# Adelic Cohomology Groups for Arithmetic Varieties and Ind-Pro Topology in Dimension Two

菅原, 弘太郎

https://doi.org/10.15017/1500519

出版情報：九州大学, 2014, 博士（数理学）, 課程博士
バージョン：

# Adelic Cohomology Groups for Arithmetic Varieties
# and
# Ind-Pro Topology in Dimension Two

Kotaro Sugahara (Kyushu University)

January 1, 2015

# Contents

# 1 Introduction

Classical adelic theory is closely related with class field theory, theory of $L$-functions and algebraic groups. Its origin may be traced back to a paper of Chevalley on the so-called ideles. In 1938, A. Weil ([17]) gave the first adelic approach to Riemann-Roch theorem for curves. Contrary to the formal sheaf theoretic approach to Riemann-Roch theorem widely adopted nowadays, the adelic one, then popular, was very concrete for curves. Most important works on adeles in this period were Tate's thesis ([16]) and Weil's works on Tamagawa numbers. Influenced by them, adeles later became a basic tool in the study of automorphic forms, Eisenstein series and trace formula.

Modern adelic theory started with A.N. Parshin's pioneer works on adeles over algebraic surfaces around 1976. In [12], Parshin introduced adelic rings and adelic complexes for divisors on algebraic surfaces, verified that his adelic cohomology groups were isomorphic to Grothendieck's sheaf theoretic cohomology groups, and established the Serre duality. In 1980, Parshin's works were generalized by A. A. Beilinson to Noetherian schemes: In [1], Beilinson constructed adelic complexes for quasi-coherent sheaves on Noetherian schemes, examined that the associated adelic cohomology groups coincided with the sheaf theoretic cohomology groups and outlined an adelic residue theory. This paper of Beilinson was very short and difficult for many to understand. To remedy this, around 1990, younger generation started to supply the unwritten details. Now we have the works of A. Huber[4], treating adelic complexes in great details, and A. Yekutieli[19], building up an adelic residue theory.

At the same time, adelic approach to Riemann-Roch theorem was moving forwards. Around 2000, in [18], L. Weng developed an adelic cohomology theory for arithmetic curves, based on [16] with a concrete construction of arithmetic cohomology groups. Furthermore, in 2011, D. V. Osipov and A. N. Parshin not only obtained an adelic proof for the Riemann-Roch theorem on algebraic surfaces in [11], but constructed arithmetic adelic rings for arithmetic surfaces in [10].

The purpose of our study is to introduce a general adelic cohomology theory for quasi-coherent sheaves on arithmetic varieties. Motivated by the works of Parshin, Beilinson, Weng and Osipov-Parshin mentioned above, we here, together with Prof. Weng, first construct adelic complexes for quasi-coherent sheaves on arithmetic varieties and hence their arithmetic cohomology groups, and then develop a general ind-pro topological theory in dimension two and hence to establish a topological duality for our arithmetic cohomology groups associated to invertible sheaves on arithmetic surfaces.

**Acknowledgements.** I would like to express my deepest gratitude and heartful thanks to my advisor Professor Lin Weng. He always gave me aspiring guidances, invaluably constructive advices and criticisms in the past seven years. Without his supervision, constant help and encouragement, this dissertation would not have been possible.

## 2 Adelic Cohomology Theory on Noetherian Schemes

In this section, we recall adelic cohomology theory on Noetherian schemes developed by Parshin ([12]) and Beilinson ([1]). We will follow [4] for the presentation.

**Definition 2.1** ([1], §2; [4], Definitions 1.3.1-3)**.** Let $X$ be a Noetherian scheme.

(1) Let $P(X)$ be the collection of scheme-theoretic points on $X$. For points $p, q \in P(X)$, we write $p \geq q$ if $q \in \overline{\{p\}}$. Clearly, $\geq$ makes $P(X)$ a poset.

(2) Let $S(X)$ be the set of simplexes corresponding to the ordered set $(P(X), \geq)$. In particular, for $m \geq 0$, let $S(X)_m$ be the set of $m$-simplexes and $S(X)_m^{\mathrm{red}}$ be the set of non-degenerate m-simplexes. That is,

$$\begin{aligned} S(X)_m &:= \left\{(p_0, \cdots, p_m) \in P(X)^{m+1} \mid p_i \geq p_{i+1}\right\}, \\ S(X)_m^{\mathrm{red}} &:= \left\{(p_0, \cdots, p_m) \in S(X)_m \mid p_i \neq p_j\ (i \neq j)\right\}. \end{aligned}$$

Accordingly, for $i \in \{0, 1, \cdots, m\}$, we define the associated boundary maps $\delta_i^m$ and degeneracy maps $\sigma_i^m$ as follows:

$$\begin{aligned} \delta_i^m &: S(X)_m \to S(X)_{m-1}; (p_0, \cdots, p_i, \cdots, p_m) \mapsto (p_0, \cdots, \check{p}_i, \cdots, p_m), \\ \sigma_i^m &: S(X)_m \to S(X)_{m+1}; (p_0, \cdots, p_i, \cdots, p_m) \mapsto (p_0, \cdots, p_i, p_i, \cdots, p_m). \end{aligned}$$

**Definition 2.2** ([1], §2; [4], §1.3)**.** For a subset $K \subset S(X)_m$ and a point $p \in P(X)$, we define a subset ${}_pK$ of $S(X)_{m-1}$ by

$${}_pK := \{(p_1, \cdots, p_m) \in S(X)_{m-1} \mid (p, p_1, \cdots, p_m) \in K\}.$$

As usual, for each point $p \in X$, let $\mathcal{O}_p$ denote the local ring of $X$ at $p$ and $m_p$ be its maximal ideal. Then we get a natural morphism $f : \operatorname{Spec} \mathcal{O}_p \to X$. Consequently, for each $\mathcal{O}_p$-modules $N$, it makes sense for us to introduce $[N]_p = f_*\widetilde{N}$. Moreover, let $AbGp$ be the category of abelian groups, and $QC(X)$ be the category of quasi-coherent sheaves on $X$.

**Proposition 2.3** (Parshin-Beilinson ([12], §2; [1], §2; see also [4], Prop. 2.1.1))**.** *For each subset $K \subset S(X)_m$, there exists an additive and exact functor*

$$\mathbb{A}(K, \cdot) : QC(X) \to AbGp$$

*determined uniquely by the following properties.*

(i) *$\mathbb{A}(K, \cdot)$ commutes with direct limits.*

(ii) *For $m = 0$ and a coherent sheaf $\mathcal{F}$ on $X$,*

$$\mathbb{A}(K, \mathcal{F}) = \prod_{p \in K} \varprojlim_l \mathcal{F}_p / m_p^l \mathcal{F}_p.$$

(iii) *For $m > 0$ and a coherent sheaf $\mathcal{F}$ on $X$,*

$$\mathbb{A}(K, \mathcal{F}) = \prod_{p \in P(X)} \varprojlim_l \mathbb{A}({}_pK, [\mathcal{F}_p / m_p^l \mathcal{F}_p]_p).$$

In the sequel, we call $\mathbb{A}(K, \cdot)$, resp. $\mathbb{A}(K, \mathcal{F})$, the adelic functor (associated to $K$), resp. the adelic group of quasi-coherent sheaf $\mathcal{F}$.

*Remark.* We cannot apply properties (i), (ii) directly for quasi-coherent but not coherent sheaves $\mathcal{F}$. In general, to calculate adelic groups for quasi-coherent sheaves, we have to use property (i). Indeed, since $X$ is a Noetherian scheme, any quasi-coherent sheaf $\mathcal{F}$ can be written as $\mathcal{F} = \varinjlim_{i\in I} \mathcal{F}_i$ with $\mathcal{F}_i$ ($i \in I$) coherent sheaves on $X$. Hence, by property (i), we can perform the following calculations: $\mathbb{A}(K,\mathcal{F}) = \mathbb{A}(K, \varinjlim_{i\in I} \mathcal{F}_i) = \varinjlim_{i\in I} \mathbb{A}(K,\mathcal{F}_i)$. Note that now all $\mathcal{F}_i$ are coherent, we can apply properties (ii), (iii) to get adelic groups $\mathbb{A}(K,\mathcal{F}_i)$.

**Definition 2.4** ([4], Definition 3.3.2)**.** Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then, for $m \geq 0$, we define the $m$-th adelic group $\mathbb{A}^m_X(\mathcal{F})$ by

$$\mathbb{A}^m_X(\mathcal{F}) \coloneqq \mathbb{A}(S(X)^{\mathrm{red}}_m, \mathcal{F}).$$

**Definition 2.5** ([13], §2)**.** Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then, for $0 \leq i_0 \leq i_1 \leq \cdots \leq i_m$, we define the type $(i_0, \cdots, i_m)$ adelic group $\mathbb{A}_{X,i_0,i_1,\cdots,i_m}(\mathcal{F})$ by

$$\mathbb{A}_{X,i_0,i_1,\cdots,i_m}(\mathcal{F}) \coloneqq \mathbb{A}_X(K_{i_0,i_1,\cdots,i_m}, \mathcal{F}),$$

where

$$K_{i_0,i_1,\cdots,i_m} \coloneqq \left\{ (p_0, p_1, \cdots, p_m) \in S(X)_m \;\middle|\; 0 \leq \forall t \leq m,\ \operatorname{codim}\overline{\{p_t\}} = i_t \right\}.$$

*Remark* ([13], §2). If $\dim X < +\infty$, we have $\mathbb{A}^m_X(\mathcal{F}) = \bigoplus_{0\leq i_0<\cdots<i_m\leq \dim X} \mathbb{A}_{X,i_0,\cdots,i_m}(\mathcal{F})$.

**Proposition 2.6** ([4], Prop. 2.1.4)**.** *For a subset $K \subset S(X)_m$ and a quasi-coherent sheaf $\mathcal{F}$, we have a natural inclusion*

$$\mathbb{A}(K,\mathcal{F}) \subset \prod_{(p_0,\cdots,p_m)\in K} \mathbb{A}((p_0, \cdots, p_m), \mathcal{F}).$$

*Notation.* By this proposition, we may write an element $f$ of $\mathbb{A}(K,\mathcal{F})$ as either $f = (f_{X_0,\cdots,X_m})$, or $f = (f_{p_0,\cdots,p_m})$, where $X_i = \overline{\{p_i\}}$ ($0 \leq i \leq m$), and $f_{X_0,\cdots,X_m} = f_{p_0,\cdots,p_m} \in \mathbb{A}((p_0, \cdots, p_m), \mathcal{F})$. If, in addition, $X$ is irreducible and $p_i$ is its generic point, we often omit the indexes $X_i, p_i$.

**Definition 2.7** ([4], Definition 2.2.2)**.** Assume that there exist $i \in \{1, \cdots, m\}$, $K \subset S(X)_m$ and $L \subset S(X)_{m-1}$ satisfying $\delta^m_i K \subset L$. Then, for a quasi-coherent sheaf $\mathcal{F}$ on $X$, we define boundary maps $d^m_i(K, L, \mathcal{F})$

$$d^m_i(K, L, \mathcal{F}) : \mathbb{A}(L,\mathcal{F}) \to \mathbb{A}(K,\mathcal{F})$$

as follows.

(a) Assume that $i = 0$ and $\mathcal{F}$ is a coherent sheaf. For each point $p \in P(X)$, we have a natural map $\mathbb{A}(L,\mathcal{F}) \to \mathbb{A}(L, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p)$ induced from the structural morphism $\mathcal{F} \to [\mathcal{F}_p/m_{p^l}\mathcal{F}_p]_p$. Then the maps $\varphi^l_p : \mathbb{A}(L,\mathcal{F}) \to \mathbb{A}({}_pK, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p)$, obtaining as the compositions of the map $\mathbb{A}(L,\mathcal{F}) \to \mathbb{A}(L, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p)$ and the projections $\mathbb{A}(L, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p) \to \mathbb{A}({}_pK, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p)$, form a projective system for $l \in \mathbb{N}$. Accordingly, we define a boundary map by $d^m_0(K, L, \mathcal{F}) = \prod_{p\in P(X)} \varprojlim_l \varphi^l_p$.

(b) Assume that $i = 1$, $m = 1$ and $\mathcal{F}$ is a coherent sheaf. For each point $p \in P(X)$, canonical maps $\pi^l_p : \Gamma(X, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p) \to \mathbb{A}({}_pK, [\mathcal{F}_p/m^l_p\mathcal{F}_p]_p)$ form a projective system for $l \in \mathbb{N}$. Accordingly, we define a boundary map by $d^1_1(K, L, \mathcal{F}) = \prod_{p\in P(X)} \varprojlim_l \pi^l_p$.

(c) Assume that $i > 0, m > 1$ and $\mathcal{F}$ is a coherent sheaf. Then we define a boundary map by

$$d_i^m(K, L, \mathcal{F}) = \prod_{p \in P(X)} \varprojlim_l d_{i-1}^{m-1}({}_pK, {}_pL, [\mathcal{F}_p / m_p^l \mathcal{F}_p]_p).$$

(d) $d_i^m(K, L, \cdot)$ commutes with direct limits.

*Notation.* For a quasi-coherent sheaf $\mathcal{F}$, set $d^m = \sum_{i=0}^m (-1)^i d_i^n(S(X)_m^{\mathrm{red}}, S(X)_{m-1}^{\mathrm{red}}), \mathcal{F})$. Then we have a boundary map $d^m : \mathbb{A}_X^{m-1}(\mathcal{F}) \to \mathbb{A}_X^m(\mathcal{F})$.

**Theorem 2.8** (Parsin-Beilinson [12], §2, Thm 1; [1], §2, Cor; see also [4], Thm 4.2.3, Prop 5.1.2)**.** *For a quasi-coherent sheaf $\mathcal{F}$ on a Noetherian scheme $X$, we have*

(i) $(\mathbb{A}_X^*(\mathcal{F}), d^*)$ *becomes a complex.*

(ii) *For each $i \geq 0$, there is a natural isomorphism, as abelian groups,*

$$H^i(\mathbb{A}_X^*(\mathcal{F}), d^*) \simeq H^i(X, \mathcal{F}),$$

*where the left side is the i-th comology group induced from the adelic complex $(\mathbb{A}_X^*(\mathcal{F}), d^*)$, and the right side is the i-th sheaf theoretic comology group of $\mathcal{F}$.*

We call the complex in this theorem an adelic complex, and their cohomology groups adelic cohomology groups.

# 3 Arithmetic Adelic Groups and Arithmetic Cohomology Groups

## 3.1 Notations

$F$ : an algebraic field,
$O_F$ : the integer ring of $F$,
$S_{\text{fin}}$ : the set of finite places of $F$,
$S_\infty$ : the set of infinite places of $F$,
$S := S_{\text{fin}} \cup S_\infty$,
$\pi : X \to Y = \operatorname{Spec} O_F$ : an arithmetic variety of dimension $n+1$,
$X_F$ : the generic fibre of $X$,
$F_v$ : the $v$-completion of $F$ ($v \in S$),
$X_\sigma := X \times_Y \operatorname{Spec} F_\sigma$ ($\sigma \in S_\infty$),
$\varphi_\sigma : X_\sigma \to X_F$.

## 3.2 Arithmetic Adelic Groups

There is a natural one-to-one correspondence between closed points on arithmetic curve $\operatorname{Spec} O_F$ and finite places of $F$. The so-called Arakelov compactification $\overline{\operatorname{Spec} O_F}$ of $\operatorname{Spec} O_F$ is obtained by adding infinite places. Associated to $\overline{\operatorname{Spec} O_F}$ is the adelic ring $\mathbb{A}_F$ of $F$ which contains much refined information on not only finite places in $S_{\text{fin}}$ but also infinite places in $S_\infty$. Similarly, when we construct adelic rings (and more generally adelic groups associated to quasi-coherent sheaves) on arithmetic varieties $X$, we have to treat scheme-theoretic points on both $X_{\text{fin}}(= X)$ and on $X_\infty := \{X_\sigma\}_{\sigma \in S_\infty}$. In parallel to Arakelov theory, (in particular, the Arakelov intersection theory,) however, we need not consider all points on $X_\sigma$, but only these corresponding to the so-called horizontal cycles of both $X_{\text{fin}}$ and $X_\infty$. Our treatment is motivated by Weng's work [18], where arithmetic curves are treated, with a use of the so-called uniformity condition: for a scheme-theoretic point $P$ on $X_F$, $\varphi_\sigma^{-1}(\overline{\{P\}})$ decomposes into finite irreducible closed varieties in $X_\sigma$. Our uniformity condition is a constrain on these irreducible components in $X_\infty$ and the induced cycles on $X_{\text{fin}}$. For our own use, we call generic points of the above irreducible closed varieties for $P$ in $X_\sigma$ the infinite points corresponding to $P$.

Before treating general arithmetic varieties, we recall the following construction of arithmetic adelic rings for arithmetic surfaces, introduced by Osipov and Parshin in [10]. For the time being, let $X$ be an arithmetic surface.

**Definition 3.1** (Osipov-Parshin ([10], §5))**.**
[Finite adelic ring]
We define the finite adelic ring $\mathbb{A}_X^{\text{fin}}$ for an arithmetic surface $X$ by

$$\mathbb{A}_X^{\text{fin}} := \mathbb{A}_{X,012}(O_X).$$

Then we have

$$\mathbb{A}_X^{\text{fin}} = \varinjlim_{D_1} \varprojlim_{D_2 : D_2 \le D_1} \mathbb{A}_{X,12}(D_1)/\mathbb{A}_{X,12}(D_2),$$

where $D_*$ are divisors on $X$ and we set $\mathbb{A}_{X,12}(D_*) = \mathbb{A}_{X,12}(O_X(D_*))$.

[∞-adelic ring]
For the adelic ring $\mathbb{A}_{X_F} = \mathbb{A}_{X_F,01}(O_{X_F})$ of the generic fibre $X_F$, we have

$$\mathbb{A}_{X_F} = \varinjlim_{C_1} \varprojlim_{C_2 : C_2 \le C_1} \mathbb{A}_{X_F,1}(C_1)/\mathbb{A}_{X_F,1}(C_2),$$

where $C_*$ are divisors on $X_F$ and we set $\mathbb{A}_{X_F,1}(C_*) = \mathbb{A}_{X_F,1}(O_X(C_*))$. We define the ∞-adelic ring $\mathbb{A}_X^\infty$ for an arithmetic surface $X$ by

$$\mathbb{A}_X^\infty := \mathbb{A}_{X_F} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \varinjlim_{C_1} \varprojlim_{C_2 : C_2 \le C_1} ((\mathbb{A}_{X_F,1}(C_1)/\mathbb{A}_{X_F,1}(C_2)) \otimes_{\mathbb{Q}} \mathbb{R}).$$

[Arithmetic adelic ring]
We define the arithmetic adelic ring $\mathbb{A}_X^{\mathrm{ar}}$ for an arithmetic surface $X$ by

$$\mathbb{A}_X^{\mathrm{ar}} := \mathbb{A}_{X,012}^{\mathrm{ar}} := \mathbb{A}_X^{\mathrm{fin}} \oplus \mathbb{A}_X^\infty.$$

*Remark.* For any divisors $C_1 \ge C_2$ on $X_F$, $\mathbb{A}_{X_F,1}(C_1)/\mathbb{A}_{X_F,1}(C_2)$ is a finite dimensional $F$-vector space, hence a finite dimensional $\mathbb{Q}$-vector space.

*Remark.* To understand meaning of $\widehat{\otimes}$, consider the following example. For the ring of Laurent series $\mathbb{Q}((t))$, we have $\mathbb{Q}((t)) \otimes_{\mathbb{Q}} \mathbb{R} \neq \mathbb{R}((t))$. On the other hand, $\mathbb{Q}((t)) = \varinjlim_{n} \varprojlim_{m : m \le n} t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]$. Since $t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]$ is a finite dimensional $\mathbb{Q}$-vector space, we have

$$\mathbb{Q}((t)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} = \varinjlim_{n} \varprojlim_{m : m \le n} (t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]) \otimes_{\mathbb{Q}} \mathbb{R} = \varinjlim_{n} \varprojlim_{m : m \le n} t^{-n}\mathbb{R}[[t]]/t^{-m}\mathbb{R}[[t]] = \mathbb{R}((t)).$$

Motivated by Osipov-Parshin's constuction, we define ∞-adelic groups for arithmetic varieties as follows. Assume in the sequel that $X$ is an arithmetic variety of dimension $n+1$, unless otherwise stated.

**Theorem 3.2** (∞-adelic groups, Sugahara-Weng [15])**.** *For each subset $K \subset S(X_F)_m$, there exists an additive and exact functor*

$$\mathbb{A}_\infty(K, \cdot) : QC(X_F) \to AbGp$$

*determined uniquely by the following properties* (i), (ii), (iii).

(i) $\mathbb{A}_\infty(K, \cdot)$ *commutes with direct limits.*

(ii) *For $m = 0$ and a coherent sheaf $\mathcal{G}$ on $X_F$,*

$$\mathbb{A}_\infty(K, \mathcal{G}) = \prod_{p \in K} \varprojlim_{l} (\mathcal{G}_p / m_p^l \mathcal{G}_p \otimes_{\mathbb{Q}} \mathbb{R}).$$

(iii) *For $m > 0$ and a coherent sheaf $\mathcal{G}$ on $X_F$,*

$$\mathbb{A}_\infty(K, \mathcal{G}) = \prod_{p \in P(X)} \varprojlim_{l} \mathbb{A}_\infty({}_pK, [\mathcal{G}_p / m_p^l \mathcal{G}_p]_p).$$

We define the arithmetic adelic groups by using ∞-adelic groups and a uniformity condition.

**Definition 3.3** (Arithmetic adelic groups, Sugahara-Weng)**.** Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$, $\mathcal{F}_F$ be the quasi-coherent sheaf on $X_F$ induced by $\mathcal{F}$. Fix an index tuple $(i_0, \cdots, i_m)$ with $0 \le i_0 \le \cdots \le i_m$.
(A) We define $(i_0, \cdots, i_m)$-type finite adelic group $\mathbb{A}^{\mathrm{fin}}_{X,i_0,\cdots,i_m}(\mathcal{F})$ and $(i_0, \cdots, i_m)$-type $\infty$-adelic group $\mathbb{A}^{\infty}_{X,i_0,\cdots,i_m}(\mathcal{F})$ by

$$\mathbb{A}^{\mathrm{fin}}_{X,i_0,\cdots,i_m}(\mathcal{F}) \coloneqq \mathbb{A}_X(K_{X,i_0,\cdots,i_m}, \mathcal{F}), \qquad \mathbb{A}^{\infty}_{X,i_0,\cdots,i_m}(\mathcal{F}) \coloneqq \mathbb{A}_\infty(K_{X_F,i_0,\cdots,i_m}, \mathcal{F}_F),$$

where for $Z = X,\ X_F$, we set

$$K_{Z,i_0,\cdots,i_m} := \left\{ (p_0, \cdots, p_m) \in S(Z)_m \;\middle|\; \operatorname{codim}\overline{\{p_t\}} = i_t \ (0 \le t \le m) \right\}.$$

(B) We define arithmetic adelic groups as follows.

(1) For $i_m = n + 1$, we define $\mathbb{A}_{\mathrm{ar};i_0,i_1,\cdots,i_m}(\mathcal{F})$ by

$$\mathbb{A}^{\mathrm{ar}}_{X;i_0,i_1,\cdots,i_m}(\mathcal{F}) \coloneqq \mathbb{A}^{\mathrm{fin}}_{X;i_0,i_1,\cdots,i_m}(\mathcal{F}) \oplus \mathbb{A}^{\infty}_{X;i_0,i_1,\cdots,i_{m-1}}(\mathcal{F}_F).$$

(2) For $i_m \ne n + 1$, we define $\mathbb{A}^{\mathrm{ar}}_{X;i_0,i_1,\cdots,i_m}(\mathcal{F})$ by

$$\mathbb{A}^{\mathrm{ar}}_{X;i_0,i_1,\cdots,i_m}(\mathcal{F}) \coloneqq \mathbb{A}_X(K^{\mathrm{nh}}_{X,i_0,i_1,\cdots,i_m}, \mathcal{F}) \oplus \mathbb{A}^{\mathrm{fin,inf}}_X(K^{\mathrm{h}}_{X,i_0,i_1,\cdot,i_m}, \mathcal{F}).$$

Here we set

$$K^{\mathrm{h}}_{X,i_0,i_1,\cdots,i_m} \coloneqq \{ (P_{i_0}, \cdots, P_{i_m}) \in K_{X,i_0,i_1,\cdots,i_m} \;|\; (P_{i_0}, \cdots, P_{i_m}) \in S(X_F)_n \}$$

$$K^{\mathrm{nh}}_{X,i_0,i_1,\cdots,i_m} \coloneqq K_{X,i_0,i_1,\cdots,i_m} \backslash K^{\mathrm{h}}_{X,i_0,i_1,\cdots,i_m}$$

and use a uniformity condition to set

$$\mathbb{A}^{\mathrm{fin,inf}}_X(K^{\mathrm{h}}_{X,i_0,i_1,\cdot,i_m}, \mathcal{F}) \coloneqq$$
$$\left\{ (f_{E_{P_{i_0}},\cdots,E_{P_{i_m}}}) \times (f_{P_{i_0},\cdots,P_{i_m}}) \in \mathbb{A}_X(K^{\mathrm{h}}_{X,i_0,i_1,\cdots,i_m}, \mathcal{F}) \oplus \mathbb{A}_{\infty;i_0,i_1,\cdots,i_m}(\mathcal{F}_F) \;\middle|\; f_{E_{P_{i_0}},\cdots,E_{P_{i_m}}} = f_{P_{i_0},\cdots,P_{i_m}} \right\}.$$

(3) For any $m \ge 0$, we define $\mathbb{A}^m_{\mathrm{ar}}(X, \mathcal{F})$ by

$$\mathbb{A}^m_{\mathrm{ar}}(X, \mathcal{F}) \coloneqq \bigoplus_{0 \le i_0 < \cdots < i_m \le \dim X} \mathbb{A}^{\mathrm{ar}}_{X,i_0,\cdots,i_m}(\mathcal{F}).$$

*Remark.* For $m = 0$, $\infty$-adelic groups $\mathbb{A}_\infty(K_{X_F,i_0,\cdots,i_{m-1}}, \mathcal{F}_F)$ makes no sense. To complete above definition, for an open subset $U \subset X$, we induce $(-1)$-simplex $\underline{1}_U$ formally (see [19], § 3.1). We set $S(X_F)_{-1} = S(X_F)^{\mathrm{red}}_{-1} = \{\underline{1}_U \mid U \subset X : \text{open sets}\}$. And for $K \subset S(X_F)_{-1}$, we define $\mathbb{A}_\infty(K, \mathcal{F}_F)$ by

$$\mathbb{A}_\infty(K, \mathcal{F}_F) \coloneqq \begin{cases} \mathcal{F}_F(U_{K,F}) \otimes_{\mathbb{Q}} \mathbb{R} & (\dim X \ge 2) \\ \{ s_\infty \in \mathcal{F}_F(U_{K,F}) \otimes_{\mathbb{Q}} \mathbb{R} \mid s \in \mathcal{F}(U_K) \} & (\dim X = 1) \end{cases}$$

where $U_K = \bigcup_{\underline{1}_U \in K} U$ and $s_\infty$ denotes an element of $\mathcal{F}_F(U_{K,F}) \otimes_{\mathbb{Q}} \mathbb{R}$ corresponding to $s$. The reason for separation of arithmetic curves with others in this latest definition is that arithmetic varieties are relative over arithmetic curves.

**Theorem 3.4** (Sugahara-Weng [15])**.** *Let $X$ be an arithmetic surface. For a Weil divisor $D$ on $X$, arithmetic adelic groups $\mathbb{A}^{\mathrm{ar}}_{X,01}$, $\mathbb{A}^{\mathrm{ar}}_{X,02}$, $\mathbb{A}^{\mathrm{ar}}_{X,12}(D)$, $\mathbb{A}^{\mathrm{ar}}_{X,0}$, $\mathbb{A}^{\mathrm{ar}}_{X,1}(D)$, $\mathbb{A}^{\mathrm{ar}}_{X,2}(D)$ as subgroups of $\mathbb{A}^{\mathrm{ar}}_{X}$ associated to the invertible sheaf $\mathcal{O}_X(D)$ are given as follows:*

(i) $\mathbb{A}^{\mathrm{ar}}_{X,01} = \{(f_{C,x}) \times (f_P) \in \mathbb{A}^{\mathrm{ar}}_X \mid (f_{C,x}) \in \mathbb{A}_{X,01},\, f_P = f_{E_P,x}\ (P \in X_F)\}$,

(ii) $\mathbb{A}^{\mathrm{ar}}_{X,02} = \mathbb{A}_{X,02} \oplus (k(X_F) \otimes_{\mathbb{Q}} \mathbb{R})$,

(iii) $\mathbb{A}^{\mathrm{ar}}_{X,12}(D) = \mathbb{A}_{X,12}(D) \oplus \varprojlim_{D'_F : D'_F \leq D_F} (\mathbb{A}_{X_F,1}(D_F)/\mathbb{A}_{X_F,1}(D'_F) \otimes_{\mathbb{Q}} \mathbb{R})$,
*where $D_F$ denote a divisor on $X_F$ induced by $D$.*

(iv) $\mathbb{A}^{\mathrm{ar}}_{X,0} = k(X) = \mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02}$,

(v) $\mathbb{A}^{\mathrm{ar}}_{X,1}(D) = \{(f_{C,x}) \times (f_P) \in \mathbb{A}^{\mathrm{ar}}_X \mid (f_{C,x}) \in \mathbb{A}_{X,1}(D),\, f_P = f_{E_P,x}\ (P \in X_F)\} = \mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D)$,

(vi) $\mathbb{A}^{\mathrm{ar}}_{X,2}(D) = \{(f_{C,x}) \times (f_P) \in \mathbb{A}^{\mathrm{ar}}_X \mid (f_{C,x}) \in \mathbb{A}_{X,2}(D), (f_P) \in H^0(X_F, D_F) \otimes_{\mathbb{Q}} \mathbb{R}\} = \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D)$.

*Proof.* These are direct consequences of our definition. □

**Corollary 3.5** ([15])**.** *Let $X$ be an arithmetic surface. Then, there are following natural ind-pro structures on the level two subspaces $\mathbb{A}^{\mathrm{ar}}_{X,01}$ and $\mathbb{A}^{\mathrm{ar}}_{X,02}$ of $\mathbb{A}^{\mathrm{ar}}_{X,012}$:*

$$\begin{aligned}\mathbb{A}^{\mathrm{ar}}_{X,01} &= \varinjlim_{D} \varprojlim_{E : E \leq D} \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(E),\\ \mathbb{A}^{\mathrm{ar}}_{X,02} &= \varinjlim_{D} \varprojlim_{E : E \leq D} \mathbb{A}^{\mathrm{ar}}_{X,2}(D)/\mathbb{A}^{\mathrm{ar}}_{X,2}(E).\end{aligned}$$

*Proof.* This follows from the ind-pro structure

$$\mathbb{A}^{\mathrm{ar}}_{X,012} = \varinjlim_{D} \varprojlim_{E : E \leq D} \mathbb{A}^{\mathrm{ar}}_{X,12}(D)/\mathbb{A}^{\mathrm{ar}}_{X,12}(E)$$

and facts that $\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) = \mathbb{A}^{\mathrm{ar}}_{X,1}(D)$, $\mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) = \mathbb{A}^{\mathrm{ar}}_{X,2}(D)$. □

### 3.3 Arithmetic Cohomology Groups

In this subsection, let $\pi : X \to Y$ be an arithmetic surface.

**Definition 3.6** ([15])**.** For $m \geq 0$ and a quasi-coherent sheaf $\mathcal{F}$, we define the boundary maps $d^m_{\mathrm{ar}}$ by

$$d^m_{\mathrm{ar}} : \mathbb{A}^m_{\mathrm{ar}}(X, \mathcal{F}) \to \mathbb{A}^{m+1}_{\mathrm{ar}}(X, \mathcal{F}); \qquad (f_{P_0,\cdots,P_m}) \mapsto (\textstyle\sum_{i=0}^{m+1} (-1)^i f_{Q_0,\cdots,\check{Q}_i,\cdots,Q_{m+1}}).$$

**Proposition 3.7** ([15])**.** *Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. $(\mathbb{A}^*_{\mathrm{ar}}(X, \mathcal{F}), d^*_{\mathrm{ar}})$ forms a complex of abelian groups.*

We call this compex an arithmetic adelic complex.

**Definition 3.8** (Adelic cohomology groups, Sugahara-Weng)**.** Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. For $i \in \{0, 1, \cdots, n+1\}$, we define the $i$-th arithmetic cohomology groups $H^i_{\mathrm{ar}}(X, \mathcal{F})$ of $\mathcal{F}$ by

$$H^i_{\mathrm{ar}}(X, \mathcal{F}) := H^i(\mathbb{A}^*_{\mathrm{ar}}(X, \mathcal{F}), d^*_{\mathrm{ar}}).$$

**Theorem 3.9** (Weng [18])**.** *Let $Y$ be an arithmetic curve. Then for any divisor $D$ on $Y$, we have*

(i) $H^0_{\mathrm{ar}}(Y,D) = \mathbb{A}^{\mathrm{ar}}_{Y,0} \cap \mathbb{A}^{\mathrm{ar}}_{Y,1}(D)$,

(ii) $H^1_{\mathrm{ar}}(Y,D) = \mathbb{A}^{\mathrm{ar}}_{Y,01}/(\mathbb{A}^{\mathrm{ar}}_{Y,0} + \mathbb{A}^{\mathrm{ar}}_{Y,1}(D))$.

*Proof.* To get arithmetic cohomology groups, calculate cohomology groups of an arithetic adelic complex

$$0 \to \mathbb{A}^{\mathrm{ar}}_{Y,0} \oplus \mathbb{A}^{\mathrm{ar}}_{Y,1}(D) \xrightarrow{d^1} \mathbb{A}^{\mathrm{ar}}_{Y,01} \to 0,$$

where

$$d^1 : (a_0, a_1) \mapsto (a_1 - a_0).$$

□

**Theorem 3.10** (Sugahara-Weng [15])**.** *Let $X$ be an arithmetic surface. Then, for a Weil divisor $D$ on $X$, the arithmetic cohomology groups of the invertible sheaf $\mathcal{O}_X(D)$ on $X$ are given by the follows:*

(i) $H^0_{\mathrm{ar}}(X,D) = \mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D)$,

(ii) $H^1_{\mathrm{ar}}(X,D)$
$\simeq ((\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02}) \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))/(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) + \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))$,
$\simeq ((\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D)) \cap \mathbb{A}^{\mathrm{ar}}_{X,02})/(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))$,
$\simeq ((\mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D)) \cap \mathbb{A}^{\mathrm{ar}}_{X,01})/(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))$,

(iii) $H^2_{\mathrm{ar}}(X,D) = \mathbb{A}^{\mathrm{ar}}_{X,012}/(\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D))$.

*Proof.* To get arithmetic cohomology groups, we calculate cohomology groups of the arithmetic adelic complex

$$0 \to \mathbb{A}^{\mathrm{ar}}_{X,0} \oplus \mathbb{A}^{\mathrm{ar}}_{X,1}(D) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D) \xrightarrow{d^1} \mathbb{A}^{\mathrm{ar}}_{X,01} \oplus \mathbb{A}^{\mathrm{ar}}_{X,02} \oplus \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \xrightarrow{d^2} \mathbb{A}^{\mathrm{ar}}_{X,012} \to 0,$$

where

$$d^1 : (a_0, a_1, a_2) \mapsto (a_1 - a_0, a_2 - a_0, a_2 - a_1), \qquad d^2 : (a_{01}, a_{02}, a_{12}) \mapsto a_{12} - a_{02} + a_{01}.$$

□

## 3.4 Inductive Long Exact Sequences

In this subsection, let $\pi : X \to Y$ be an arithmetic surface.
There are two types of irreducible curves on $X$. Namely,

(a) horizontal curves $H$, where $H = \overline{\{P\}}$ for an algebraic point $P$ of $X_F$. In this case we write $H = E_P$.

(b) vertical curves $V$, where $\pi(V) = \{v\}$ consisting of a closed point $v$ on $Y$.

For an irreducible curve $C$ on $X$, we define a map $\varphi$

$$\mathbb{A}^{\mathrm{ar}}_{X,1}(D+C)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D+C)/\mathbb{A}^{\mathrm{ar}}_{X,2}(D) \xrightarrow{\varphi} \mathbb{A}^{\mathrm{ar}}_{X,12}(D+C)/\mathbb{A}^{\mathrm{ar}}_{X,12}(D),$$

where $\varphi : (a_1, a_2) \mapsto a_2 - a_1$.

**Proposition 3.11** ([15])**.** *Let $V \subset X$ be a vertical curve. We have the following long exact sequence of cohomology groups*

$$\begin{aligned} 0 \;&\to\; H^0_{\mathrm{ar}}(X,D) \to H^0_{\mathrm{ar}}(X,D+V) \to H^0(V,(D+V)\mid_V) \\ &\to\; H^1_{\mathrm{ar}}(X,D) \to H^1_{\mathrm{ar}}(X,D+V) \to H^1(V,(D+V)\mid_V) \\ &\to\; H^2_{\mathrm{ar}}(X,D) \to H^2_{\mathrm{ar}}(X,D+V) \to 0. \end{aligned}$$

*Proof.* This is a direct consequence of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{A}^{\mathrm{ar}}_{X,0} \oplus \mathbb{A}^{\mathrm{ar}}_{X,1}(D) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D) \arrow[r, "d_1"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X,01} \oplus \mathbb{A}^{\mathrm{ar}}_{X,02} \oplus \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \arrow[r, "d_2"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X} \arrow[d, equal] \\
\mathbb{A}^{\mathrm{ar}}_{X,0} \oplus \mathbb{A}^{\mathrm{ar}}_{X,1}(D+V) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D+V) \arrow[r, "d_1"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X,01} \oplus \mathbb{A}^{\mathrm{ar}}_{X,02} \oplus \mathbb{A}^{\mathrm{ar}}_{X,12}(D+V) \arrow[r, "d_2"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X} \arrow[d] \\
\mathbb{A}^{\mathrm{ar}}_{X,1}(D+V)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D+V)/\mathbb{A}^{\mathrm{ar}}_{X,2}(D) \arrow[r, "\varphi"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X,12}(D+V)/\mathbb{A}^{\mathrm{ar}}_{X,12}(D) \arrow[r] \arrow[d] & 0 \\
0 & 0.
\end{tikzcd}$$

with exact columns and facts that

$$\begin{aligned} &\mathbb{A}^{\mathrm{ar}}_{X,1}(D+V)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D) \simeq \mathbb{A}_{V,0}, \\ &\mathbb{A}^{\mathrm{ar}}_{X,2}(D+V)/\mathbb{A}^{\mathrm{ar}}_{X,2}(D) \simeq \mathbb{A}_{V,1}((D+V)\mid_V), \\ &\mathbb{A}^{\mathrm{ar}}_{X,12}(D+V)/\mathbb{A}^{\mathrm{ar}}_{X,12}(D) \simeq \mathbb{A}_{V,01}. \end{aligned}$$

□

**Proposition 3.12** ([15])**.** *Let $E_P \subset X$ be a horizontal curve. We have the following long exact sequence of cohomology groups*

$$\begin{aligned} 0 \;&\to\; H^0_{\mathrm{ar}}(X,D) \to H^0_{\mathrm{ar}}(X,D+E_P) \to \mathrm{Ker}\,\varphi \\ &\to\; H^1_{\mathrm{ar}}(X,D) \to H^1_{\mathrm{ar}}(X,D+E_P) \to \mathrm{Coker}\,\varphi \\ &\to\; H^2_{\mathrm{ar}}(X,D) \to H^2_{\mathrm{ar}}(X,D+E_P) \to 0. \end{aligned}$$

*Proof.* This is a direct consequence of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{A}^{\mathrm{ar}}_{X,0} \oplus \mathbb{A}^{\mathrm{ar}}_{X,1}(D) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D) \arrow[r, "d_1"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X,01} \oplus \mathbb{A}^{\mathrm{ar}}_{X,02} \oplus \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \arrow[r, "d_2"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X} \arrow[d, equal] \\
\mathbb{A}^{\mathrm{ar}}_{X,0} \oplus \mathbb{A}^{\mathrm{ar}}_{X,1}(D+E_P) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D+V) \arrow[r, "d_1"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X,01} \oplus \mathbb{A}^{\mathrm{ar}}_{X,02} \oplus \mathbb{A}^{\mathrm{ar}}_{X,12}(D+E_P) \arrow[r, "d_2"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X} \arrow[d] \\
\mathbb{A}^{\mathrm{ar}}_{X,1}(D+E_P)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D) \oplus \mathbb{A}^{\mathrm{ar}}_{X,2}(D+E_P)/\mathbb{A}^{\mathrm{ar}}_{X,2}(D) \arrow[r, "\varphi"] \arrow[d] & \mathbb{A}^{\mathrm{ar}}_{X,12}(D+E_P)/\mathbb{A}^{\mathrm{ar}}_{X,12}(D) \arrow[r] \arrow[d] & 0 \\
0 & 0.
\end{tikzcd}$$

with exact columns. $\square$

*Remark.* Unlike for vertical curves, for horizontal curves, we do not have the group isomorphisms between Ker $\varphi$, resp. Coker $\varphi$, and $H^0_{\mathrm{ar}}(E_P, (D + E_P)\mid_{E_P})$, resp. $H^1_{\mathrm{ar}}(E_P, (D + E_P)\mid_{E_P})$. This is in fact not surprising: different from vertical curves, for the arithmetic cohomology, there is no simple additive law with respect to horizontal curves when count these arithmetic groups: In Arakeloc theory, we have the following formulas.

(i) ([6], V, § 3, Proof of Lem 3.8) For any vertical curve $V$,

$$\chi_{X/Y}(O_X(D+V)) - \chi_{X/Y}(O_X(D)) = \chi(O_X(D+V)\mid_V).$$

(ii) ([6], V, § 3, Theorem 3.4) For any horizontal curve $E_P$,

$$\chi_{X/Y}(O_X(D+E_P)) - \chi_{X/Y}(O_X(D)) = \chi_{E_P/Y}(O_X(D+E_P)\mid_{E_P}) - \frac{1}{2}d_\lambda(E_P),$$

where let $\lambda = \frac{1}{2}g$ be a Neron function resulting from a Green function $g$, $d_\lambda(E_P)$ be a logarithmic $\lambda-$discriminant, $\chi$ be a Euler characteristic (see [6], pp. 3, 21, 99 and 112).

# 4 Residue Pairings

In this section, let $\pi : X \to Y$ be a regular arithmetic surface.

**Definition 4.1** (Morrow ([8], Definition 2.5))**.** Let $(A, m_A)$ be a Noetherian local ring and $N$ be an $A$-module. Then we define the maximal Hausdorff quotient $N^{\mathrm{sep}}$ of $N$ by

$$N^{\mathrm{sep}} := N / \bigcap_{n \geq 1} m_A^n N.$$

Let $F$ be a complete discrete valuation field, $O_F$ be its valuation ring and $K$ be a subfield of $F$ such that $K = \mathrm{Frac}(O_F \cap K)$. Then for the differential module $\Omega_{O_F/K \cap O_F}$ we define the continuous differential module $\Omega^{\mathrm{cts}}_{F/K}$

$$\Omega^{\mathrm{cts}}_{F/K} := \Omega^{\mathrm{sep}}_{O_F/K \cap O_F} \otimes_{O_F} F.$$

**Definition 4.2** (Residue maps: equal characteristic zero, Morrow ([8], § 2.2))**.** Let $N$ be a 2-dimensional local field of equal characteristic zero. Assume that $N$ includes local field $L$. Then the following (1)-(4) hold.

(1) $N$ have a unique coefficient field $k_N$ including $L$,

(2) $k_N/L$ is a finite extension,

(3) $k_N$ is an algebraic closure of $L$ in $N$,

(4) There exists a uniformizer $t \in N$ such that $N \simeq k_N((t))$.

We define residue map $\mathrm{res}_N$ for $N$ by

$$\mathrm{res}_N : \Omega^{\mathrm{cts}}_{N/L} = N dt \to k_N, (\sum_n a_n t^n) dt \mapsto a_{-1}.$$

**Definition 4.3** (Residue maps: mixed characteristic, Morrow ([8], § 2.3))**.** Let $N$ be a 2-dimensional local field of mixed characteristics with the residue field of characteristic $p$. Then the following (1)-(3) hold.

(1) $N$ contains $\mathbb{Q}_p$.

(2) Let $k_N$ be an algebraic closure $\mathbb{Q}_p$ in $N$. $k_N$ corresponds to a coefficient field of $N$.

(3) There exists a 2-dimensional local field $M \subset N$ satisfying conditions (i)-(iv):

  (i) $N/M$ is a finite extension.
  (ii) $\overline{M} = \overline{N}$, where $\overline{M}$ and $\overline{N}$ denote residue fields of $M$ and $N$ respectively.
  (iii) $k_M = k_N$.
  (iv) $M$ is $k_M$-isomorphic to $k_M\{\{t\}\}$, where

$$k_M\{\{t\}\} := \{\textstyle\sum_{n \in \mathbb{Z}} a_n t^n \mid a_n \in k_M; \inf_i v_{k_M}(a_i) > -\infty; a_n \to 0 \text{ as } n \to -\infty\}.$$

We define the residue map $\mathrm{res}_N$ for $N$ by

$$\mathrm{res}_N : \Omega^{\mathrm{cts}}_{N/L} = \Omega^{\mathrm{cts}}_{M/L} \otimes_M N \xrightarrow{\mathrm{Tr}_{N/M}} \Omega^{\mathrm{cts}}_{M/L} \xrightarrow{\mathrm{res}_M} k_M = k_N,$$
$$\mathrm{res}_M : \Omega^{\mathrm{cts}}_{M/L} = M dt \to k_M, (\sum_n a_n t^n) dt \mapsto -a_{-1}.$$

For the finite adelic ring $\mathbb{A}_X^{\mathrm{fin}}$, it is known that $\mathbb{A}_X^{\mathrm{fin}} \subset \prod_{\{P_0,P_1,P_2\}\in S(X)_2^{\mathrm{red}}} \mathbb{A}(\{P_0,P_1,P_2\},O_X)$ and $k(X)_{C,x} := \mathbb{A}(\{P_0,P_1,P_2\},O_X)$ is a finite direct sum of 2-dimensional local fields (see [1], 2; [13], Prop. 1). Hence $\mathbb{A}_X^{\mathrm{fin}}$ is a subgroup of direct product of 2-dimensional local fields.

Similary, for $\infty$-adelic ring $\mathbb{A}_X^{\infty}$, we have a natural inclusion $\mathbb{A}_X^{\infty} \subset \prod_{P\in X_K:\mathrm{closed}} \mathbb{A}(\{\eta,P\},O_{X_F})\,\widehat{\otimes}_{\mathbb{Q}}\,\mathbb{R}$ and $k(X_F)_P\,\widehat{\otimes}_{\mathbb{Q}}\,\mathbb{R} := \mathbb{A}(\{\eta,P\},O_{X_F})\,\widehat{\otimes}_{\mathbb{Q}}\,\mathbb{R}$ is a finite direct sum of fields of the form $\mathbb{R}((t))$ or $\mathbb{C}((t))$. Hence $\mathbb{A}_X^{\infty}$ is a subgroup of direct product of fields of the form $\mathbb{R}((t))$ or $\mathbb{C}((t))$.

From the above, to define residue paring for an arithmetic adelic ring, we can use residue maps for 2-dimensional local fields above and the natural residue maps $\mathrm{res}_{K((t))} : K((t)) \to K;\ \ \sum_n a_n t^n \mapsto a_{-1}$ (where $K$ denotes either $\mathbb{R}$ or $\mathbb{C}$).

**Definition 4.4** (Tate ([16], § 2.2))**.** We define residue maps $\lambda_\infty,\ \lambda_p$ ($p$ a prime) by

$$\begin{aligned} \lambda_\infty : \mathbb{R} \to \mathbb{R}/\mathbb{Z}; &\qquad x \mapsto -x \bmod \mathbb{Z}, \\ \lambda_p : \mathbb{Q}_p \to \mathbb{R}/\mathbb{Z}; &\qquad \sum_n a_n p^n \mapsto \sum_{n<0} a_n p^n \bmod \mathbb{Z}. \end{aligned}$$

*Notation.* Set $\mathrm{Res}_N := \lambda_v \circ \mathrm{Tr}_{k_N/\mathbb{Q}_v} \circ \mathrm{res}_N$. Since $\mathbb{A}(\{P_0,P_1,P_2\},O_X)$ can be written as a finite direct sum $\bigoplus_N N$ of 2-dimensional local fields $N$, we define residue map $\mathrm{Res}_{C,x}$ by $\mathrm{Res}_{C,x} := \sum_N \mathrm{Res}_N$, where $C = \overline{\{P_1\}}$, $x = P_2$.

Similarly, since $\mathbb{A}(\{\eta,P\},O_{X_F})\,\widehat{\otimes}_{\mathbb{Q}}\,\mathbb{R}$ can be written as a finite direct sum fields $\bigoplus_L L$ of fields $L$ of the form $\mathbb{R}((t))$ or $\mathbb{C}((t))$, we define the residue map $\mathrm{Res}_P$ by $\mathrm{Res}_P := \sum_L \mathrm{Res}_L$.

**Definition 4.5** (Residue parings, Sugahara-Weng [15])**.** Let $0 \neq \omega \in \Omega_{k(X)/F}$. We define the global residue pairing $\langle\cdot,\cdot\rangle_\omega$ on the arithmetic adelic ring $\mathbb{A}_X^{\mathrm{ar}}$ by

$$\begin{aligned} \langle\cdot,\cdot\rangle_\omega : \mathbb{A}_X^{\mathrm{ar}} \times \mathbb{A}_X^{\mathrm{ar}} &\to \mathbb{R}/\mathbb{Z}; \\ ((f_{C,x})\times(f_P)),(g_{C,x})\times(g_P)) &\mapsto \sum_{(C,x)} \mathrm{Res}_{C,x}(f_{C,x}g_{C,x}\omega) + \sum_P \mathrm{Res}_P(f_P g_P \omega). \end{aligned}$$

For the global residue pairing, we have the following fundamental result.

**Theorem 4.6** (Non-Degenercy of Residue Pairing, Sugahara-Weng [15])**.** *Let* $0 \neq \omega \in \Omega_{k(X)/F}$. $\langle\cdot,\cdot\rangle_\omega$ *on* $\mathbb{A}_X^{\mathrm{ar}}$ *is a non-degenerate pairing.*

For later use, we also recall the following

**Theorem 4.7** (Residue formulas, Morrow ([8], Thm 4.1; [9], Thm 5.4))**.** *Let* $0 \neq \omega \in \Omega_{k(X)/F}$. *We have*

(i) *For a fixed closed point* $x \in X$,

$$\sum_{C:x\in C} \mathrm{Res}_{C,x}(\omega) = 0.$$

(ii) *For a fixed horizontal curve* $E_P \subset X$,

$$\sum_{x:x\in E_P} \mathrm{Res}_{E_P,x}(\omega) + \sum_P \mathrm{Res}_P(\omega) = 0.$$

(iii) *For a fixed vertical curve* $V \subset X$,

$$\sum_{x:x\in V} \mathrm{Res}_{V,x}(\omega) = 0.$$

# 5 Topological Structures of Adelic Groups

In this section, let $\pi : X \to Y$ be a regular arithmetic surface.

## 5.1 Topological Structures of Arithmetic Adelic Rings

To introduce a natural topology on the adelic ring $\mathbb{A}_X^{\mathrm{ar}}$, we follow [15] using ind-pro structures on our space starting from locally compact topologies. More precisely, it goes as follows.

For all divisors $D \geq E$ on $X$, $\mathbb{A}_X(D)/\mathbb{A}_X(E)$ are locally compact, Hausdorff topological groups. Indeed, setting $D - E = \sum_{i=1}^{s} a_i C_i$ $(a_i \geq 0)$, we have

$$\mathbb{A}_{X,12}(D)/\mathbb{A}_{X,12}(E) = \prod_{i=1}^{s} \mathbb{A}({}_{\eta_{C_i}}K_{12}, [\mathcal{O}_X(D)_{\eta_{C_i}}/m_{\eta_{C_i}}^{a_i}\mathcal{O}_X(D)_{\eta_{C_i}}]_{\eta_{C_i}}).$$

Setting $\mathcal{F} = [\mathcal{O}_X(D)_{\eta_{C_i}}/m_{\eta_{C_i}}^{a_i}\mathcal{O}_X(D)_{\eta_{C_i}}]_{\eta_{C_i}}$, since $\mathcal{F}$ is a quasi-coherent sheaf, we can express $\mathcal{F} = \varinjlim_{j\in J} \mathcal{F}_j$ in terms of direct limit of certain coherent sheaves $\mathcal{F}_i$ $(i \in I)$. Consequently, we have

$$\begin{aligned} \mathbb{A}({}_{\eta_{C_i}}K_{12}, \mathcal{F}) &= \mathbb{A}(\{x \mid x \in C_i\}, \varinjlim_{j\in J} \mathcal{F}_j) \\ &= \varinjlim_{j\in J} \mathbb{A}(\{x \mid x \in C_i\}, \mathcal{F}_j) \\ &= \varinjlim_{j\in J} \prod_{x:x\in C_i} (\varprojlim_{l} \mathcal{F}_{j,x}/m_x^l \mathcal{F}_{j,x}). \end{aligned}$$

Note that $\mathcal{F}_j$ $(j \in J)$ are coherent sheaves and $x$ are closed points. So, $\mathcal{F}_{j,x}/m_x^l\mathcal{F}_{j,x}$ are finite groups. In particular, we can endow all $\mathcal{F}_{j,x}/m_x^l\mathcal{F}_{j,x}$ with discrete and compact topologies. Accordingly, by using product topology, inductive limit topology and projective limit topology, we obtain a natural topology on the space $\mathbb{A}_X^{\mathrm{fin}}(D)/\mathbb{A}_X^{\mathrm{fin}}(E)$. This topology on $\mathbb{A}_X^{\mathrm{fin}}(D)/\mathbb{A}_X^{\mathrm{fin}}(E)$ is well-known to be Hausdorff and locally compact.

Moreover, since we have

$$\mathbb{A}_X^{\mathrm{fin}} = \varinjlim_{D} \varprojlim_{E:E\leq D} \mathbb{A}_{X,12}(D)/\mathbb{A}_{X,12}(E).$$

Again by using projective limit topology first and then inductive limit topology, we obtain a natural topology on $\mathbb{A}_X^{\mathrm{fin}}$.

On the other hand, for all divisors $D \geq E$ on $X_F$, since $\mathbb{A}_{X,1}^{\infty}(D)/\mathbb{A}_{X,1}^{\infty}(E)$ are finite $\mathbb{R}$-vector spaces, $\mathbb{A}_{X,1}^{\infty}(D)/\mathbb{A}_{X,1}^{\infty}(E)$ are locally compact, Hausdorff topological spaces. Furthermore, since

$$\mathbb{A}_X^{\infty} = \varinjlim_{D} \varprojlim_{E:E\leq D} \mathbb{A}_{X,1}^{\infty}(D)/\mathbb{A}_{X,1}^{\infty}(E),$$

similarly, we obtain a natural topology on $\mathbb{A}_X^{\infty}$ by using inductive limit and projective limit.

Recall that $\mathbb{A}_X^{\mathrm{ar}} = \mathbb{A}_X^{\mathrm{fin}} \oplus \mathbb{A}_X^{\infty}$. Using product structure, we obtain a natural topology on our adelic ring $\mathbb{A}_X^{\mathrm{ar}}$, which we call the ind-pro topology.

We may realize the above formal definition of the ind-pro topology on $\mathbb{A}_X^{\mathrm{ar}}$ in a more concrete term following [15]. To explain this, fix a Madunts-Zhukov lifiting

$$h_C : \mathbb{A}_{C,01} \simeq (\prod_{x:x\in C}{}' \mathcal{O}_{C,x})/(\pi_C \prod_{x:x\in C}{}' \mathcal{O}_{C,x}) \xrightarrow{\text{lifiting}} \prod_{x:x\in C}{}' \mathcal{O}_{C,x}$$

(see [7]). Then, following Parshin ([12, 13]),

$$\mathbb{A}_X^{\mathrm{fin}} = \left\{ \Big( \sum_{i_C=-\infty}^{\infty} h_C(a_{i_C})\, \pi_C^{i_C} \Big)_C \in \prod_C \prod_{x:x\in C} k(X)_{C,x} \;\middle|\; \begin{array}{l} a_{i_C} \in \mathbb{A}_{C,01}, \\ a_{i_C} = 0 \text{ for sufficiently small } i_C, \\ \min\{i_C \mid a_{i_C} \neq 0\} \geq 0 \text{ for almost all } C \end{array} \right\}.$$

Moreover, one checks that a fundamental system of open neighborhoods of 0 for the ind-pro topology on $\mathbb{A}_X^{\mathrm{fin}}$ may be described as follows:

$$\left\{ \Big( \sum_{i_C=-\infty}^{\infty} h_C(U_{i_C})\, \pi_C^{i_C} \Big)_C \in \mathbb{A}_X^{\mathrm{fin}} \;\middle|\; \begin{array}{l} U_{i_C} \subset \mathbb{A}_{C,01} \text{ open subsets}, \\ U_{i_C} = \mathbb{A}_{C,01} \text{ for sufficiently large } i_C, \\ \max\{i_C \mid U_{i_C} \neq \mathbb{A}_{C,01}\} < 0 \text{ for almost all } C \end{array} \right\}.$$

## 5.2 Ind-Pro Topological Spaces and Their Duals

For a topological space $T$, denote by $\widehat{T} := \{f : T \to \mathbb{S}^1 \mid \text{continuous}\}$. There is a natural topology on $\widehat{T}$, namely, the compact-open topology generated by open subsets of the form $W(K, U) := \{f \in \widehat{T} \mid f(K) \subset U\}$, where $K \subset T$ are compact, $U \subset \mathbb{S}^1$ are open. We call $\widehat{T}$ the (Pontryagin) dual of $T$.

**Proposition 5.1** ([15]). *Let $\{P_n\}_n$ be a projective system of Hausdorff topological groups with structural maps $\pi_{n,m} : P_m \to P_n$ and $\pi_n : \varprojlim_n P_n \to P_n$. Assume that all $\pi_n$ and $\pi_{n,m}$ are surjective and open, and that for any $n, n'$, there exists an $n''$ such that $n'' \leq n$ and $n'' \leq n'$. Then, as topological groups,*

$$\widehat{\varprojlim_n P_n} \simeq \varinjlim_n \widehat{P_n}.$$

*Proof.* Denote by $\widehat{\pi_{n,m}} : \widehat{P_n} \to \widehat{P_m}$, $f_n \mapsto f_n \circ \pi_{n,m}$, the dual of $\pi_{n,m}$. Then, for an element $\varinjlim_n f_n \in \varinjlim_n \widehat{P_n}$, we have $\widehat{\pi_{n,m}}(f_n) = f_m$, or equivalently, $f_n \circ \pi_{n,m} = f_m$. Hence, if $x = \varprojlim_n x_n \in \varprojlim_n P_n$, we have $f_n(x_n) = f_n(\pi_{n,m}(x_m)) = f_m(x_m)$ for sufficiently small $n \geq m$. Based on this, we define a natural map

$$\varphi : \varinjlim_n \widehat{P_n} \to \widehat{\varprojlim_n P_n}, \qquad \varinjlim_n f_n \mapsto f$$

where $f : \varprojlim_n P_n \to \mathbb{S}^1$, $x = \varprojlim_n x_n \mapsto f_n(x_n)$. From above discussion, $f$ is well-defined.

**Lemma 5.2** ([15]). *Concerning $\varphi$ and $f$, the following holds.*

(1) *$f$ is continuous. In particular, $\varphi$ is well-defined.*

(2) *$\varphi$ is a bijection.*

(3) *$\varphi$ is continuous.*

(4) *$\varphi$ is open.*

*Proof.* (1) For sufficiently small $n$, $f = f_n \circ \pi_n$. Since $f_n$ and $\pi_n$ is continuous, $f$ is continuous, and hence $\varphi$ is well-defined.

(2) To prove that $\varphi$ is injective, we assume that $\varphi(\varinjlim_n f_n) = 0$. Thus $f_n(x_n) = 0$ for sufficiently small $n$ and for all $\varprojlim_n x_n \in \varprojlim_n P_n$. Note that $\pi_n$ are surjective. So $f_n(x_n) = 0$ for all $x_n \in P_n$. This means $f_n = 0$ for sufficiently small $n$. Consequently, $\varprojlim_n f_n = 0$, and hence $\varphi$ is injective.

To show that $\varphi$ is surjective, let $f : \varprojlim_n P_n \to \mathbb{S}^1$ be a continuous map. Then, for any open subset $U \subset \mathbb{S}^1$ containing 1, $f^{-1}(U)$ is an open neighborhood of 0 in $\varprojlim_n P_n$. Hence, we can write $f^{-1}(U)$ as $f^{-1}(U) = \varprojlim_n P_n \cap \prod_n K_n$ where $K_n \subset P_n$ are open subsets and $K_n = P_n$ for almost all $n$. By our assumptions, for $n_1, \cdots, n_r$ such that $K_{n_i} \subsetneq P_n$, there exists an $N$ such that $N \le n_i$. Hence, $f(\mathrm{Ker}\, \pi_N) = 1$. So, $f(\mathrm{Ker}\, \pi_n) = 1$ for all $n \le N$. Built on this, we define, for $n \le N$, the maps $f_n : P_n \to \mathbb{S}^1, x_n \mapsto f(x)$ if $\pi_n(x) = x_n$. Note that $f(x)$ always make sense, since $\pi_n$ is surjective. Moreover, $f_n$'s are well-defined. Indeed, if $y \in \varprojlim_n P_n$ such that $\pi_n(y) = x_n$, then $\pi_n(y) = x_n = \pi_n(x)$ for $n \le N$. Hence $x - y \in \mathrm{Ker}\, \pi_n$. This implies that $f(y) = f(x)$. Clearly, by definition, $\varphi(\varinjlim_n f_n) = f$. So $\varphi$ is surjecitive.

(3) To prove that $\varphi$ is continuous, it suffices to show that for open subsets of $\widehat{\varprojlim_n P_n}$ in the form $W(K, V)$, $\varphi^{-1}(W(K, V))$ is open in $\varinjlim_n \widehat{P_n}$, where $K$ is a compact subset of $\varprojlim_{P_n}$ and $V$ is open subset of $\mathbb{S}^1$. Since $\pi_n$ are continuous, $K_n = \pi_n(K)$ are compact. In this way, we get an inductive system of open subsets $\{W(K_n, V)\}_n$. Set $U = \varinjlim_n W(K_n, V)$. Note that we have $f(U) = W(K, V)$. This shows that $f$ is continuous.

(4) To prove that $\varphi$ is open, let $U$ be an open subset of $\varinjlim_n \widehat{P_n}$ such that $\iota^{-1}(U) = W(K_n, V)$ for a compact subset $K_n$ of $P_n$ for any $n$ where $\iota_n : \widehat{P_n} \to \varinjlim_n \widehat{P_n}$. $K := \varprojlim_n K_n$ is compact in $\varprojlim_n P_n$. Consequently, $W(K, V)$ is open in $\widehat{\varprojlim_n P_n}$. Note that we have $\varphi(U) = W(K, V)$. So $\varphi$ is open. This proves the lemma.

$\square$

Clearly, our proposition is a direct consequence of the lemma above. Hence, the proposition is proven. $\square$

Let $\{D_n\}_n$ be an inductive system of Hausdorff topological groups such that all $\iota_{n,m}$ are injective and closed. By definition, $\{D_n\}_n$ is called compact oriented, if, for any compact subset $K \subset \varinjlim_n D_n$, there exists an index $n_0$ such that $K \subset D_{n_0}$.

**Proposition 5.3** ([15])**.** *Let $\{D_n\}_n$ be a compact oriented inductive system of topological groups with structural maps $\iota_{n,m} : D_n \to D_m$ and $\iota_n : D_n \to \varinjlim_n D_n$. Assume (that all $\iota_{n,m}$ are injective and closed, and) that for any $n, n'$, there exists an $n''$ such that $n'' \ge n$ and $n'' \ge n'$. Then, as topological groups,*

$$\widehat{\varinjlim_n D_n} \simeq \varprojlim_n \widehat{D_n}.$$

*Proof.* Denote by $\widehat{\iota_{n,m}} : \widehat{D_m} \to \widehat{D_n}$, $f_m \mapsto f_m \circ \iota_{n,m}$, the dual of $\iota_{n,m}$. Then, for $\varprojlim_n f_n \in \varprojlim_n \widehat{D_n}$, we have $\widehat{\iota_{n,m}}(f_m) = f_n$, or equivalently, $f_m \circ \iota_{n,m} = f_n$. Hence, if $x = \varinjlim_n x_n \in \varinjlim_n D_n$, $f_m(x_m) = f_m(\iota_{n,m}(x_n)) = f_n(x_n)$ for sufficiently large $n \geq m$. Based on this, we define a natural map

$$\psi : \varprojlim_n \widehat{D_n} \to \widehat{\varinjlim_n D_n}, \qquad \varprojlim_n f_n \mapsto f$$

where $f : \varinjlim_n D_n \to \mathbb{S}^1, \varinjlim_n x_n \mapsto f_n(x_n)$. From above discussion, $f$ is well-defined.

**Lemma 5.4** ([15])**.** *For $\psi$ and $f$, the following holds.*

(1) *$f$ is continuous. In particular, $\psi$ is well-defined.*

(2) *$\psi$ is a bijection.*

(3) *$\psi$ is continuous.*

(4) *$\psi$ is open.*

*Proof.* (1) To prove that $f$ is continuous, let $U \subset \mathbb{S}^1$ be an open subset. For any $n$, $\iota_n^{-1}(f^{-1}(U)) = f_n^{-1}(U)$ is open in $D_n$. Hence $f^{-1}(U)$ is open. So $f$ is continuous.

(2) To prove that $\psi$ is injective, we assume that $\psi(\varprojlim_n f_n) = 0$. Then $f(x_n) = f_n(x_n) = 0$ for all $x_n \in D_n$. This means that $f_n = 0$ for any $n$. Consequently, $\varinjlim_n f_n = 0$, and hence $\psi$ is injective.

To show that $\psi$ is surjective, let $f : \varinjlim_n D_n \to \mathbb{S}^1$ be a continuous map. Set $f_n = f \circ \iota_n$. Clearly, $f_n$ is continuous. So $f_n \in \widehat{D_n}$. Moreover, for all $n' \geq n$, $f_n = f \circ \iota_n = f \circ \iota_{n'} \circ \iota_{n,n'} = f_{n'} \circ \iota_{n,n'}$. That is, $\{f_n\}_n$ forms a projective limit. Obviously, $\psi(\varprojlim_n f_n) = f$.

(3) To prove that $\psi$ is continuous, it suffices to show that for open subsets of $\widehat{\varinjlim_n D_n}$ in the form $W(K, V)$, $\psi^{-1}(W(K, V))$ is open in $\varprojlim_n \widehat{D_n}$, where $K$ is a compact subset of $\varinjlim_n D_n$ and $V$ is an open subset of $\mathbb{S}^1$. By our assumptions, for any $n'$, $\iota_{n'}^{-1}(D_n)$ is closed. Hence $D_n \subset \varinjlim_n D_n$ are closed. So $K_n := K \cap D_n$ are compact. Since $\psi^{-1}(W(K, V)) = \varprojlim_n W(K_n, V)$, it suffices to show that $\varprojlim_n W(K_n, V)$ is open. This is a consequence of our assumptions. Indeed, since our system is compact oriented, there exists a certain $n_0$ such that $K = \varinjlim_n K_n \subset D_{n_0}$. Hence, $K_n = K_{n_0} = K$ for all $n \geq n_0$. $\varprojlim_n W(K_n, V) = \pi_{n_0}^{-1}(W(K_{n_0}, V))$. So, $\varprojlim_n W(K_n, V)$ is open.

(4) To prove that $\psi$ is open, let $U$ be an open subset of $\varprojlim_n \widehat{D_n}$ such that $U = \pi_n^{-1}(W(K_n, V))$ for a compact subset $K_n$ of $P_n$ for some $n$, where $\pi_n : \varinjlim_n \widehat{D_n} \to \widehat{D_n}$. Then $W(K_n, V)$ is open in $\widehat{\varinjlim_n D_n}$. Note that we have $\psi(U) = W(K_n, V)$. So $\psi$ is open. This proves the lemma.

□

Clearly, our proposition is a direct consequence of the lemma above. Hence, the proposition is proven. □

## 5.3 Completeness and Compact Orientedness of Arithmetic Adelic Groups

For basis of complete topological groups, please refer to [2] and [3].

In the sequel, we use simply $\mathbb{A}$ to denote arithmetic adelic groups $\mathbb{A}_X^{\mathrm{fin}}$, or $\mathbb{A}_{X,01}$, or $\mathbb{A}_{X,02}$ or $\mathbb{A}_X^\infty$. Similarly, we use $\mathbb{A}(D)$ to denote arithmetic adelic groups $\mathbb{A}_{X,12}(D)$, or $\mathbb{A}_{X,1}(D)$, or $\mathbb{A}_{X,2}(D)$ or $\mathbb{A}_{X,1}^\infty(D)$. When $\mathbb{A}$ and/or $\mathbb{A}(D)$ represent what will be clear according to the text involved below.

**Proposition 5.5** ([15])**.** *The subgroups $\mathbb{A}(D)$ of $\mathbb{A}$ are complete and hence closed.*

*Proof.* Since $\mathbb{A}(D)/\mathbb{A}(E)$ are complete Hausdorff locally compact, as a projective limit of complete spaces, $\mathbb{A}(D) = \varprojlim_{E:E\leq D} \mathbb{A}(D)/\mathbb{A}(E)$ is complete. It is also closed since $\mathbb{A}$ is Hausdorff. □

**Lemma 5.6** ([15])**.** *Let $\{\mathbb{A}(D_n)\}_n$ be a strictly increasing sequence and $\{a_n\}_n$ be a sequence of elements of $\mathbb{A}$. Assume that $a_n \in \mathbb{A}(D_n) - \mathbb{A}(D_{n-1})$ for all $n \geq 1$. Then there exists an open subset $U$ of $\varinjlim_n \mathbb{A}(D_n)$ such that $a_1, \cdots, a_n, \cdots \notin U$ and $a_{m+1}, \cdots, a_n, \cdots \notin U + \mathbb{A}(D_m)$ for all $m < n$.*

*Proof.* We separate the finite and infinite adeles.

[Finite Adeles] Since $\mathbb{A}(D_1)/\mathbb{A}(D_0)$ is Hausdorff, there exists an open, and hence closed, subgroup $U_1 \subset \mathbb{A}(D_1)$ such that $a_1 \notin U_1$ and $U_1 \supset \mathbb{A}(D_0)$. Since $\mathbb{A}(D_1)$ is complete and $U_1$ is closed in $\mathbb{A}(D_1)$, $U_1$ is complete as well. Now, viewing in $\mathbb{A}(D_2)$, since $\mathbb{A}(D_2)$ is Hausdorff, $U_1$ is a complete subgroup, so $U_1$ is closed in $\mathbb{A}(D_2)$. Hence $\mathbb{A}(D_2)/U_1$ is Hausdorff too. Therefore, there exists an open and hence closed subgroup $V_{2,0}$ of $\mathbb{A}(D_2)$ such that $a_1, a_2 \notin V_{2,0}$ and $V_{2,0} \supset U_1$. In addition, $\mathbb{A}(D_2)/\mathbb{A}(D_1)$ is Hausdorff, there exists an open subgroup $V_{2,1}$ such that $a_2 \notin V_{2,1}$ and $V_{2,1} \supset \mathbb{A}(D_2)$. Consequently, if we set $U_2 = V_{2,0} \cap V_{2,1}$, $U_2$ is an open hence closed subgroup of $\mathbb{A}(D_2)$ such that $a_1$, $a_2 \notin U_2$, $a_2 \notin U_2 + \mathbb{A}(D_1)$ and $U_2 \supset U_1$. So, inductively, we may assume that there exists an increasing sequence of open subgroups $U_1, \cdots, U_{n-1}$ satisfying the properties required. In particular, the following quotient groups

$$\mathbb{A}(D_n)/U_{n-1} + \mathbb{A}(D_0)\ (= \mathbb{A}(D_n)/U_{n-1}), \cdots, \mathbb{A}(D_n)/U_{n-1} + \mathbb{A}(D_{n-1})\ (= \mathbb{A}(D_{n-1}))$$

are Hausdorff. Hence there are open subgroups $V_{n,m}, 0 \leq m \leq n-1$ of $\mathbb{A}(D_n)$ such that $a_{m+1}, \cdots, a_n \notin V_{n,m}$ and $V_{n,m} \supset U_{n-1} + \mathbb{A}(D_m)$. Define $U_n := \bigcap_{m=1}^{n-1} V_{n,m}$. Then $U_n$ is an open subgroup of $\mathbb{A}(D_n)$ satisfying $a_1, \cdots, a_n \notin U_n$, $a_{m+1}, \cdots, a_n \notin U_n + \mathbb{A}(D_m), 1 \leq m \leq n-1$ and $U_n \supset U_{n-1}$. Accordingly, if we let $U = \varinjlim_n U_n$, by definition, $U$ is an open subgroup of $\varinjlim_n \mathbb{A}(D_n)$, and from our construction, $a_1, \cdots, a_n, \cdots \notin U$ and $a_{m+1}, \cdots, a_n, \cdots \notin U + \mathbb{A}(D_m), m \geq 1$.

[Infinite Adeles] Since $\mathbb{A}(D_1)$ is Hausdorff, there exists an open subset $U_1$ of $\mathbb{A}(D_1)$ such that $a_1 \notin \mathbb{A}(D_1)$. Moreover, since $\mathbb{A}(D_2) \simeq \mathbb{A}(D_2)/\mathbb{A}(D_1) \oplus \mathbb{A}(D_1)$ and $\mathbb{A}(D_2)/\mathbb{A}(D_1)$ is Hausdorff, there exists an open subset $U_2$ of $\mathbb{A}(D_2)$ such that $a_1, a_2 \notin U_2$ and $U_2 \cap \mathbb{A}(D_1) = U_1$. In particular, $a_2 \notin U_2 + \mathbb{A}(D_1)$. Similarly, as above, with an inductive process, based on the fact that $\mathbb{A}(D_n) \simeq \mathbb{A}(D_n)/\mathbb{A}(D_{n-1}) \oplus \mathbb{A}(D_{n-1})$ and $\mathbb{A}(D_n)/\mathbb{A}(D_{n-1})$ is Hausdorff, there exists an open subset $U_n$ of $\mathbb{A}(D_n)$ such that $a_1, \cdots, a_n \notin U_n$ and $U_n \cap \mathbb{A}(D_n) = U_{n-1}$. Consequently, $a_{m+1}, \cdots, a_n \notin U_n + \mathbb{A}(D_m), 1 \leq m \leq n-1$. In this way, we obtain an infinite increasing sequence of open subsets $U_n$. Let $U = \varinjlim_n U_n$. Then we have $a_1, \cdots, a_n, \cdots \notin U$ and $a_{m+1}, \cdots, a_n, \cdots \notin U + \mathbb{A}(D_m), m \geq 1$. This proves the lemma. □

**Proposition 5.7** ([15])**.** *$\mathbb{A}$ is complete.*

*Proof.* Let $\{a_n\}_n$ be a Cauchy sequence of $\mathbb{A}$. We will show that these exists a divisor $D$ such that $\{a_n\}_n \subset \mathbb{A}(D)$. Assume that, on the contrary, for all divisors $D$, $\{a_n\}_n \subset \mathbb{A}(D)$. Then there exists a subsequence $\{a_{k_n}\}_n$ of $\{a_n\}_n$, a (strictly increasing) sequence $\{D_n\}_n$, and an open neighborhood $U$ of 0 in $\varinjlim_n \mathbb{A}(D_n)$ such that

(i) $a_{k_n} \in \mathbb{A}(D_n) - \mathbb{A}(D_{n-1})$ $(n \geq 2)$,

(ii) $a_{k_1}, \cdots, a_{k_n}, \cdots \notin U$,

(iii) $a_{k_{i+1}}, \cdots, a_{k_n}, \cdots \notin U + \mathbb{A}(D_i)$ $(i \geq 1)$.

Since $\{a_{k_n}\}_n$ is not a Cauchy sequence of $\varinjlim_n \mathbb{A}(D_n)$, we the get a contradiction. Therefore, there exists a divisor $D$ such that $\{a_n\} \subset \mathbb{A}(D)$. But $\mathbb{A}(D)$ is complete, the Cauchy sequence $\{a_n\}_n$ is convergent in $\mathbb{A}(D)$. □

**Proposition 5.8** ([15])**.** *$\mathbb{A}$ is compact oriented.*

*Proof.* Assume that for all divisors $D$, $K \not\subset \mathbb{A}(D)$. Then there exists a sequence $\{a_n\}_n$ in $K$, a (strictly increasing) sequence $\{D_n\}_n$, and an open neighborhood $U$ of 0 in $\varinjlim_n \mathbb{A}(D_n)$ such that

(i) $a_n \in \mathbb{A}(D_n) - \mathbb{A}(D_{n-1})$ $(n \geq 2)$,

(ii) $a_1, \cdots, a_n, \cdots \notin U$,

(iii) $a_{i+1}, \cdots, a_n, \cdots \notin U + \mathbb{A}(D_i)$ $(i \geq 1)$.

$\{U + \mathbb{A}(D_n)\}_n$ is a open covering of $K \cap \varinjlim_n \mathbb{A}(D_n)$ and admits no finite sub-covering. On the other hand, $K \cap \varinjlim_n \mathbb{A}(D_n)$ is compact, a contradiction. Indeed, since $\varinjlim_n \mathbb{A}(D_n)$ is complete and $\mathbb{A}$ is Hausdorff, $\varinjlim_n \mathbb{A}(D_n)$ is closed in $\mathbb{A}$ and hence $K \cap \varinjlim_n \mathbb{A}(D_n)$ is compact. This completes the proof. □

### 5.4 Double Dual of Arithmetic Adelic Rings

**Proposition 5.9** ([15])**.** *As topological groups, we have the following isomorphisms.*

(i) $\widehat{\mathbb{A}} \simeq \varprojlim_D \varinjlim_{E:E\leq D} \widehat{(\mathbb{A}(D)/\mathbb{A}(E))}$,

(ii) $\widehat{\widehat{\mathbb{A}}} \simeq \mathbb{A}$.

*Proof.* (i)

$$\begin{aligned}
\widehat{\mathbb{A}} &\simeq \widehat{(\varinjlim_D \varprojlim_{E:E\leq D} \mathbb{A}(D)/\mathbb{A}(E))} \\
&\simeq \widehat{\varprojlim_D( \varprojlim_{E:E\leq D} \mathbb{A}(D)/\mathbb{A}(E))} \\
&\simeq \varprojlim_D \varinjlim_{E:E\leq D} \widehat{(\mathbb{A}(D)/\mathbb{A}(E))}.
\end{aligned}$$

(ii) Remarking that $(\mathbb{A}(D)/\mathbb{A}(E))\widehat{\phantom{x}}\widehat{\phantom{x}} \simeq \mathbb{A}(D)/\mathbb{A}(E)$ since $\mathbb{A}(D)/\mathbb{A}(E)$ are Hausdorff locally compact groups,

$$\begin{aligned}
\widehat{\widehat{\mathbb{A}}} &\simeq (\varprojlim_{D} \varinjlim_{E:E\le D} (\mathbb{A}(D)/\mathbb{A}(E))\widehat{\phantom{x}})\widehat{\phantom{x}} \\
&\simeq \varinjlim_{D}( \varinjlim_{E:E\le D} (\mathbb{A}(D)/\mathbb{A}(E))\widehat{\phantom{x}})\widehat{\phantom{x}} \\
&\simeq \varinjlim_{D} \varinjlim_{E:E\le D} (\mathbb{A}(D)/\mathbb{A}(E))\widehat{\widehat{\phantom{x}}} \\
&\simeq \varinjlim_{D} \varinjlim_{E:E\le D} \mathbb{A}(D)/\mathbb{A}(E) = \mathbb{A}.
\end{aligned}$$

□

## 5.5 Continuity of Scalar Product

**Proposition 5.10** ([15]). *For a fixed element $a$ of $\mathbb{A}_X^{\mathrm{fin}}$, resp. of $\mathbb{A}_X^{\infty}$, the induced scalar product map: $\phi_a^{\mathrm{fin}} : \mathbb{A}_X^{\mathrm{fin}} \xrightarrow{\times a} \mathbb{A}_X^{\mathrm{fin}}$, resp. $\phi_a^{\infty} : \mathbb{A}_X^{\infty} \xrightarrow{\times a} \mathbb{A}_X^{\infty}$ is continuous.*

*Proof.* If $a = 0$, there is nothing to prove. Assume, from now on, that

$$0 \neq a = (a_C) = (\sum_{i_C=i_{C,0}}^{\infty} h_C(a_{i_C})\, \pi_C^{i_C}) \in \mathbb{A}_X^{\mathrm{fin}}.$$

Here, for each $C$, we assume that $a_{i_{C,0}} \neq 0$. To prove that $\phi_a^{\mathrm{fin}}$ is continuous, it suffices to show that for an open subgroup

$$U = (U_C) = (\sum_{j_C=-\infty}^{r_C-1} h_C(\mathbb{A}_{C,1}(D_{j_C}))\, \pi_C^{j_C} + \sum_{j_C=r_C}^{\infty} h_C(\mathbb{A}_{C,01})\, \pi_C^{j_C}) \cap \mathbb{A}_X^{\mathrm{fin}},$$

as an open neighborhood of 0, its inverse image $(\phi_a^{\mathrm{fin}})^{-1}(U)$ contains an open subgroup. For later use, set $I_C := r_C - i_{C,0}$.

Let

$$b = (b_C) = (\sum_{k_C=-\infty}^{\infty} h_C(b_{k_C})\, \pi_C^{k_C}) \in (\phi_a^{\mathrm{fin}})^{-1}(U) \subset \mathbb{A}_X^{\mathrm{fin}}.$$

Then, for each fixed $C$,

$$a_C b_C = \sum_{l_C=-\infty}^{\infty} (\sum_{i_C=i_{C,0}}^{\infty} h_C(a_{i_C}) h_C(b_{l_C-i_C}))\, \pi_C^{l_C}.$$

Recall that $h_C$ is the Madunts-Zhukov lifting map

$$h_C : \mathbb{A}_{C,01} \simeq (\prod_{x:x\in C}{}' \mathcal{O}_{C,x})/(\pi_C \prod_{x:x\in C}{}' \mathcal{O}_{C,x}) \xrightarrow{\text{lifiting}} \prod_{x:x\in C}{}' \mathcal{O}_{C,x}.$$

Thus if $b_{k_C} \in \mathbb{A}_{C,01}$, we always have

$$h_C(a_{i_C}) h_C(b_{l_C-i_C}) \in \sum_{m_C=0}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{m_C}.$$

Moreover, if we write, as we can, $a_{i_C} \in \mathbb{A}_{C,1}(F_{i_C}), b_{k_C} \in \mathbb{A}_{C,1}(E_{k_C})$ for some divisors $F_{i_C}$ and $E_{k_C}$, we have

$$h_C(a_{i_C})h_C(b_{l_C-i_C}) \in \sum_{m_C=0}^{\infty} h_C(\mathbb{A}_{C,1}(F_{i_C}+E_{l_C-i_C}))\,\pi_C^{m_C}.$$

Now write

$$b_C = (\sum_{k_C=-\infty}^{I_C-1} + \sum_{k_C=I_C}^{\infty})h_C(b_{k_C})\,\pi_C^{k_C}.$$

We will construct the required open subgroup according to the range of the degree index $k_C$.

(i) If

$$b_C \in (\sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01})\,\pi_C^{k_C}) \cap (\prod_{x:x\in C}{}' k(X)_{C,x}),$$

we have $a_C b_C \in U_C$.

(ii) To extend the range including to also the degree $I_C-1$, choose a divisor $E_{I_C-1}$ such that

$$h_C(\mathbb{A}_{C,1}(F_{i_{C,0}}+E_{I_C-1})) \subset h_C(\mathbb{A}_{C,1}(D_{r_C-1})).$$

Then if we choose

$$b_C \in (h_C(\mathbb{A}_{C,1}(E_{I_C-1})\,\pi_C^{I_C-1} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01})\,\pi_C^{k_C}) \cap (\prod_{x:x\in C}{}' k(X)_{C,x}),$$

we also have $a_C b_C \in U_C$.

(iii) Similarly, to extend the range including the degree $I_C-2$, choose a divisor $E_{I_C-2}$ such that

$$\begin{aligned}&h_C(\mathbb{A}_{C,1}(F_{i_{C,0}}+E_{I_C-2})) \subset h_C(\mathbb{A}_{C,1}(D_{r_C-2})) \cap h_C(\mathbb{A}_{C,1}(D_{r_C-1}))\\ &h_C(\mathbb{A}_{C,1}(F_{i_{C,0}+1}+E_{I_C-2})) \subset h_C(\mathbb{A}_{C,1}(D_{r_C-1})).\end{aligned}$$

Then, if we choose

$$b_C \in (\sum_{k_C=I_C-2}^{I_C-1} h_C(\mathbb{A}_{C,1}(E_{k_C}))\,\pi_C^{k_C} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01})\,\pi_C^{k_C}) \cap (\prod_{x:x\in C}{}' k(X)_{C,x}),$$

then we have $a_C b_C \in U_C$.

Continuing this process repeatedly, we obtain divisors $E_{k_C}$'s such that, for

$$b_C \in V_C := (\sum_{k_C=-\infty}^{\mathrm{I}_C-1} h_C(\mathbb{A}_{C,1}(E_{k_C}))\,\pi_C^{k_C} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01})\,\pi_C^{k_C}) \cap (\prod_{x:x\in C}{}' k(X)_{C,x}),$$

we have $a_C b_C \in U_C$.

Since, for all but finitely many $C$, $r_C \leq 0$ and $i_{C,0} \geq 0$, or better, $I_C \leq 0$. Therefore, from above discussions, we conclude that $\prod_C V_C \cap \mathbb{A}_X^{\mathrm{fin}}$ is an open subgroup of $\mathbb{A}_X^{\mathrm{fin}}$ and $a(\prod_C V_C \cap \mathbb{A}_X^{\mathrm{fin}}) \subset U$. In particular, $\phi_a^{\mathrm{fin}}$ is continuous.

A similar proof works for $\phi_a^{\infty}$. □

## 5.6 Continuity of Residue Maps

Fix a non-zero rational differential $\omega$ on $X$. Then for an element $a$ of $\mathbb{A}_X^{\mathrm{fin}}$, resp. $\mathbb{A}_X^{\infty}$, induced from the natural residue pairing $\langle\cdot,\cdot\rangle_\omega$, we get a natural map $\phi_a^{\mathrm{fin}} := \langle a,\cdot\rangle_\omega : \mathbb{A}_X^{\mathrm{fin}} \to \mathbb{R}/\mathbb{Z}$, resp. $\phi_a^{\infty} := \langle a,\cdot\rangle_\omega : \mathbb{A}_X^{\infty} \to \mathbb{R}/\mathbb{Z}$.

**Lemma 5.11** ([15])**.** *Let $a$ be a fixed element in $\mathbb{A}_X^{\mathrm{fin}}$, resp. $\mathbb{A}_X^{\infty}$. Then $\phi_a^{\infty} := \langle a,\cdot\rangle_\omega : \mathbb{A}_X^{\infty} \to \mathbb{R}/\mathbb{Z}$ is continuous. In particular, the residue map on arithmetic adeles $\mathbb{A}_X^{\mathrm{ar}}$ is continuous.*

*Proof.* We prove only for $\phi_a^{\mathrm{fin}}$, as a similar proof works $\phi_a^{\infty}$. Write $\mathbb{A}_X^{\mathrm{fin}} = \prod_F{}' F$ where the product ranges over 2-dimensional local fields. And, for each 2-dimensional local field $F$, fix an element $t_F$ of $F$ such that for equal characteristic field $F$, $t_F$ is a uniformizer of $F$, while for mixed characteristic field $F$, $t_F$ is a lift of a uniformizer of its residue field. Since the scalar product is continuous, to prove the continuity of $\langle a,\cdot\rangle_\omega$ , it suffices to show that the residue map $\mathrm{Res} : \mathbb{A}_X^{\mathrm{fin}} \to \mathbb{R}/\mathbb{Z}$, $(x_F) \mapsto \sum_F \mathrm{res}_F(x_F dt_F)$ is continuous. (Note that, by the definition of $\mathbb{A}_X^{\mathrm{fin}}$, the above summation is a finite sum.) Since the open subgroup

$$\Big(\sum_{i_C=-\infty}^{-1} h_C(\mathbb{A}_{C,1}(0))\,\pi_C^{i_C} + \sum_{i_C=0}^{\infty} h_C(\mathbb{A}_{C,01})\,\pi_C^{i_C}\Big) \cap \mathbb{A}_X^{\mathrm{fin}}$$

is contained, the kernel of the residue map is an open subgroup. This proves the lemma. □

## 5.7 Self-Duality of Arithmetic Adelic Rings

We will treat both $\mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_X^{\infty}$ simultaneously. So as before, we use $\mathbb{A}$ to represent them.

Recall that, for a fixed $a \in \mathbb{A}$, the map $\langle a,\cdot\rangle_\omega : \mathbb{A} \to \mathbb{S}^1$ is continuous. Accordingly, we define a map $\varphi : \mathbb{A} \to \widehat{\mathbb{A}}$, $a \mapsto \varphi_a := \langle a,\cdot\rangle_\omega$.

**Proposition 5.12** ([15])**.** *For the map $\varphi : \mathbb{A} \to \widehat{\mathbb{A}}$, $a \mapsto \varphi_a := \langle a,\cdot\rangle_\omega$, we have the follows.*

(1) *$\varphi$ is continuous.*

(2) *$\varphi$ is injective.*

(3) *The image of $\varphi$ is dense.*

(4) *$\varphi$ is open.*

*Proof.* (1) For an open subset $W(K,V)$ of $\widehat{\mathbb{A}}$, where $K$ is a compact subset of $\mathbb{A}$ and $V$ is an open subset of $\mathbb{S}^1$, let $U := \varphi^{-1}(W(K,V))$. Since $\widehat{\mathbb{A}} = \varprojlim_D \varinjlim_{E:E\le D} (\mathbb{A}(D)/\mathbb{A}(E))\widehat{\ }$, we may write $\chi_0 := \langle 1,\cdot\rangle_\omega$ as $\varprojlim_D \varinjlim_{E:E\le D} \chi_{D/E}$ with $\chi_{D/E} \in (\mathbb{A}(D)/\mathbb{A}(E))\widehat{\ }$. Accordingly, write

$$\begin{aligned} A_{D/E} &:= \mathbb{A}(D)/\mathbb{A}(E),\\ K_{D/E} &:= K\cap\mathbb{A}(D)/K\cap\mathbb{A}(E),\\ U_{D/E} &:= \{a_{D/E}\in A_{D/E} \mid \chi_{D/E}(a_{D/E}K_{D/E}) \subset V\ (V:\text{open})\}. \end{aligned}$$

Since, for a fixed divisor $D$, $\mathbb{A}(D)$ is closed in $\mathbb{A}$, $K\cap\mathbb{A}(D)$ is a compact subset. So, for $E \le D$, $K_{D/E}$ is compact in $A_{D/E}$. Consequently, from the non-degeneracy of $\chi_{D/E}$ on locally compact spaces, $U_{D/E}$ is an open subset of $\mathbb{A}$, and $U := \varinjlim_D \varprojlim_{E:E\le D} U_{D/E}$. We claim that $U$ is open. Indeed, since $\mathbb{A}$ is compact oriented, for compact $K$, there exists a divisor $D_1$ such that

$K \subset \mathbb{A}(D_1)$. On the other hand, since $\chi_0$ is continuous, there exists a divisor $D_2$ such that $\mathbb{A}(D_1 + D_2) \subset \mathrm{Ker}(\chi_0)$. Hence $U \supset \mathbb{A}(D_2)$. Thus, for a fixed $D$, with respect to sufficiently small $E \leq D$, we have $U_{D/E} = A_{D/E}$. This verifies that $U$ is open, and hence proves (1), since the topology of $\widehat{\mathbb{A}}$ is generated by the open subsets of the form $W(K, V)$.

(2) This is a direct consequence of the non-degeneracy of the residue pairing. So we have (2).

(3) To prove (3), we use the fact that $\psi : \mathbb{A} \simeq \widehat{\widehat{\mathbb{A}}}$, where, for $a \in \mathbb{A}$, $\psi_a$ is given by $\psi_a : \widehat{\mathbb{A}} \to \mathbb{S}^1, \chi \mapsto \chi(a)$. Thus to show that the image of $\varphi$ is dense, it suffices to show that the annihilator subgroup $\mathrm{Ann}(\mathrm{Im}(\varphi))$ of $\mathrm{Im}(\varphi)$ is zero. Let then $x \in \mathrm{Ann}(\mathrm{Im}(\varphi))$ be an annihilator of $\mathrm{Im}(\varphi)$. Then, by definition, $\{0\} = \psi_x(\{\varphi_a \mid a \in \mathbb{A}\}) = \{\varphi_a(x) \mid a \in \mathbb{A}\}$. That is to say, $\langle a, x\rangle_\omega = 0$ for all $a \in \mathbb{A}$. But the residue pairing is non-degenerate. So, $x = 0$.

(4) This is the dual of (2). Indeed, let $U \subset \mathbb{A}$ be an open subset of $\mathbb{A}$. Then $U \cap \mathbb{A}(D)$ is open in $\mathbb{A}(D)$. Write

$$\begin{aligned} U_{D/E} &\coloneqq U \cap \mathbb{A}(D)/U \cap \mathbb{A}(E) \\ K_{D/E} &\coloneqq \{a_{D/E} \in A_{D/E} \mid \chi_{D/E}(a_{D/E} U_{D/E}) \subset V \ (V : \text{open})\} \end{aligned}$$

Since $\mathbb{A}(D)$ is closed, $U_{D/E}$ is open in $A_{D/E}$. This, together with the fact that $\chi_{D/E}$ is non-degenerate on its locally compact base space, implies that $K_{D/E}$ is a compact subset. Let $K \coloneqq \varinjlim_{D} \varprojlim_{E:E\leq D} K_{D/E}$. Since $U$ is open, there exists a divisor $E$ such that $\mathbb{A}(E) \subset U$. This implies that there exists a divisor $D$ such that $K \subset \mathbb{A}(D)$. Otherwise, assume that, for any $D$, we have $K \not\subset \mathbb{A}(D)$. Then, there exists an element $k \in K$ such that $k \notin \mathbb{A}((\omega) - E)$. Hence we have $\chi(k\mathbb{A}(E)) \neq \{0\}$, a contradiction. This then completes the proof of (4).

□

**Theorem 5.13** (Sugahara-Weng [15]). *Let $X$ be an arithmetic surface. Then, as topological groups, we have the following canonical isomorphisms.*

(i) $\widehat{\mathbb{A}_X^{\mathrm{fin}}} \simeq \mathbb{A}_X^{\mathrm{fin}}$.

(ii) $\widehat{\mathbb{A}_X^{\infty}} \simeq \mathbb{A}_X^{\infty}$.

(iii) $\widehat{\mathbb{A}_X^{\mathrm{ar}}} \simeq \mathbb{A}_X^{\mathrm{ar}}$.

*Proof.* We have an injective continuous open morphism $\varphi : \mathbb{A} \to \widehat{\mathbb{A}}$. So it suffices to show that $\varphi$ is surjective. But this is a direct consequence of the fact that $\varphi$ is dense, since both $\mathbb{A}$ and $\widehat{\mathbb{A}}$ are complete and Hausdorff. This proves the theorem. □

## 6 Duality Theorem for Arithmetic Curves

**Proposition 6.1** (Tate ([16], Lem 4.1.5)). *For any elment $x \in F$, residue formula*

$$\sum_{v \in S} \lambda_v(\mathrm{Tr}_{F_v/\mathbb{Q}_v}(x)) = 0$$

*holds.*

**Definition 6.2** (Tate ([16], § 4.1)). We define a pairing $\langle \cdot, \cdot \rangle$ for an adelic ring $\mathbb{A}_F$

$$\langle \cdot, \cdot \rangle : \mathbb{A}_F \times \mathbb{A}_F \to \mathbb{R}/\mathbb{Z}; \; ((x_v), (y_v)) \mapsto \sum_{v \in S} \lambda_v(\mathrm{Tr}_{F_v/\mathbb{Q}_v}(x_v y_v)).$$

**Theorem 6.3** (Tate ([16], Theorems 4.1.1, 4.1.4)). *For the above pairing $\langle \cdot, \cdot \rangle$ of an adelic ring $\mathbb{A}_F$, we have the follows.*

(i) $\langle \cdot, \cdot \rangle$ *is perfect.*

(ii) *(Self-duality) $\widehat{\mathbb{A}_F} = \mathbb{A}_F$, where $\widehat{X}$ denote the Pontryagin duality of $X$.*

(iii) $F^\perp = F$.

(iv) *(*Weng ([18], §1.3)*) For any divisor $D$ on $Y$, $\mathbb{A}_{Y,1}(D)^\perp = \mathbb{A}_{Y,1}(K_F - D)$, where $K_F$ denote codifferent of $F$ on $\mathbb{Q}$.*

**Theorem 6.4** (Weng ([18], Prop. 3)). *Let $D$ be an Arakelov divisor on $Y$, we have the following canonical isomorphism as topological groups:*

$$\widehat{H^1_{\mathrm{ar}}(Y, D)} \simeq H^0_{\mathrm{ar}}(Y, K_F - D).$$

Since $H^0_{\mathrm{ar}}(Y, D)$ is discrete and $H^1_{\mathrm{ar}}(Y, D)$ is compact, using Fourier analysis for locally compact groups, we obtain their arithmetic counts $h^0_{\mathrm{ar}}(Y, D)$ and $h^1_{\mathrm{ar}}(Y, D)$ (See [18], Defs 2, 3).

**Theorem 6.5** (Weng ([18], Theorem 2)). *For any divisor $D$ on $Y$, we have the follows.*

(i) *(Arithmetic duality)*

$$h^1_{\mathrm{ar}}(Y, K_F - D) = h^0_{\mathrm{ar}}(Y, D).$$

(ii) *(Arithmetic Riemann-Roch theorem)*

$$h^0_{\mathrm{ar}}(Y, D) - h^1_{\mathrm{ar}}(Y, D) = \deg(D) - \frac{1}{2} \log |\Delta_F|,$$

*where $\Delta_F$ denote the discriminant of $F$.*

## 7 Duality Theorem for Arithmetic Surfaces

**Theorem 7.1** (Sugahara-Weng [15])**.** *Let $X$ be an arithmetic surface. Fix a rational differential $0 \neq \omega \in \Omega_{k(X)/F}$, and denote by $\langle \cdot, \cdot \rangle_\omega$ the natural residue pairing on the arithmetic adelic ring $\mathbb{A}_X^{\mathrm{ar}}$ induced by $\omega$. Then the following holds.*

(i) *For a divisor $D$ on $X$, $(\mathbb{A}^{\mathrm{ar}}_{X,12}(D))^\perp = \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega) - D)$.*

(ii) $(\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp = \mathbb{A}^{\mathrm{ar}}_{X,01}$.

(iii) $(\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp = \mathbb{A}^{\mathrm{ar}}_{X,02}$.

*Proof.* (i) To prove $(\mathbb{A}^{\mathrm{ar}}_{X,12}(D))^\perp = \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega) - D)$, we go as follows. Set $\tilde{\pi} : X \to Y \to \operatorname{Spec} \mathbb{Z}$. Then, for an open subset $U$ of $X$, the dualizing sheaf $\omega_{\tilde{\pi}}$ of $\tilde{\pi}$ can be written as

$$\omega_\pi(U) = \{\omega \in \Omega_{k(X)/\mathbb{Q}} \mid \mathrm{Res}_{C,x}(f\omega) = 0 \ (x \in C \ (\subset U),\ f \in \mathcal{O}_{X,C})\}$$

(See e.g., [8], Theorem 5.7). By a similar argument as in the proof of this result, we have, for a fixed irreducible curve $C_0$,

$$\omega_{\pi,C_0} = \{\omega \in \Omega_{k(X)/\mathbb{Q}} \mid \mathrm{Res}_{C_0,x}(f\omega) = 0 \ (x \in C_0,\ f \in \mathcal{O}_{X,C_0})\}.$$

This is just the set of differentials $\omega$ satisfying $\mathrm{ord}_{C_0}((\omega)) \geq 0$. Moreover, we also have, a fixed pair $x \in C_0$,

$$\omega_{\pi,C_0} \otimes_{\mathcal{O}_{C_0}} \mathcal{O}_{C_0,x_0} = \{\omega \in \Omega_{k(X)_{C_0,x_0}/\mathbb{Q}_{\tilde{\pi}(x_0)}} \mid \mathrm{Res}_{C_0,x_0}(f\omega) = 0 \ (f \in \mathcal{O}_{C_0,x_0})\}.$$

These results implies that for a fixed pair $(x_0, C_0)$ the following conditions (1), (2) are equivalent.

(1) For any $f \in \mathcal{O}_{X,C_0}$, $\mathrm{Res}_{C_0,x_0}(f\omega) = 0$.

(2) $\mathrm{ord}_{C_0}((\omega)) \geq 0$.

By a similar argument, we have that for a fixed closed point $P_0$ on $X_F$ the following conditions (3), (4) are equivalent.

(3) For any $f \in \mathcal{O}_{X_F,P_0}$, $\mathrm{Res}_{P_0}(f\omega) = 0$.

(4) $\mathrm{ord}_{P_0}((\omega)) \geq 0$.

Given these facts, we conclude $(\mathbb{A}^{\mathrm{ar}}_{X,12}(D))^\perp = \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega) - D)$.

(ii) To show $\mathbb{A}^{\mathrm{ar}}_{X,01} \subset (\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp$, we use Theorem 4.6. By definition, elements of $\mathbb{A}^{\mathrm{ar}}_{X,01}$ are independent of closed points $x \in X$. So we may write components of $f = (f_{C,x}) \times (f_P) \in \mathbb{A}^{\mathrm{ar}}_{X,01}$ as $f_{C,x} = f_C$ and $f_P = f_{E_P}$, which are independent of closed points of $X$. For $f, g \in \mathbb{A}^{\mathrm{ar}}_{X,01}$, we have then

$$\langle f, g \rangle_\omega = \sum_{V:\text{vertical}} \sum_{x:x\in V} \mathrm{Res}_{V,x}(f_V g_V \omega) + \sum_{P \in X_F} \Big( \sum_{x:x\in E_P} \mathrm{Res}_{E_P,x}(f_{E_P} g_{E_P} \omega) + \mathrm{Res}_P(f_{E_P} g_{E_P} \omega)\Big) = 0.$$

Now, for vertical curves $V$, by Theorem 4.6(iii), we have

$$\sum_{x:x\in V} \mathrm{Res}_{V,x}(f_V g_V \omega) = 0.$$

Furthermore, by Theorem 4.6(ii), we have, for closed points $P \in X_F$,

$$\sum_{x:x\in E_P} \mathrm{Res}_{E_P,x}(f_{E_P} g_{E_P} \omega) + \mathrm{Res}_P(f_{E_P} g_{E_P} \omega) = 0.$$

This then proves that $\mathbb{A}^{\mathrm{ar}}_{X,01} \subset (\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp$.

Next we prove that $(\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp \subset \mathbb{A}^{\mathrm{ar}}_{X,01}$. For this purpose, we use the ind-pro structure of $\mathbb{A}^{\mathrm{ar}}_X$. For divisors $D$ and vertical curves $C$, using (i), we obtain perfect pairings

$$\mathbb{A}^{\mathrm{ar}}_{X,12}(D)/\mathbb{A}^{\mathrm{ar}}_{X,12}(D-C) \times \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D+C)/\mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D) \to \mathbb{R}/\mathbb{Z} \tag{1}$$

induced by the residue pairing $\langle\cdot,\cdot\rangle_\omega$. By above result $\mathbb{A}^{\mathrm{ar}}_{X,01} \subset (\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp$, we see that the perfect residue pairing $\langle\cdot,\cdot\rangle_\omega$ annihilates $\mathbb{A}^{\mathrm{ar}}_{X,01} \times \mathbb{A}^{\mathrm{ar}}_{X,01}$, hence the above perfect pairing (1) annihilates $\mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C) \times \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D+C)/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D)$. Note that we have isomorphisms

$$\begin{cases} \mathbb{A}^{\mathrm{ar}}_{X,12}(D)/\mathbb{A}^{\mathrm{ar}}_{X,12}(D-C) \simeq \mathbb{A}_{C,01}, \\ \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C) \simeq \mathbb{A}_{C,0} = k(C), \\ \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D+C)/\mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D) \simeq \mathbb{A}_{C,01}, \\ \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D+C)/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D) \simeq \mathbb{A}_{C,0} = k(C). \end{cases}$$

Hence the above pairing (1) can be regarded as a perfect pairing $\mathbb{A}_{C,01} \times \mathbb{A}_{C,01}$ $(\to \mathbb{F}_p) \to \mathbb{R}/\mathbb{Z}$ on the vertical curve $C/\mathbb{F}_p$ ($\{p\} = \tilde{\pi}(C)$), which annihilates $\mathbb{A}_{C,0} \times \mathbb{A}_{C,0}$. In particular, for some non-zero $\omega_C \in \Omega_{k(C)/\mathbb{F}_p}$ and a certain $(a_x) \in \mathbb{A}_{C,01}$, we can write this perfect pairing as a residue pairing

$$\langle\cdot,\cdot\rangle_{\omega_C,(a_x)} : \mathbb{A}_{C,01} \times \mathbb{A}_{C,01} \to \mathbb{R}/\mathbb{Z}; \qquad ((f_x),(g_x)) \mapsto \sum_x \mathrm{Res}_x(f_x g_x a_x \omega_C). \tag{2}$$

It is well-known that for a perfect pairing of $\mathbb{A}_{C,01}$ annihilating $\mathbb{A}_{C,0} \times \mathbb{A}_{C,0}$, we have $\mathbb{A}_{C,0}^\perp = \mathbb{A}_{C,0}$ (See e.g., [5], §4). Therefore, if we set $G(D) = (\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D)$, we have that

$$\begin{aligned} G(D)/G(D-C) &= (\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D+C)/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D))^\perp \\ &= \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C) \end{aligned} \tag{3}$$

with respect to the perfect pairing (1) and for a certain $(a_x) \in \mathbb{A}_{C,0}$ used in the residue pairing (2). When necessary, with a possible modification on $\omega_C$, without loss of generality, we may and will assume that $(a_x) = 1$, and write $\langle\cdot,\cdot\rangle_{\omega_C,(a_x)}$ simply as $\langle\cdot,\cdot\rangle_{\omega_C}$. In parallel, when $C$ is horizontal, by a similar argument as above, we get the same conclusion. Consequently, using (3), we have, for any irreducible curves $C_1, C_2$, the following commutative diagram with exact columns

$$\begin{tikzcd} \mathbb{A}^{\mathrm{ar}}_{X,1}(D-C_1)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C_1-C_2) \arrow[r, equal] \arrow[d] & G(D-C_1)/G(D-C_1-C_2) \arrow[d] \\ \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C_1-C_2) \arrow[r] \arrow[d] & G(D)/G(D-C_1-C_2) \arrow[d] \\ \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C_1) \arrow[r, equal] & G(D)/G(D-C_1). \end{tikzcd}$$

This then proves that the horizontal map in the middle is surjective. Furthermore, the horizontal map in the middle is injective, since, from above, $\mathbb{A}^{\mathrm{ar}}_{X,01} \subset (\mathbb{A}^{\mathrm{ar}}_{X,01})^\perp$. Therefore, we have

$$G(D)/G(D-C_1-C_2) = \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C_1-C_2).$$

Repeating a similar argument as above, we have, for any divisors $D \geq E$

$$G(D)/G(E) = \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(E). \tag{4}$$

Since we have

$$\mathbb{A}^{\mathrm{ar}}_{X} = \varinjlim_{D} \varprojlim_{E:E\leq D} \mathbb{A}^{\mathrm{ar}}_{X,12}(D)/\mathbb{A}^{\mathrm{ar}}_{X,12}(E),$$

using (4), we conclude that

$$\begin{aligned}(\mathbb{A}^{\mathrm{ar}}_{X,01})^{\perp} &= \varinjlim_{D} \varprojlim_{E:E\leq D} G(D)/G(E) \\ &= \varinjlim_{D} \varprojlim_{E:E\leq D} \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(E) = \mathbb{A}^{\mathrm{ar}}_{X,01}.\end{aligned}$$

This proves that $(\mathbb{A}^{\mathrm{ar}}_{X,01})^{\perp} = \mathbb{A}^{\mathrm{ar}}_{X,01}$.

(iii) Next we prove $\mathbb{A}^{\mathrm{ar}}_{X,02} \subset (\mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp}$. By definition, since elements of $\mathbb{A}^{\mathrm{ar}}_{X,02}$ are independent of irreducible curves $C$ of $X$ and closed points $P \in X_F$, we may write components of $f = (f_{C,x}) \times (f_P) \in \mathbb{A}^{\mathrm{ar}}_{X,02}$ as $f_{C,x} = f_x$ and $f_P = f_\infty$, which are independent of irreducible curves $C$ and closed points $P$. Hence, for $f, g \in \mathbb{A}^{\mathrm{ar}}_{X,02}$, we have

$$\langle f, g\rangle_\omega = \sum_{x} \sum_{C:x\in C} \mathrm{Res}_{C,x}(f_x g_x \omega) + \sum_{P} \mathrm{Res}_P(f_\infty g_\infty \omega) = 0.$$

Now, by Theorem 4.6(i), we have, for any closed point $x$,

$$\sum_{C:x\in C} \mathrm{Res}_{C,x}(f_x g_x \omega) = 0.$$

Furthermore,

$$\sum_{P} \mathrm{Res}_P(f_\infty g_\infty \omega) = 0$$

since we can use a standard residue formula for a curve $X_F/F$ (See [14], §II. 7, Prop. 6). Therefore, $\mathbb{A}^{\mathrm{ar}}_{X,02} \subset (\mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp}$.

Finally, we show that $(\mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp} \subset \mathbb{A}^{\mathrm{ar}}_{X,02}$. Fix a vertical curve $C$. Then, for any divisor $D$ on $X$, we have

$$\begin{cases} \mathbb{A}^{\mathrm{ar}}_{X,2}(D)/\mathbb{A}^{\mathrm{ar}}_{X,2}(D-C) \simeq \mathbb{A}_{C,1}(D\,|_C), \\ \mathbb{A}^{\mathrm{ar}}_{X,2}((\omega)-D+C)/\mathbb{A}^{\mathrm{ar}}_{X,2}((\omega)-D) \simeq \mathbb{A}_{C,1}((\omega'_C)-D\,|_C), \end{cases}$$

for a certain $\omega'_C \in \Omega_{k(C)/\mathbb{F}_p}$ satisfying $(\omega'_C) = ((\omega)+C)\,|_C$ by adjunction formula (see [6], Theorem 3.6). We claim that $(\omega_C) = (\omega'_C)$. Indeed, since $\mathbb{A}_{C,1}(D\,|_C)^{\perp} = \mathbb{A}_{C,1}((\omega_C)-D\,|_C)$ and $\mathbb{A}^{\mathrm{ar}}_{X,02} \subset (\mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp}$, $\mathbb{A}_{C,1}((\omega'_C)-D\,|_C) \subset \mathbb{A}_{C,1}((\omega_C)-D\,|_C)$. This implies that $(\omega_C) \geq (\omega'_C)$ and hence $(\omega_C) = (\omega'_C)$, because there is no $f \in k(C)$ such that $(f) > 0$. Thus, for a residue pairing (2), we have $(\mathbb{A}_{C,1}((\omega'_C)-D\,|_C))^{\perp} = \mathbb{A}_{C,1}(D\,|_C)$. By a similar argument as a proof of (4), we have that for any integers $n \geq m \in \mathbb{Z}$,

$$H(nC)/H(mC) = \mathbb{A}^{\mathrm{ar}}_{X,2}(nC)/\mathbb{A}^{\mathrm{ar}}_{X,2}(mC), \tag{5}$$

where we set $H(D) = (\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D)$. Since we have

$$\mathbb{A}^{\mathrm{ar}}_X = \varinjlim_D \varprojlim_{E:E\le D} \mathbb{A}^{\mathrm{ar}}_{X,12}(D)/\mathbb{A}^{\mathrm{ar}}_{X,12}(E),$$

using (5), we get

$$\begin{aligned}(\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp &= \varinjlim_D \varprojlim_{E:E\le D} H(D)/H(E)\\ &\supset \varinjlim_n \varprojlim_{m\le n} H(nC)/H(mC)\\ &= \varinjlim_n \varprojlim_{m\le n} \mathbb{A}^{\mathrm{ar}}_{X,2}(nC)/\mathbb{A}^{\mathrm{ar}}_{X,2}(mC)\\ &= \textstyle\prod'_{x:x\in C} k(X)_x.\end{aligned}$$

Hence for any $x \in C$, a $(x, C)$-component of $(\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp$ is an element of $k(X)_x$ $(*)$. To prove the rest, we take a classical approach with a use of Chinese reminder theorem, using an idea in the proof of Prop. 1 of [12]. To be more precise, fix $x_0 \in X$ and a hyperplane $H$ on X satisfying that $x_0 \in H$. Setting $V = \mathrm{Spec}\, \mathcal{O}_{X,x_0} - H$, $V$ is affine. We claim that for any family of prime divisors $D_j$ on $V$ such that $D_i \cap D_j = \phi$ if $i \neq j$, and any rational functions $f_0, f_1, \cdots, f_n$ on $V$, and any fixed divisor $D$ supported on $D_i$'s, there exists a rational function $g$ such that

$$\begin{cases} \mathrm{ord}_{D_i}(f_i - g) \ge \mathrm{ord}_{D_i}(D) & (i = 1, \cdots, n)\\ \mathrm{ord}_{D_i}(g) \ge \mathrm{ord}_{D_i}(D) & (i \neq 1, \cdots, n).\end{cases}$$

Indeed, by clearing the common denominators for $f_i$'s (with a modification of $f_i$'s if necessary), we may assume that $f_i$'s are all regular. Then by applying the Chinese reminder theorem to the fractional ideals $\mathcal{P}_i^{\mathrm{ord}_{D_i}(D)}$, $i = 0, \cdots, n$, and $\bigcap_{i\notin\{0,1,\cdots,n\}} \mathcal{P}_i^{\mathrm{ord}_{D_i}(D)}$, where $\mathcal{P}_i$ are the prime ideals associated to the prime divisors $D_i$, we see the existence of such a $g$. Associated to $f \in (\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp$, form a new adele $f' \in \mathbb{A}^{\mathrm{ar}}_{X,012}$ by setting $f'_{C,x} = f_{C,x} - f_{H,x}$ where H is a fixed vertical hyperplane through $x$. By $(*)$, the definition of $f'_{C,x}$ is well-defined. Then

$$\begin{aligned}\sum_{C:x\in C} \mathrm{Res}_{C,x}(f'_{C,x} g_x \omega) &= \sum_{C:x\in C} \mathrm{Res}_{C,x}(f_{C,x} g_x \omega) - \sum_{C:x\in C} \mathrm{Res}_{C,x}(f'_{H,x} g_x \omega)\\ &= 0 - 0 = 0.\end{aligned}$$

Now, applying the above existence to obtain a $g$ satisfying that for any fixed rational function $f_0$ and any fixed curve $C_0 \ni x$, we have

$$\begin{cases} \mathrm{ord}_C(f'_{C,x}) + \mathrm{ord}_C(f_0 - g) + \mathrm{ord}_C((\omega)) \ge 0 & (C = C_0)\\ \mathrm{ord}_C(f'_{C,x}) + \mathrm{ord}_C(g) + \mathrm{ord}_C((\omega)) \ge 0 & (C \neq C_0, H).\end{cases}$$

Consequently, by the definition of the residue map, with $f'_{H,x} = 0$ in mind, we get, for any $f_0 \in k(X)$ and the corresponding $g$ just chosen,

$$0 = \sum_{C:x\in C} \mathrm{Res}_{C,x}(f'_{C,x} g\omega) = \mathrm{Res}_{C_0,x}(f'_{C_0,x} g\omega) = \mathrm{Res}_{C_0,x}(f'_{C_0,x} f_0 \omega).$$

Since the last quantity is always zero for all $f_0$, this then implies that $f'_{C_0,x} = 0$, namely, $f_{C_0,x} = f_{H,x_0}$. To end the proof of (iii), we still need to show that $f_P = f_{P_0}$ for a fixed $P_0 \in X_F$ and all $P \in X_F$. But this is amount to a use of a similar argument just said again, based on Chinese reminder theorem (See [5], §4). Thus, if $f = (f_{C,x})\times(f_P) \in (\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp$, then $f_{C,x} = f_{C_0,x}$ and $f_P = f_{P_0}$ for fixed $C_0$ and $P_0$. Therefore $f \in \mathbb{A}^{\mathrm{ar}}_{X,02}$. That is to say, $(\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp = \mathbb{A}^{\mathrm{ar}}_{X,02}$. This proves (iii).

□

**Theorem 7.2** (Duality theorem, Sugahara-Weng [15]). *Let $0 \neq \omega \in \Omega_{k(X)/F}$. For any $i \in \{0,1,2\}$, we have isomorphisms as topological groups*

$$\widehat{H^i_{\mathrm{ar}}(X,D)} \simeq H^{2-i}_{\mathrm{ar}}((\omega)-D).$$

*Proof.*

(1) Duality between $H^0_{\mathrm{ar}}$ and $H^2_{\mathrm{ar}}$

$$\begin{aligned}
\widehat{H^0_{\mathrm{ar}}(X,D)} &= (\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))\,\widehat{} \\
&\simeq \mathbb{A}^{\mathrm{ar}}_{X}/(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))^{\perp} \\
&= \mathbb{A}^{\mathrm{ar}}_{X}/(\mathbb{A}^{\mathrm{ar}}_{X,01})^{\perp} + (\mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp} + (\mathbb{A}^{\mathrm{ar}}_{X,12}(D))^{\perp} \\
&= \mathbb{A}^{\mathrm{ar}}_{X}/\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D) = H^2_{\mathrm{ar}}(X,(\omega)-D),
\end{aligned}$$

(2) Duality among $H^1_{\mathrm{ar}}$

$$\begin{aligned}
\widehat{H^1_{\mathrm{ar}}(X,D)} &\simeq \left(\frac{\mathbb{A}^{\mathrm{ar}}_{X,02} \cap (\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D))}{\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D)}\right)\widehat{} \\
&\simeq \frac{(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp} \cap (\mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D))^{\perp}}{(\mathbb{A}^{\mathrm{ar}}_{X,02})^{\perp} + (\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D))^{\perp}} \\
&= \frac{(\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02}) \cap (\mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D))}{\mathbb{A}^{\mathrm{ar}}_{X,02} + (\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D))} \\
&\simeq \frac{(\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02}) \cap \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D)}{(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02}) + (\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}((\omega)-D))} \simeq H^1_{\mathrm{ar}}(X,(\omega)-D).
\end{aligned}$$

□

## References


[1] A. A. Beilinson, Residues and adeles, Funct. Anal. Pril., 14 (1980), no. 1, 44-45; English transl. in Func. Anal. Appl., **14**, no. 1 (1980), 34-35.

[2] N. Bourbaki, *Elements of Mathematics: General Topology*, Springer, 1987.

[3] S.A. Gaal, *Point set topology*, Pure and Applied Mathematics, Vol. XVI, Academic Press,1964

[4] A. Huber, On the Parshin-Beilinson Adeles for Schemes. *Abh. Math. Sem.* Univ. Hamburg, **61**(1991), 249-273.

[5] K. Iwasawa, *Algebraic Theory of Algebraic Functions*, Lecture notes at Princeton university, 1975.

[6] S. Lang, *Introduction to Arakelov theory*, 1988, Springer Verlag.

[7] A. I. Madunts and I. B. Zhukov, Multidimensional complete fields: topology and other basic constructions, Trudy S.-Peterb. Mat. Obshch. (1995); English translation in Amer. Math. Soc. Transl. (Ser. 2) 166 (1995), 1-34.

[8] M. Morrow, An explicit approach to residues on and dualizing sheaves of arithmetic surfaces. *New York J. Math.*, **16** (2010), 575-627.

[9] M. Morrow, Grothendieck's trace map for arithmetic surfaces via residues and higher adeles, Journal of Algebra and Number Theory, **6** (2012) 1503-1536

[10] D. V. Osipov and A. N. Parshin, Harmonic analysis on local fields and adelic spaces. II. *Izv. Ross. Akad. Nauk Ser. Mat.*, **75** (2011), no. 4, 91–164.

[11] D. V. Osipov and A. N. Parshin, Harmonic analysis and the Riemann-Roch theorem, arXiv:1107.0408.

[12] A. N. Parshin, On the arithmetic of two-dimensional schemes. I. Distributions and residues. *Izv. Akad. Nauk SSSR Ser. Mat.* **40** (1976), no. 4, 736-773.

[13] A. N. Parshin, Chern classes, adeles and $L$-functions. *J. Reine Angew. Math.* **341**(1983), 174-192.

[14] J. P. Serre, Algebraic groups and class fields, GTM 117, 1988, Springer-Verlag.

[15] K. Sugahara and L. Weng, Arithmetic Cohomology Groups, preprint, 2014 (submitted) available at http://www2.math.kyushu-u.ac.jp/~weng/AriCohRes.pdf

[16] J. Tate, Fourier analysis in number fields, and Hecke's zeta-functions. 1967 Algebraic Number Theory (*Proc. Instructional Conf.*, Brighton, 1965) pp. 305-347 Thompson, Washington, D.C.

[17] A. Weil, Zur algebraischen Theorie der algebraischen Funktionen, *J. Reine Angew. Math.* **179**(1938), 129-133.

[18] L. Weng, Geometry of numbers, arXiv:1102.1302, see also arXiv:math/0007146.

[19] A. Yekutieli, *An Explicit Construction of the Grothendieck Residue Complex*, With an appendix by Pramathanath Sastry. Asterisque No. 208 (1992)