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RECENT PROGRESS IN THE VALUE DISTRIBUTION OF THE HYPERBOLIC GAUSS MAP

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Abstract. We give a brief survey of our work on the value distribution of the hyperbolic Gauss map. In particular, we define an algebraic class for constant mean curvature one surfaces in the hyperbolic three-space and give a ramification estimate for the hyperbolic Gauss map. Moreover, we give an effective estimate for the number of exceptional values of the hyperbolic Gauss maps of flat fronts in the hyperbolic three-space.

Introduction

One revealed the geometric meaning of the best possible upper bound for the number of exceptional values (for more precisely, defects) of meromorphic functions by using the Nevanlinna theory. Indeed, Ahlfors [1] showed that the geometric meaning of the best possible upper bound “2” for the number of exceptional values of nonconstant meromorphic functions on the complex plane $\mathbb{C}$ is the Euler number of the Riemann sphere. Moreover, he showed that the upper bound for the number of exceptional values (defects) of nonconstant holomorphic maps from $\mathbb{C}$ to a closed Riemann surface corresponds to the Euler number of the target space.

On the other hand, the author, Kobayashi and Miyaoka [9] refined the Osserman argument [19, 20] and gave a ramification estimate for the Gauss map of pseudo-algebraic and algebraic minimal surfaces in Euclidean three-space $\mathbb{R}^3$. We also gave a new proof of the Fujimoto theorem [5] in the pseudo-algebraic case, and revealed behind the Osserman result. In [10], we treated the case of Euclidean four-space $\mathbb{R}^4$.

The purpose of this survey is to give a geometric meaning of the upper bound for a ramification estimate for the hyperbolic Gauss map of nonflat algebraic constant mean curvature one (CMC-1, for short) surfaces and an effective estimate for the number of exceptional values of the hyperbolic Gauss maps for flat fronts in the hyperbolic three-space $\mathbb{H}^3$. In Section 1, we recall some fundamental facts and notations about CMC-1 surfaces. In Section 2, we give a ramification estimate for the hyperbolic Gauss map of algebraic CMC-1 surfaces and reveal geometric meaning behind it. In

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Section 3, we give an effective estimate for the number of exceptional values of the hyperbolic Gauss maps of flat fronts in $\mathcal{H}^3$.

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1. Preliminaries

First, we recall some basic facts and notations on CMC-1 surfaces in $\mathcal{H}^3$. Let $\mathbb{R}^4_1$ be the Lorentz-Minkowski 4-space with the Lorentz metric
\[
\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.
\]
Then the hyperbolic 3-space is given by
\[
\mathcal{H}^3 = \{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid -x_0^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = -1, x_0 > 0 \}
\]
corresponding isometrically to
\[
\mathbb{H}^3 = \{ a a^* \mid a \in \text{SL}(2, \mathbb{C}) \}
\]
with the metric
\[
\langle X, Y \rangle = -\frac{1}{2} \text{trace}(X \tilde{Y}), \quad \langle X, X \rangle = -\det(X),
\]
where $\tilde{Y}$ is the cofactor matrix of $Y$. The complex Lie group $\text{PSL}(2, \mathbb{C}) := \text{SL}(2, \mathbb{C})/\{ \pm \text{id} \}$ acts isometrically on $\mathcal{H}^3$ by
\[
\mathcal{H}^3 \ni X \longmapsto a X a^*,
\]
where $a \in \text{PSL}(2, \mathbb{C})$.

There exists a representation formula for CMC-1 surfaces in $\mathcal{H}^3$ as an analogy of the Enneper-Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$.

**Theorem 1.1** (Bryant [2], Umehara and Yamada [23]). Let $\tilde{M}$ be a simply connected Riemann surface with a reference point $z_0 \in \tilde{M}$. Let $g$ be a meromorphic function and $\omega$ be a holomorphic 1-form on $\tilde{M}$ such that
\[
ds^2 = (1 + |g|^2)^2 |\omega|^2
\]
is a Riemannian metric on $\tilde{M}$. Take a holomorphic immersion $F = (F_{ij}) : \tilde{M} \to \text{SL}(2, \mathbb{C})$ satisfying $F(z_0) = \text{id}$ and
\[
F^{-1} dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.
\]
Then $f : \tilde{M} \to \mathcal{H}^3$ defined by
\[
f = FF^*
\]
is a CMC-1 surface and the induced metric of $f$ is $ds^2$. Moreover, the second fundamental form $h$ and the Hopf differential $Q$ of $f$ are given by

$$h = -Q - Q^2 + ds^2, \quad Q = \omega d\gamma.$$  

(1.8)

Conversely, for any CMC-1 surface $f : \tilde{M} \to \mathcal{H}^3$, there exist a meromorphic function $g$ and a holomorphic 1-form $\omega$ on $\tilde{M}$ such that the induced metric of $f$ is given by (1.5) and (1.7) holds, where the map $F : \tilde{M} \to SL(2, \mathbb{C})$ is a holomorphic null (“null” means $\det (F^{-1}dF) = 0$) immersion satisfying (1.6).

**Remark 1.2.** The map $g$ is called a secondary Gauss map of $f$ [23]. The pair $(g, \omega)$ is called the Weierstrass data of $f$, and $F$ is called a holomorphic null lift of $f$.

Let $f : M \to \mathcal{H}^3$ be a CMC-1 surface of a (not necessarily simply connected) Riemann surface $M$. Then the holomorphic null lift $F$ is defined on the universal cover $\tilde{M}$ of $M$. Thus, the Weierstrass data $(g, \omega)$ is not single-valued on $M$. However, the Hopf differential $Q$ of $f$ is well-defined on $M$. By (1.6), the secondary Gauss map $g$ satisfies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}, \quad \text{where} \quad F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}.$$  

(1.9)

The hyperbolic Gauss map $G$ of $f$ is defined by

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}.$$  

(1.10)

Identifying the ideal boundary $S^2_{\infty}$ of $\mathcal{H}^3$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$, we give a geometric meaning of $G$ as follows (cf. [2]): The hyperbolic Gauss map $G$ sends each $p \in M$ to the terminal point $G(p)$ at $S^2_{\infty}$ of the oriented normal geodesics of $\mathcal{H}^3$ that starting $f(p)$. In particular, $G$ is a meromorphic function on $M$.

The inverse matrix $F^{-1}$ is also a holomorphic null immersion, and produce a new CMC-1 surface $f^* = F^{-1}(F^{-1})^* : \tilde{M} \to \mathcal{H}^3$, called the dual of $f$ [26]. By definition, the Weierstrass data $(g^*, \omega^*)$ of $f^*$ satisfies

$$(F^*)^{-1}dF^* = \begin{pmatrix} g^* & -(g^*)^2 \\ 1 & -g^* \end{pmatrix} \omega^*.$$  

(1.11)

Umehara and Yamada [26, Proposition 4] proved that the Weierstrass data, the Hopf differential $Q^*$, and the hyperbolic Gauss map $G^*$ of $f^*$ are given by

$$g^* = G, \quad \omega^* = -\frac{Q}{dG}, \quad Q^* = -Q, \quad G^* = g.$$  

(1.12)

So this duality between $f$ and $f^*$ exchanges the roles of the hyperbolic Gauss map and the secondary Gauss map. We call the pair $(G, \omega^*)$ the dual Weierstrass data of $f$. Moreover, these invariants are related by

$$S(g) - S(G) = 2Q,$$  

(1.13)

where $S(\cdot)$ denotes the Schwarzian derivative

$$S(h) = \left[\left(\frac{h''}{h'}\right)' - \frac{1}{2} \left(\frac{h''}{h'}\right)^2\right] dz^2, \quad (') = \frac{d}{dz}.$$
with respect to a complex local coordinate $z$ on $M$. By Theorem 1.1 and (1.12), the induced metric $ds^2$ of $f^\circ$ is given by
\[ds^2 = (1 + |g^\circ|^2)\omega^2 = (1 + |G|^2)^2 Q \frac{dG}{d\omega}.\] (1.14)

We call the metric $ds^2$ the dual metric of $f$. There exists the following relation between the dual metric $ds^2$ and the metric $ds^2$.

**Lemma 1.3** (Umehara-Yamada [26], Z. Yu [27]). The Riemannian metric $ds^2$ is complete (resp. nondegenerate) if and only if $ds^2$ is complete (resp. nondegenerate).

Since $G$ and $Q$ are single-valued on $M$, the dual metric $ds^2$ is also single-valued on $M$. So we can define the dual total absolute curvature by
\[\text{TA}(f^\circ) := \int_M (-K^\circ) dA^\circ = \int_M \frac{4|dG|^2}{(1 + |G|^2)^2}.\]
where $K^\circ(\leq 0)$ and $dA^\circ$ are the Gaussian curvature and the area element of $ds^2$, respectively. Note that $\text{TA}(f^\circ)$ is the area of $M$ with respect to the (singular) metric induced from the Fubini-Study metric on the complex projective line $\mathbb{P}^1(\mathbb{C}) (= \mathbb{C} \cup \{\infty\})$. When the dual total absolute curvature of a complete CMC-$1$ surface is finite, the surface is called an algebraic CMC-$1$ surface.

**Theorem 1.4** (Bryant, Huber, Z. Yu). An algebraic CMC-$1$ surface $f: M \to \mathcal{H}^3$ satisfies:

(i) $M$ is biholomorphic to $\bar{M}_\gamma \setminus \{p_1, \ldots, p_k\}$, where $\bar{M}_\gamma$ is a closed Riemann surface of genus $\gamma$ and $p_j \in M_\gamma$ ($j = 1, \ldots, k$). ([7])

(ii) The dual Weierstrass data $(G, \omega^\circ)$ can be extended meromorphically to $\bar{M}_\gamma$. ([2],[27])

We call the points $p_j$ the ends of $f$. An end $p_j$ of $f$ is called regular if the hyperbolic Gauss map $G$ has at most a pole at $p_j$ [23]. By Theorem 1.4, each end of an algebraic CMC-$1$ surface is regular.

2. Ramification estimate for the hyperbolic Gauss map of algebraic CMC-$1$ surfaces

Next, we give a ramification estimate for the hyperbolic Gauss map of algebraic CMC-$1$ surfaces.

**Theorem 2.1** ([11]). Let $f: M = \bar{M}_\gamma \setminus \{p_1, \ldots, p_k\} \to \mathcal{H}^3$ be a nonflat algebraic CMC-$1$ surface, $G: M \to \mathbb{C} \cup \{\infty\}$ be the hyperbolic Gauss map of $f$ and $d$ be the degree of $G$ considered as a map from $M_\gamma$. Assume that, for some fixed distinct values $a_1, \ldots, a_q \in \mathbb{C} \cup \{\infty\}$, $\nu_j$ be the minimum of multiplicities of $G$ at $G^{-1}(a_j)$ (If $a_j$ is the exceptional value of $G$, we set $\nu_j = \infty$). Then we have
\[\sum_{j=1}^q \left(1 - \frac{1}{\nu_j}\right) \leq 2 + \frac{2}{R}, \quad \frac{1}{R} = \frac{\gamma - 1 + k/2}{d} < 1.\] (2.1)

As a corollary, we obtain the following result.
Corollary 2.2 ([4], [11]). The hyperbolic Gauss map of a nonflat algebraic CMC-1 surface in $\mathcal{H}^3$ can omit at most three values.

However, we do not know the best possible upper bound for the number of exceptional values of $G$ in this class. Indeed, we do not find an algebraic CMC-1 surface whose hyperbolic Gauss map omits 3 values, but there are some algebraic CMC-1 surfaces whose hyperbolic Gauss map omits 2 values (for instance, catenoid cousins, examples in [21, Theorem 4.7] and [11, Proposition 2.7]).

The upper bound “$2 + 2/R$” of (2.1) has a geometric meaning. The geometric meaning of “2” is the Euler number of the target Riemann sphere. Moreover, the geometric meaning of “$R−1$” is as follows: We assume that the universal covering of $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$ is the unit disk $\mathbb{D}$. Let $A_{hyp}(M)$ be the hyperbolic area of $M$ with respect to the hyperbolic metric with Gaussian curvature $-4\pi$ on $\mathbb{D}$, $A_{FS}(M)$ be the area of $M$ with respect to the induced Fubini-Study metric $G^\ast \omega_{FS}$ with Gaussian curvature $4\pi$ on the Riemann sphere. Then we have

$$\frac{1}{R} = \frac{\gamma - 1 + k/2}{d} = \frac{A_{hyp}(M)}{A_{FS}(M)}.$$  

(2.2)

For more detail, see [9, Section 6].

The proof of Theorem 2.1 consists of two parts. One is “$R−1 < 1$”. This corresponds to the Oseemann type inequality [26] for algebraic CMC-1 surfaces in $\mathcal{H}^3$. Another is the left side of the system of inequalities (2.1). Its proof is based on [9, Theorem 3.3]. For more detail of the proof of Theorem 2.1, see [11].

3. Value distribution of the hyperbolic Gauss maps of flat fronts

In this section, we give an effective estimate for the number of exceptional values of the hyperbolic Gauss maps of flat fronts in $\mathcal{H}^3$. First, we briefly recall definitions and basic facts on flat fronts in $\mathcal{H}^3$. For more details, refer to [6], [13], [14] and [15]. A smooth map $f: M \to \mathcal{H}^3$ from a 2-manifold $M$ is called a front if there exists a Legendrian immersion $L_f: M \to T^\ast T^\ast \mathcal{H}^3$ into the unit cotangent bundle $T^\ast T^\ast \mathcal{H}^3$ of $\mathcal{H}^3$ whose projection is $f$, which may have singular points (points $x \in M$ where rank$(df)_x < 2$). A point which is not singular is said to be regular, where the first fundamental form is positive definite. The Gaussian curvature is well-defined at regular points. A front is said to be flat if the Gaussian curvature vanishes at each regular point (However, this definition of “flat” is not suitable when all points of the front are degenerate. For more details, see [15, Definition 2.1 and Remark 2.2]).

A front $f$ is said to be complete if there is a symmetric tensor $T$ on $M$ which has compact support such that $T + ds^2$ is a complete Riemannian metric on $M$, where $ds^2$ is the first fundamental form of $f$. For a flat front $f: M \to \mathcal{H}^3$, there exists a (unique) complex structure of $M$. Then we regard $M$ as a Riemann surface. Moreover if $f$ is complete, then there exists a closed Riemann surface $\overline{M}_{\gamma}$ of genus $\gamma$ such that $M$ is biholomorphic to $\overline{M}_{\gamma} \setminus \{p_1, \ldots, p_k\}$ ([6], [15]). The points $p_1, \ldots, p_k$ are called the ends of $f$.

For each point $p \in M$, there exists a pair $(G(p), G_\ast(p)) \in S^2 \times S^2_\infty$ of distinct points on the ideal boundary $S^2_\infty$ such that the geodesic in $\mathcal{H}^3$ starting from $G_\ast(p)$ towards $G(p)$
coincides with the oriented normal geodesic at \( p \). The pair of the maps 
\[ G, G_* : M \to S^2_\infty \]
are called the hyperbolic Gauss maps of \( f \). If a front \( f \) is flat and we regard \( S^2_\infty \) as the Riemann sphere, then both \( G \) and \( G_* \) are holomorphic. An end \( p_j \) of a flat front 
\[ f : \tilde{M} \setminus \{p_1, \ldots, p_k\} \to \mathcal{H}^3 \]
is said to be regular, if both \( G \) and \( G_* \) can be extended meromorphically across it. Kokubu, Umehara and Yamada showed the following global property for this class.

**Theorem 3.1** ([15], Theorem 3.13). Let \( f : \tilde{M} \setminus \{p_1, \ldots, p_k\} \to \mathcal{H}^3 \) be a complete flat front whose ends are all regular. Then
\[ d + d_* \geq k \]
where \( d \) is the degree of \( G \) considered as maps \( \tilde{M} \) (if \( G \) has essential singularities, then we define \( d = \infty \)) and \( d_* \) is the degree of \( G_* \) considered as the same way. Furthermore, equality holds if and only if all ends are embedded.

As an application of the inequality, we show the following effective estimate for the number of exceptional values of the hyperbolic Gauss maps for this class.

**Theorem 3.2** ([12]). Let \( f : \tilde{M} \setminus \{p_1, \ldots, p_k\} \to \mathcal{H}^3 \) be a complete flat front and the maps \( G \) and \( G_* \) be the hyperbolic Gauss maps of \( f \). If \( G \) omits \( p \) values and \( G_* \) omits \( q \) values, then \( p \leq 2 \) or \( q \leq 2 \) or,
\[ \frac{1}{p-1} + \frac{1}{q-1} \geq \frac{k}{2\gamma-2+2}. \]

Note that the right side of the inequality is given by topological invariants of \( M = \tilde{M} \setminus \{p_1, \ldots, p_k\} \) without any data of the hyperbolic Gauss map. As a corollary, we give more sharp results on the number of exceptional values of them for some topological cases.

**Corollary 3.3** ([12]). For complete flat fronts in \( \mathcal{H}^3 \), we have the following:

1. There does not exist a complete flat front with \( \gamma = 0, p \geq 4 \) and \( q \geq 4 \).
2. There does not exist a complete flat front with \( \gamma = 1, p \geq 5 \) and \( q \geq 5 \).

**References**


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