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The module of lowerable vector fields for a multigerm

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Abstract

Arnol'd introduced the notion of lowerable and liftable vector fields for a mapping in 1976. These vector fields have various interesting applications; in particular, they have applications to classification problems of singularities. However, no general theory has been constructed as far as the author knows, and little is known about the modules of lowerable and liftable vector fields.

In this thesis, we prove that the module of lowerable vector fields is always finitely generated for a finitely $\mathcal{L}$-determined multigerm. We also present some examples of non-finitely $\mathcal{L}$-determined multigerms for which we explicitly construct generators for the modules of lowerable vector fields.

Contents

1 Introduction

The history of lowerable and liftable vector fields dates back to 1970's. Arnol'd [1] introduced the notion of lowerable and liftable vector fields for a mapping for studying bifurcations of wave front singularities.

In the following, let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Throughout the thesis, all mappings are of class C^∞ for $\mathbb{K} = \mathbb{R}$, and are holomorphic for $\mathbb{K} = \mathbb{C}$, unless otherwise stated.

Let S be a finite set in $\mathbb{K}^n$. For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, a vector field ξ on the source is said to be *lowerable* if there exists a vector field η on the target such that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{K}^n, S) & \xrightarrow{\xi} & T\mathbb{K}^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle df} \\ (\mathbb{K}^p, 0) & \xrightarrow{\eta} & T\mathbb{K}^p, \end{array}$$

where df is the differential of f. A vector field η on the target is said to be *liftable* if there exists a vector field ξ on the source such that the above diagram is commutative. See Section 2 for details.

The set of liftable vector fields has the natural structure of a module over the ring of function-germs on the target, while that of lowerable vector fields has the structure of a module over the same ring "via f".

Lowerable and liftable vector fields have been shown to have applications to classification problems of singularities. Bruce and West [3] obtained generators for the module of liftable vector fields for a crosscap in the complex analytic case and classified certain functions on a crosscap. Ishikawa [5] used lowerable vector fields to construct versal openings, which have applications to classifications of tangential singularities.

Unfortunately, no general theory has been constructed on the modules of lowerable or liftable vector fields as far as the author knows. It is usually difficult to prove finite generation or to find explicit generators for their modules.

In the thesis, we investigate the module of lowerable vector fields for a multigerm. Let $C_{p,0}$ be the ring of all function-germs on $(\mathbb{K}^p, 0)$. The main result is Corollary 4.4 as follows:

Main Result. *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a finitely $\mathcal{L}$-determined multigerm. Then, the module of lowerable vector fields is finitely generated as a $C_{p,0}$-module via f.*

Here, a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is said to be *finitely $\mathcal{L}$-determined* if there exists a positive integer k such that every $g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ which has the same Taylor series up to degree k as f, is $\mathcal{L}$-equivalent to f in the sense that g coincides with f up to a change of coordinates of the target.

When f is a finitely $\mathcal{L}$-determined multigerm, we can prove finite generation of the module of lowerable vector fields in a constructive way in both the real C^∞ case and the complex analytic case. Note that it is usually difficult to judge whether a module over the ring of function-germs is finitely generated in the real C^∞ case due to "flat functions".

The thesis is organized as follows. In Section 2, we explain basic materials necessary for the rest of the thesis with many examples for the reader's convenience. In Section 3, we present a lot of examples of finitely $\mathcal{L}$-determined multigerms by introducing the linear map ${}_i^{\mathcal{C}}\overline{\omega}f$ for a multigerm f and a non-negative integer i.

In Section 4, we prove Theorem 4.2, which is the key for proving the main result. We then show that the module of lowerable vector fields is finitely generated for a finitely $\mathcal{L}$-determined multigerm (Corollary 4.4). Our proof is, in principle, constructive and gives a method for constructing explicit generators for the module of lowerable vector fields. We also give some examples in which we construct explicit generators for the module of lowerable vector fields. In Section 5, we consider the module of lowerable vector fields for some non-finitely $\mathcal{L}$-determined multigerms.

In Section 6, we mention some open problems. Section 7 is devoted to acknowledgements.

The contents of Section 4 are mainly based on [8], which is a joint work with Takashi Nishimura.

Throughout the thesis, we sometimes omit the brackets of equivalence classes and the terminology "germ", when there is no confusion.

2 Preliminaries

In this section, we present some basic materials necessary for the rest of the thesis.

Let $(x_1, x_2, \ldots, x_n)$ (resp., $(X_1, X_2, \ldots, X_p)$) be the standard local coordinates of $\mathbb{K}^n$ (resp., $\mathbb{K}^p$) around the origin. Sometimes, (x_1, x_2) (resp., (X_1, X_2)) is denoted by (x, y) (resp., (X, Y)), and (x_1, x_2, x_3) (resp., (X_1, X_2, X_3)) is denoted by (x, y, z) (resp., (X, Y, Z)).

Let S be a finite set consisting of r distinct points in $\mathbb{K}^n$ and T a subset in $\mathbb{K}^p$. Let $f : U \to \mathbb{K}^p$ and $g : V \to \mathbb{K}^p$ be mappings defined on open sets containing S, such that $f(S) \subset T$ and $g(S) \subset T$ hold. We say that f and g are *equivalent at* S if there exists an open set W containing S in $U \cap V$ such that the restrictions of f and g to W coincide. We call the equivalence class of f a *map-germ* f *at* S, and denote it by $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, T)$. We often denote $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, \mathbb{K}^p)$ simply by $f : (\mathbb{K}^n, S) \to \mathbb{K}^p$ when $T = \mathbb{K}^p$. A map-germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ (that is, $T = \{0\}$) is called a *multigerm*. When $r = 1$, f is called a *monogerm*. A multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ can be identified with an ordered set $\{f_k : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0) | \, 1 \leq k \leq r\}$ of a finite number of monogerms. Each f_k is called a *branch* of f.

Example 2.1. Set $S = \{s_1, s_2\} \subset \mathbb{K}$, where $s_1 \neq s_2$. Let $\widetilde{f} : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by the monogerms $\widetilde{f}_i : (\mathbb{K}, \{s_i\}) \to (\mathbb{K}^2, 0)$, $i = 1, 2$, defined by

$$\widetilde{f}_1(x) = ((x - s_1)^2, (x - s_1)^3) \text{ and } \widetilde{f}_2(x) = ((x - s_2)^3, (x - s_2)^2).$$

This multigerm is identified with the ordered set of monogerms $f_1, f_2 : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ $\{f_1(x) = (x^2, x^3),\ f_2(x) = (x^3, x^2)\}$.

In the following, a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ given by

$$f_1(x), f_2(x), \ldots, f_r(x) \quad (f_k : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0),\ k = 1, 2, \ldots, r),$$

means a multigerm $\widetilde{f} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ with S consisting of r distinct points $s_1, s_2, \ldots, s_r \in \mathbb{K}^n$, given by the monogerms $\widetilde{f}_i : (\mathbb{K}^n, \{s_i\}) \to (\mathbb{K}^2, 0)$, $i = 1, 2, \ldots, r$, defined by

$$\widetilde{f}_1(x) = f_1(x - s_1),\ \widetilde{f}_2(x) = f_2(x - s_2), \ldots, \widetilde{f}_r(x) = f_r(x - s_r)$$

for $x \in \mathbb{K}^n$.

For multigerms $f, g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, f is said to be $\mathcal{A}$-*equivalent* to g if there exist diffeomorphism germs for $\mathbb{K} = \mathbb{R}$, or biholomorphism germs for $\mathbb{K} = \mathbb{C}$, $\varphi : (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and $\psi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$, such that we have $\varphi(s) = s$ for every $s \in S$ and

$$f = \psi \circ g \circ \varphi.$$

If, in addition, φ can be chosen to be the germ of the identity mapping of $(\mathbb{K}^n, S)$, then f is said to be $\mathcal{L}$-*equivalent* to g.

Let $C_{n,S}$ (resp., $C_{p,0}$) be the $\mathbb{K}$-algebra of all function-germs on $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$) and $m_{n,S}$ (resp., $m_{p,0}$) be the ideal of $C_{n,S}$ (resp., $C_{p,0}$) consisting of the function-germs $(\mathbb{K}^n, S) \to (\mathbb{K}, 0)$ (resp., $(\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$). For a non-negative integer i, let $m_{n,S}^i$ (resp., $m_{p,0}^i$) denote the ideal of $C_{n,S}$ (resp., $C_{p,0}$) consisting of those function-germs on $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$) whose Taylor series vanish up to degree $i-1$. Thus, we have

$$m_{n,S}^0 = C_{n,S},\ m_{p,0}^0 = C_{p,0},\ m_{n,S}^1 = m_{n,S}, \text{ and } m_{p,0}^1 = m_{p,0}.$$

Let R be a ring. For a finite subset $A = \{a_1, a_2, \ldots, a_m\}$ of an R-module, set

$$A_R = \left\{ \sum_{i=1}^{m} r_i a_i \;\middle|\; r_i \in R \right\},$$

which is the smallest R-submodule containing A. The R-module A_R is sometimes denoted by $\langle a_1, a_2, \ldots, a_m \rangle_R$.

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, let $f^* : C_{p,0} \to C_{n,S}$ be the $\mathbb{K}$-algebra homomorphism defined by $f^*(\psi) = \psi \circ f$, $\psi \in C_{p,0}$, and for a non-negative integer i and a $C_{n,S}$-module B, we define the $C_{n,S}$-submodule $f^* m_{p,0}^i B$ of B by

$$f^* m_{p,0}^i B = \left\{ \sum_{j=1}^{N} (\psi_j \circ f) b_j \;\middle|\; N \in \mathbb{N},\ \psi_j \in m_{p,0}^i,\ b_j \in B \right\}.$$

For a p-tuple of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, an n-tuple of non-negative integers $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, and a monogerm $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, set

$$\begin{aligned} |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p, \quad |\beta| = \beta_1 + \beta_2 + \cdots + \beta_n, \\ X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_p^{\alpha_p}, \quad x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}, \\ f^\alpha = (X_1 \circ f)^{\alpha_1} (X_2 \circ f)^{\alpha_2} \cdots (X_p \circ f)^{\alpha_p}. \end{aligned}$$

We use the following proposition in Section 3.

Proposition 2.2. *Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^p, 0)$ be given by*

$$f(x) = (x^k + x^{k+1}\varphi_1(x),\ x^{k+1}\varphi_2(x), \ldots,\ x^{k+1}\varphi_p(x)),$$

where $k \geq 1$ and $\varphi_1, \varphi_2, \ldots, \varphi_p \in C_{1,0}$. Then, for every non-negative integer i, we have

$$f^* m_{p,0}^i C_{1,0} = \left\langle x^{ik} \right\rangle_{C_{1,0}}.$$

Proof. Let us take any element $\varphi \in f^* m_{p,0}^i C_{1,0}$. Then, we have

$$\begin{aligned}
\varphi &= \sum_{j=1}^{N} (\psi_j \circ f)\widetilde{\varphi}_j \\
&= \sum_{j=1}^{N} \left(\sum_{|\alpha|=i} (\psi_{j,\alpha} \circ f) f^{\alpha} \right) \widetilde{\varphi}_j \\
&= \left(\sum_{j=1}^{N} \sum_{|\alpha|=i} (\psi_{j,\alpha} \circ f)(1 + x\varphi_1)^{\alpha_1} (x\varphi_2)^{\alpha_2} \cdots (x\varphi_p)^{\alpha_p} \widetilde{\varphi}_j \right) x^{ik}
\end{aligned}$$

for some $N \in \mathbb{N}$, $\psi_j \in m_{p,0}^i$, $\psi_{j,\alpha} \in C_{p,0}$ and $\widetilde{\varphi}_j \in C_{1,0}$. Therefore, $\varphi \in \left\langle x^{ik} \right\rangle_{C_{1,0}}$ holds.

Conversely, let us take any element $\varphi \in \left\langle x^{ik} \right\rangle_{C_{1,0}}$. Then, we have

$$\varphi = \widetilde{\varphi}(x) x^{ik} = (X_1^i \circ f)(x) \frac{\widetilde{\varphi}(x)}{(1 + x\varphi_1(x))^i}$$

for some $\widetilde{\varphi} \in C_{1,0}$. Therefore, $\varphi \in f^* m_{p,0}^i C_{1,0}$ holds. Thus, we have

$$f^* m_{p,0}^i C_{1,0} = \left\langle x^{ik} \right\rangle_{C_{1,0}}$$

for every non-negative integer i. □

Set

$$Q(f) = \frac{C_{n,S}}{f^* m_{p,0} C_{n,S}},$$

which is a $\mathbb{K}$-vector space, and set $\delta(f) = \dim_{\mathbb{K}} Q(f)$. We call $\delta(f)$ the *multiplicity* of f. Note that the multiplicity is invariant under $\mathcal{A}$-equivalence. Finiteness of the multiplicity of f implies $n \leq p$ (see [10, p. 494]). Note that we have

$$f^* m_{p,0} C_{n,S} = \bigoplus_{k=1}^{r} f_k^* m_{p,0} C_{n,0},$$

where f_k are the branches of f. Thus, we have

$$Q(f) = \bigoplus_{k=1}^{r} Q(f_k) \text{ and } \delta(f) = \sum_{k=1}^{r} \delta(f_k).$$

Example 2.3. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be given by

$$f(x) = (x^2, x^3).$$

Then, we have $f^*(X) = x^2$, $f^*(Y) = x^3$, and $f^* m_{2,0} C_{1,0} = \langle x^2 \rangle_{C_{1,0}}$. Therefore, the 2 functions $[1]$ and $[x]$ form a basis of $Q(f)$. Thus, we have $\delta(f) = 2$.

Proposition 2.4. *For a monogerm $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, we have $\delta(f) < \infty$ if and only if there exists a positive integer ℓ such that $x_i^\ell \in f^* m_{p,0} C_{n,0}$ holds for every $i = 1, 2, \ldots, n$.*

Proof. Suppose that $\delta(f) < \infty$ holds. Set $\ell = \delta(f)$. Then, we have

$$\dim_{\mathbb{K}} \frac{C_{n,0}}{m_{n,0}^{\ell+1} C_{n,0} + f^* m_{p,0} C_{n,0}} \leq \dim_{\mathbb{K}} \frac{C_{n,0}}{f^* m_{p,0} C_{n,0}} = \ell.$$

Since $C_{n,0}$ is a finitely generated $C_{n,0}$-module and $f^* m_{p,0} C_{n,0}$ is a submodule of $C_{n,0}$, we have $m_{n,0}^\ell C_{n,0} \subset f^* m_{p,0} C_{n,0}$ by Corollary (1.6) in [6, p. 130]. Thus, $x_i^\ell \in f^* m_{p,0} C_{n,0}$ holds for every $i = 1, 2, \ldots, n$.

Conversely, suppose that there exists a positive integer ℓ such that $x_i^\ell \in f^* m_{p,0} C_{n,0}$ holds for every $i = 1, 2, \ldots, n$. Then, we have $x^\beta \in f^* m_{p,0} C_{n,0}$ for every n-tuple of non-negative integers β with $|\beta| = n\ell$. Thus, we have $\delta(f) < \infty$, since $m_{n,0}^{n\ell} \subset f^* m_{p,0} C_{n,0}$ holds. □

Example 2.5. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ be given by

$$f(x, y) = (x^2 - y^2, xy).$$

Then, we have

$$x^3 = x(x^2 - y^2) + y(xy) \text{ and } y^3 = -y(x^2 - y^2) + x(xy).$$

Thus, $\delta(f) < \infty$ holds by Proposition 2.4.

Let us obtain a basis of the $\mathbb{K}$-vector space $Q(f)$. Since

$$x^2 y = x(xy) \text{ and } xy^2 = y(xy)$$

hold, we have $m_{2,0}^3 \subset f^* m_{2,0} C_{2,0}$. Therefore, we have

$$f^* m_{2,0} C_{2,0} = \langle x^2 - y^2, xy \rangle_{\mathbb{K}} + m_{2,0}^3.$$

Then, we can show that the 4 functions $[1]$, $[x]$, $[y]$, and $[x^2 + y^2]$ form a basis of $Q(f)$ as follows.

First, let us take any element $[\varphi] \in Q(f)$, $\varphi \in C_{2,0}$. Then, we have

$$\begin{aligned} [\varphi(x, y)] &= [c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2] \\ &= c_1[1] + c_2[x] + c_3[y] + \left(\frac{c_4 + c_6}{2}\right)[x^2 + y^2] + \left(\frac{c_4 - c_6}{2}\right)[x^2 - y^2] \\ &= c_1[1] + c_2[x] + c_3[y] + \left(\frac{c_4 + c_6}{2}\right)[x^2 + y^2] \end{aligned}$$

for some $c_1, c_2, \ldots, c_6 \in \mathbb{K}$.

Moreover, the functions $[1]$, $[x]$, $[y]$, and $[x^2 + y^2]$ are linearly independent over $\mathbb{K}$. In fact, suppose that

$$d_1[1] + d_2[x] + d_3[y] + d_4[x^2 + y^2] = [0]$$

holds for some d_1, d_2, d_3, $d_4 \in \mathbb{K}$. Since

$$d_1 + d_2 x + d_3 y + d_4(x^2 + y^2) = d_5(x^2 - y^2) + d_6 xy + \psi_1(x, y)$$

holds for some d_5, $d_6 \in \mathbb{K}$ and $\psi_1 \in m_{2,0}^3$, we have $d_1 = d_2 = d_3 = d_4 = 0$. Therefore, the 4 functions $[1]$, $[x]$, $[y]$, and $[x^2 + y^2]$ form a basis of $Q(f)$. Thus, we have $\delta(f) = 4$.

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and a subset B of a $C_{n,S}$-module A, the set B has a *$C_{p,0}$-module structure via f* (or *$f^*C_{p,0}$-module structure*) if B has the module structure by the addition induced from that of A and the multiplication defined as follows:

$$\psi\eta := f^*(\psi)\eta \quad (\psi \in C_{p,0}, \eta \in B).$$

Note that $f^*(\psi)\eta$ denotes the multiplication of $f^*(\psi)$ and η in A. A $C_{n,S}$-module naturally has a $C_{p,0}$-module structure via f.

The following theorem is a profound result and is useful in singularity theory (see [6, p. 132]).

Preparation Theorem. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and a $C_{n,S}$-module A, suppose that A is finitely generated as a $C_{n,S}$-module and $A/f^*m_{p,0}A$ is a finite dimensional $\mathbb{K}$-vector space. Then, A is finitely generated as a $C_{p,0}$-module via f.*

In this thesis, Preparation Theorem is always used in the form as in the following theorem.

Theorem 2.6. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and a $C_{n,S}$-module A, suppose that A is a finitely generated $C_{n,S}$-module and that $[p_1], [p_2], \ldots, [p_k]$ span the $\mathbb{K}$-vector space $A/f^*m_{p,0}A$. Then, $p_1, p_2, \ldots, p_k$ generate A as a $C_{p,0}$-module via f.*

Proof. Note that A is finitely generated as a $C_{p,0}$-module via f by Preparation Theorem. Since $[p_1], [p_2], \ldots, [p_k]$ span the $\mathbb{K}$-vector space $A/f^*m_{p,0}A$, we have

$$\begin{aligned} A &= \langle p_1, p_2, \ldots, p_k \rangle_{\mathbb{K}} + f^*m_{p,0}A \\ &= \langle p_1, p_2, \ldots, p_k \rangle_{f^*C_{p,0}} + f^*m_{p,0}A. \end{aligned}$$

Here, $f^*C_{p,0}$ is a commutative ring with identity, $\langle p_1, p_2, \ldots, p_k \rangle_{f^*C_{p,0}}$ is a f^*C_p-submodule of A, and $f^*m_{p,0}$ is an ideal in $f^*C_{p,0}$ such that $1 + \varphi$ is invertible for every $\varphi \in f^*m_{p,0}$. Thus, by Nakayama's Lemma (see [6, p. 130]), we have $A = \langle p_1, p_2, \ldots, p_k \rangle_{f^*C_{p,0}}$. $\square$

For a map-germ $f : (\mathbb{K}^n, S) \to \mathbb{K}^p$, a map-germ $\xi : (\mathbb{K}^n, S) \to T\mathbb{K}^p$ is a *vector field along f* if the following diagram is commutative:

$$\begin{array}{ccc} T\mathbb{K}^p & = & T\mathbb{K}^p \\ {\scriptstyle \xi}\uparrow & & \downarrow{\scriptstyle \pi} \\ (\mathbb{K}^n, S) & \xrightarrow{f} & \mathbb{K}^p, \end{array}$$

where $\pi : T\mathbb{K}^p \to \mathbb{K}^p$ is the projection. Set

$$\theta(f) = \{\xi : (\mathbb{K}^n, S) \to T\mathbb{K}^p \,|\, \pi \circ \xi = f\}.$$

The set $\theta(f)$ has the natural structure of a $C_{n,S}$-module and we have

$$\theta(f) = \bigoplus_{k=1}^{r} \theta(f_k),$$

where f_k are the branches of f. Note that each $\theta(f_k)$ is identified with the direct sum of p copies of $C_{n,0}$. Set

$$\theta_S(n) = \theta(\mathrm{id}_{(\mathbb{K}^n,S)}) \text{ and } \theta_0(p) = \theta(\mathrm{id}_{(\mathbb{K}^p,0)}),$$

where $\mathrm{id}_{(\mathbb{K}^n,S)}$ (resp., $\mathrm{id}_{(\mathbb{K}^p,0)}$) is the germ of the identity mapping of $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$).

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, following Mather [6, p. 141], let $tf : \theta_S(n) \to \theta(f)$ and $\omega f : \theta_0(p) \to \theta(f)$ be defined by

$$tf(\xi) = df \circ \xi \text{ and } \omega f(\eta) = \eta \circ f$$

for $\xi \in \theta_S(n)$ and $\eta \in \theta_0(p)$, respectively, where df is the differential of f. The map tf is a $C_{n,S}$-module homomorphism, while ωf is a $C_{p,0}$-module homomorphism, where $\theta(f)$ is considered to be a $C_{p,0}$-module via f. Following Wall [10, p. 485], set

$$T\mathcal{R}_e(f) = tf(\theta_S(n)) \text{ and } T\mathcal{L}_e(f) = \omega f(\theta_0(p)).$$

We call $T\mathcal{R}_e(f)$ (resp., $T\mathcal{L}_e(f)$) the *extended $\mathcal{R}$-tangent space* (resp., *extended $\mathcal{L}$-tangent space*) of f. Note that

$$T\mathcal{R}_e(f) = \bigoplus_{k=1}^{r} T\mathcal{R}_e(f_k)$$

always holds, while in general we have

$$T\mathcal{L}_e(f) \neq \bigoplus_{k=1}^{r} T\mathcal{L}_e(f_k),$$

although we always have

$$T\mathcal{L}_e(f) \subset \bigoplus_{k=1}^{r} T\mathcal{L}_e(f_k),$$

where f_k are the branches of f.

Example 2.7. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by

$$f_1(x) = (x^2, x^3),\ f_2(x) = (x^2, x^3).$$

Then, we have $\theta(f) = \theta(f_1) \oplus \theta(f_2) = (C_{1,0} \oplus C_{1,0}) \oplus (C_{1,0} \oplus C_{1,0})$ and we see that

$$\begin{aligned} T\mathcal{R}_e(f) &= \left\{ \left(\begin{pmatrix} 2x\varphi_1(x) \\ 3x^2\varphi_1(x) \end{pmatrix}, \begin{pmatrix} 2x\varphi_2(x) \\ 3x^2\varphi_2(x) \end{pmatrix} \right) \middle| \; \varphi_1, \varphi_2 \in C_{1,0} \right\}, \\ T\mathcal{L}_e(f) &= \left\{ \left(\begin{pmatrix} \psi_1(x^2, x^3) \\ \psi_2(x^2, x^3) \end{pmatrix}, \begin{pmatrix} \psi_1(x^2, x^3) \\ \psi_2(x^2, x^3) \end{pmatrix} \right) \middle| \; \psi_1, \psi_2 \in C_{2,0} \right\} \end{aligned}$$

hold. Moreover, $T\mathcal{L}_e(f) \neq T\mathcal{L}_e(f_1) \oplus T\mathcal{L}_e(f_2)$ holds, since we have

$$\left(\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \notin T\mathcal{L}_e(f) \text{ and } \left(\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in T\mathcal{L}_e(f_1) \oplus T\mathcal{L}_e(f_2).$$

We use the following proposition in Sections 4 and 5 .

Proposition 2.8. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, suppose that $\delta(f) < \infty$ holds. Then, tf is injective.*

Proof. It suffices to show that if a monogerm $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ satisfies $\delta(f) < \infty$, then tf is injective.

Suppose that for $\xi \in \theta_0(n)$, we have $tf(\xi) = 0$ on an open set U_1 containing 0. Then, f is constant along any integral curve of ξ on U_1. Since $\delta(f) < \infty$ holds, each integral curve of ξ on an open set U_2 containing 0 must consist of a single point by Propositions 2.2 and 2.3 in [4, pp. 167–168]. Therefore, we have $\xi = 0$ on $U_1 \cap U_2$. Thus, tf is injective. □

We cannot, in general, remove the assumption that $\delta(f) < \infty$ holds in Proposition 2.8.

Example 2.9. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ be given by

$$f(x, y) = (x, 0).$$

Since $[1], [y], [y^2], \ldots$ are linearly independent in $Q(f)$ over $\mathbb{K}$, the multiplicity $\delta(f)$ is not finite. On the other hand, tf is not injective, since we have

$$tf\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0.$$

Let us now define lowerable and liftable vector fields.

Definition 2.10. A vector field $\xi \in \theta_S(n)$ is said to be *lowerable* for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ if $df \circ \xi$ belongs to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. A vector field $\eta \in \theta_0(p)$ is said to be *liftable* for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ if $\eta \circ f$ belongs to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$.

A vector field $\xi \in \theta_S(n)$ (resp., $\eta \in \theta_0(p)$) is lowerable (resp., liftable) for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ if and only if there exist a vector field $\eta \in \theta_S(n)$

(resp., $\xi \in \theta_0(p)$) such that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{K}^n, S) & \xrightarrow{\xi} & T\mathbb{K}^n \\ f\big\downarrow & & \big\downarrow df \\ (\mathbb{K}^p, 0) & \xrightarrow{\eta} & T\mathbb{K}^p. \end{array}$$

Let $\mathrm{Lower}(f)$ (resp., $\mathrm{Lift}(f)$) be the set of all lowerable (resp., liftable) vector fields for the multigerm f. Then, $\mathrm{Lower}(f)$ has a $C_{p,0}$-module structure via f and $\mathrm{Lift}(f)$ has a natural $C_{p,0}$-module structure. By the definitions, we have

$$tf(\mathrm{Lower}(f)) = T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \text{ and } \omega f(\mathrm{Lift}(f)) = T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f).$$

Example 2.11. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be given by

$$f(x) = (x^2, x^3).$$

Then, the Jacobian matrix of f at 0 is the transpose of $\left(2x, 3x^2\right)$. We see that

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix} \circ f = \begin{pmatrix} 2x^2 \\ 3x^3 \end{pmatrix} = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix} \begin{pmatrix} x \end{pmatrix}$$

holds. Thus, we have

$$\begin{pmatrix} x \end{pmatrix} \in \mathrm{Lower}(f) \text{ and } \begin{pmatrix} 2X \\ 3Y \end{pmatrix} \in \mathrm{Lift}(f).$$

The liftable vector field for f given above is depicted in red in Figure 1. The blue curve corresponds to the image of f.

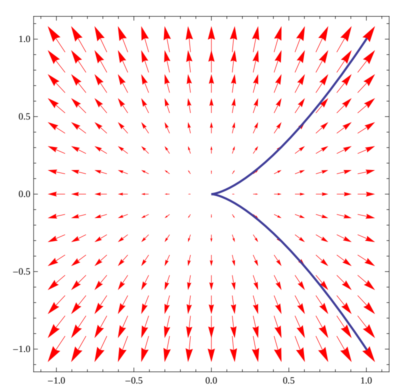

Figure 1: A liftable vector field

Let us finally recall the notion of finite $\mathcal{L}$-determinacy.

Definition 2.12. A multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is said to be *finitely $\mathcal{L}$-determined* if there exists a positive integer ℓ such that

$$m_{n,S}^{\ell}\theta(f) \subset T\mathcal{L}_e(f)$$

holds.

Remark 2.13. Originally, $\mathcal{L}$-determinacy of a multigerm f is defined as follows. A multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is said to be *finitely $\mathcal{L}$-determined* if there exists a positive integer k such that every $g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ which has the same Taylor series up to degree k as f, is $\mathcal{L}$-equivalent to f. It is known that this definition is equivalent to Definition 2.12 (see [6, pp. 140–141]).

Proposition 2.14. *If $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is a finitely $\mathcal{L}$-determined multigerm, then we have $\delta(f) < \infty$.*

Proof. There exists a positive integer ℓ such that $m_{n,S}^{\ell}\theta(f) \subset T\mathcal{L}_e(f)$ holds. Then, for every branch f_k, $k = 1, 2, \ldots, r$, we have

$$m_{n,0}^{\ell} \subset f_k^* C_{p,0} \subset \mathbb{K} + f_k^* m_{p,0} C_{n,0}.$$

Therefore, we see that

$$m_{n,0}^{\ell} \subset f_k^* m_{p,0} C_{n,0}$$

holds. Thus, we have $\delta(f) < \infty$, since $\delta(f_k) < \infty$ holds for every $k = 1, 2, \ldots, r$. □

A monogerm $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is said to be *non-singular* if $\operatorname{rank} Jf(0) = \min\{n, p\}$ holds, where $Jf(0)$ is the Jacobian matrix of f at 0.

Example 2.15. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $n \leq p$, be given by

$$f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, \ldots, 0).$$

Then, f is non-singular, since $\operatorname{rank} Jf(0) = n$ holds.

Moreover, every $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ which has the same Taylor series up to degree 1 as f, is $\mathcal{L}$-equivalent to f by the implicit function theorem. Thus, f is finitely $\mathcal{L}$-determined by Remark 2.13.

A lot more examples of finitely $\mathcal{L}$-determined multigerms are given in Section 3.

3 Examples of finitely $\mathcal{L}$-determined multigerms

It is, in general, difficult to check $\mathcal{L}$-determinacy for a given multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, since $T\mathcal{L}_e(f)$ does not have a $C_{n,S}$-module structure.

In this section, we present a lot of examples of finitely $\mathcal{L}$-determined multigerms. Although the contents of this section seem folklore for experts, we give complete proofs for all the results, since there is no literature in which the contents of this section are written explicitly as far as the author knows.

For a non-negative integer i and a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, we define the $\mathbb{K}$-linear map ${}^{\mathcal{C}}_{i}\overline{\omega}f$ as follows:

$$
{}^{\mathcal{C}}_{i}\overline{\omega}f : \frac{m_{p,0}^{i}\theta_0(p)}{m_{p,0}^{i+1}\theta_0(p)} \to \frac{f^*m_{p,0}^{i}\theta(f)}{f^*m_{p,0}^{i+1}\theta(f)},
$$

$$
{}^{\mathcal{C}}_{i}\overline{\omega}f([\eta]) = [\omega f(\eta)], \quad \eta \in m_{p,0}^{i}\theta_0(p).
$$

This map is well-defined. We call ${}^{\mathcal{C}}_{i}\overline{\omega}f$ the *i-th reduced version* of ωf. A reduced version of ωf was first introduced as $\overline{\omega}f$ in [7, p. 227] to characterize stability of f with respect to $\mathcal{A}$-equivalence. Our reduced version ${}^{\mathcal{C}}_{i}\overline{\omega}f$ is useful in studying $\mathcal{L}$-determinacy.

Proposition 3.1. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and a non-negative integer i, ${}^{\mathcal{C}}_{i}\overline{\omega}f$ is surjective if and only if $f^*m_{p,0}^{i}\theta(f) \subset \omega f\left(m_{p,0}^{i}\theta_0(p)\right)$ holds.*

Proof. Suppose that $f^*m_{p,0}^{i}\theta(f) \subset \omega f\left(m_{p,0}^{i}\theta_0(p)\right)$ holds. Let us take any element $[\xi] \in f^*m_{p,0}^{i}\theta(f)/f^*m_{p,0}^{i+1}\theta(f)$. Then, there exists an $\eta \in m_{p,0}^{i}\theta_0(p)$ such that we have $[\xi] = [\omega f(\eta)] = {}^{\mathcal{C}}_{i}\overline{\omega}f([\eta])$. Thus, ${}^{\mathcal{C}}_{i}\overline{\omega}f$ is surjective.

Conversely, suppose that ${}^{\mathcal{C}}_{i}\overline{\omega}f$ is surjective. Then, we have

$$
\begin{aligned}
f^*m_{p,0}^{i}\theta(f) &= f^*m_{p,0}^{i+1}\theta(f) + \omega f\left(m_{p,0}^{i}\theta_0(p)\right) \\
&= f^*m_{p,0}^{i+1}\theta(f) + \left\{ \sum_{j=1}^{p} \sum_{|\alpha|=i} c_{j,\alpha}\left(\eta_{j,\alpha} \circ f\right) \,\middle|\, c_{j,\alpha} \in \mathbb{K} \right\},
\end{aligned}
$$

where

$$
\eta_{j,\alpha} = X^{\alpha}\frac{\partial}{\partial X_j}.
$$

Therefore, the $\mathbb{K}$-vector space $f^*m_{p,0}^{i}\theta(f)/f^*m_{p,0}^{i+1}\theta(f)$ is spanned by the classes represented by the vector fields along f

$$
[\eta_{j,\alpha} \circ f] \quad (j = 1, 2, \ldots, p,\ |\alpha| = i).
$$

Note that we have

$$
\frac{f^*m_{p,0}^{i}\theta(f)}{f^*m_{p,0}^{i+1}\theta(f)} = \frac{f^*m_{p,0}^{i}\theta(f)}{f^*m_{p,0}\left(f^*m_{p,0}^{i}\theta(f)\right)}
$$

and that $f^*m_{p,0}^{i}\theta(f)$ is a finitely generated $C_{n,S}$-module.

Let us take any element $\xi \in f^* m_{p,0}^i \theta(f)$. Then, by Theorem 2.6, we have

$$\begin{aligned} \xi &= \sum_{j=1}^{p} \sum_{|\alpha|=i} (\psi_{j,\alpha} \circ f)(\eta_{j,\alpha} \circ f) \\ &= \left(\sum_{j=1}^{p} \sum_{|\alpha|=i} \psi_{j,\alpha} \eta_{j,\alpha} \right) \circ f \end{aligned}$$

for some $\psi_{j,\alpha} \in C_{p,0}$. Thus, $f^* m_{p,0}^i \theta(f) \subset \omega f \left(m_{p,0}^i \theta_0(p) \right)$ holds. □

Proposition 3.2. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and a non-negative integer i, suppose that ${}_i^{\mathcal{C}}\overline{\omega} f$ is surjective. Then, ${}_\ell^{\mathcal{C}}\overline{\omega} f$ is surjective for every positive integer ℓ with $\ell > i$.*

Proof. Suppose that ${}_i^{\mathcal{C}}\overline{\omega} f$ is surjective. It suffices to show that ${}_{i+1}^{\mathcal{C}}\overline{\omega} f$ is surjective. Note that

$$f^* m_{p,0}^i \theta(f) \subset \omega f \left(m_{p,0}^i \theta_0(p) \right)$$

holds by Proposition 3.1.

Let us take any element $\xi \in f^* m_{p,0}^{i+1} \theta(f)$. Since

$$f^* m_{p,0}^{i+1} \theta(f) = f^* m_{p,0} (f^* m_{p,0}^i \theta(f))$$

holds, we have

$$\begin{aligned} \xi &= \sum_{j=1}^{N_1} (\psi_j \circ f) \left(\left(\sum_{k=1}^{N_2} \psi_{j,k} \eta_{j,k} \right) \circ f \right) \\ &= \left(\sum_{j=1}^{N_1} \sum_{k=1}^{N_2} (\psi_j \psi_{j,k}) \eta_{j,k} \right) \circ f \end{aligned}$$

for some $N_1, N_2 \in \mathbb{N}$, $\psi_j \in m_{p,0}$, $\psi_{j,k} \in m_{p,0}^i$ and $\eta_{j,k} \in \theta_0(p)$. Since $\psi_j \psi_{j,k}$ belongs to $m_{p,0}^{i+1}$, the vector field ξ belongs to $\omega f \left(m_{p,0}^{i+1} \theta_0(p) \right)$. Thus, ${}_{i+1}^{\mathcal{C}}\overline{\omega} f$ is surjective by Proposition 3.1. □

Lemma 3.3. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a monogerm satisfying $\delta(f) < \infty$. For every positive integer k, there exists a positive integer ℓ such that $m_{n,0}^\ell \subset f^* m_{p,0} (m_{n,0}^k)$ holds.*

Proof. Since we have $\delta(f) < \infty$, there exists a positive integer ℓ' such that $x_i^{\ell'} \in f^* m_{p,0} C_{n,0}$ holds for every $i = 1, 2, \ldots, n$ by Proposition 2.4. Then, for every positive integer k, we have $x_i^{\ell'+k} \in f^* m_{p,0} (m_{n,0}^k)$. Thus, $m_{n,0}^\ell \subset f^* m_{p,0} (m_{n,0}^k)$ holds for $\ell = n(\ell' + k)$. □

By the following proposition, we can find examples of finitely $\mathcal{L}$-determined multigerms.

Proposition 3.4. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ satisfying $\delta(f) < \infty$, suppose that there exists a non-negative integer i such that ${}^{\mathcal{C}}_{i}\overline{\omega} f$ is surjective. Then, f is finitely $\mathcal{L}$-determined.*

Proof. Since $\delta(f) < \infty$ holds, by Propsotion 3.1 and Lemma 3.3, we have

$$m_{n,S}^{\ell}\theta(f) \subset f^* m_{p,0}^{i}\theta(f) \subset \omega f \left(m_{p,0}^{i}\theta_0(p)\right) \subset T\mathcal{L}_e(f)$$

for some positive integer ℓ. Thus, f is finitely $\mathcal{L}$-determined. $\square$

Remark 3.5. Proposition 3.4 does not hold, in general, for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ such that $\delta(f)$ is not finite. For example, let us consider the zero map-germ $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ given by $f(x) = (0, 0)$. The multiplicity $\delta(f)$ is not finite, since $f^* m_{2,0}\theta(f)$ vanishes. We have $f^* m_{2,0}^{i}\theta(f) \subset \omega f \left(m_{2,0}^{i}\theta_0(2)\right)$ for every positive integer i. Thus, ${}^{\mathcal{C}}_{i}\overline{\omega} f$ is surjective for every positive integer i by Proposition 3.1. However, f is not finitely $\mathcal{L}$-determined, since $T\mathcal{L}_e(f)$ consists only of constant vector fields.

Lemma 3.6. *For multigerms $f, g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and a non-negative integer i, suppose that f is $\mathcal{A}$-equivalent to g and ${}^{\mathcal{C}}_{i}\overline{\omega} f$ is surjective. Then, ${}^{\mathcal{C}}_{i}\overline{\omega} g$ is also surjective.*

Proof. Since f is $\mathcal{A}$-equivalent to g, there exist diffeomorphism germs for $\mathbb{K} = \mathbb{R}$, or biholomorphism germs for $\mathbb{K} = \mathbb{C}$, $\varphi : (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and $\psi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$, such that we have $\varphi(s) = s$ for every $s \in S$ and $f = \psi \circ g \circ \varphi$. By Proposition 3.1, it is enough to show that

$$g^* m_{p,0}^{i} C_{n,S} \subset g^* m_{p,0}^{i}.$$

Let us take any element $\overline{\varphi} \in g^* m_{p,0}^{i} C_{n,S}$. Then, we have $\overline{\varphi} \circ \varphi \in f^* m_{p,0}^{i} C_{n,S}$. Since ${}^{\mathcal{C}}_{i}\overline{\omega} f$ is surjective, there exists a $\widetilde{\psi} \in m_{p,0}^{i}$ such that $\overline{\varphi} \circ \varphi = \widetilde{\psi} \circ f$. Then, we have

$$\overline{\varphi} = \widetilde{\psi} \circ f \circ \varphi^{-1} = \widetilde{\psi} \circ \psi \circ g.$$

Thus, $\overline{\varphi} \in g^* m_{p,0}^{i}$ holds, since we have $\widetilde{\psi} \circ \psi \in m_{p,0}^{i}$. $\square$

Proposition 3.7. *For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, ${}^{\mathcal{C}}_{0}\overline{\omega} f$ is surjective if and only if $n \leq p$, $r = 1$ and f is non-singular, where r is the cardinality of S.*

Proof. Suppose that ${}^{\mathcal{C}}_{0}\overline{\omega} f$ is surjective. We see that $\theta(f) = T\mathcal{L}_e(f)$ holds by Proposition 3.1. Therefore, f is finitely $\mathcal{L}$-determined and we have $\delta(f) < \infty$ by Proposition 2.14.

The dimension of the $\mathbb{K}$-vector space $\theta_0(p)/m_{p,0}\theta_0(p)$ is clearly equal to p. On the other hand, the dimension of the $\mathbb{K}$-vector space $\theta(f)/f^* m_{p,0}\theta(f)$ is equal to $p\delta(f)$. Since ${}^{\mathcal{C}}_{0}\overline{\omega} f$ is surjective, we see that $\delta(f) \leq 1$ holds. Furthermore, since $1 \leq r \leq \delta(f)$ and $\delta(f) < \infty$ hold, we see that $r = 1$ and $n \leq p$ hold. In order to see that f is non-singular, let us consider the mono-germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $n \leq p$, given by

$$g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, \ldots, 0).$$

We see that g is non-singular. Moreover, the $\mathbb{K}$-vector space $\theta_0(p)/m_{p,0}\theta_0(p)$ is spanned by the classes represented by the p vector fields on the target of g

$$\left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}\right], \dots, \left[\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right].$$

On the other hand, the $\mathbb{K}$-vector space $\theta(g)/g^* m_{p,0}\theta(g)$ is spanned by the classes represented by the p vector fields along g

$$\left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}\right], \dots, \left[\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right].$$

Therefore, we also see that ${}^{\mathcal{C}}_{0}\overline{\omega}g$ is surjective. We see that

$$\theta(f) = T\mathcal{L}_e(f) \text{ and } \theta(g) = T\mathcal{L}_e(g)$$

hold by Proposition 3.1. These facts imply that $f^*C_{p,0} = g^*C_{p,0}$ holds. Therefore, f is $\mathcal{L}$-equivalent to g by Proposition A (a) in [2, p. 303]. Thus, f is a non-singular monogerm.

The converse follows from the implicit function theorem and Lemma 3.6. □

Proposition 3.8. *For a curve-germ $f : (\mathbb{K}, 0) \to (\mathbb{K}^p, 0)$, $p \geq 2$, satisfying $\delta(f) < \infty$, ${}^{\mathcal{C}}_{1}\overline{\omega}f$ is surjective if and only if f is $\mathcal{L}$-equivalent to one of the map-germs g_k given by*

$$g_k(x) = (x^k, x^{k+1}, \dots, x^{2k-1}, 0, \dots, 0), \quad k = 1, 2, \dots, p.$$

Moreover, g_{k_1} is not $\mathcal{L}$-equivalent to g_{k_2} if $k_1 \neq k_2$.

Proof. Suppose that ${}^{\mathcal{C}}_{1}\overline{\omega}f$ is surjective. Since we have $\delta(f) < \infty$, f is $\mathcal{L}$-equivalent to the map-germ g of the form

$$g(x) = (x^{\delta(f)} + x^{\delta(f)+1}\varphi_1(x), x^{\delta(f)+1}\varphi_2(x), \dots, x^{\delta(f)+1}\varphi_p(x))$$

for some $\varphi_1, \varphi_2, \dots, \varphi_p \in C_{1,0}$.

We see that

$$g^* m_{p,0} C_{1,0} = \left\langle x^{\delta(f)} \right\rangle_{C_{1,0}} \text{ and } g^* m_{p,0}^2 C_{1,0} = \left\langle x^{2\delta(f)} \right\rangle_{C_{1,0}}$$

hold by Proposition 2.2. Therefore, we have

$$\frac{g^* m_{p,0} C_{1,0}}{g^* m_{p,0}^2 C_{1,0}} = \left\langle \left[x^{\delta(f)}\right], \left[x^{\delta(f)+1}\right], \dots, \left[x^{2\delta(f)-1}\right] \right\rangle_{\mathbb{K}}.$$

Thus, the dimension of the $\mathbb{K}$-vector space $g^*m_{p,0}\theta(g)/g^*m_{p,0}^2\theta(g)$ is equal to $p\delta(f)$. The dimension of the $\mathbb{K}$-vector space $m_{p,0}\theta_0(p)/m_{p,0}^2\theta_0(p)$ is clearly equal to p^2. Since ${}_1^{\mathcal{C}}\overline{\omega}g$ is also surjective by Lemma 3.6, we see that $\delta(f) \leq p$ holds.

First, suppose that $\delta(f) = 1$ holds. Then, g is $\mathcal{L}$-equivalent to

$$g_1(x) = (x, 0, \ldots, 0)$$

by the implicit function theorem.

Second, suppose that $2 \leq \delta(f) \leq p$ holds. Here, note that ${}_1^{\mathcal{C}}\overline{\omega}g_k$ is surjective for every k with $1 \leq k \leq p$. In fact, the $\mathbb{K}$-vector space $m_{p,0}\theta_0(p)/m_{p,0}^2\theta_0(p)$ is spanned by the classes represented by the p^2 vector fields

$$\left[\begin{pmatrix} X^\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ X^\alpha \\ \vdots \\ 0 \end{pmatrix}\right], \ldots, \left[\begin{pmatrix} 0 \\ 0 \\ \vdots \\ X^\alpha \end{pmatrix}\right]$$

$$(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p = 1).$$

On the other hand, the $\mathbb{K}$-vector space $g_k^*m_{p,0}\theta(g_k)/g_k^*m_{p,0}^2\theta(g_k)$ is spanned by the classes represented by the pk vector fields

$$\left[\begin{pmatrix} x^{k+j_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ x^{k+j_2} \\ \vdots \\ 0 \end{pmatrix}\right], \ldots, \left[\begin{pmatrix} 0 \\ 0 \\ \vdots \\ x^{k+j_p} \end{pmatrix}\right]$$

$$(j_1 = 0, 1, \ldots, k-1;\ j_2 = 0, 1, \ldots, k-1; \ldots;\ j_p = 0, 1, \ldots, k-1)$$

by Proposition 2.2. Since $X_{j+1} \circ g_k = x^{k+j}$ holds for every $j = 0, 1, \ldots, k-1$, we see that ${}_1^{\mathcal{C}}\overline{\omega}g_k$ is surjective. Therefore, we have

$$\omega g_k(m_{p,0}\theta_0(p)) = g_k^*m_{p,0}\theta(g_k)$$

for every k with $1 \leq k \leq p$ by Proposition 3.1. Note that

$$g^*m_{p,0}C_{1,0} = g_{\delta(f)}^*m_{p,0}C_{1,0}$$

by Proposition 2.2. Moreover, we have

$$\omega g(m_{p,0}\theta_0(p)) = g^*m_{p,0}\theta(g)$$

by Proposition 3.1 and Lemma 3.6. Therefore, we have

$$\omega g(m_{p,0}\theta_0(p)) = \omega g_{\delta(f)}(m_{p,0}\theta_0(p)).$$

This fact implies that $g^*C_{p,0} = g_{\delta(f)}^*C_{p,0}$ holds. Thus, g is $\mathcal{L}$-equivalent to $g_{\delta(f)}$ by Proposition A (a) in [2, p. 303].

Conversely, suppose that $f : (\mathbb{K}, 0) \to (\mathbb{K}^p, 0)$, $p \geq 2$, is $\mathcal{L}$-equivalent to the map-germ g_k for some $k = 1, 2, \ldots, p$. Since ${}_1^{\mathcal{C}}\overline{\omega}g_k$ is surjective for every $k = 1, 2, \ldots, p$, ${}_1^{\mathcal{C}}\overline{\omega}f$ is also surjective by Lemma 3.6.

Finally, if $k_1 \neq k_2$, then we have $\delta(f_{k_1}) \neq \delta(f_{k_2})$. Thus, f_{k_1} is not $\mathcal{L}$-equivalent to f_{k_2}. $\square$

Remark 3.9. Proposition 3.8 gives the complete $\mathcal{L}$-classification of curve-germs $f : (\mathbb{K}, 0) \to (\mathbb{K}^p, 0)$, $p \geq 2$, satisfying $\omega f(m_{p,0}\theta_0(p)) = f^* m_{p,0}\theta(f)$ and $\delta(f) < \infty$ by Proposition 3.1. This proposition was first mentioned in [9, p. 61], although the precise proof was not given there.

Let us give other examples of finitely $\mathcal{L}$-determined multigerms.

Example 3.10. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be given by

$$f(x) = (x^k + x^{k+1}\varphi_1(x),\, x^{k+1} + x^{k+2}\varphi_2(x)),$$

where $k \geq 2, \varphi_1,\, \varphi_2 \in C_{1,0}$. We see that $\delta(f) = k$ holds.

The $\mathbb{K}$-vector space $m_{2,0}^{k-1}\theta_0(2)/m_{2,0}^{k}\theta_0(2)$ is spanned by the classes represented by the $2k$ vector fields

$$\left[\begin{pmatrix} X^{k-1-i}Y^i \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ X^{k-1-j}Y^j \end{pmatrix}\right]$$
$$(i = 0, 1, \ldots, k-1;\; j = 0, 1, \ldots, k-1).$$

On the other hand, the $\mathbb{K}$-vector space $f^* m_{2,0}^{k-1}\theta(f)/f^* m_{2,0}^{k}\theta(f)$ is spanned by the classes represented by the $2k$ vector fields

$$\left[\begin{pmatrix} x^{k(k-1)+i} \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ x^{k(k-1)+j} \end{pmatrix}\right]$$
$$(i = 0, 1, \ldots, k-1;\; j = 0, 1, \ldots, k-1)$$

by Proposition 2.2.

We see that

$$\begin{aligned}
{}_{k-1}^{\;\mathcal{C}}\overline{\omega} f\left(\left[\begin{pmatrix} X^{k-1-i}Y^i \\ 0 \end{pmatrix}\right]\right) &= \left[\begin{pmatrix} x^{k(k-1)+i}(1 + x\widetilde{\varphi}_{1,i}(x)) \\ 0 \end{pmatrix}\right], \\
{}_{k-1}^{\;\mathcal{C}}\overline{\omega} f\left(\left[\begin{pmatrix} 0 \\ X^{k-1-j}Y^j \end{pmatrix}\right]\right) &= \left[\begin{pmatrix} 0 \\ x^{k(k-1)+j}(1 + x\widetilde{\varphi}_{2,j}(x)) \end{pmatrix}\right]
\end{aligned}$$
$$(i = 0, 1, \ldots, k-1;\; j = 0, 1, \ldots, k-1)$$

hold for some $\widetilde{\varphi}_{1,i},\, \widetilde{\varphi}_{2,j} \in C_{1,0}$. Therefore, a matrix representative of ${}_{k-1}^{\;\mathcal{C}}\overline{\omega} f$ is regular. Thus, ${}_{k-1}^{\;\mathcal{C}}\overline{\omega} f$ is surjective. By Proposition 3.4, f is finitely $\mathcal{L}$-determined.

We have

$$f^* m_{2,0}^{k-1}\theta(f) \subset \omega f(m_{2,0}^{k-1}\theta_0(2))$$

by Proposition 3.1 and we have

$$f^* m_{2,0}^{k-1}\theta(f) = m_{1,0}^{k(k-1)}\theta(f)$$

by Proposition 2.2. Thus,

$$m_{1,0}^{k(k-1)}\theta(f) \subset T\mathcal{L}_e(f)$$

holds.

Example 3.11. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by

$$f_1(x) = (x^2,\, x^3), \quad f_2(x) = (x^3,\, x^2).$$

We see that $\delta(f) = \delta(f_1) + \delta(f_2) = 2 + 2 = 4$ holds.

The $\mathbb{K}$-vector space $m_{2,0}^3\theta_0(2)/m_{2,0}^4\theta_0(2)$ is spanned by the classes represented by the 8 vector fields

$$\left[\begin{pmatrix} X^{3-i}Y^i \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ X^{3-j}Y^j \end{pmatrix}\right]$$

$$(i = 0, 1, 2, 3;\ j = 0, 1, 2, 3).$$

On the other hand, the $\mathbb{K}$-vector space $f^*m_{2,0}^3\theta(f)/f^*m_{2,0}^4\theta(f)$ is spanned by the classes represented by the 8 vector fields

$$\left[\left(\begin{pmatrix} x^{6+i_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right], \left[\left(\begin{pmatrix} 0 \\ x^{6+i_2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right],$$

$$\left[\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x^{6+i_3} \\ 0 \end{pmatrix}\right)\right], \left[\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^{6+i_4} \end{pmatrix}\right)\right]$$

$$(i_1 = 0, 1;\ i_2 = 0, 1;\ i_3 = 0, 1;\ i_4 = 0, 1)$$

by Proposition 2.2.

We see that

$$\begin{aligned} {}_3^{\mathcal{C}}\overline{\omega}f\left(\left[\begin{pmatrix} X^{3-i}Y^i \\ 0 \end{pmatrix}\right]\right) &= \left[\left(\begin{pmatrix} x^{6+i} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{9-i} \\ 0 \end{pmatrix}\right)\right], \\ {}_3^{\mathcal{C}}\overline{\omega}f\left(\left[\begin{pmatrix} 0 \\ X^{3-j}Y^j \end{pmatrix}\right]\right) &= \left[\left(\begin{pmatrix} 0 \\ x^{6+j} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{9-j} \end{pmatrix}\right)\right] \end{aligned}$$

$$(i = 0, 1, 2, 3;\ j = 0, 1, 2, 3).$$

Since $x^8,\, x^9 \in f_1^*m_{2,0}^4C_{1,0} \cap f_2^*m_{2,0}^4C_{1,0}$ holds, a matrix representative of ${}_3^{\mathcal{C}}\overline{\omega}f$ is regular. Thus, ${}_3^{\mathcal{C}}\overline{\omega}f$ is surjective. By Proposition 3.4, f is finitely $\mathcal{L}$-determined.

We have

$$f^*m_{2,0}^3\theta(f) \subset \omega f(m_{2,0}^3\theta_0(2))$$

by Proposition 3.1 and we have

$$f^*m_{2,0}^3\theta(f) = m_{1,0}^6\theta(f)$$

by Proposition 2.2. Thus,

$$m_{1,0}^6\theta(f) \subset T\mathcal{L}_e(f)$$

holds.

Example 3.12. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^7, 0)$ be given by

$$f(x, y) = (x^2, xy, y^2, x^3, x^2y, xy^2, y^3).$$

We see that $\delta(f) = 3$ holds.

The $\mathbb{K}$-vector space $m_{7,0}\theta_0(7)/m_{7,0}^2\theta_0(7)$ is spanned by the classes represented by the 49 vector fields

$$\left[\begin{pmatrix} X^\alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ X^\alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right], \ldots, \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ X^\alpha \end{pmatrix}\right]$$

$$(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_7 = 1).$$

On the other hand, the $\mathbb{K}$-vector space $f^*m_{7,0}\theta(f)/f^*m_{7,0}^2\theta(f)$ is spanned by the classes represented by the 49 vector fields

$$\left[\begin{pmatrix} f^\alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ f^\alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right], \ldots, \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f^\alpha \end{pmatrix}\right]$$

$$(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_7 = 1).$$

Since we see that a matrix representative of ${}_1^{\mathcal{C}}\overline{\omega}f$ is regular, ${}_1^{\mathcal{C}}\overline{\omega}f$ is surjective. By Proposition 3.4, f is finitely $\mathcal{L}$-determined.

We have

$$f^*m_{7,0}\theta(f) \subset \omega f(m_{7,0}\theta_0(7))$$

by Proposition 3.1 and we have

$$f^*m_{7,0}\theta(f) = m_{2,0}^2\theta(f).$$

Thus,

$$m_{2,0}^2\theta(f) \subset T\mathcal{L}_e(f)$$

holds.

4 The module of lowerable vector fields for a finitely $\mathcal{L}$-determined multigerm

The main theme of this thesis is the following problem.

Problem 4.1. Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a multigerm satisfying $\delta(f) < \infty$. Then, is the module $\mathrm{Lower}(f)$ of lowerable vector fields finitely generated? In the case that $\mathrm{Lower}(f)$ is finitely generated, prove it in a constructive way.

Problem 4.1 is reduced to that of the finite generation of $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ by Proposition 2.8 and the fact that $tf(\mathrm{Lower}(f)) = T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. We see that, in the complex analytic case, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is finitely generated, since $C_{p,0}$ is Noetherian and $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is a $C_{p,0}$-submodule of the finitely generated module $\theta(f)$. The algebraic argument, however, gives no constructive proof. Moreover, the finite generation of $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ has been an open problem in the real C^∞ case, as far as the author knows.

In this section, we give a constructive proof of the following theorem, which works well in both the real C^∞ case and the complex analytic case.

Theorem 4.2. *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a finitely $\mathcal{L}$-determined multigerm. Then, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is finitely generated as a $C_{p,0}$-module via f.*

Proof. There exists a positive integer ℓ such that

$$m_{n,S}^{\ell}\theta(f) \subset T\mathcal{L}_e(f) \tag{1}$$

holds and we have $\delta(f) < \infty$ by Proposition 2.14. Thus, $Q(f_k)$ is a finite dimensional $\mathbb{K}$-vector space of dimension $\delta(f_k)$ for every k with $1 \le k \le r$, where f_k are the branches of f. Then, there exist $\varphi_{k,j} \in C_{n,0}$, $1 \le j \le \delta(f_k)$, such that we have

$$Q(f_k) = \left\langle [\varphi_{k,1}], [\varphi_{k,2}], \ldots, [\varphi_{k,\delta(f_k)}] \right\rangle_{\mathbb{K}}.$$

We would like to find a finite set of generators for $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Let us take any element $\overline{\eta} = (\overline{\eta}_1, \overline{\eta}_2, \ldots, \overline{\eta}_r) \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. For every $k = 1, 2, \ldots, r$, the vector field $\overline{\eta}_k$ can be expressed as

$$\overline{\eta}_k = \begin{pmatrix} \dfrac{\partial(X_1 \circ f_k)}{\partial x_1} & \dfrac{\partial(X_1 \circ f_k)}{\partial x_2} & \cdots & \dfrac{\partial(X_1 \circ f_k)}{\partial x_n} \\ \dfrac{\partial(X_2 \circ f_k)}{\partial x_1} & \dfrac{\partial(X_2 \circ f_k)}{\partial x_2} & \cdots & \dfrac{\partial(X_2 \circ f_k)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial(X_p \circ f_k)}{\partial x_1} & \dfrac{\partial(X_p \circ f_k)}{\partial x_2} & \cdots & \dfrac{\partial(X_p \circ f_k)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \widetilde{\varphi}_{1,k} \\ \widetilde{\varphi}_{2,k} \\ \vdots \\ \widetilde{\varphi}_{n,k} \end{pmatrix}$$

for some $\widetilde{\varphi}_{1,k}, \widetilde{\varphi}_{2,k}, \ldots, \widetilde{\varphi}_{n,k} \in C_{n,0}$.

By Theorem 2.6, for every $i = 1, 2, \ldots, n$, there exist $\psi_{k,i,j} \in C_{p,0}$ such that we have

$$\widetilde{\varphi}_{i,k} = \sum_{1 \le j \le \delta(f_k)} (\psi_{k,i,j} \circ f_k)\varphi_{k,j}.$$

Thus, $\overline{\eta}_k$ can be simplified as follows:

$$\overline{\eta}_k = \sum_{i,\,j} (\psi_{k,i,j} \circ f_k)\xi_{k,i,j},$$

where the symbol $\displaystyle\sum_{i,\,j}$ means the summation taken over all i and j with $1 \le i \le n$ and $1 \le j \le \delta(f_k)$, respectively, and $\xi_{k,i,j}$ is the transpose of

$$\left(\frac{\partial(X_1 \circ f_k)}{\partial x_i}\varphi_{k,j}, \frac{\partial(X_2 \circ f_k)}{\partial x_i}\varphi_{k,j}, \ldots, \frac{\partial(X_p \circ f_k)}{\partial x_i}\varphi_{k,j} \right).$$

Note that $\xi_{k,i,j} \in T\mathcal{R}_e(f_k)$ holds.

The function-germ $\psi_{k,i,j} \in C_{p,0}$ can be written in the form

$$\psi_{k,i,j}(X_1, X_2, \ldots, X_p) = \sum_{0 \le |\alpha| \le \ell-1} c_{k,i,j,\alpha} X^\alpha + \sum_{|\alpha|=\ell} \widetilde{\psi}_{k,i,j,\alpha} X^\alpha$$

for some $c_{k,i,j,\alpha} \in \mathbb{K}$ and $\widetilde{\psi}_{k,i,j,\alpha} \in C_{p,0}$. Recall that ℓ is the positive integer given in (1). Then, we have

$$\overline{\eta}_k = \sum_{i,j} \sum_{0 \le |\alpha| \le \ell-1} c_{k,i,j,\alpha}(f_k^\alpha \xi_{k,i,j}) + \sum_{i,j} \sum_{|\alpha|=\ell} (\widetilde{\psi}_{k,i,j,\alpha} \circ f_k)(f_k^\alpha \xi_{k,i,j}).$$

Set

$$\overline{\xi}_{k,i,j,\alpha} = (\underbrace{0,\, 0, \ldots,\, 0,\, f_k^\alpha \xi_{k,i,j}}_{k \text{ entries}},\, 0, \ldots,\, 0).$$

Note that $\overline{\xi}_{k,i,j,\alpha} \in T\mathcal{R}_e(f)$ holds. Then, we have

$$\overline{\eta} = \sum_{1 \le k \le r} \sum_{i,j} \sum_{0 \le |\alpha| \le \ell-1} c_{k,i,j,\alpha} \overline{\xi}_{k,i,j,\alpha} + \sum_{1 \le k \le r} \sum_{i,j} \sum_{|\alpha|=\ell} (\widetilde{\psi}_{k,i,j,\alpha} \circ f)\overline{\xi}_{k,i,j,\alpha}.$$

We define the finite sets L and H of $T\mathcal{R}_e(f)$ as follows:

$$L = \left\{ \overline{\xi}_{k,i,j,\alpha} \,|\, 0 \le |\alpha| \le \ell - 1,\, 1 \le k \le r,\, 1 \le i \le n,\, 1 \le j \le \delta(f_k) \right\},$$

$$H = \left\{ \overline{\xi}_{k,i,j,\alpha} \,|\, |\alpha| = \ell,\, 1 \le k \le r,\, 1 \le i \le n,\, 1 \le j \le \delta(f_k) \right\}.$$

Then, $H \subset T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ by (1). Therefore,

$$\sum_{1 \le k \le r} \sum_{i,j} \sum_{0 \le |\alpha| \le \ell-1} c_{k,i,j,\alpha} \overline{\xi}_{k,i,j,\alpha}$$

belongs to $V = T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$.

The set V is a finite dimensional $\mathbb{K}$-vector space. Set $\dim_{\mathbb{K}} V = m$. Then, there exist $\underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ such that we have

$$V = \langle \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \rangle_{\mathbb{K}}.$$

Clearly, we have $V \subset \langle \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \rangle_{f^*C_{p,0}}$. Therefore, we see that

$$\overline{\eta} \in \langle \underline{\xi}_1, \underline{\xi}_2 \ldots, \underline{\xi}_m \rangle_{f^*C_{p,0}} + H_{f^*C_{p,0}}.$$

Thus, we have

$$T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \subset \langle \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \rangle_{f^*C_{p,0}} + H_{f^*C_{p,0}}.$$

The converse inclusion also holds, since $\{\underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m\} \cup H$ is contained in $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Thus, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is finitely generated as a $C_{p,0}$-module via f. □

Remark 4.3. By our proof of Theorem 4.2, we can estimate the minimal number of generators for $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ from above as follows.

We have

$$\dim_{\mathbb{K}} T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}} \leq \dim_{\mathbb{K}} L_{\mathbb{K}}.$$

Thus, the minimal number of generators for $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is less than or equal to the cardinality of $L \cup H$, that is,

$$n \sum_{k=1}^{r} \delta(f_k) \left(\frac{(\ell + p)!}{\ell \,! \, p\,!} \right),$$

where ℓ is a positive integer such that $m_{n,S}^{\ell} \theta(f) \subset T\mathcal{L}_e(f)$ holds.

We have the following partial affirmative answer to Problem 4.1 as a corollary to Theorem 4.2.

Corollary 4.4. *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a finitely $\mathcal{L}$-determined multigerm. Then, the module* Lower(f) *of lowerable vector fields is finitely generated as a $C_{p,0}$-module via f.*

Proof. Since $tf(\mathrm{Lower}(f)) = T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ holds, the required conclusion follows directly from Propositions 2.8, 2.14, and Theorem 4.2. □

The proof of Theorem 4.2 gives a method for finding explicit generators for the module $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. The required data are the following:

1. a positive integer ℓ such that $m_{n,S}^{\ell} \theta(f) \subset T\mathcal{L}_e(f)$ holds,
2. a spanning set of the $\mathbb{K}$-vector space $Q(f)$,
3. a spanning set of the $\mathbb{K}$-vector space $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$, where $L_{\mathbb{K}}$ is determined by the data given in the above item 2.

We present some examples in which we give explicit generators for the module of lowerable vector fields. Note that these examples are new as far as the author knows.

Example 4.5. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be given by

$$f(x) = (x^2, x^3).$$

By Example 3.10, $m_{1,0}^2\theta(f) \subset T\mathcal{L}_e(f)$ holds. We see that

$$Q(f) = \langle [1], [x] \rangle_{\mathbb{K}}$$

holds.

We first look for a spanning set of the $\mathbb{K}$-vector space $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$. The set L consists of the 6 vector fields

$$\begin{aligned}
\overline{\xi}_{1,1,1,(0,0)} &= \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}, & \overline{\xi}_{1,1,1,(1,0)} &= \begin{pmatrix} 2x^3 \\ 3x^4 \end{pmatrix}, & \overline{\xi}_{1,1,1,(0,1)} &= \begin{pmatrix} 2x^4 \\ 3x^5 \end{pmatrix}, \\
\overline{\xi}_{1,1,2,(0,0)} &= \begin{pmatrix} 2x^2 \\ 3x^3 \end{pmatrix}, & \overline{\xi}_{1,1,2,(1,0)} &= \begin{pmatrix} 2x^4 \\ 3x^5 \end{pmatrix}, & \overline{\xi}_{1,1,2,(0,1)} &= \begin{pmatrix} 2x^5 \\ 3x^6 \end{pmatrix}.
\end{aligned}$$

Let us show that $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$ is spanned by the 4 vector fields

$$\overline{\xi}_{1,1,1,(1,0)}, \overline{\xi}_{1,1,1,(0,1)}, \overline{\xi}_{1,1,2,(0,0)}, \overline{\xi}_{1,1,2,(0,1)}.$$

It is clear that these vector fields belong to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$. Let us take any element $\eta \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$. Then, since $\eta \in L_{\mathbb{K}}$ holds, there exist $c_1, c_2, \ldots, c_5 \in \mathbb{K}$ such that we have

$$\eta = c_1 \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix} + c_2 \begin{pmatrix} 2x^2 \\ 3x^3 \end{pmatrix} + c_3 \begin{pmatrix} 2x^3 \\ 3x^4 \end{pmatrix} + c_4 \begin{pmatrix} 2x^4 \\ 3x^5 \end{pmatrix} + c_5 \begin{pmatrix} 2x^5 \\ 3x^6 \end{pmatrix}.$$

Since $\eta \in T\mathcal{L}_e(f)$ holds, there exists a $\psi \in C_{2,0}$ such that we have

$$2c_1x + 2c_2x^2 + 2c_3x^3 + 2c_4x^4 + 2c_5x^5 = \psi(x^2, x^3).$$

Therefore, we have $c_1 = 0$. Thus, the $\mathbb{K}$-vector space $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$ is spanned by the 4 vector fields

$$\overline{\xi}_{1,1,1,(1,0)}, \overline{\xi}_{1,1,1,(0,1)}, \overline{\xi}_{1,1,2,(0,0)}, \overline{\xi}_{1,1,2,(0,1)}.$$

On the other hand, the set H consists of the 6 vector fields

$$\begin{aligned}
\overline{\xi}_{1,1,1,(2,0)} &= \begin{pmatrix} 2x^5 \\ 3x^6 \end{pmatrix}, & \overline{\xi}_{1,1,1,(1,1)} &= \begin{pmatrix} 2x^6 \\ 3x^7 \end{pmatrix}, & \overline{\xi}_{1,1,1,(0,2)} &= \begin{pmatrix} 2x^7 \\ 3x^8 \end{pmatrix}, \\
\overline{\xi}_{1,1,2,(2,0)} &= \begin{pmatrix} 2x^6 \\ 3x^7 \end{pmatrix}, & \overline{\xi}_{1,1,2,(1,1)} &= \begin{pmatrix} 2x^7 \\ 3x^8 \end{pmatrix}, & \overline{\xi}_{1,1,2,(0,2)} &= \begin{pmatrix} 2x^8 \\ 3x^9 \end{pmatrix}.
\end{aligned}$$

According to the proof of Theorem 4.2, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is generated as a $C_{2,0}$-module via f by the 10 vector fields

$$\overline{\xi}_{1,1,1,(1,0)}, \overline{\xi}_{1,1,1,(0,1)}, \overline{\xi}_{1,1,2,(0,0)}, \overline{\xi}_{1,1,2,(0,1)}$$

and

$$\overline{\xi}_{1,1,1,(2,0)}, \overline{\xi}_{1,1,1,(1,1)}, \overline{\xi}_{1,1,1,(0,2)}, \overline{\xi}_{1,1,2,(2,0)}, \overline{\xi}_{1,1,2,(1,1)}, \overline{\xi}_{1,1,2,(0,2)}.$$

By easy calculations, we see that $\overline{\xi}_{1,1,1,(1,0)}$ and $\overline{\xi}_{1,1,2,(0,0)}$ generate $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. The corresponding lowerable vector fields are (x) and (x^2), respectively. Thus, Lower(f) is generated as a $C_{2,0}$-module via f by the 2 vector fields

$$(x),\ (x^2).$$

Example 4.6. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^7, 0)$ be given by

$$f(x, y) = (x^2, xy, y^2, x^3, x^2y, xy^2, y^3).$$

By Example 3.12, $m_{2,0}^2\theta(f) \subset T\mathcal{L}_e(f)$ holds. We see that

$$Q(f) = \langle [1], [x], [y] \rangle_{\mathbb{K}}$$

holds.

We first look for a basis of $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$. The set L consists of 48 elements, say, $\xi_1, \xi_2, \ldots, \xi_{48}$. The elements $\xi_3, \xi_4, \ldots, \xi_{48}$ all belong to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ except for the following ξ_1 and ξ_2:

$$\xi_1 = \begin{pmatrix} 2x \\ y \\ 0 \\ 3x^2 \\ 2xy \\ y^2 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ x \\ 2y \\ 0 \\ x^2 \\ 2xy \\ 3y^2 \end{pmatrix}.$$

Therefore, we have

$$\langle \xi_3, \xi_4, \ldots, \xi_{48} \rangle_{\mathbb{K}} \subset T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}.$$

Conversely, let us take any element $\eta \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$. Then, there exist $c_1, c_2, \ldots, c_{48} \in \mathbb{K}$ such that we have

$$\eta = c_1 \begin{pmatrix} 2x \\ y \\ 0 \\ 3x^2 \\ 2xy \\ y^2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ x \\ 2y \\ 0 \\ x^2 \\ 2xy \\ 3y^2 \end{pmatrix} + c_3\xi_3 + \cdots + c_{48}\xi_{48}.$$

Then, there exist $\psi_1, \psi_2 \in C_{7,0}$ such that we have

$$2c_1x = \psi_1(x^2, xy, y^2, x^3, x^2y, xy^2, y^3), \quad 2c_2y = \psi_2(x^2, xy, y^2, x^3, x^2y, xy^2, y^3).$$

Therefore, $c_1 = c_2 = 0$ holds. Thus, the $\mathbb{K}$-vector space $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cap L_{\mathbb{K}}$ is spanned by the 46 vector fields $\xi_3, \xi_4, \ldots, \xi_{48}$. By easy calculations, we see that

$$\langle \xi_3, \xi_4, \ldots, \xi_{48} \rangle_{f^*C_{7,0}}$$

is generated as a $C_{7,0}$-module via f by the 10 vector fields

$$x\xi_1,\ y\xi_1,\ x^2\xi_1,\ xy\xi_1,\ y^2\xi_1,\ x\xi_2,\ y\xi_2,\ x^2\xi_2,\ xy\xi_2,\ y^2\xi_2.$$

On the other hand, we have

$$H_{f^*C_{7,0}} \subset \left\langle x\xi_1, y\xi_1, x^2\xi_1, xy\xi_1, y^2\xi_1, x\xi_2, y\xi_2, x^2\xi_2, xy\xi_2, y^2\xi_2 \right\rangle_{f^*C_{7,0}}.$$

Therefore, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is generated as a $C_{7,0}$-module via f by the 10 vector fields

$$x\xi_1,\ y\xi_1,\ x^2\xi_1,\ xy\xi_1,\ y^2\xi_1,\ x\xi_2,\ y\xi_2,\ x^2\xi_2,\ xy\xi_2,\ y^2\xi_2.$$

Thus, $\mathrm{Lower}(f)$ is generated as a $C_{7,0}$-module via f by the 10 vector fields

$$\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}.$$

5 Non-finitely $\mathcal{L}$-determined cases

In this section, we give some examples of multigerms f which are not finitely $\mathcal{L}$-determined and for which we can nevertheless calculate $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ explicitly. We also present explicit generators for the module Lower(f) of lowerable vector fields in these examples.

Example 5.1. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ be given by

$$f(x_1, x_2, \ldots, x_n) = (x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}),$$

where $m_1, m_2, \ldots, m_n$ are positive integers such that $m_i \geq 2$ holds for some $i = 1, 2, \ldots, n$.

We see that

$$\begin{pmatrix} x_i^{km_i - 1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \notin T\mathcal{L}_e(f)$$

for every positive integer k. Thus, f is not finitely $\mathcal{L}$-determined.

Let us show that $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is generated as a $C_{n,0}$-module via f by the n vector fields

$$\begin{pmatrix} \overline{\varphi}_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{\varphi}_2(x) \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \overline{\varphi}_n(x) \end{pmatrix} \quad \left(\overline{\varphi}_j(x) = \begin{cases} 1 & (m_j = 1) \\ x_j^{m_j} & (m_j \geq 2) \end{cases} \right).$$

It is clear that these vector fields belong to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$.

Let us take any element $\xi \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Then, there exist

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_n \in C_{n,0}$$

such that we have

$$\begin{aligned}
\xi &= \begin{pmatrix} m_1 x_1^{m_1 - 1} & 0 & \cdots & 0 \\ 0 & m_2 x_2^{m_2 - 1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n x_n^{m_n - 1} \end{pmatrix} \begin{pmatrix} \varphi_1(x_1, x_2, \ldots, x_n) \\ \varphi_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ \varphi_n(x_1, x_2, \ldots, x_n) \end{pmatrix} \\
&= \begin{pmatrix} m_1 x_1^{m_1 - 1} \varphi_1(x_1, x_2, \ldots, x_n) \\ m_2 x_2^{m_2 - 1} \varphi_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ m_n x_n^{m_n - 1} \varphi_n(x_1, x_2, \ldots, x_n) \end{pmatrix} \\
&= \begin{pmatrix} \psi_1(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}) \\ \psi_2(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}) \\ \vdots \\ \psi_n(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}) \end{pmatrix}.
\end{aligned}$$

Therefore, we have

$$m_j x_j^{m_j-1}\varphi_j(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}) = \psi_j(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n})$$

for every $j = 1, 2, \ldots, n$. If $m_j = 1$, then we have

$$\psi_j(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}) = (\psi_j \circ f)1.$$

If $m_j \geq 2$, then there exists a $\widetilde{\psi}_j \in C_{n,0}$ such that

$$\psi_j(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}) = (\widetilde{\psi}_j \circ f)x_j^{m_j}$$

holds, since we have

$$\psi_j(x_1^{m_1}, x_2^{m_2}, \ldots, x_{j-1}^{m_{j-1}}, 0, x_{j+1}^{m_{j+1}}, \ldots, x_n^{m_n}) = 0.$$

Therefore, Thus, ξ is expressed as follows:

$$\xi = (\overline{\psi}_1 \circ f)\begin{pmatrix} \overline{\varphi}_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (\overline{\psi}_2 \circ f)\begin{pmatrix} 0 \\ \overline{\varphi}_2(x) \\ \vdots \\ 0 \end{pmatrix} + \cdots + (\overline{\psi}_n \circ f)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \overline{\varphi}_n(x) \end{pmatrix}$$

$$\left(\overline{\varphi}_j(x) = \left\{ \begin{array}{ll} 1 & (m_j = 1) \\ x_j^{m_j} & (m_j \geq 2) \end{array} \right. , \quad \overline{\psi_j}(x) = \left\{ \begin{array}{ll} \psi_j & (m_j = 1) \\ \widetilde{\psi}_j & (m_j \geq 2) \end{array} \right. \right).$$

Thus, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is generated as a $C_{n,0}$-module via f by the n vector fields

$$\begin{pmatrix} \overline{\varphi}_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{\varphi}_2(x) \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \overline{\varphi}_n(x) \end{pmatrix} \quad \left(\overline{\varphi}_j(x) = \left\{ \begin{array}{ll} 1 & (m_j = 1) \\ x_j^{m_j} & (m_j \geq 2) \end{array} \right. \right).$$

Moreover, by Proposition 2.8, Lower(f) is generated as a $C_{n,0}$-module via f by the n vector fields

$$\begin{pmatrix} \widehat{\varphi}_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \widehat{\varphi}_2(x) \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \widehat{\varphi}_n(x) \end{pmatrix} \quad \left(\widehat{\varphi}_i(x) = \left\{ \begin{array}{ll} 1 & (m_j = 1) \\ x_j & (m_j \geq 2) \end{array} \right. \right).$$

For $n = 2$, $m_1 = m_2 = 2$, and $\mathbb{K} = \mathbb{R}$, explicit generators for Lower(f) are given in [5, p. 109].

Example 5.2. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ be given by

$$f(x, y) = (x, y^2, xy),$$

which is the so-called the crosscap or Whitney umbrella. Note that we have

$$Q(f) = \langle [1], [y] \rangle_{\mathbb{K}}.$$

We see that

$$\begin{pmatrix} 0 \\ y^{2i-1} \\ 0 \end{pmatrix} \notin T\mathcal{L}_e(f)$$

for every positive integer i. Therefore, f is not finitely $\mathcal{L}$-determined.

Let us show that $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is generated as a $C_{3,0}$-module via f by the 4 vector fields

$$\begin{pmatrix} x \\ 0 \\ xy \end{pmatrix}, \begin{pmatrix} xy \\ 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2xy \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2y^2 \\ xy \end{pmatrix}.$$

It is clear that these vector fields belong to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$.

Let us take any element $\xi \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Then, by Theorem 2.6, there exist $\psi_1, \psi_2, \psi_3, \psi_4 \in C_{3,0}$ such that we have

$$\begin{aligned} \xi &= \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \begin{pmatrix} (\psi_1 \circ f) + (\psi_2 \circ f)y \\ (\psi_3 \circ f) + (\psi_4 \circ f)y \end{pmatrix} \\ &= (\psi_1 \circ f) \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} + (\psi_2 \circ f) \begin{pmatrix} y \\ 0 \\ y^2 \end{pmatrix} \\ &\quad + (\psi_3 \circ f) \begin{pmatrix} 0 \\ 2y \\ x \end{pmatrix} + (\psi_4 \circ f) \begin{pmatrix} 0 \\ 2y^2 \\ xy \end{pmatrix}. \end{aligned}$$

Since $\xi \in T\mathcal{L}_e(f)$ holds, there exist $\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3 \in C_{3,0}$ such that we have

$$\begin{aligned} (\psi_1(x, y^2, xy))y &= \widetilde{\psi}_1(x, y^2, xy), \\ (\psi_2(x, y^2, xy))y &= \widetilde{\psi}_2(x, y^2, xy), \\ (\psi_3(x, y^2, xy))y &= \widetilde{\psi}_3(x, y^2, xy). \end{aligned}$$

Here, set $x = 0$. Then, we see that

$$\psi_1(0, y^2, 0) = \psi_2(0, y^2, 0) = \psi_3(0, y^2, 0) = 0$$

holds. Note that there exist $\widetilde{\psi}_4, \widetilde{\psi}_5, \ldots, \widetilde{\psi}_9 \in C_{3,0}$ such that we have

$$\begin{aligned} \psi_1(x, y^2, xy) &= \psi_1(0, y^2, 0) + x\widetilde{\psi}_4(x, y^2, xy) + xy\widetilde{\psi}_5(x, y^2, xy), \\ \psi_2(x, y^2, xy) &= \psi_2(0, y^2, 0) + x\widetilde{\psi}_6(x, y^2, xy) + xy\widetilde{\psi}_7(x, y^2, xy), \\ \psi_3(x, y^2, xy) &= \psi_3(0, y^2, 0) + x\widetilde{\psi}_8(x, y^2, xy) + xy\widetilde{\psi}_9(x, y^2, xy). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
\xi &= ((\widetilde{\psi}_4 + Y\widetilde{\psi}_7)\circ f)\begin{pmatrix} x \\ 0 \\ xy \end{pmatrix} + ((\widetilde{\psi}_5 + \widetilde{\psi}_6)\circ f)\begin{pmatrix} xy \\ 0 \\ xy^2 \end{pmatrix} \\
&\quad +(\widetilde{\psi}_8 \circ f)\begin{pmatrix} 0 \\ 2xy \\ x^2 \end{pmatrix} + ((X\widetilde{\psi}_9 + \psi_4)\circ f)\begin{pmatrix} 0 \\ 2y^2 \\ xy \end{pmatrix}.
\end{aligned}$$

Thus, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is generated as a $C_{3,0}$-module via f by the 4 vector fields

$$\begin{pmatrix} x \\ 0 \\ xy \end{pmatrix}, \begin{pmatrix} xy \\ 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2xy \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2y^2 \\ xy \end{pmatrix}.$$

Moreover, by Proposition 2.8, $\mathrm{Lower}(f)$ is generated as a $C_{3,0}$-module via f by the 4 vector fields

$$\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

This example is new as far as the author knows.

6 Problems

We have proved that the module $\mathrm{Lower}(f)$ of lowerable vector fields is finitely generated for a finitely $\mathcal{L}$-determined multigerm f in the thesis. The key point is that $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is finitely generated if f is finitely $\mathcal{L}$-determined. We expect to weaken the hypothesis of Theorem 4.2. In fact, we have not found a multigerm f satisfying $\delta(f) < \infty$ such that $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is not finitely generated yet.

It is difficult to investigate $\mathrm{Lift}(f)$, the module of liftable vector fields for a multigerm f. Since $\omega f(\mathrm{Lift}(f)) = T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ holds, we have the following isomorphism as $C_{p,0}$-modules:

$$T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f) \cong \frac{\mathrm{Lift}(f)}{\ker \omega f}.$$

However, even if f is finitely $\mathcal{L}$-determined, $\ker \omega f$ does not vanish in general. Therefore, we need to study $\ker \omega f$, although it seems to be difficult in the real C^∞ case.

In Section 3, we have classified curve germs $f : (\mathbb{K}, 0) \to (\mathbb{K}^p, 0)$, $p \geq 2$, such that $\delta(f)$ are finite and ${}^{\mathcal{C}}_{1}\overline{\omega} f$ are surjective up to $\mathcal{L}$-equivalence. The next problem would be to classify multigerms $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ such that $\delta(f)$ are finite and ${}^{\mathcal{C}}_{i}\overline{\omega} f$ are surjective for higher i or for more general pairs of dimensions (n, p).

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