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Kawakami, Yu
Graduate School of Mathematics, Kyushu University

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RECENT PROGRESS IN THE VALUE DISTRIBUTION OF THE HYPERBOLIC GAUSS MAP

YU KAWAKAMI

Abstract. We give a brief survey of our work on the value distribution of the hyperbolic Gauss map. In particular, we define an algebraic class for constant mean curvature one surfaces in the hyperbolic three-space and give a ramification estimate for the hyperbolic Gauss map. Moreover, we give an effective estimate for the number of exceptional values of the hyperbolic Gauss maps of flat fronts in the hyperbolic three-space.

Introduction

One revealed the geometric meaning of the best possible upper bound for the number of exceptional values (for more precisely, defects) of meromorphic functions by using the Nevanlinna theory. Indeed, Ahlfors [1] showed that the geometric meaning of the best possible upper bound "2" for the number of exceptional values of nonconstant meromorphic functions on the complex plane $\mathbb{C}$ is the Euler number of the Riemann sphere. Moreover, he showed that the upper bound for the number of exceptional values (defects) of nonconstant holomorphic maps from $\mathbb{C}$ to a closed Riemann surface corresponds to the Euler number of the target space.

On the other hand, the author, Kobayashi and Miyaoka [9] refined the Osserman argument [19, 20] and gave a ramification estimate for the Gauss map of pseudo-algebraic and algebraic minimal surfaces in Euclidean three-space $\mathbb{R}^3$. We also gave a new proof of the Fujimoto theorem [5] in the pseudo-algebraic case, and revealed behind the Osserman result. In [10], we treated the case of Euclidean four-space $\mathbb{R}^4$.

The purpose of this survey is to give a geometric meaning of the upper bound for a ramification estimate for the hyperbolic Gauss map of nonflat algebraic constant mean curvature one (CMC-1, for short) surfaces and an effective estimate for the number of exceptional values of the hyperbolic Gauss maps for flat fronts in the hyperbolic three-space $\mathcal{H}^3$. In Section 1, we recall some fundamental facts and notations about CMC-1 surfaces in $\mathcal{H}^3$. In particular, using the notion of "duality", we give a definition of algebraic CMC-1 surfaces. In Section 2, we give a ramification estimate for the hyperbolic Gauss map of algebraic CMC-1 surfaces and reveal geometric meaning behind it. In

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Section 3, we give an effective estimate for the number of exceptional values of the hyperbolic Gauss maps of flat fronts in $\mathcal{H}^3$.

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1. Preliminaries

First, we recall some basic facts and notations on CMC-1 surfaces in $\mathcal{H}^3$. Let $\mathbb{R}^4_1$ be the Lorentz-Minkowski 4-space with the Lorentz metric

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3)\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3\,. \tag{1.1}$$

Then the hyperbolic 3-space is given by

$$\mathcal{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \,|\, -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = -1, x_0 > 0\}$$

with the induced metric from $\mathbb{R}^4_1$, which is a simply connected Riemannian 3-manifold with constant sectional curvature -1. We identify $\mathbb{R}^4_1$ with the set of 2×2 Hermitian matrices Herm(2)= $\{X^* = X\}$ $(X^* := {}^t\overline{X}\,)$ by

$$(x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}, \tag{1.2}$$

where $i = \sqrt{-1}$. In this identification, $\mathcal{H}^3$ is represented as

$$\mathcal{H}^3 = \{aa^* \,|\, a \in SL(2, \mathbb{C})\} \tag{1.3}$$

with the metric

$$\langle X, Y\rangle = -\frac{1}{2}\,\mathrm{trace}\,(X\widetilde{Y}), \quad \langle X, X\rangle = -\det(X)\,,$$

where $\widetilde{Y}$ is the cofactor matrix of Y. The complex Lie group $PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/\{\pm \mathrm{id}\}$ acts isometrically on $\mathcal{H}^3$ by

$$\mathcal{H}^3 \ni X \longmapsto aXa^*\,, \tag{1.4}$$

where $a \in PSL(2, \mathbb{C})$.

There exists a representation formula for CMC-1 surfaces in $\mathcal{H}^3$ as an analogy of the Enneper-Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$.

Theorem 1.1 (Bryant [2], Umehara and Yamada [23]). *Let $\tilde{M}$ be a simply connected Riemann surface with a reference point $z_0 \in \tilde{M}$. Let g be a meromorphic function and ω be a holomorphic 1-form on $\tilde{M}$ such that*

$$ds^2 = (1 + |g|^2)^2|\omega|^2 \tag{1.5}$$

is a Riemannian metric on $\tilde{M}$. Take a holomorphic immersion $F = (F_{ij})\colon \tilde{M} \to SL(2, \mathbb{C})$ satisfying $F(z_0) = id$ and

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega\,. \tag{1.6}$$

Then $f\colon \tilde{M} \to \mathcal{H}^3$ defined by

$$f = FF^* \tag{1.7}$$

is a CMC-1 surface and the induced metric of f is ds^2. Moreover, the second fundamental form h and the Hopf differential Q of f are given by

$$h = -Q - \overline{Q} + ds^2, \quad Q = \omega dg\,. \tag{1.8}$$

Conversely, for any CMC-1 surface $f\colon \tilde{M} \to \mathcal{H}^3$, there exist a meromorphic function g and a holomorphic 1-form ω on $\tilde{M}$ such that the induced metric of f is given by (1.5) *and* (1.7) *holds, where the map $F\colon \tilde{M} \to SL(2,\mathbb{C})$ is a holomorphic null (“null” means* $\det(F^{-1}dF) = 0$) *immersion satisfying* (1.6).

Remark 1.2. The map g is called a *secondary* Gauss map of f [23]. The pair (g,ω) is called the *Weierstrass data* of f, and F is called a *holomorphic null lift* of f.

Let $f\colon M \to \mathcal{H}^3$ be a CMC-1 surface of a (not necessarily simply connected) Riemann surface M. Then the holomorphic null lift F is defined on the universal cover $\tilde{M}$ of M. Thus, the Weierstrass data (g,ω) is not single-valued on M. However, the Hopf differential Q of f is well-defined on M. By (1.6), the secondary Gauss map g satisfies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}, \quad \text{where } F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}. \tag{1.9}$$

The *hyperbolic Gauss map* G of f is defined by

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}\,. \tag{1.10}$$

Identifying the ideal boundary $\mathbb{S}^2_\infty$ of $\mathcal{H}^3$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$, we give a geometric meaning of G as follows (cf. [2]): The hyperbolic Gauss map G sends each $p \in M$ to the terminal point $G(p)$ at $\mathbb{S}^2_\infty$ of the oriented normal geodesics of $\mathcal{H}^3$ that starting $f(p)$. In particular, G is a meromorphic function on M.

The inverse matrix F^{-1} is also a holomorphic null immersion, and produce a new CMC-1 surface $f^\sharp = F^{-1}(F^{-1})^*\colon \tilde{M} \to \mathcal{H}^3$, called the *dual* of f [26]. By definition, the Weierstrass data $(g^\sharp, \omega^\sharp)$ of $f^\sharp$ satisfies

$$(F^\sharp)^{-1} dF^\sharp = \begin{pmatrix} g^\sharp & -(g^\sharp)^2 \\ 1 & -g^\sharp \end{pmatrix} \omega^\sharp\,. \tag{1.11}$$

Umehara and Yamada [26, Proposition 4] proved that the Weierstrass data, the Hopf differential $Q^\sharp$, and the hyperbolic Gauss map $G^\sharp$ of $f^\sharp$ are given by

$$g^\sharp = G, \quad \omega^\sharp = -\frac{Q}{dG}, \quad Q^\sharp = -Q, \quad G^\sharp = g\,. \tag{1.12}$$

So this duality between f and $f^\sharp$ exchanges the roles of the hyperbolic Gauss map and the secondary Gauss map. We call the pair $(G, \omega^\sharp)$ the *dual Weierstrass data* of f. Moreover, these invariants are related by

$$S(g) - S(G) = 2Q\,, \tag{1.13}$$

where $S(\cdot)$ denotes the Schwarzian derivative

$$S(h) = \left[\left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2\right] dz^2, \ \left({}' = \frac{d}{dz}\right)$$

with respect to a complex local coordinate z on M. By Theorem 1.1 and (1.12), the induced metric $ds^{2\sharp}$ of $f^\sharp$ is given by

$$ds^{2\sharp} = (1+|g^\sharp|^2)^2|\omega^\sharp|^2 = (1+|G|^2)^2\left|\frac{Q}{dG}\right|^2. \tag{1.14}$$

We call the metric $ds^{2\sharp}$ the *dual metric* of f. There exists the following relation between the dual metric $ds^{2\sharp}$ and the metric ds^2.

Lemma 1.3 (Umehara-Yamada [26], Z. Yu [27]). *The Riemannian metric $ds^{2\sharp}$ is complete (resp. nondegenerate) if and only if ds^2 is complete (resp. nondegenerate).*

Since G and Q are single-valued on M, the dual metric $ds^{2\sharp}$ is also single-valued on M. So we can define the *dual total absolute curvature* by

$$\mathrm{TA}(f^\sharp) := \int_M (-K^\sharp)dA^\sharp = \int_M \frac{4|dG|^2}{(1+|G|^2)^2},$$

where $K^\sharp(\leq 0)$ and $dA^\sharp$ are the Gaussian curvature and the area element of $ds^{2\sharp}$, respectively. Note that $\mathrm{TA}(f^\sharp)$ is the area of M with respect to the (singular) metric induced from the Fubini-Study metric on the complex projective line $\mathbb{P}^1(\mathbb{C})(=\mathbb{C}\cup\{\infty\})$. When the dual total absolute curvature of a complete CMC-1 surface is finite, the surface is called *an algebraic CMC-1 surface*.

Theorem 1.4 (Bryant, Huber, Z. Yu). *An algebraic CMC-1 surface $f\colon M \to \mathcal{H}^3$ satisfies:*

(i) *M is biholomorphic to $\bar{M}_\gamma \setminus \{p_1,\dots,p_k\}$, where $\bar{M}_\gamma$ is a closed Riemann surface of genus γ and $p_j \in \bar{M}_\gamma$ $(j=1,\dots,k)$.* ([7])

(ii) *The dual Weierstrass data $(G,\omega^\sharp)$ can be extended meromorphically to $\bar{M}_\gamma$.* ([2],[27])

We call the points p_j the *ends* of f. An end p_j of f is called *regular* if the hyperbolic Gauss map G has at most a pole at p_j [23]. By Theorem 1.4, each end of an algebraic CMC-1 surface is regular.

2. Ramification estimate for the hyperbolic Gauss map of algebraic CMC-1 surfaces

Next, we give a ramification estimate for the hyperbolic Gauss map of algebraic CMC-1 surfaces.

Theorem 2.1 ([11]). *Let $f\colon M = \bar{M}_\gamma\setminus\{p_1,\dots,p_k\} \to \mathcal{H}^3$ be a nonflat algebraic CMC-1 surface, $G\colon M \to \mathbb{C}\cup\{\infty\}$ be the hyperbolic Gauss map of f and d be the degree of G considered as a map from $\bar{M}_\gamma$. Assume that, for some fixed distinct values $a_1,\dots,a_q \in \mathbb{C}\cup\{\infty\}$, ν_j be the minimum of multiplicities of G at $G^{-1}(a_j)$ (If a_j is the exceptional value of G, we set $\nu_j=\infty$) . Then we have*

$$\sum_{j=1}^{q}\left(1-\frac{1}{\nu_j}\right) \leq 2+\frac{2}{R}, \quad \frac{1}{R} = \frac{\gamma-1+k/2}{d} < 1. \tag{2.1}$$

As a corollary, we obtain the following result.

Corollary 2.2 ([4], [11]). *The hyperbolic Gauss map of a nonflat algebraic CMC-1 surface in $\mathcal{H}^3$ can omit at most three values.*

However, we do not know the best possible upper bound for the number of exceptional values of G in this class. Indeed, we do not find an algebraic CMC-1 surface whose hyperbolic Gauss map omits 3 values, but there are some algebraic CMC-1 surfaces whose hyperbolic Gauss map omits 2 values (for instance, catenoid cousins, examples in [21, Theorem 4.7] and [11, Proposition 2.7]).

The upper bound "$2 + 2/R$" of (2.1) has a geometric meaning. The geometric meaning of "2" is the Euler number of the target Riemann sphere. Moreover, the geometric meaning of "R^{-1}" is as follows: We assume that the universal covering of $M = \bar{M}_\gamma \backslash \{p_1, \dots, p_k\}$ is the unit disk $\mathbb{D}$. Let $A_{hyp}(M)$ be the hyperbolic area of M with respect to the hyperbolic metric with Gaussian curvature -4π on $\mathbb{D}$, $A_{FS}(M)$ be the area of M with respect to the induced Fubini-Study metric $G^*\omega_{FS}$ with Gaussian curvature 4π on the Riemann sphere. Then we have

$$\frac{1}{R} = \frac{\gamma - 1 + k/2}{d} = \frac{A_{hyp}(M)}{A_{FS}(M)} \,. \tag{2.2}$$

For more detail, see [9, Section 6].

The proof of Theorem 2.1 consists of two parts. One is "$R^{-1} < 1$". This corresponds to the Oseerman type inequality [26] for algebraic CMC-1 surfaces in $\mathcal{H}^3$. Another is the left side of the system of inequalities (2.1). Its proof is based on [9, Theorem 3.3]. For more detail of the proof of Theorem 2.1, see [11].

3. Value distribution of the hyperbolic Gauss maps of flat fronts

In this section, we give an effective estimate for the number of exceptional values of the hyperbolic Gauss maps of flat fronts in $\mathcal{H}^3$. First, we briefly recall definitions and basic facts on flat fronts in $\mathcal{H}^3$. For more details, refer to [6], [13], [14] and [15]. A smooth map $f\colon M \to \mathcal{H}^3$ from a 2-manifold M is called a *front* if there exists a Legendrian immersion $L_f\colon M \to T_1^*\mathcal{H}^3$ into the unit cotangent bundle $T_1^*\mathcal{H}^3$ of $\mathcal{H}^3$ whose projection is f, which may have singular points (points $x \in M$ where $\mathrm{rank}(df)_x < 2$). A point which is not singular is said to be *regular*, where the first fundamental form is positive definite. The Gaussian curvature is well-defined at regular points. A front is said to be *flat* if the Gaussian curvature vanishes at each regular point (However, this definition of "flat" is not suitable when all points of the front are degenerate. For more details, see [15, Definition 2.1 and Remark 2.2].) .

A front f is said to be *complete* if there is a symmetric tensor T on M which has compact support such that $T + ds^2$ is a complete Riemannian metric on M, where ds^2 is the first fundamental form of f. For a flat front $f\colon M \to \mathcal{H}^3$, there exists a (unique) complex structure of M. Then we regard M as a Riemann surface. Moreover if f is complete, then there exists a closed Riemann surface $\bar{M}_\gamma$ of genus γ such that M is biholomorphic to $\bar{M}_\gamma \backslash \{p_1, \dots, p_k\}$ ([6], [15]). The points $p_1, \dots, p_k$ are called the *ends* of f.

For each point $p \in M$, there exists a pair $(G(p), G_*(p)) \in \mathbb{S}^2_\infty \times \mathbb{S}^2_\infty$ of distinct points on the ideal boundary $\mathbb{S}^2_\infty$ such that the geodesic in $\mathcal{H}^3$ starting from $G_*(p)$ towards $G(p)$

coincides with the oriented normal geodesic at p. The pair of the maps

$$G,\, G_*\colon M \to \mathbb{S}^2_\infty$$

are called the *hyperbolic Gauss maps* of f. If a front f is flat and we regard $\mathbb{S}^2_\infty$ as the Riemann sphere, then both G and G_* are holomorphic. An end p_j of a flat front $f\colon \bar{M}\backslash\{p_1,\dots,p_k\} \to \mathcal{H}^3$ is said to be *regular*, if both G and G_* can be extended meromorphically across it. Kokubu, Umehara and Yamada showed the following global property for this class.

Theorem 3.1 ([15], Theorem 3.13). *Let $f\colon \bar{M}_\gamma\backslash\{p_1,\dots,p_k\} \to \mathcal{H}^3$ be a complete flat front whose ends are all regular. Then*

$$d + d_* \geq k$$

where d is the degree of G considered as maps $\bar{M}_\gamma$ (if G has essential singularities, then we define $d=\infty$) and d_* is the degree of G_* considered as the same way. Furthermore, equality holds if and only if all ends are embedded.

As an application of the inequality, we show the following effective estimate for the number of exceptional values of the hyperbolic Gauss maps for this class.

Theorem 3.2 ([12]). *Let $f\colon \bar{M}_\gamma\backslash\{p_1,\dots,p_k\} \to \mathcal{H}^3$ be a complete flat front and the maps G and G_* be the hyperbolic Gauss maps of f. If G omits p values and G_* omits q values, then $p \leq 2$ or $q \leq 2$ or,*

$$\frac{1}{p-2} + \frac{1}{q-2} \geq \frac{k}{2\gamma - 2 + k}\,.$$

Note that the right side of the inequality is given by topological invariants of $M = \bar{M}_\gamma\backslash\{p_1,\dots,p_k\}$ without any data of the hyperbolic Gauss map. As a corollary, we give more sharp results on the number of exceptional values of them for some topological cases

Corollary 3.3 ([12]). *For complete flat fronts in $\mathcal{H}^3$, we have the following:*

(i) *There does not exist a complete flat front with $\gamma = 0$, $p \geq 4$ and $q \geq 4$.*
(ii) *There does not exist a complete flat front with $\gamma = 1$, $p \geq 5$ and $q \geq 5$.*

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(Y. Kawakami) Faculty of Mathematics, Kyushu university, 744, Motooka, Nishiku, Fukuoka-city, 819-0395, Japan

E-mail address: `kawakami@math.kyushu-u.ac.jp`