{H_0^2(\Lambda)}. \tag{12}$$

As a result of the orthogonality of $\{\phi_n\}$ in $H_0^2(\Lambda)$, the H_0^2-projection coincides with the truncation up to N. Hence, we have

$$P_N^2 u = \sum_{n=4}^{N} a_n \phi_n.$$

Therefore, the Parseval equality implies the following:

$$\begin{aligned} \left\| u - P_N^2 u \right\|_{H_0^2(\Lambda)}^2 &= \left\| \sum_{n=N+1}^{\infty} a_n \phi_n \right\|_{H_0^2(\Lambda)}^2 \\ &= \sum_{n=N+1}^{\infty} a_n^2. \end{aligned} \tag{13}$$

On the other hand, since $\{P_n\}$ is a complete orthogonal system in $L^2(\Lambda)$, $u'''' \in L^2(\Lambda)$ can also be expanded as

$$u'''' = \sum_{n=0}^{\infty} b_n \frac{P_n}{\|P_n\|_{L^2(\Lambda)}} \tag{14}$$

$$\text{with} \quad b_n = \left(u'''', \frac{P_n}{\|P_n\|_{L^2(\Lambda)}} \right)_{L^2(\Lambda)}. \tag{15}$$

Taking into account that $P_n / \|P_n\|_{L^2(\Lambda)}$ is a complete orthonormal system in $L^2(\Lambda)$, by the Parseval equality, we have

$$|u|_{H^4(\Lambda)}^2 = \|u''''\|_{L^2(\Lambda)}^2 = \sum_{n=0}^{\infty} b_n^2. \tag{16}$$

Now, for any integer $n \geq 4$, observe that by using Lemma 2.1

$$
\begin{aligned}
a_n &= (u, \phi_n)_{H_0^2(\Lambda)} \\
&= (u'', \phi_n'')_{L^2(\Lambda)} \\
&= \sqrt{2n-3}\,|\Lambda|^{-1/2}\,(u'', P_{n-2})_{L^2(\Lambda)} \\
&= \frac{|\Lambda|^{1/2}}{2\sqrt{2n-3}}\,(u''', P_{n-3} - P_{n-1})_{L^2(\Lambda)} \\
&= \frac{|\Lambda|^{1/2}}{2\sqrt{2n-3}}\frac{|\Lambda|}{2(2n-5)}\,(u'''', P_{n-4} - P_{n-2})_{L^2(\Lambda)} - \frac{|\Lambda|^{1/2}}{2\sqrt{2n-3}}\frac{|\Lambda|}{2(2n-1)}\,(u'''', P_{n-2} - P_n)_{L^2(\Lambda)} \\
&= \frac{|\Lambda|^{3/2}}{4}\frac{1}{(2n-5)\sqrt{2n-3}}\,(u'''', P_{n-4})_{L^2(\Lambda)} - \frac{|\Lambda|^{3/2}}{4}\frac{2\sqrt{2n-3}}{(2n-5)(2n-1)}\,(u'''', P_{n-2})_{L^2(\Lambda)} \\
&\quad + \frac{|\Lambda|^{3/2}}{4}\frac{1}{\sqrt{2n-3}(2n-1)}\,(u'''', P_n)_{L^2(\Lambda)} \\
&=: \left(\frac{|\Lambda|}{2}\right)^2 \big(\alpha_{n-4}b_{n-4} - \beta_{n-2}b_{n-2} + \gamma_n b_n\big). \qquad (17)
\end{aligned}
$$

Here, α_n, β_n, and γ_n are defined, respectively, as follows:

$$
\alpha_{n-4} = \frac{\|P_{n-4}\|_{L^2(\Lambda)}}{\sqrt{|\Lambda|}(2n-5)\sqrt{2n-3}} = \frac{1}{\sqrt{2n-7}(2n-5)\sqrt{2n-3}},
$$

$$
\text{namely,} \quad \alpha_n = \frac{1}{\sqrt{2n+1}(2n+3)\sqrt{2n+5}}, \qquad (18)
$$

$$
\beta_{n-2} = \frac{2\sqrt{2n-3}\,\|P_{n-2}\|_{L^2(\Lambda)}}{\sqrt{|\Lambda|}(2n-5)(2n-1)} = \frac{2}{(2n-5)(2n-1)},
$$

$$
\text{namely,} \quad \beta_n = \frac{2}{(2n-1)(2n+3)}, \qquad (19)
$$

$$
\gamma_n = \frac{\|P_n\|_{L^2(\Lambda)}}{\sqrt{|\Lambda|}\sqrt{2n-3}(2n-1)} = \frac{1}{\sqrt{2n-3}(2n-1)\sqrt{2n+1}}. \qquad (20)
$$

Note that α_n, β_n, and $\gamma_n \approx O(n^{-2})$ and are monotonically decreasing sequences in n. Then, we obtain the following estimates for each term of the final equality in (13)

$$
\begin{aligned}
a_n^2 &= \left(\frac{|\Lambda|}{2}\right)^4 \big(\alpha_{n-4}b_{n-4} - \beta_{n-2}b_{n-2} + \gamma_n b_n\big)^2 \\
&= \left(\frac{|\Lambda|}{2}\right)^4 \big(\alpha_{n-4}^2 b_{n-4}^2 + \beta_{n-2}^2 b_{n-2}^2 + \gamma_n^2 b_n^2 \\
&\qquad - 2\alpha_{n-4}b_{n-4}\beta_{n-2}b_{n-2} - 2\beta_{n-2}b_{n-2}\gamma_n b_n + 2\alpha_{n-4}b_{n-4}\gamma_n b_n\big) \\
&\le 3\left(\frac{|\Lambda|}{2}\right)^4 \big(\alpha_{n-4}^2 b_{n-4}^2 + \beta_{n-2}^2 b_{n-2}^2 + \gamma_n^2 b_n^2\big).
\end{aligned}
$$

Therefore, from (13), we have the estimates

$$
\begin{aligned}
\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}^2 &= \sum_{n=N+1}^{\infty} a_n^2 \\
&\le 3\left(\frac{|\Lambda|}{2}\right)^4 \sum_{n=N+1}^{\infty} \big(\alpha_{n-4}^2 b_{n-4}^2 + \beta_{n-2}^2 b_{n-2}^2 + \gamma_n^2 b_n^2\big) \\
&= 3\left(\frac{|\Lambda|}{2}\right)^4 \Bigg(\alpha_{N-3}^2 b_{N-3}^2 + \alpha_{N-2}^2 b_{N-2}^2 + \big(\alpha_{N-1}^2 + \beta_{N-1}^2\big) b_{N-1}^2 \\
&\qquad + \big(\alpha_N^2 + \beta_N^2\big) b_N^2 + \sum_{n=N+1}^{\infty} \left(\alpha_n^2 + \beta_n^2 + \gamma_n^2\right) b_n^2 \Bigg) \\
&\le 3\left(\frac{|\Lambda|}{2}\right)^4 \max\big\{\alpha_{N-3}^2,\ \alpha_{N-1}^2 + \beta_{N-1}^2,\ \alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2\big\} \sum_{n=N-3}^{\infty} b_n^2 \\
&\le 3\left(\frac{|\Lambda|}{2}\right)^4 \max\big\{\alpha_{N-3}^2,\ \alpha_{N-1}^2 + \beta_{N-1}^2,\ \alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2\big\} \left|u\right|_{H^4(\Lambda)}^2 \\
&=: \tilde{C}(|\Lambda|\,, N)^2 \left|u\right|_{H^4(\Lambda)}^2 .
\end{aligned}
$$

Finally, estimating the terms operated on by $\max\{\cdots\}$ in the above expression, we obtain for $N = 3$ then $\alpha_{N-1}^2 + \beta_{N-1}^2 \le \alpha_{N-3}^2$ and $\alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2 \le \alpha_{N-3}^2$, which implies the following:

$$
\tilde{C}(|\Lambda|\,, N) = \sqrt{3}\left(\frac{|\Lambda|}{2}\right)^2 \frac{1}{\sqrt{2N-5}(2N-3)\sqrt{2N-1}},
$$

for $4 \le N \le 38$. Thus, $\alpha_{N-3}^2 \le \alpha_{N-1}^2 + \beta_{N-1}^2$ and $\alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2 \le$

$\alpha_{N-1}^2 + \beta_{N-1}^2$, which implies the following:

$$\tilde{C}(|\Lambda|, N) = \sqrt{3}\left(\frac{|\Lambda|}{2}\right)^2 \frac{\sqrt{10N-3}}{(2N-3)\sqrt{(2N-1)(2N+1)(2N+3)}},$$

for $39 \leq N$ then $\alpha_{N-3}^2 \leq \alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2$ and $\alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2$, which implies

$$\tilde{C}(|\Lambda|, N) = 3\sqrt{2}\left(\frac{|\Lambda|}{2}\right)^2 \frac{1}{\sqrt{(2N-1)(2N+1)(2N+5)(2N+7)}}$$

Thus, we have the desired result. □

Based on the estimates in Theorem 3.4, we can obtain a smaller constant by using a method similar to that described in [4].

Lemma 3.5. *Let a_n, $b_n \in \mathbb{R}$ be as given in the proof of Theorem 3.4. In addition, let α_n, β_n, γ_n be positive numbers defined by* (18)*,* (19)*, and* (20)*, respectively. Then, for any integers $N \geq 3$ and $M \geq N+5$, there exists a constant $\sigma_{N,M} > 0$ such that*

$$\sum_{n=N+1}^{M} a_n^2 \leq \left(\frac{|\Lambda|}{2}\right)^4 \sigma_{N,M} \sum_{n=N-3}^{M} b_n^2, \tag{21}$$

where $\sigma_{N,M} := \max\{c_1(N), c_2(N), c_3(N), d_1(M), d_2(M)\}$ and $c_i(N), d_i(M)$ are defined as follows:

$$\begin{aligned}
c_1(N) &= \alpha_{N-3}^2 + \alpha_{N-3}\beta_{N-1} + \alpha_{N-3}\gamma_{N+1},\\
c_2(N) &= \alpha_{N-3}\beta_{N-1} + \alpha_{N-1}^2 + \beta_{N-1}^2 + \alpha_{N-1}\beta_{N+1} + \beta_{N-1}\gamma_{N+1} + \alpha_{N-1}\gamma_{N+3},\\
c_3(N) &= \alpha_{N-3}\gamma_{N+1} + \alpha_{N-1}\beta_{N+1} + \beta_{N-1}\gamma_{N+1}\\
&\quad + \alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2 + \alpha_{N+1}\beta_{N+3} + \beta_{N+1}\gamma_{N+3} + \alpha_{N+1}\gamma_{N+5},\\
d_1(M) &= \alpha_{M-7}\gamma_{M-3} + \alpha_{M-5}\beta_{M-3} + \beta_{M-5}\gamma_{M-3} + \beta_{M-3}^2 + \gamma_{M-3}^2 + \beta_{M-3}\gamma_{M-1},\\
d_2(M) &= \alpha_{M-5}\gamma_{M-1} + \beta_{M-3}\gamma_{M-1} + \gamma_{M-1}^2.
\end{aligned}$$

Proof: Setting $\vec{b} \equiv (b_{N-3}, b_{N-2}, \cdots, b_M)^T \in \mathbb{R}^{M-N+4}$ and taking (17) into account, reveals that there exists a symmetric and positive definite matrix A satisfying

$$\begin{aligned}
\sum_{n=N+1}^{M} a_n^2 &= \left(\frac{|\Lambda|}{2}\right)^4 \sum_{n=N+1}^{M} \left(\alpha_{n-4}b_{n-4} - \beta_{n-2}b_{n-2} + \gamma_n b_n\right)^2 \qquad (22)\\
&= \left(\frac{|\Lambda|}{2}\right)^4 \vec{b}^T A \vec{b}.
\end{aligned}$$

Here, $A = \left(A_{ij}\right)_{1\le i,j\le M-N+4}$ can be explicitly written as

$$A = \begin{pmatrix}
\alpha_{N-3}^2 & & & & & \text{symmetry} & & \\
0 & \alpha_{N-2}^2 & & & & & & \\
-\alpha_{N-3}\beta_{N-1} & 0 & \alpha_{N-1}^2+\beta_{N-1}^2 & & & & & \\
0 & -\alpha_{N-2}\beta_N & 0 & \alpha_N^2+\beta_N^2 & & & & \\
\alpha_{N-3}\gamma_{N+1} & 0 & -\alpha_{N-1}\beta_{N+1}-\beta_{N-1}\gamma_{N+1} & 0 & \alpha_{N+1}^2+\beta_{N+1}^2+\gamma_{N+1}^2 & & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\alpha_{M-7}\gamma_{M-3} & 0 & -\alpha_{M-5}\beta_{M-3}-\beta_{M-5}\gamma_{M-3} & 0 & \beta_{M-3}^2+\gamma_{M-3}^2 & & & \\
 & \alpha_{M-6}\gamma_{M-2} & 0 & -\alpha_{M-4}\beta_{M-2}-\beta_{M-4}\gamma_{M-2} & 0 & \beta_{M-2}^2+\gamma_{M-2}^2 & & \\
\text{\huge 0} & 0 & \alpha_{M-5}\gamma_{M-1} & 0 & -\beta_{M-3}\gamma_{M-1} & 0 & \gamma_{M-1}^2 & \\
 & & 0 & \alpha_{M-4}\gamma_M & 0 & -\beta_{M-2}\gamma_M & 0 & \gamma_M^2
\end{pmatrix}$$

The symmetry and positivity of A are clearly followed by the property of the quadratic form (22). Using Gerschgorin's theorem, the maximum eigenvalue of A is bounded by

$$\max\sigma(A) \le \max_{1\le j\le M-N+4} \sum_{i=1}^{M-N+4} |A_{ij}| =: \sigma_{N,M},$$

where $\sigma(A)$ denotes the set of eigenvalues of A.

Moreover, from the monotonically decreasing property of α_n, β_n, and γ_n in n, we have

$$\begin{aligned}
\sigma_{N,M} = \max\bigl\{&\alpha_{N-3}^2 + \alpha_{N-3}\beta_{N-1} + \alpha_{N-3}\gamma_{N+1},\\
&\alpha_{N-3}\beta_{N-1} + \alpha_{N-1}^2 + \beta_{N-1}^2 + \alpha_{N-1}\beta_{N+1} + \beta_{N-1}\gamma_{N+1} + \alpha_{N-1}\gamma_{N+3},\\
&\alpha_{N-3}\gamma_{N+1} + \alpha_{N-1}\beta_{N+1} + \beta_{N-1}\gamma_{N+1} + \alpha_{N+1}^2 + \beta_{N+1}^2 + \gamma_{N+1}^2\\
&\qquad + \alpha_{N+1}\beta_{N+3} + \beta_{N+1}\gamma_{N+3} + \alpha_{N+1}\gamma_{N+5},\\
&\alpha_{M-7}\gamma_{M-3} + \alpha_{M-5}\beta_{M-3} + \beta_{M-5}\gamma_{M-3} + \beta_{M-3}^2 + \gamma_{M-3}^2 + \beta_{M-3}\gamma_{M-1},\\
&\alpha_{M-5}\gamma_{M-1} + \beta_{M-3}\gamma_{M-1} + \gamma_{M-1}^2\bigr\}.
\end{aligned}$$

Thus, we obtain

$$\sum_{n=N+1}^{M} a_n^2 = \left(\frac{|\Lambda|}{2}\right)^4 \vec{b}^T A\vec{b} \le \left(\frac{|\Lambda|}{2}\right)^4 \sigma_{N,M}\left|\vec{b}\right|^2 = \left(\frac{|\Lambda|}{2}\right)^4 \sigma_{N,M}\sum_{n=N-3}^{M} b_n^2,$$

which proves the lemma. □

The following theorem gives alternative estimates to that given in Theorem 3.4, which enables better estimates of each constant to be obtained.

Theorem 3.6. *For each integer $N \geq 3$, there exists a constant $C_0(|\Lambda|, N) > 0$ such that*

$$\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)} \leq C_0(|\Lambda|, N)\,|u|_{H^4(\Lambda)}, \quad \forall u \in H_0^2(\Lambda) \cap H^4(\Lambda), \tag{23}$$

where

$$C_0(|\Lambda|, N) = \begin{cases} \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{c_1(N)}, & \text{if } N = 3, \\ \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{c_2(N)}, & \text{if } 4 \leq N \leq 19, \\ \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{c_3(N)}, & \text{if } 20 \leq N, \end{cases}$$

where $c_i(N)$ are the constants given in Lemma 3.5.
Moreover, $c_i(N)$ are explicitly written as

$$\begin{aligned} c_1(N) &= \frac{1}{(2N-5)(2N-3)^2(2N-1)} + \frac{2}{\sqrt{2N-5}(2N-3)^2\sqrt{2N-1}(2N+1)} \\ &\quad + \frac{1}{\sqrt{2N-5}(2N-3)(2N-1)(2N+1)\sqrt{2N+3}}, \\ c_2(N) &= \frac{2}{\sqrt{2N-5}(2N-3)^2\sqrt{2N-1}(2N+1)} + \frac{4}{(2N-3)\sqrt{2N-1}(2N+1)\sqrt{2N+3}(2N+5)} \\ &\quad + \frac{1}{\sqrt{2N-1}(2N+1)(2N+3)(2N+5)\sqrt{2N+7}} + \frac{10N-3}{(2N-3)^2(2N-1)(2N+1)(2N+3)}, \\ c_3(N) &= \frac{1}{\sqrt{2N-5}(2N-3)(2N-1)(2N+1)\sqrt{2N+3}} + \frac{4}{(2N-3)\sqrt{2N-1}(2N+1)\sqrt{2N+3}(2N+5)} \\ &\quad + \frac{6}{(2N-1)(2N+1)(2N+5)(2N+7)} + \frac{4}{(2N+1)\sqrt{2N+3}(2N+5)\sqrt{2N+7}(2N+9)} \\ &\quad + \frac{1}{\sqrt{2N+3}(2N+5)(2N+7)(2N+9)\sqrt{2N+11}}. \end{aligned}$$

Proof: For any $M \geq N + 5$, using Lemma 3.5 and arguments similar to those presented in the proof of Theorem 3.4, we have

$$\begin{aligned} \left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}^2 &= \sum_{n=N+1}^{M} a_n^2 + \sum_{n=M+1}^{\infty} a_n^2 \\ &\leq \left(\frac{|\Lambda|}{2}\right)^4 \sigma_{N,M} \sum_{n=N-3}^{M} b_n^2 + \tilde{C}(|\Lambda|, M)^2 \sum_{n=M-3}^{\infty} b_n^2 \\ &\leq \left(\left(\frac{|\Lambda|}{2}\right)^4 \sigma_{N,M} + \tilde{C}(|\Lambda|, M)^2\right) |u|_{H^4(\Lambda)}^2. \end{aligned}$$

For arbitrary $\varepsilon > 0$, there exists an M such that $\tilde{C}(|\Lambda|, M)^2 < \varepsilon$ and $d_i(M) < c_j(N)$, $(i = 1, 2,\ j = 1, 2, 3)$. We now fix such an M. Then, we have $\sigma_{N,M} = \max\{c_1(N), c_2(N), c_3(N)\}$. From the definition of $c_i(N)$ in Lemma 3.5, it is easily seen that

$$\begin{aligned} N = 3 &\Longrightarrow c_2(N) < c_1(N), \quad c_3(N) < c_1(N), \\ 4 \le N \le 19 &\Longrightarrow c_1(N) < c_2(N), \quad c_3(N) < c_2(N), \\ 20 \le N &\Longrightarrow c_1(N) < c_3(N), \quad c_2(N) < c_3(N). \end{aligned}$$

Hence, setting

$$C_0(|\Lambda|, N) \equiv \begin{cases} \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{c_1(N)}, & \text{if } \ N = 3, \\ \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{c_2(N)}, & \text{if } \ 4 \le N \le 19, \\ \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{c_3(N)}, & \text{if } \ 20 \le N, \end{cases}$$

we have $\|u - P_N^2 u\|_{H_0^2(\Lambda)}^2 \le \left(C_0(|\Lambda|, N)^2 + \varepsilon\right) |u|_{H^4(\Lambda)}^2$. Since ε is an arbitrary positive number, it holds that

$$\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)} \le C_0(|\Lambda|, N)\, |u|_{H^4(\Lambda)}.$$

Thus, explicit expressions of $c_j(N)$ yield the desired results. □

Now, we also obtain the following estimates, which further improve the constant, by a computer-assisted approach.

Theorem 3.7. *For any $N \ge 3$ and $M \ge \max\{N + 5, 20\}$, there exists a constant $C_M(|\Lambda|, N) > 0$ such that*

$$\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)} \le C_M(|\Lambda|, N)\, |u|_{H^4(\Lambda)}, \quad \forall u \in H_0^2(\Lambda) \cap H^4(\Lambda). \tag{24}$$

Here,

$$C_M(|\Lambda|, N) \equiv \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{\max \sigma(A) + c_3(M)}, \tag{25}$$

where A is as defined in Lemma 3.5, and $c_3(M)$ is a constant given in Lemma 3.5.

Proof: By the same argument in Theorem 3.6, using slightly different estimates, we have, for arbitrary integer $L \geq M+5$,

$$
\begin{aligned}
\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}^2 &= \sum_{n=N+1}^{M} a_n^2 + \sum_{n=M+1}^{L} a_n^2 + \sum_{n=L+1}^{\infty} a_n^2 \\
&\leq \left(\frac{|\Lambda|}{2}\right)^4 \vec{b}^T A \vec{b} + \left(\frac{|\Lambda|}{2}\right)^4 \sigma_{M,L} \sum_{n=M-3}^{L} b_n^2 + \tilde{C}(|\Lambda|, L)^2 \sum_{n=L-3}^{\infty} b_n^2 \\
&\leq \left(\left(\frac{|\Lambda|}{2}\right)^4 \max \sigma(A) + \left(\frac{|\Lambda|}{2}\right)^4 \sigma_{M,L} + \tilde{C}(|\Lambda|, L)^2\right) |u|_{H^4(\Lambda)}^2 .
\end{aligned}
$$

Here, we used the vector $\vec{b}$ and matrix A given in Lemma 3.5. For arbitrary $\varepsilon > 0$, there exists an integer L such that $\tilde{C}(|\Lambda|, L)^2 < \varepsilon$ and $d_i(L) < c_j(M)$, $(i = 1,2,\ j = 1,2,3)$. For such a fixed L, based on the assumption that $M \geq 20$, we have $\sigma_{M,L} = c_3(M)$. Therefore, we obtain

$$
\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}^2 \leq \left(\left(\frac{|\Lambda|}{2}\right)^4 \max \sigma(A) + \left(\frac{|\Lambda|}{2}\right)^4 c_3(M) + \varepsilon\right) |u|_{H^4(\Lambda)}^2 .
$$

Since ε is arbitrary, the theorem is proven. □

Now, let $C(|\Lambda|, N)$ denote the smallest constant satisfying the estimates given by (10) in Theorem 3.4. Then, we have the following enclosure of the optimal constant.

Theorem 3.8. *Under the assumptions of Theorem 3.7, we have*

$$
\left(\frac{|\Lambda|}{2}\right)^2 \sqrt{\max \sigma(A)} \leq C(|\Lambda|, N) \leq C_M(|\Lambda|, N). \tag{26}
$$

Proof: In the error estimates, we take a particular $u \in H_0^2(\Lambda) \cap H^4(\Lambda)$ such that $\vec{b} \equiv (b_{N-3}, \cdots b_M)$ coincides with an eigenvector corresponding to the maximum eigenvalue of the matrix A, as well as $b_n = 0$,or other n. Then, we have

$$
\begin{aligned}
\left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}^2 &= \sum_{n=N+1}^{M} a_n^2 + \sum_{n=M+1}^{\infty} a_n^2 \\
&= \left(\frac{|\Lambda|}{2}\right)^4 \max\sigma(A) \sum_{n=N-3}^{M} b_n^2 + \sum_{n=M+1}^{\infty} a_n^2 \\
&\geq \left(\frac{|\Lambda|}{2}\right)^4 \max\sigma(A) \sum_{n=N-3}^{M} b_n^2 \\
&= \left(\frac{|\Lambda|}{2}\right)^4 \max\sigma(A) \left|u\right|_{H^4(\Lambda)}^2 .
\end{aligned}
$$

Therefore, the optimal constant $C(|\Lambda|, N)$ satisfies

$$
C(|\Lambda|, N) \geq \left(\frac{|\Lambda|}{2}\right)^2 \sqrt{\max\sigma(A)},
$$

which proves the theorem.□

Numerical Verification Results

In this subsection, we present the verified intervals that enclose the optimal constant $C(|\Lambda|, N)$ computed by expression given in Theorem 3.8. We used the following environment for verified numerical computations.

Computer environment. CPU: Intel Core2 Quad Q6700, Memory: DDR2 8GB, OS: Ubuntu Linux 7.10 AMD64, Compiler: Intel Fortran 10.1, LAPACK: version 3.1.1, BLAS: Goto BLAS 1.26, Interval arithmetic: INTLIB [3].

Table 1 shows the validated computational results of the lower bound $\frac{1}{4}\sqrt{\max\sigma(A)}$ and upper bounds $C_M(|\Lambda|, N)\,|\Lambda|^{-2} = \frac{1}{4}\sqrt{\max\sigma(A) + c_3(M)}$ of the optimal constants $C(|\Lambda|, N)\,|\Lambda|^{-2}$ for $3 \leq N \leq 32$. Here, we use the parameter $M = N + 10,000$. The real numbers in each column in the table are the lower and upper bounds of intervals given in abbreviated form. For example, in case of $N = 3$, there exists an optimal constant $C(|\Lambda|, 3)\,|\Lambda|^{-2}$ in the interval $[\,0.04469616240857,\ 0.04469616240858\,]$. We also have the following L^2 and H_0^1 estimates.

Table 1: Verification results of $C(|\Lambda|, N)\, |\Lambda|^{-2}$.

N	$C(\|\Lambda\|, N)\, \|\Lambda\|^{-2}$	N	$C(\|\Lambda\|, N)\, \|\Lambda\|^{-2}$	N	$C(\|\Lambda\|, N)\, \|\Lambda\|^{-2}$
3	4.46961624085^{8}_{7}E-02	13	1.16345321482^{4}_{0}E-03	23	3.8117279298^{9}_{0}E-04
4	1.62145975082^{4}_{3}E-02	14	1.00465830406^{4}_{0}E-03	24	3.510274719^{41}_{31}E-04
5	9.00958826647^{3}_{2}E-03	15	8.7692840453^{5}_{1}E-04	25	3.243743962^{93}_{83}E-04
6	5.86304210222^{3}_{1}E-03	16	7.725589432^{54}_{49}E-04	26	3.00689176^{801}_{789}E-04
7	4.16384630084^{9}_{7}E-03	17	6.8611413019^{7}_{2}E-04	27	2.795429334^{57}_{45}E-04
8	3.128902248^{40}_{38}E-03	18	6.1366503096^{6}_{0}E-04	28	2.605820527^{65}_{52}E-04
9	2.446716930^{45}_{39}E-03	19	5.5231170046^{8}_{2}E-04	29	2.435127985^{92}_{79}E-04
10	1.97099111568^{9}_{6}E-03	20	4.99874033^{201}_{194}E-04	30	2.280894866^{51}_{37}E-04
11	1.6248684798^{91}_{88}E-03	21	4.546862824^{96}_{88}E-04	31	2.141053068^{63}_{48}E-04
12	1.36454584426^{9}_{6}E-03	22	4.15457197^{406}_{397}E-04	32	2.013851350^{36}_{20}E-04

Theorem 3.9. *Let $C_M(|\Lambda|, N) > 0$ be the constant given in Theorem 3.7. Then, it holds that*

$$\left\|u - P_N^2 u\right\|_{L^2(\Lambda)} \leq C_M(|\Lambda|, N) \left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}, \quad \forall u \in H_0^2(\Lambda), \tag{27}$$

$$\left\|u - P_N^2 u\right\|_{H_0^1(\Lambda)} \leq \sqrt{C_M(|\Lambda|, N)} \left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}, \quad \forall u \in H_0^2(\Lambda). \tag{28}$$

Proof: The estimates given in (27) are obtained by applying Aubin-Nitsche's trick. Next, for arbitrary $u \in H_0^2(\Lambda)$, using (27), observe that

$$\begin{aligned}
\left\|u - P_N^2 u\right\|_{H_0^1(\Lambda)}^2 &= \left(\frac{d}{dx}\big(u - P_N^2 u\big), \frac{d}{dx}\big(u - P_N^2 u\big)\right)_{L^2(\Lambda)} \\
&= \left(\frac{d^2}{dx^2}\big(u - P_N^2 u\big), u - P_N^2 u\right)_{L^2(\Lambda)} \\
&\leq \left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)} \left\|u - P_N^2 u\right\|_{L^2(\Lambda)} \\
&\leq C_M(|\Lambda|, N) \left\|u - P_N^2 u\right\|_{H_0^2(\Lambda)}^2,
\end{aligned}$$

which implies (28). $\square$

4. Error estimates for the finite element method in the one-dimensional case

In this section, applying Theorem 3.7 and Theorem 3.9, we derive the constructive a priori error estimates for the finite element method on one-dimensional intervals. Let Ω be a finite interval $\Omega = (\omega_0, \omega_1)$, $(\omega_0 < \omega_1)$ on $\mathbb{R}$. Let $\omega_0 = x_0 < x_1 < \cdots < x_k = \omega_1$ be a mesh of Ω and set $\Omega_i = (x_{i-1}, x_i)$. In addition, we set $h_i \equiv |\Omega_i| = x_i - x_{i-1}$ and $h \equiv (h_1, \cdots, h_k) \in \mathbb{R}^k$. For an integer vector $N = (N_1, \cdots, N_k) \in \mathbb{Z}^k$ with $(N_i \geq 3)$, let $S_{h,N}$ be a finite-dimensional subspace of $H^2(\Omega)$ constituted of piecewise polynomials of degree N_i on Ω_i. Then, $S_{h,N}$ is generated by two types of bases, namely, a piecewise cubic Hermite polynomial whose support is two consecutive elements and a function whose support is a single element corresponding to a polynomial of degree ≥ 4 that satisfies (8).

Definition 4.1 (Hermite interpolation). *Let Π_h denote a cubic Hermite interpolation from $H^2(\Omega)$ to $S_{h,N}$. That is, for each $u \in H^2(\Omega)$, $\Pi_h u \in S_{h,N}$ satisfies*

$$u(x_i) = \Pi_h u(x_i), \qquad \frac{du}{dx}(x_i) = \frac{d\Pi_h u}{dx}(x_i), \qquad \forall i = 0, \cdots, k. \tag{29}$$

Definition 4.2 (H_0^2-projection). *Let $P_{h,N}^2$ denote an H_0^2-projection from $H_0^2(\Omega)$ to $S_{h,N}$. That is, for each $u \in H_0^2(\Omega)$, $P_{h,N}^2 u \in S_{h,N}$ is defined as*

$$\left(u - P_{h,N}^2 u, v_{h,N}\right)_{H_0^2(\Omega)} = 0, \quad \forall v_{h,N} \in S_{h,N}. \tag{30}$$

It follows that the Definition 4.2 is well defined, because $(\cdot,\cdot)_{H_0^2(\Omega)}$ is a bounded and coercive bilinear form on $S_{h,N}$, and Definition 4.2 ensures the unique existence of $P_{h,N}^2 u$ satisfying (30). Moreover, for each $v \in H^2(\Omega)$ with $v|_{\Omega_i} \in H_0^2(\Omega_i)$, we define $P_{N_i} v \in S_{h,N}$ such that $\operatorname{supp} P_{N_i} v = \overline{\Omega_i}$ and $P_{N_i} v|_{\Omega_i}$ is a polynomial of degree N_i on Ω_i that satisfies (9) for $\Lambda \equiv \Omega_i$.

Theorem 4.3. *Let $C_M(\cdot,\cdot)$ denote the positive constant defined in Theorem 3.7. Then, we have the following a priori error estimates for the H_0^2-projection:*

$$\left\|u - P_{h,N}^2 u\right\|_{H_0^2(\Omega)} \leq \max_{1 \leq i \leq k} C_M(h_i, N_i)\, |u|_{H^4(\Omega)}, \quad \forall u \in H_0^2(\Omega) \cap H^4(\Omega). \tag{31}$$

Proof: For each $u \in H_0^2(\Omega) \cap H^4(\Omega)$, from (30), we have

$$\begin{aligned}\left\|u - P_{h,N}^2 u\right\|_{H_0^2(\Omega)}^2 &= \left(u - P_{h,N}^2 u, u\right)_{H_0^2(\Omega)} \\ &= \left(u - P_{h,N}^2 u, u - \Pi_h u - \sum_{i=1}^{k} P_{N_i}^2 (u - \Pi_h u)\right)_{H_0^2(\Omega)}.\end{aligned}$$

Therefore,

$$\left\|u - P_{h,N}^2 u\right\|_{H_0^2(\Omega)} \le \left\|u - \Pi_h u - \sum_{i=1}^{k} P_{N_i}^2 (u - \Pi_h u)\right\|_{H_0^2(\Omega)}.$$

Setting $\tilde{u} := u - \Pi_h u$, note that $P_{N_i}^2 \tilde{u}$ is uniquely determined by the definition of Π_h. In addition, taking into account that the support of $P_{N_i}^2 \tilde{u}$ coincides with $\overline{\Omega}_i$, we have

$$\left\|\tilde{u} - \sum_{i=1}^{k} P_{N_i}^2 \tilde{u}\right\|_{H_0^2(\Omega)}^2 = \sum_{i=1}^{k} \left\|\tilde{u} - P_{N_i}^2 \tilde{u}\right\|_{H_0^2(\Omega_i)}^2.$$

Thus, by Theorem 3.7, we have

$$\begin{aligned}\sum_{i=1}^{k} \left\|\tilde{u} - P_{N_i}^2 \tilde{u}\right\|_{H_0^2(\Omega_i)}^2 &\le \sum_{i=1}^{k} C_M(h_i, N_i)^2 \left|\tilde{u}\right|_{H^4(\Omega_i)}^2 \\ &= \sum_{i=1}^{k} C_M(h_i, N_i)^2 \left\|\left(\frac{d}{dx}\right)^4 \left(u - \Pi_h u\right)\right\|_{L^2(\Omega_i)}^2 \\ &= \sum_{i=1}^{k} C_M(h_i, N_i)^2 \left\|u''''\right\|_{L^2(\Omega_i)}^2 \\ &\le \max_{1 \le i \le k} C_M(h_i, N_i)^2 \left|u\right|_{H^4(\Omega)}^2.\end{aligned}$$

Here, since $(\Pi_h u)'''' = 0$, we obtain the estimates (31). $\square$

We also have the following L^2 and H_0^1 error estimates.

Theorem 4.4. *Under the same assumption given in Theorem, the following error estimates hold:*

$$\left\|u - P^2_{h,N}u\right\|_{L^2(\Omega)} \leq \max_{1\leq i\leq k} C_M(h_i, N_i) \left\|u - P^2_{h,N}u\right\|_{H^2_0(\Omega)}, \quad \forall u \in H^2_0(\Omega), \tag{32}$$

$$\left\|u - P^2_{h,N}u\right\|_{H^1_0(\Omega)} \leq \max_{1\leq i\leq k} \sqrt{C_M(h_i, N_i)} \left\|u - P^2_{h,N}u\right\|_{H^2_0(\Omega)}, \quad \forall u \in H^2_0(\Omega). \tag{33}$$

Since the proof of Theorem 4.4 is similar to that of Theorem 3.9, it is not presented in the present paper.

Remark 4.5. *Theorems 4.3 and 4.4 indicate a refinement of the estimates given in [7]. Namely, as shown in Table 1, the values of* $C(|\Lambda|, N)\,|\Lambda|^{-2}$ *obtained in the present study are approximately half that of the constant in the error estimates for the cubic Hermite interpolation (*$1/\pi^2 \approx 0.101321$*) presented in [7]*

Remark 4.6. *For the two-dimensional case, several constructive error estimates for* H^2_0*-projection on a rectangular domain are presented in [2], [6]. Since, in these studies, the error estimates are used for the one-dimensional case, which error estimation can be improved by applying Theorem 4.3.*

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