

Adelic Cohomology Groups for Arithmetic Varieties and Ind-Pro Topology in Dimension Two

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<https://doi.org/10.15017/1500519>

出版情報 : 九州大学, 2014, 博士 (数理学), 課程博士
バージョン :
権利関係 : 全文ファイル公表済

Adelic Cohomology Groups for Arithmetic Varieties
and
Ind-Pro Topology in Dimension Two

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January 1, 2015

Contents

1	Introduction	3
2	Adelic Cohomology Theory on Noetherian Schemes	4
3	Arithmetic Adelic Groups and Arithmetic Cohomology Groups	7
3.1	Notations	7
3.2	Arithmetic Adelic Groups	7
3.3	Arithmetic Cohomology Groups	10
3.4	Inductive Long Exact Sequences	11
4	Residue Pairings	14
5	Topological Structures of Adelic Groups	16
5.1	Topological Structures of Arithmetic Adelic Rings	16
5.2	Ind-Pro Topological Spaces and Their Duals	17
5.3	Completeness and Compact Orientedness of Arithmetic Adelic Groups	20
5.4	Double Dual of Arithmetic Adelic Rings	21
5.5	Continuity of Scalar Product	22
5.6	Continuity of Residue Maps	24
5.7	Self-Duality of Arithmetic Adelic Rings	24
6	Duality Theorem for Arithmetic Curves	26
7	Duality Theorem for Arithmetic Surfaces	27

1 Introduction

Classical adelic theory is closely related with class field theory, theory of L -functions and algebraic groups. Its origin may be traced back to a paper of Chevalley on the so-called ideles. In 1938, A. Weil ([17]) gave the first adelic approach to Riemann-Roch theorem for curves. Contrary to the formal sheaf theoretic approach to Riemann-Roch theorem widely adopted nowadays, the adelic one, then popular, was very concrete for curves. Most important works on adeles in this period were Tate's thesis ([16]) and Weil's works on Tamagawa numbers. Influenced by them, adeles later became a basic tool in the study of automorphic forms, Eisenstein series and trace formula.

Modern adelic theory started with A.N. Parshin's pioneer works on adeles over algebraic surfaces around 1976. In [12], Parshin introduced adelic rings and adelic complexes for divisors on algebraic surfaces, verified that his adelic cohomology groups were isomorphic to Grothendieck's sheaf theoretic cohomology groups, and established the Serre duality. In 1980, Parshin's works were generalized by A. A. Beilinson to Noetherian schemes: In [1], Beilinson constructed adelic complexes for quasi-coherent sheaves on Noetherian schemes, examined that the associated adelic cohomology groups coincided with the sheaf theoretic cohomology groups and outlined an adelic residue theory. This paper of Beilinson was very short and difficult for many to understand. To remedy this, around 1990, younger generation started to supply the unwritten details. Now we have the works of A. Huber[4], treating adelic complexes in great details, and A. Yekutieli[19], building up an adelic residue theory.

At the same time, adelic approach to Riemann-Roch theorem was moving forwards. Around 2000, in [18], L. Weng developed an adelic cohomology theory for arithmetic curves, based on [16] with a concrete construction of arithmetic cohomology groups. Furthermore, in 2011, D. V. Osipov and A. N. Parshin not only obtained an adelic proof for the Riemann-Roch theorem on algebraic surfaces in [11], but constructed arithmetic adelic rings for arithmetic surfaces in [10].

The purpose of our study is to introduce a general adelic cohomology theory for quasi-coherent sheaves on arithmetic varieties. Motivated by the works of Parshin, Beilinson, Weng and Osipov-Parshin mentioned above, we here, together with Prof. Weng, first construct adelic complexes for quasi-coherent sheaves on arithmetic varieties and hence their arithmetic cohomology groups, and then develop a general ind-pro topological theory in dimension two and hence to establish a topological duality for our arithmetic cohomology groups associated to invertible sheaves on arithmetic surfaces.

Acknowledgements. I would like to express my deepest gratitude and heartfelt thanks to my advisor Professor Lin Weng. He always gave me aspiring guidances, invaluable constructive advices and criticisms in the past seven years. Without his supervision, constant help and encouragement, this dissertation would not have been possible.

2 Adelic Cohomology Theory on Noetherian Schemes

In this section, we recall adelic cohomology theory on Noetherian schemes developed by Parshin ([12]) and Beilinson ([1]). We will follow [4] for the presentation.

Definition 2.1 ([1], §2; [4], Definitions 1.3.1-3). Let X be a Noetherian scheme.

- (1) Let $P(X)$ be the collection of scheme-theoretic points on X . For points $p, q \in P(X)$, we write $p \geq q$ if $q \in \overline{\{p\}}$. Clearly, \geq makes $P(X)$ a poset.
- (2) Let $S(X)$ be the set of simplexes corresponding to the ordered set $(P(X), \geq)$. In particular, for $m \geq 0$, let $S(X)_m$ be the set of m -simplexes and $S(X)_m^{\text{red}}$ be the set of non-degenerate m -simplexes. That is,

$$S(X)_m := \{(p_0, \dots, p_m) \in P(X)^{m+1} \mid p_i \geq p_{i+1}\},$$

$$S(X)_m^{\text{red}} := \{(p_0, \dots, p_m) \in S(X)_m \mid p_i \neq p_j \ (i \neq j)\}.$$

Accordingly, for $i \in \{0, 1, \dots, m\}$, we define the associated boundary maps δ_i^m and degeneracy maps σ_i^m as follows:

$$\delta_i^m : S(X)_m \rightarrow S(X)_{m-1}; (p_0, \dots, p_i, \dots, p_m) \mapsto (p_0, \dots, \check{p}_i, \dots, p_m),$$

$$\sigma_i^m : S(X)_m \rightarrow S(X)_{m+1}; (p_0, \dots, p_i, \dots, p_m) \mapsto (p_0, \dots, p_i, p_i, \dots, p_m).$$

Definition 2.2 ([1], §2; [4], §1.3). For a subset $K \subset S(X)_m$ and a point $p \in P(X)$, we define a subset ${}_pK$ of $S(X)_{m-1}$ by

$${}_pK := \{(p_1, \dots, p_m) \in S(X)_{m-1} \mid (p, p_1, \dots, p_m) \in K\}.$$

As usual, for each point $p \in X$, let \mathcal{O}_p denote the local ring of X at p and m_p be its maximal ideal. Then we get a natural morphism $f : \text{Spec } \mathcal{O}_p \rightarrow X$. Consequently, for each \mathcal{O}_p -modules N , it makes sense for us to introduce $[N]_p = f_* \widetilde{N}$. Moreover, let $AbGp$ be the category of abelian groups, and $QC(X)$ be the category of quasi-coherent sheaves on X .

Proposition 2.3 (Parshin-Beilinson ([12], §2; [1], §2; see also [4], Prop. 2.1.1)). *For each subset $K \subset S(X)_m$, there exists an additive and exact functor*

$$\mathbb{A}(K, \cdot) : QC(X) \rightarrow AbGp$$

determined uniquely by the following properties.

- (i) $\mathbb{A}(K, \cdot)$ commutes with direct limits.
- (ii) For $m = 0$ and a coherent sheaf \mathcal{F} on X ,

$$\mathbb{A}(K, \mathcal{F}) = \prod_{p \in K} \varprojlim_l \mathcal{F}_p / m_p^l \mathcal{F}_p.$$

- (iii) For $m > 0$ and a coherent sheaf \mathcal{F} on X ,

$$\mathbb{A}(K, \mathcal{F}) = \prod_{p \in P(X)} \varprojlim_l \mathbb{A}({}_pK, [\mathcal{F}_p / m_p^l \mathcal{F}_p]_p).$$

In the sequel, we call $\mathbb{A}(K, \cdot)$, resp. $\mathbb{A}(K, \mathcal{F})$, the adelic functor (associated to K), resp. the adelic group of quasi-coherent sheaf \mathcal{F} .

Remark. We cannot apply properties (i), (ii) directly for quasi-coherent but not coherent sheaves \mathcal{F} . In general, to calculate adelic groups for quasi-coherent sheaves, we have to use property (i). Indeed, since X is a Noetherian scheme, any quasi-coherent sheaf \mathcal{F} can be written as $\mathcal{F} = \varinjlim_{i \in I} \mathcal{F}_i$ with \mathcal{F}_i ($i \in I$) coherent sheaves on X . Hence, by property (i), we can perform the following calculations: $\mathbb{A}(K, \mathcal{F}) = \mathbb{A}(K, \varinjlim_{i \in I} \mathcal{F}_i) = \varinjlim_{i \in I} \mathbb{A}(K, \mathcal{F}_i)$. Note that now all \mathcal{F}_i are coherent, we can apply properties (ii), (iii) to get adelic groups $\mathbb{A}(K, \mathcal{F}_i)$.

Definition 2.4 ([4], Definition 3.3.2). Let \mathcal{F} be a quasi-coherent sheaf on X . Then, for $m \geq 0$, we define the m -th adelic group $\mathbb{A}_X^m(\mathcal{F})$ by

$$\mathbb{A}_X^m(\mathcal{F}) := \mathbb{A}(S(X)_m^{\text{red}}, \mathcal{F}).$$

Definition 2.5 ([13], §2). Let \mathcal{F} be a quasi-coherent sheaf on X . Then, for $0 \leq i_0 \leq i_1 \leq \dots \leq i_m$, we define the type (i_0, \dots, i_m) adelic group $\mathbb{A}_{X, i_0, i_1, \dots, i_m}(\mathcal{F})$ by

$$\mathbb{A}_{X, i_0, i_1, \dots, i_m}(\mathcal{F}) := \mathbb{A}_X(K_{i_0, i_1, \dots, i_m}, \mathcal{F}),$$

where

$$K_{i_0, i_1, \dots, i_m} := \left\{ (p_0, p_1, \dots, p_m) \in S(X)_m \mid 0 \leq \forall t \leq m, \text{codim} \overline{\{p_t\}} = i_t \right\}.$$

Remark ([13], §2). If $\dim X < +\infty$, we have $\mathbb{A}_X^m(\mathcal{F}) = \bigoplus_{0 \leq i_0 < \dots < i_m \leq \dim X} \mathbb{A}_{X, i_0, \dots, i_m}(\mathcal{F})$.

Proposition 2.6 ([4], Prop. 2.1.4). For a subset $K \subset S(X)_m$ and a quasi-coherent sheaf \mathcal{F} , we have a natural inclusion

$$\mathbb{A}(K, \mathcal{F}) \subset \prod_{(p_0, \dots, p_m) \in K} \mathbb{A}((p_0, \dots, p_m), \mathcal{F}).$$

Notation. By this proposition, we may write an element f of $\mathbb{A}(K, \mathcal{F})$ as either $f = (f_{X_0, \dots, X_m})$, or $f = (f_{p_0, \dots, p_m})$, where $X_i = \overline{\{p_i\}}$ ($0 \leq i \leq m$), and $f_{X_0, \dots, X_m} = f_{p_0, \dots, p_m} \in \mathbb{A}((p_0, \dots, p_m), \mathcal{F})$. If, in addition, X is irreducible and p_i is its generic point, we often omit the indexes X_i, p_i .

Definition 2.7 ([4], Definition 2.2.2). Assume that there exist $i \in \{1, \dots, m\}$, $K \subset S(X)_m$ and $L \subset S(X)_{m-1}$ satisfying $d_i^m K \subset L$. Then, for a quasi-coherent sheaf \mathcal{F} on X , we define boundary maps $d_i^m(K, L, \mathcal{F})$

$$d_i^m(K, L, \mathcal{F}) : \mathbb{A}(L, \mathcal{F}) \rightarrow \mathbb{A}(K, \mathcal{F})$$

as follows.

- (a) Assume that $i = 0$ and \mathcal{F} is a coherent sheaf. For each point $p \in P(X)$, we have a natural map $\mathbb{A}(L, \mathcal{F}) \rightarrow \mathbb{A}(L, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p)$ induced from the structural morphism $\mathcal{F} \rightarrow [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p$. Then the maps $\varphi_p^l : \mathbb{A}(L, \mathcal{F}) \rightarrow \mathbb{A}(K, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p)$, obtaining as the compositions of the map $\mathbb{A}(L, \mathcal{F}) \rightarrow \mathbb{A}(L, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p)$ and the projections $\mathbb{A}(L, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p) \rightarrow \mathbb{A}(K, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p)$, form a projective system for $l \in \mathbb{N}$. Accordingly, we define a boundary map by $d_0^m(K, L, \mathcal{F}) = \prod_{p \in P(X)} \varprojlim_l \varphi_p^l$.
- (b) Assume that $i = 1$, $m = 1$ and \mathcal{F} is a coherent sheaf. For each point $p \in P(X)$, canonical maps $\pi_p^l : \Gamma(X, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p) \rightarrow \mathbb{A}(K, [\mathcal{F}_p/m_p^l \mathcal{F}_p]_p)$ form a projective system for $l \in \mathbb{N}$. Accordingly, we define a boundary map by $d_1^1(K, L, \mathcal{F}) = \prod_{p \in P(X)} \varprojlim_l \pi_p^l$.

(c) Assume that $i > 0, m > 1$ and \mathcal{F} is a coherent sheaf. Then we define a boundary map by

$$d_i^m(K, L, \mathcal{F}) = \prod_{p \in P(X)} \varprojlim_l d_{i-1}^{m-1}({}_p K, {}_p L, [\mathcal{F}_p/m_p^l \mathcal{F}_p]).$$

(d) $d_i^m(K, L, \cdot)$ commutes with direct limits.

Notation. For a quasi-coherent sheaf \mathcal{F} , set $d^m = \sum_{i=0}^m (-1)^i d_i^m(S(X)_m^{\text{red}}, S(X)_{m-1}^{\text{red}}, \mathcal{F})$. Then we have a boundary map $d^m : \mathbb{A}_X^{m-1}(\mathcal{F}) \rightarrow \mathbb{A}_X^m(\mathcal{F})$.

Theorem 2.8 (Parsin-Beilinson [12], §2, Thm 1; [1], §2, Cor; see also [4], Thm 4.2.3, Prop 5.1.2). *For a quasi-coherent sheaf \mathcal{F} on a Noetherian scheme X , we have*

- (i) $(\mathbb{A}_X^*(\mathcal{F}), d^*)$ becomes a complex.
- (ii) For each $i \geq 0$, there is a natural isomorphism, as abelian groups,

$$H^i(\mathbb{A}_X^*(\mathcal{F}), d^*) \simeq H^i(X, \mathcal{F}),$$

where the left side is the i -th cohomology group induced from the adelic complex $(\mathbb{A}_X^*(\mathcal{F}), d^*)$, and the right side is the i -th sheaf theoretic cohomology group of \mathcal{F} .

We call the complex in this theorem an adelic complex, and their cohomology groups adelic cohomology groups.

3 Arithmetic Adelic Groups and Arithmetic Cohomology Groups

3.1 Notations

F : an algebraic field,
 \mathcal{O}_F : the integer ring of F ,
 S_{fin} : the set of finite places of F ,
 S_{∞} : the set of infinite places of F ,
 $S := S_{\text{fin}} \cup S_{\infty}$,
 $\pi : X \rightarrow Y = \text{Spec } \mathcal{O}_F$: an arithmetic variety of dimension $n + 1$,
 X_F : the generic fibre of X ,
 F_v : the v -completion of F ($v \in S$),
 $X_{\sigma} := X \times_Y \text{Spec } F_{\sigma}$ ($\sigma \in S_{\infty}$),
 $\varphi_{\sigma} : X_{\sigma} \rightarrow X_F$.

3.2 Arithmetic Adelic Groups

There is a natural one-to-one correspondence between closed points on arithmetic curve $\text{Spec } \mathcal{O}_F$ and finite places of F . The so-called Arakelov compactification $\overline{\text{Spec } \mathcal{O}_F}$ of $\text{Spec } \mathcal{O}_F$ is obtained by adding infinite places. Associated to $\overline{\text{Spec } \mathcal{O}_F}$ is the adelic ring \mathbb{A}_F of F which contains much refined information on not only finite places in S_{fin} but also infinite places in S_{∞} . Similarly, when we construct adelic rings (and more generally adelic groups associated to quasi-coherent sheaves) on arithmetic varieties X , we have to treat scheme-theoretic points on both $X_{\text{fin}} (= X)$ and on $X_{\infty} := \{X_{\sigma}\}_{\sigma \in S_{\infty}}$. In parallel to Arakelov theory, (in particular, the Arakelov intersection theory,) however, we need not consider all points on X_{σ} , but only these corresponding to the so-called horizontal cycles of both X_{fin} and X_{∞} . Our treatment is motivated by Weng's work [18], where arithmetic curves are treated, with a use of the so-called uniformity condition: for a scheme-theoretic point P on X_F , $\varphi_{\sigma}^{-1}(\overline{\{P\}})$ decomposes into finite irreducible closed varieties in X_{σ} . Our uniformity condition is a constrain on these irreducible components in X_{∞} and the induced cycles on X_{fin} . For our own use, we call generic points of the above irreducible closed varieties for P in X_{σ} the infinite points corresponding to P .

Before treating general arithmetic varieties, we recall the following construction of arithmetic adelic rings for arithmetic surfaces, introduced by Osipov and Parshin in [10]. For the time being, let X be an arithmetic surface.

Definition 3.1 (Osipov-Parshin ([10], §5)).

[Finite adelic ring]

We define the finite adelic ring $\mathbb{A}_X^{\text{fin}}$ for an arithmetic surface X by

$$\mathbb{A}_X^{\text{fin}} := \mathbb{A}_{X,012}(\mathcal{O}_X).$$

Then we have

$$\mathbb{A}_X^{\text{fin}} = \varinjlim_{D_1} \varprojlim_{D_2: D_2 \leq D_1} \mathbb{A}_{X,12}(D_1) / \mathbb{A}_{X,12}(D_2),$$

where D_* are divisors on X and we set $\mathbb{A}_{X,12}(D_*) = \mathbb{A}_{X,12}(\mathcal{O}_X(D_*))$.

[∞ -adelic ring]

For the adelic ring $\mathbb{A}_{X_F} = \mathbb{A}_{X_F,01}(\mathcal{O}_{X_F})$ of the generic fibre X_F , we have

$$\mathbb{A}_{X_F} = \varinjlim_{C_1} \varprojlim_{C_2: C_2 \leq C_1} \mathbb{A}_{X_F,1}(C_1)/\mathbb{A}_{X_F,1}(C_2),$$

where C_* are divisors on X_F and we set $\mathbb{A}_{X_F,1}(C_*) = \mathbb{A}_{X_F,1}(\mathcal{O}_X(C_*))$. We define the ∞ -adelic ring \mathbb{A}_X^∞ for an arithmetic surface X by

$$\mathbb{A}_X^\infty := \mathbb{A}_{X_F} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \varinjlim_{C_1} \varprojlim_{C_2: C_2 \leq C_1} ((\mathbb{A}_{X_F,1}(C_1)/\mathbb{A}_{X_F,1}(C_2)) \otimes_{\mathbb{Q}} \mathbb{R}).$$

[Arithmetic adelic ring]

We define the arithmetic adelic ring \mathbb{A}_X^{ar} for an arithmetic surface X by

$$\mathbb{A}_X^{\text{ar}} := \mathbb{A}_{X,012}^{\text{ar}} := \mathbb{A}_X^{\text{fin}} \oplus \mathbb{A}_X^\infty.$$

Remark. For any divisors $C_1 \geq C_2$ on X_F , $\mathbb{A}_{X_F,1}(C_1)/\mathbb{A}_{X_F,1}(C_2)$ is a finite dimensional F -vector space, hence a finite dimensional \mathbb{Q} -vector space.

Remark. To understand meaning of $\widehat{\otimes}$, consider the following example. For the ring of Laurent series $\mathbb{Q}((t))$, we have $\mathbb{Q}((t)) \otimes_{\mathbb{Q}} \mathbb{R} \neq \mathbb{R}((t))$. On the other hand, $\mathbb{Q}((t)) = \varinjlim_n \varprojlim_{m: m \leq n} t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]$.

Since $t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]$ is a finite dimensional \mathbb{Q} -vector space, we have

$$\mathbb{Q}((t)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} = \varinjlim_n \varprojlim_{m: m \leq n} (t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]) \otimes_{\mathbb{Q}} \mathbb{R} = \varinjlim_n \varprojlim_{m: m \leq n} t^{-n}\mathbb{R}[[t]]/t^{-m}\mathbb{R}[[t]] = \mathbb{R}((t)).$$

Motivated by Osipov-Parshin's construction, we define ∞ -adelic groups for arithmetic varieties as follows. Assume in the sequel that X is an arithmetic variety of dimension $n + 1$, unless otherwise stated.

Theorem 3.2 (∞ -adelic groups, Sugahara-Weng [15]). *For each subset $K \subset S(X_F)_m$, there exists an additive and exact functor*

$$\mathbb{A}_\infty(K, \cdot) : \mathcal{QC}(X_F) \rightarrow \text{AbGp}$$

determined uniquely by the following properties (i), (ii), (iii).

(i) $\mathbb{A}_\infty(K, \cdot)$ commutes with direct limits.

(ii) For $m = 0$ and a coherent sheaf \mathcal{G} on X_F ,

$$\mathbb{A}_\infty(K, \mathcal{G}) = \prod_{p \in K} \varprojlim_l (\mathcal{G}_p / m_p^l \mathcal{G}_p \otimes_{\mathbb{Q}} \mathbb{R}).$$

(iii) For $m > 0$ and a coherent sheaf \mathcal{G} on X_F ,

$$\mathbb{A}_\infty(K, \mathcal{G}) = \prod_{p \in P(X)} \varprojlim_l \mathbb{A}_\infty({}_p K, [\mathcal{G}_p / m_p^l \mathcal{G}_p]_p).$$

We define the arithmetic adelic groups by using ∞ -adelic groups and a uniformity condition.

Definition 3.3 (Arithmetic adelic groups, Sugahara-Weng). Let \mathcal{F} be a quasi-coherent sheaf on X , \mathcal{F}_F be the quasi-coherent sheaf on X_F induced by \mathcal{F} . Fix an index tuple (i_0, \dots, i_m) with $0 \leq i_0 \leq \dots \leq i_m$.

(A) We define (i_0, \dots, i_m) -type finite adelic group $\mathbb{A}_{X, i_0, \dots, i_m}^{\text{fin}}(\mathcal{F})$ and (i_0, \dots, i_m) -type ∞ -adelic group $\mathbb{A}_{X, i_0, \dots, i_m}^{\infty}(\mathcal{F})$ by

$$\mathbb{A}_{X, i_0, \dots, i_m}^{\text{fin}}(\mathcal{F}) := \mathbb{A}_X(K_{X, i_0, \dots, i_m}, \mathcal{F}), \quad \mathbb{A}_{X, i_0, \dots, i_m}^{\infty}(\mathcal{F}) := \mathbb{A}_{\infty}(K_{X_F, i_0, \dots, i_m}, \mathcal{F}_F),$$

where for $Z = X, X_F$, we set

$$K_{Z, i_0, \dots, i_m} := \left\{ (p_0, \dots, p_m) \in S(Z)_m \mid \text{codim}\{\overline{p_t}\} = i_t \ (0 \leq t \leq m) \right\}.$$

(B) We define arithmetic adelic groups as follows.

(1) For $i_m = n + 1$, we define $\mathbb{A}_{\text{ar}; i_0, i_1, \dots, i_m}(\mathcal{F})$ by

$$\mathbb{A}_{X; i_0, i_1, \dots, i_m}^{\text{ar}}(\mathcal{F}) := \mathbb{A}_{X; i_0, i_1, \dots, i_m}^{\text{fin}}(\mathcal{F}) \oplus \mathbb{A}_{X; i_0, i_1, \dots, i_{m-1}}^{\infty}(\mathcal{F}_F).$$

(2) For $i_m \neq n + 1$, we define $\mathbb{A}_{X; i_0, i_1, \dots, i_m}^{\text{ar}}(\mathcal{F})$ by

$$\mathbb{A}_{X; i_0, i_1, \dots, i_m}^{\text{ar}}(\mathcal{F}) := \mathbb{A}_X(K_{X, i_0, i_1, \dots, i_m}^{\text{nh}}, \mathcal{F}) \oplus \mathbb{A}_X^{\text{fin, inf}}(K_{X, i_0, i_1, \dots, i_m}^{\text{h}}, \mathcal{F}).$$

Here we set

$$K_{X, i_0, i_1, \dots, i_m}^{\text{h}} := \{ (P_{i_0}, \dots, P_{i_m}) \in K_{X, i_0, i_1, \dots, i_m} \mid (P_{i_0}, \dots, P_{i_m}) \in S(X_F)_n \}$$

$$K_{X, i_0, i_1, \dots, i_m}^{\text{nh}} := K_{X, i_0, i_1, \dots, i_m} \setminus K_{X, i_0, i_1, \dots, i_m}^{\text{h}}$$

and use a uniformity condition to set

$$\mathbb{A}_X^{\text{fin, inf}}(K_{X, i_0, i_1, \dots, i_m}^{\text{h}}, \mathcal{F}) := \left\{ (f_{E_{P_{i_0}, \dots, E_{P_{i_m}}}}) \times (f_{P_{i_0}, \dots, P_{i_m}}) \in \mathbb{A}_X(K_{X, i_0, i_1, \dots, i_m}^{\text{h}}, \mathcal{F}) \oplus \mathbb{A}_{\infty; i_0, i_1, \dots, i_m}(\mathcal{F}_F) \mid f_{E_{P_{i_0}, \dots, E_{P_{i_m}}}} = f_{P_{i_0}, \dots, P_{i_m}} \right\}.$$

(3) For any $m \geq 0$, we define $\mathbb{A}_{\text{ar}}^m(X, \mathcal{F})$ by

$$\mathbb{A}_{\text{ar}}^m(X, \mathcal{F}) := \bigoplus_{0 \leq i_0 < \dots < i_m \leq \dim X} \mathbb{A}_{X, i_0, \dots, i_m}^{\text{ar}}(\mathcal{F}).$$

Remark. For $m = 0$, ∞ -adelic groups $\mathbb{A}_{\infty}(K_{X_F, i_0, \dots, i_{m-1}}, \mathcal{F}_F)$ makes no sense. To complete above definition, for an open subset $U \subset X$, we induce (-1) -simplex $\underline{1}_U$ formally (see [19], § 3.1). We set $S(X_F)_{-1} = S(X_F)_{-1}^{\text{red}} = \{\underline{1}_U \mid U \subset X : \text{open sets}\}$. And for $K \subset S(X_F)_{-1}$, we define $\mathbb{A}_{\infty}(K, \mathcal{F}_F)$ by

$$\mathbb{A}_{\infty}(K, \mathcal{F}_F) := \begin{cases} \mathcal{F}_F(U_{K, F}) \otimes_{\mathbb{Q}} \mathbb{R} & (\dim X \geq 2) \\ \{s_{\infty} \in \mathcal{F}_F(U_{K, F}) \otimes_{\mathbb{Q}} \mathbb{R} \mid s \in \mathcal{F}(U_K)\} & (\dim X = 1) \end{cases}$$

where $U_K = \bigcup_{\underline{1}_U \in K} U$ and s_{∞} denotes an element of $\mathcal{F}_F(U_{K, F}) \otimes_{\mathbb{Q}} \mathbb{R}$ corresponding to s . The reason for separation of arithmetic curves with others in this latest definition is that arithmetic varieties are relative over arithmetic curves.

Theorem 3.4 (Sugahara-Weng [15]). *Let X be an arithmetic surface. For a Weil divisor D on X , arithmetic adelic groups $\mathbb{A}_{X,01}^{\text{ar}}$, $\mathbb{A}_{X,02}^{\text{ar}}$, $\mathbb{A}_{X,12}^{\text{ar}}(D)$, $\mathbb{A}_{X,0}^{\text{ar}}$, $\mathbb{A}_{X,1}^{\text{ar}}(D)$, $\mathbb{A}_{X,2}^{\text{ar}}(D)$ as subgroups of \mathbb{A}_X^{ar} associated to the invertible sheaf $\mathcal{O}_X(D)$ are given as follows:*

- (i) $\mathbb{A}_{X,01}^{\text{ar}} = \{(f_{C,x}) \times (f_P) \in \mathbb{A}_X^{\text{ar}} \mid (f_{C,x}) \in \mathbb{A}_{X,01}, f_P = f_{E_P,x} (P \in X_F)\}$,
- (ii) $\mathbb{A}_{X,02}^{\text{ar}} = \mathbb{A}_{X,02} \oplus (k(X_F) \otimes_{\mathbb{Q}} \mathbb{R})$,
- (iii) $\mathbb{A}_{X,12}^{\text{ar}}(D) = \mathbb{A}_{X,12}(D) \oplus \varprojlim_{D'_F: D'_F \leq D} (\mathbb{A}_{X_F,1}(D_F)/\mathbb{A}_{X_F,1}(D'_F) \otimes_{\mathbb{Q}} \mathbb{R})$,
where D_F denote a divisor on X_F induced by D .
- (iv) $\mathbb{A}_{X,0}^{\text{ar}} = k(X) = \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}}$,
- (v) $\mathbb{A}_{X,1}^{\text{ar}}(D) = \{(f_{C,x}) \times (f_P) \in \mathbb{A}_X^{\text{ar}} \mid (f_{C,x}) \in \mathbb{A}_{X,1}(D), f_P = f_{E_P,x} (P \in X_F)\} = \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)$,
- (vi) $\mathbb{A}_{X,2}^{\text{ar}}(D) = \{(f_{C,x}) \times (f_P) \in \mathbb{A}_X^{\text{ar}} \mid (f_{C,x}) \in \mathbb{A}_{X,2}(D), (f_P) \in H^0(X_F, D_F) \otimes_{\mathbb{Q}} \mathbb{R}\} = \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)$.

Proof. These are direct consequences of our definition. \square

Corollary 3.5 ([15]). *Let X be an arithmetic surface. Then, there are following natural ind-pro structures on the level two subspaces $\mathbb{A}_{X,01}^{\text{ar}}$ and $\mathbb{A}_{X,02}^{\text{ar}}$ of $\mathbb{A}_{X,012}^{\text{ar}}$:*

$$\begin{aligned} \mathbb{A}_{X,01}^{\text{ar}} &= \varinjlim_D \varprojlim_{E:E \leq D} \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(E), \\ \mathbb{A}_{X,02}^{\text{ar}} &= \varinjlim_D \varprojlim_{E:E \leq D} \mathbb{A}_{X,2}^{\text{ar}}(D)/\mathbb{A}_{X,2}^{\text{ar}}(E). \end{aligned}$$

Proof. This follows from the ind-pro structure

$$\mathbb{A}_{X,012}^{\text{ar}} = \varinjlim_D \varprojlim_{E:E \leq D} \mathbb{A}_{X,12}^{\text{ar}}(D)/\mathbb{A}_{X,12}^{\text{ar}}(E)$$

and facts that $\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) = \mathbb{A}_{X,1}^{\text{ar}}(D)$, $\mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) = \mathbb{A}_{X,2}^{\text{ar}}(D)$. \square

3.3 Arithmetic Cohomology Groups

In this subsection, let $\pi : X \rightarrow Y$ be an arithmetic surface.

Definition 3.6 ([15]). For $m \geq 0$ and a quasi-coherent sheaf \mathcal{F} , we define the boundary maps d_{ar}^m by

$$d_{\text{ar}}^m : \mathbb{A}_{\text{ar}}^m(X, \mathcal{F}) \rightarrow \mathbb{A}_{\text{ar}}^{m+1}(X, \mathcal{F}); \quad (f_{P_0, \dots, P_m}) \mapsto (\sum_{i=0}^{m+1} (-1)^i f_{Q_0, \dots, \check{Q}_i, \dots, Q_{m+1}}).$$

Proposition 3.7 ([15]). *Let \mathcal{F} be a quasi-coherent sheaf on X . $(\mathbb{A}_{\text{ar}}^*(X, \mathcal{F}), d_{\text{ar}}^*)$ forms a complex of abelian groups.*

We call this complex an arithmetic adelic complex.

Definition 3.8 (Adelic cohomology groups, Sugahara-Weng). Let \mathcal{F} be a quasi-coherent sheaf on X . For $i \in \{0, 1, \dots, n+1\}$, we define the i -th arithmetic cohomology groups $H_{\text{ar}}^i(X, \mathcal{F})$ of \mathcal{F} by

$$H_{\text{ar}}^i(X, \mathcal{F}) := H^i(\mathbb{A}_{\text{ar}}^*(X, \mathcal{F}), d_{\text{ar}}^*).$$

Theorem 3.9 (Weng [18]). *Let Y be an arithmetic curve. Then for any divisor D on Y , we have*

- (i) $H_{\text{ar}}^0(Y, D) = \mathbb{A}_{Y,0}^{\text{ar}} \cap \mathbb{A}_{Y,1}^{\text{ar}}(D)$,
- (ii) $H_{\text{ar}}^1(Y, D) = \mathbb{A}_{Y,0,1}^{\text{ar}} / (\mathbb{A}_{Y,0}^{\text{ar}} + \mathbb{A}_{Y,1}^{\text{ar}}(D))$.

Proof. To get arithmetic cohomology groups, calculate cohomology groups of an arithmetic adelic complex

$$0 \rightarrow \mathbb{A}_{Y,0}^{\text{ar}} \oplus \mathbb{A}_{Y,1}^{\text{ar}}(D) \xrightarrow{d^1} \mathbb{A}_{Y,0,1}^{\text{ar}} \rightarrow 0,$$

where

$$d^1 : (a_0, a_1) \mapsto (a_1 - a_0).$$

□

Theorem 3.10 (Sugahara-Weng [15]). *Let X be an arithmetic surface. Then, for a Weil divisor D on X , the arithmetic cohomology groups of the invertible sheaf $\mathcal{O}_X(D)$ on X are given by the follows:*

- (i) $H_{\text{ar}}^0(X, D) = \mathbb{A}_{X,0,1}^{\text{ar}} \cap \mathbb{A}_{X,0,2}^{\text{ar}} \cap \mathbb{A}_{X,1,2}^{\text{ar}}(D)$,
- (ii) $H_{\text{ar}}^1(X, D) \simeq ((\mathbb{A}_{X,0,1}^{\text{ar}} + \mathbb{A}_{X,0,2}^{\text{ar}}) \cap \mathbb{A}_{X,1,2}^{\text{ar}}(D)) / (\mathbb{A}_{X,0,1}^{\text{ar}} \cap \mathbb{A}_{X,1,2}^{\text{ar}}(D) + \mathbb{A}_{X,0,2}^{\text{ar}} \cap \mathbb{A}_{X,1,2}^{\text{ar}}(D))$,
 $\simeq ((\mathbb{A}_{X,0,1}^{\text{ar}} + \mathbb{A}_{X,1,2}^{\text{ar}}(D)) \cap \mathbb{A}_{X,0,2}^{\text{ar}}) / (\mathbb{A}_{X,0,1}^{\text{ar}} \cap \mathbb{A}_{X,0,2}^{\text{ar}} + \mathbb{A}_{X,0,2}^{\text{ar}} \cap \mathbb{A}_{X,1,2}^{\text{ar}}(D))$,
 $\simeq ((\mathbb{A}_{X,0,2}^{\text{ar}} + \mathbb{A}_{X,1,2}^{\text{ar}}(D)) \cap \mathbb{A}_{X,0,1}^{\text{ar}}) / (\mathbb{A}_{X,0,1}^{\text{ar}} \cap \mathbb{A}_{X,0,2}^{\text{ar}} + \mathbb{A}_{X,0,1}^{\text{ar}} \cap \mathbb{A}_{X,1,2}^{\text{ar}}(D))$,
- (iii) $H_{\text{ar}}^2(X, D) = \mathbb{A}_{X,0,1,2}^{\text{ar}} / (\mathbb{A}_{X,0,1}^{\text{ar}} + \mathbb{A}_{X,0,2}^{\text{ar}} + \mathbb{A}_{X,1,2}^{\text{ar}}(D))$.

Proof. To get arithmetic cohomology groups, we calculate cohomology groups of the arithmetic adelic complex

$$0 \rightarrow \mathbb{A}_{X,0}^{\text{ar}} \oplus \mathbb{A}_{X,1}^{\text{ar}}(D) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D) \xrightarrow{d^1} \mathbb{A}_{X,0,1}^{\text{ar}} \oplus \mathbb{A}_{X,0,2}^{\text{ar}} \oplus \mathbb{A}_{X,1,2}^{\text{ar}}(D) \xrightarrow{d^2} \mathbb{A}_{X,0,1,2}^{\text{ar}} \rightarrow 0,$$

where

$$d^1 : (a_0, a_1, a_2) \mapsto (a_1 - a_0, a_2 - a_0, a_2 - a_1), \quad d^2 : (a_{01}, a_{02}, a_{12}) \mapsto a_{12} - a_{02} + a_{01}.$$

□

3.4 Inductive Long Exact Sequences

In this subsection, let $\pi : X \rightarrow Y$ be an arithmetic surface.

There are two types of irreducible curves on X . Namely,

- (a) horizontal curves H , where $H = \overline{\{P\}}$ for an algebraic point P of X_F . In this case we write $H = E_P$.
- (b) vertical curves V , where $\pi(V) = \{v\}$ consisting of a closed point v on Y .

For an irreducible curve C on X , we define a map φ

$$\mathbb{A}_{X,1}^{\text{ar}}(D + C) / \mathbb{A}_{X,1}^{\text{ar}}(D) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D + C) / \mathbb{A}_{X,2}^{\text{ar}}(D) \xrightarrow{\varphi} \mathbb{A}_{X,1,2}^{\text{ar}}(D + C) / \mathbb{A}_{X,1,2}^{\text{ar}}(D),$$

where $\varphi : (a_1, a_2) \mapsto a_2 - a_1$.

Proposition 3.11 ([15]). *Let $V \subset X$ be a vertical curve. We have the following long exact sequence of cohomology groups*

$$\begin{aligned} 0 &\rightarrow H_{\text{ar}}^0(X, D) \rightarrow H_{\text{ar}}^0(X, D + V) \rightarrow H^0(V, (D + V)|_V) \\ &\rightarrow H_{\text{ar}}^1(X, D) \rightarrow H_{\text{ar}}^1(X, D + V) \rightarrow H^1(V, (D + V)|_V) \\ &\rightarrow H_{\text{ar}}^2(X, D) \rightarrow H_{\text{ar}}^2(X, D + V) \rightarrow 0. \end{aligned}$$

Proof. This is a direct consequence of the following commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}_{X,0}^{\text{ar}} \oplus \mathbb{A}_{X,1}^{\text{ar}}(D) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D) & \xrightarrow{d_1} & \mathbb{A}_{X,01}^{\text{ar}} \oplus \mathbb{A}_{X,02}^{\text{ar}} \oplus \mathbb{A}_{X,12}^{\text{ar}}(D) & \xrightarrow{d_2} & \mathbb{A}_X^{\text{ar}} \\ \downarrow & & \downarrow & & \parallel \\ \mathbb{A}_{X,0}^{\text{ar}} \oplus \mathbb{A}_{X,1}^{\text{ar}}(D + V) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D + V) & \xrightarrow{d_1} & \mathbb{A}_{X,01}^{\text{ar}} \oplus \mathbb{A}_{X,02}^{\text{ar}} \oplus \mathbb{A}_{X,12}^{\text{ar}}(D + V) & \xrightarrow{d_2} & \mathbb{A}_X^{\text{ar}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}_{X,1}^{\text{ar}}(D + V)/\mathbb{A}_{X,1}^{\text{ar}}(D) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D + V)/\mathbb{A}_{X,2}^{\text{ar}}(D) & \xrightarrow{\varphi} & \mathbb{A}_{X,12}^{\text{ar}}(D + V)/\mathbb{A}_{X,12}^{\text{ar}}(D) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

with exact columns and facts that

$$\begin{aligned} \mathbb{A}_{X,1}^{\text{ar}}(D + V)/\mathbb{A}_{X,1}^{\text{ar}}(D) &\simeq \mathbb{A}_{V,0}, \\ \mathbb{A}_{X,2}^{\text{ar}}(D + V)/\mathbb{A}_{X,2}^{\text{ar}}(D) &\simeq \mathbb{A}_{V,1}((D + V)|_V), \\ \mathbb{A}_{X,12}^{\text{ar}}(D + V)/\mathbb{A}_{X,12}^{\text{ar}}(D) &\simeq \mathbb{A}_{V,01}. \end{aligned}$$

□

Proposition 3.12 ([15]). *Let $E_P \subset X$ be a horizontal curve. We have the following long exact sequence of cohomology groups*

$$\begin{aligned} 0 &\rightarrow H_{\text{ar}}^0(X, D) \rightarrow H_{\text{ar}}^0(X, D + E_P) \rightarrow \text{Ker } \varphi \\ &\rightarrow H_{\text{ar}}^1(X, D) \rightarrow H_{\text{ar}}^1(X, D + E_P) \rightarrow \text{Coker } \varphi \\ &\rightarrow H_{\text{ar}}^2(X, D) \rightarrow H_{\text{ar}}^2(X, D + E_P) \rightarrow 0. \end{aligned}$$

Proof. This is a direct consequence of the following commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}_{X,0}^{\text{ar}} \oplus \mathbb{A}_{X,1}^{\text{ar}}(D) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D) & \xrightarrow{d_1} & \mathbb{A}_{X,01}^{\text{ar}} \oplus \mathbb{A}_{X,02}^{\text{ar}} \oplus \mathbb{A}_{X,12}^{\text{ar}}(D) & \xrightarrow{d_2} & \mathbb{A}_X^{\text{ar}} \\ \downarrow & & \downarrow & & \parallel \\ \mathbb{A}_{X,0}^{\text{ar}} \oplus \mathbb{A}_{X,1}^{\text{ar}}(D + E_P) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D + V) & \xrightarrow{d_1} & \mathbb{A}_{X,01}^{\text{ar}} \oplus \mathbb{A}_{X,02}^{\text{ar}} \oplus \mathbb{A}_{X,12}^{\text{ar}}(D + E_P) & \xrightarrow{d_2} & \mathbb{A}_X^{\text{ar}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}_{X,1}^{\text{ar}}(D + E_P)/\mathbb{A}_{X,1}^{\text{ar}}(D) \oplus \mathbb{A}_{X,2}^{\text{ar}}(D + E_P)/\mathbb{A}_{X,2}^{\text{ar}}(D) & \xrightarrow{\varphi} & \mathbb{A}_{X,12}^{\text{ar}}(D + E_P)/\mathbb{A}_{X,12}^{\text{ar}}(D) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

with exact columns. □

Remark. Unlike for vertical curves, for horizontal curves, we do not have the group isomorphisms between $\text{Ker } \varphi$, resp. $\text{Coker } \varphi$, and $H_{\text{ar}}^0(E_P, (D + E_P) |_{E_P})$, resp. $H_{\text{ar}}^1(E_P, (D + E_P) |_{E_P})$. This is in fact not surprising: different from vertical curves, for the arithmetic cohomology, there is no simple additive law with respect to horizontal curves when count these arithmetic groups: In Arakeloc theory, we have the following formulas.

(i) ([6], V, § 3, Proof of Lem 3.8) For any vertical curve V ,

$$\chi_{X/Y}(O_X(D + V)) - \chi_{X/Y}(O_X(D)) = \chi(O_X(D + V) |_V).$$

(ii) ([6], V, § 3, Theorem 3.4) For any horizontal curve E_P ,

$$\chi_{X/Y}(O_X(D + E_P)) - \chi_{X/Y}(O_X(D)) = \chi_{E_P/Y}(O_X(D + E_P) |_{E_P}) - \frac{1}{2}d_\lambda(E_P),$$

where let $\lambda = \frac{1}{2}g$ be a Neron function resulting from a Green function g , $d_\lambda(E_P)$ be a logarithmic λ -discriminant, χ be a Euler characteristic (see [6], pp. 3, 21, 99 and 112).

4 Residue Pairings

In this section, let $\pi : X \rightarrow Y$ be a regular arithmetic surface.

Definition 4.1 (Morrow ([8], Definition 2.5)). Let (A, m_A) be a Noetherian local ring and N be an A -module. Then we define the maximal Hausdorff quotient N^{sep} of N by

$$N^{\text{sep}} := N / \bigcap_{n \geq 1} m_A^n N.$$

Let F be a complete discrete valuation field, \mathcal{O}_F be its valuation ring and K be a subfield of F such that $K = \text{Frac}(\mathcal{O}_F \cap K)$. Then for the differential module $\Omega_{\mathcal{O}_F/K \cap \mathcal{O}_F}$ we define the continuous differential module $\Omega_{F/K}^{\text{cts}}$

$$\Omega_{F/K}^{\text{cts}} := \Omega_{\mathcal{O}_F/K \cap \mathcal{O}_F}^{\text{sep}} \otimes_{\mathcal{O}_F} F.$$

Definition 4.2 (Residue maps: equal characteristic zero, Morrow ([8], § 2.2)). Let N be a 2-dimensional local field of equal characteristic zero. Assume that N includes local field L . Then the following (1)-(4) hold.

- (1) N have a unique coefficient field k_N including L ,
- (2) k_N/L is a finite extension,
- (3) k_N is an algebraic closure of L in N ,
- (4) There exists a uniformizer $t \in N$ such that $N \simeq k_N(\!(t)\!)$.

We define residue map res_N for N by

$$\text{res}_N : \Omega_{N/L}^{\text{cts}} = N dt \rightarrow k_N, \left(\sum_n a_n t^n \right) dt \mapsto a_{-1}.$$

Definition 4.3 (Residue maps: mixed characteristic, Morrow ([8], § 2.3)). Let N be a 2-dimensional local field of mixed characteristics with the residue field of characteristic p . Then the following (1)-(3) hold.

- (1) N contains \mathbb{Q}_p .
- (2) Let k_N be an algebraic closure \mathbb{Q}_p in N . k_N corresponds to a coefficient field of N .
- (3) There exists a 2-dimensional local field $M \subset N$ satisfying conditions (i)-(iv):
 - (i) N/M is a finite extension.
 - (ii) $\overline{M} = \overline{N}$, where \overline{M} and \overline{N} denote residue fields of M and N respectively.
 - (iii) $k_M = k_N$.
 - (iv) M is k_M -isomorphic to $k_M\{\{t\}\}$, where

$$k_M\{\{t\}\} := \{ \sum_{n \in \mathbb{Z}} a_n t^n \mid a_n \in k_M; \inf_i v_{k_M}(a_i) > -\infty; a_n \rightarrow 0 \text{ as } n \rightarrow -\infty \}.$$

We define the residue map res_N for N by

$$\begin{aligned} \text{res}_N : \Omega_{N/L}^{\text{cts}} &= \Omega_{M/L}^{\text{cts}} \otimes_M N \xrightarrow{\text{Tr}_{N/M}} \Omega_{M/L}^{\text{cts}} \xrightarrow{\text{res}_M} k_M = k_N, \\ \text{res}_M : \Omega_{M/L}^{\text{cts}} &= M dt \rightarrow k_M, \left(\sum_n a_n t^n \right) dt \mapsto -a_{-1}. \end{aligned}$$

For the finite adelic ring $\mathbb{A}_X^{\text{fin}}$, it is known that $\mathbb{A}_X^{\text{fin}} \subset \prod_{\{P_0, P_1, P_2\} \in \mathcal{S}(X)_2^{\text{red}}} \mathbb{A}(\{P_0, P_1, P_2\}, \mathcal{O}_X)$ and

$k(X)_{C,x} := \mathbb{A}(\{P_0, P_1, P_2\}, \mathcal{O}_X)$ is a finite direct sum of 2-dimensional local fields (see [1], 2; [13], Prop. 1). Hence $\mathbb{A}_X^{\text{fin}}$ is a subgroup of direct product of 2-dimensional local fields.

Similarly, for ∞ -adelic ring \mathbb{A}_X^∞ , we have a natural inclusion $\mathbb{A}_X^\infty \subset \prod_{P \in X_K: \text{closed}} \mathbb{A}(\{\eta, P\}, \mathcal{O}_{X_F}) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}$

and $k(X_F)_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \mathbb{A}(\{\eta, P\}, \mathcal{O}_{X_F}) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}$ is a finite direct sum of fields of the form $\mathbb{R}((t))$ or $\mathbb{C}((t))$. Hence \mathbb{A}_X^∞ is a subgroup of direct product of fields of the form $\mathbb{R}((t))$ or $\mathbb{C}((t))$.

From the above, to define residue pairing for an arithmetic adelic ring, we can use residue maps for 2-dimensional local fields above and the natural residue maps $\text{res}_{K((t))} : K((t)) \rightarrow K$; $\sum_n a_n t^n \mapsto a_{-1}$ (where K denotes either \mathbb{R} or \mathbb{C}).

Definition 4.4 (Tate ([16], § 2.2)). We define residue maps $\lambda_\infty, \lambda_p$ (p a prime) by

$$\begin{aligned} \lambda_\infty : \mathbb{R} &\rightarrow \mathbb{R}/\mathbb{Z}; & x &\mapsto -x \bmod \mathbb{Z}, \\ \lambda_p : \mathbb{Q}_p &\rightarrow \mathbb{R}/\mathbb{Z}; & \sum_n a_n p^n &\mapsto \sum_{n < 0} a_n p^n \bmod \mathbb{Z}. \end{aligned}$$

Notation. Set $\text{Res}_N := \lambda_v \circ \text{Tr}_{k_N/\mathbb{Q}_v} \circ \text{res}_N$. Since $\mathbb{A}(\{P_0, P_1, P_2\}, \mathcal{O}_X)$ can be written as a finite direct sum $\bigoplus_N N$ of 2-dimensional local fields N , we define residue map $\text{Res}_{C,x}$ by $\text{Res}_{C,x} := \sum_N \text{Res}_N$, where $C = \overline{\{P_1\}}$, $x = P_2$.

Similarly, since $\mathbb{A}(\{\eta, P\}, \mathcal{O}_{X_F}) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}$ can be written as a finite direct sum fields $\bigoplus_L L$ of fields L of the form $\mathbb{R}((t))$ or $\mathbb{C}((t))$, we define the residue map Res_P by $\text{Res}_P := \sum_L \text{Res}_L$.

Definition 4.5 (Residue pairings, Sugahara-Weng [15]). Let $0 \neq \omega \in \Omega_{k(X)/F}$. We define the global residue pairing $\langle \cdot, \cdot \rangle_\omega$ on the arithmetic adelic ring \mathbb{A}_X^{ar} by

$$\begin{aligned} \langle \cdot, \cdot \rangle_\omega : \mathbb{A}_X^{\text{ar}} \times \mathbb{A}_X^{\text{ar}} &\rightarrow \mathbb{R}/\mathbb{Z}; \\ ((f_{C,x}) \times (f_P)), (g_{C,x}) \times (g_P) &\mapsto \sum_{(C,x)} \text{Res}_{C,x}(f_{C,x} g_{C,x} \omega) + \sum_P \text{Res}_P(f_P g_P \omega). \end{aligned}$$

For the global residue pairing, we have the following fundamental result.

Theorem 4.6 (Non-Degeneracy of Residue Pairing, Sugahara-Weng [15]). *Let $0 \neq \omega \in \Omega_{k(X)/F}$. $\langle \cdot, \cdot \rangle_\omega$ on \mathbb{A}_X^{ar} is a non-degenerate pairing.*

For later use, we also recall the following

Theorem 4.7 (Residue formulas, Morrow ([8], Thm 4.1; [9], Thm 5.4)). *Let $0 \neq \omega \in \Omega_{k(X)/F}$. We have*

(i) *For a fixed closed point $x \in X$,*

$$\sum_{C:x \in C} \text{Res}_{C,x}(\omega) = 0.$$

(ii) *For a fixed horizontal curve $E_P \subset X$,*

$$\sum_{x:x \in E_P} \text{Res}_{E_P,x}(\omega) + \sum_P \text{Res}_P(\omega) = 0.$$

(iii) *For a fixed vertical curve $V \subset X$,*

$$\sum_{x:x \in V} \text{Res}_{V,x}(\omega) = 0.$$

5 Topological Structures of Adelic Groups

In this section, let $\pi : X \rightarrow Y$ be a regular arithmetic surface.

5.1 Topological Structures of Arithmetic Adelic Rings

To introduce a natural topology on the adelic ring \mathbb{A}_X^{ar} , we follow [15] using ind-pro structures on our space starting from locally compact topologies. More precisely, it goes as follows.

For all divisors $D \geq E$ on X , $\mathbb{A}_X(D)/\mathbb{A}_X(E)$ are locally compact, Hausdorff topological groups.

Indeed, setting $D - E = \sum_{i=1}^s a_i C_i$ ($a_i \geq 0$), we have

$$\mathbb{A}_{X,12}(D)/\mathbb{A}_{X,12}(E) = \prod_{i=1}^s \mathbb{A}_{(\eta_{C_i} K_{12}, [O_X(D)_{\eta_{C_i}}/m_{\eta_{C_i}}^{a_i} O_X(D)_{\eta_{C_i}}]_{\eta_{C_i}})}.$$

Setting $\mathcal{F} = [O_X(D)_{\eta_{C_i}}/m_{\eta_{C_i}}^{a_i} O_X(D)_{\eta_{C_i}}]_{\eta_{C_i}}$, since \mathcal{F} is a quasi-coherent sheaf, we can express $\mathcal{F} = \varinjlim_{j \in J} \mathcal{F}_j$ in terms of direct limit of certain coherent sheaves \mathcal{F}_i ($i \in I$). Consequently, we have

$$\begin{aligned} \mathbb{A}_{(\eta_{C_i} K_{12}, \mathcal{F})} &= \mathbb{A}(\{x \mid x \in C_i\}, \varinjlim_{j \in J} \mathcal{F}_j) \\ &= \varinjlim_{j \in J} \mathbb{A}(\{x \mid x \in C_i\}, \mathcal{F}_j) \\ &= \varinjlim_{j \in J} \prod_{x: x \in C_i} (\varinjlim_l \mathcal{F}_{j,x}/m_x^l \mathcal{F}_{j,x}). \end{aligned}$$

Note that \mathcal{F}_j ($j \in J$) are coherent sheaves and x are closed points. So, $\mathcal{F}_{j,x}/m_x^l \mathcal{F}_{j,x}$ are finite groups. In particular, we can endow all $\mathcal{F}_{j,x}/m_x^l \mathcal{F}_{j,x}$ with discrete and compact topologies. Accordingly, by using product topology, inductive limit topology and projective limit topology, we obtain a natural topology on the space $\mathbb{A}_X^{\text{fin}}(D)/\mathbb{A}_X^{\text{fin}}(E)$. This topology on $\mathbb{A}_X^{\text{fin}}(D)/\mathbb{A}_X^{\text{fin}}(E)$ is well-known to be Hausdorff and locally compact.

Moreover, since we have

$$\mathbb{A}_X^{\text{fin}} = \varinjlim_D \varprojlim_{E: E \leq D} \mathbb{A}_{X,12}(D)/\mathbb{A}_{X,12}(E).$$

Again by using projective limit topology first and then inductive limit topology, we obtain a natural topology on $\mathbb{A}_X^{\text{fin}}$.

On the other hand, for all divisors $D \geq E$ on X_F , since $\mathbb{A}_{X,1}^{\infty}(D)/\mathbb{A}_{X,1}^{\infty}(E)$ are finite \mathbb{R} -vector spaces, $\mathbb{A}_{X,1}^{\infty}(D)/\mathbb{A}_{X,1}^{\infty}(E)$ are locally compact, Hausdorff topological spaces. Furthermore, since

$$\mathbb{A}_X^{\infty} = \varinjlim_D \varprojlim_{E: E \leq D} \mathbb{A}_{X,1}^{\infty}(D)/\mathbb{A}_{X,1}^{\infty}(E),$$

similarly, we obtain a natural topology on \mathbb{A}_X^{∞} by using inductive limit and projective limit.

Recall that $\mathbb{A}_X^{\text{ar}} = \mathbb{A}_X^{\text{fin}} \oplus \mathbb{A}_X^{\infty}$. Using product structure, we obtain a natural topology on our adelic ring \mathbb{A}_X^{ar} , which we call the ind-pro topology.

We may realize the above formal definition of the ind-pro topology on \mathbb{A}_X^{ar} in a more concrete term following [15]. To explain this, fix a Madunts-Zhukov lifting

$$h_C : \mathbb{A}_{C,01} \simeq \left(\prod_{x: x \in C} ' \mathcal{O}_{C,x} \right) / (\pi_C \prod_{x: x \in C} ' \mathcal{O}_{C,x}) \xrightarrow{\text{lifting}} \prod_{x: x \in C} ' \mathcal{O}_{C,x}$$

(see [7]). Then, following Parshin ([12, 13]),

$$\mathbb{A}_X^{\text{fin}} = \left\{ \left(\sum_{i_C=-\infty}^{\infty} h_C(a_{i_C}) \pi_C^{i_C} \right)_C \in \prod_C \prod_{x \in C} k(X)_{C,x} \left| \begin{array}{l} a_{i_C} \in \mathbb{A}_{C,01}, \\ a_{i_C} = 0 \text{ for sufficiently small } i_C, \\ \min\{i_C \mid a_{i_C} \neq 0\} \geq 0 \text{ for almost all } C \end{array} \right. \right\}.$$

Moreover, one checks that a fundamental system of open neighborhoods of 0 for the ind-pro topology on $\mathbb{A}_X^{\text{fin}}$ may be described as follows:

$$\left\{ \left(\sum_{i_C=-\infty}^{\infty} h_C(U_{i_C}) \pi_C^{i_C} \right)_C \in \mathbb{A}_X^{\text{fin}} \left| \begin{array}{l} U_{i_C} \subset \mathbb{A}_{C,01} \text{ open subsets,} \\ U_{i_C} = \mathbb{A}_{C,01} \text{ for sufficiently large } i_C, \\ \max\{i_C \mid U_{i_C} \neq \mathbb{A}_{C,01}\} < 0 \text{ for almost all } C \end{array} \right. \right\}.$$

5.2 Ind-Pro Topological Spaces and Their Duals

For a topological space T , denote by $\widehat{T} := \{f : T \rightarrow \mathbb{S}^1 \mid \text{continuous}\}$. There is a natural topology on \widehat{T} , namely, the compact-open topology generated by open subsets of the form $W(K, U) := \{f \in \widehat{T} \mid f(K) \subset U\}$, where $K \subset T$ are compact, $U \subset \mathbb{S}^1$ are open. We call \widehat{T} the (Pontryagin) dual of T .

Proposition 5.1 ([15]). *Let $\{P_n\}_n$ be a projective system of Hausdorff topological groups with structural maps $\pi_{n,m} : P_m \rightarrow P_n$ and $\pi_n : \varprojlim_n P_n \rightarrow P_n$. Assume that all π_n and $\pi_{n,m}$ are surjective and open, and that for any n, n' , there exists an n'' such that $n'' \leq n$ and $n'' \leq n'$. Then, as topological groups,*

$$\widehat{\varprojlim_n P_n} \simeq \varinjlim_n \widehat{P_n}.$$

Proof. Denote by $\widehat{\pi_{n,m}} : \widehat{P_n} \rightarrow \widehat{P_m}$, $f_n \mapsto f_n \circ \pi_{n,m}$, the dual of $\pi_{n,m}$. Then, for an element $\varinjlim_n f_n \in \varinjlim_n \widehat{P_n}$, we have $\widehat{\pi_{n,m}}(f_n) = f_m$, or equivalently, $f_n \circ \pi_{n,m} = f_m$. Hence, if $x = \varprojlim_n x_n \in \varprojlim_n P_n$, we have $f_n(x_n) = f_n(\pi_{n,m}(x_m)) = f_m(x_m)$ for sufficiently small $n \geq m$. Based on this, we define a natural map

$$\varphi : \varinjlim_n \widehat{P_n} \rightarrow \widehat{\varprojlim_n P_n}, \quad \varinjlim_n f_n \mapsto f$$

where $f : \varprojlim_n P_n \rightarrow \mathbb{S}^1$, $x = \varprojlim_n x_n \mapsto f_n(x_n)$. From above discussion, f is well-defined.

Lemma 5.2 ([15]). *Concerning φ and f , the following holds.*

- (1) f is continuous. In particular, φ is well-defined.
- (2) φ is a bijection.
- (3) φ is continuous.
- (4) φ is open.

Proof. (1) For sufficiently small n , $f = f_n \circ \pi_n$. Since f_n and π_n is continuous, f is continuous, and hence φ is well-defined.

(2) To prove that φ is injective, we assume that $\varphi(\varinjlim f_n) = 0$. Thus $f_n(x_n) = 0$ for sufficiently small n and for all $\varprojlim x_n \in \varprojlim P_n$. Note that π_n are surjective. So $f_n(x_n) = 0$ for all $x_n \in P_n$. This means $f_n = 0$ for sufficiently small n . Consequently, $\varprojlim f_n = 0$, and hence φ is injective.

To show that φ is surjective, let $f : \varprojlim P_n \rightarrow \mathbb{S}^1$ be a continuous map. Then, for any open subset $U \subset \mathbb{S}^1$ containing 1, $f^{-1}(U)$ is an open neighborhood of 0 in $\varprojlim P_n$. Hence, we can write $f^{-1}(U)$ as $f^{-1}(U) = \varprojlim P_n \cap \prod K_n$ where $K_n \subset P_n$ are open subsets and $K_n = P_n$ for almost all n . By our assumptions, for n_1, \dots, n_r such that $K_{n_i} \subseteq P_{n_i}$, there exists an N such that $N \leq n_i$. Hence, $f(\text{Ker } \pi_N) = 1$. So, $f(\text{Ker } \pi_n) = 1$ for all $n \leq N$. Built on this, we define, for $n \leq N$, the maps $f_n : P_n \rightarrow \mathbb{S}^1, x_n \mapsto f(x)$ if $\pi_n(x) = x_n$. Note that $f(x)$ always make sense, since π_n is surjective. Moreover, f_n 's are well-defined. Indeed, if $y \in \varprojlim P_n$ such that $\pi_n(y) = x_n$, then $\pi_n(y) = x_n = \pi_n(x)$ for $n \leq N$. Hence $x - y \in \text{Ker } \pi_n$. This implies that $f(y) = f(x)$. Clearly, by definition, $\varphi(\varinjlim f_n) = f$. So φ is surjective.

(3) To prove that φ is continuous, it suffices to show that for open subsets of $\widehat{\varprojlim P_n}$ in the form $W(K, V), \varphi^{-1}(W(K, V))$ is open in $\varinjlim \widehat{P_n}$, where K is a compact subset of $\varprojlim P_n$ and V is open subset of \mathbb{S}^1 . Since π_n are continuous, $K_n = \pi_n(K)$ are compact. In this way, we get an inductive system of open subsets $\{W(K_n, V)\}_n$. Set $U = \varinjlim W(K_n, V)$. Note that we have $f(U) = W(K, V)$. This shows that f is continuous.

(4) To prove that φ is open, let U be an open subset of $\varinjlim \widehat{P_n}$ such that $\iota^{-1}(U) = W(K_n, V)$ for a compact subset K_n of P_n for any n where $\iota_n : \widehat{P_n} \rightarrow \varinjlim \widehat{P_n}$. $K := \varprojlim K_n$ is compact in $\varprojlim P_n$. Consequently, $W(K, V)$ is open in $\widehat{\varprojlim P_n}$. Note that we have $\varphi(U) = W(K, V)$. So φ is open. This proves the lemma. □

Clearly, our proposition is a direct consequence of the lemma above. Hence, the proposition is proven. □

Let $\{D_n\}_n$ be an inductive system of Hausdorff topological groups such that all $\iota_{n,m}$ are injective and closed. By definition, $\{D_n\}_n$ is called compact oriented, if, for any compact subset $K \subset \varinjlim D_n$, there exists an index n_0 such that $K \subset D_{n_0}$.

Proposition 5.3 ([15]). *Let $\{D_n\}_n$ be a compact oriented inductive system of topological groups with structural maps $\iota_{n,m} : D_n \rightarrow D_m$ and $\iota_n : D_n \rightarrow \varinjlim D_n$. Assume (that all $\iota_{n,m}$ are injective and closed, and) that for any n, n' , there exists an n'' such that $n'' \geq n$ and $n'' \geq n'$. Then, as topological groups,*

$$\varinjlim \widehat{D_n} \simeq \widehat{\varinjlim D_n}.$$

Proof. Denote by $\widehat{\iota_{n,m}} : \widehat{D}_m \rightarrow \widehat{D}_n, f_m \mapsto f_m \circ \iota_{n,m}$, the dual of $\iota_{n,m}$. Then, for $\varprojlim_n f_n \in \varprojlim_n \widehat{D}_n$, we have $\widehat{\iota_{n,m}}(f_m) = f_n$, or equivalently, $f_m \circ \iota_{n,m} = f_n$. Hence, if $x = \varinjlim_n x_n \in \varinjlim_n D_n$, $f_m(x_m) = f_m(\iota_{n,m}(x_n)) = f_n(x_n)$ for sufficiently large $n \geq m$. Based on this, we define a natural map

$$\psi : \varprojlim_n \widehat{D}_n \rightarrow \varinjlim_n \widehat{D}_n, \quad \varprojlim_n f_n \mapsto f$$

where $f : \varinjlim_n D_n \rightarrow \mathbb{S}^1, \varinjlim_n x_n \mapsto f(x_n)$. From above discussion, f is well-defined.

Lemma 5.4 ([15]). *For ψ and f , the following holds.*

- (1) f is continuous. In particular, ψ is well-defined.
- (2) ψ is a bijection.
- (3) ψ is continuous.
- (4) ψ is open.

Proof. (1) To prove that f is continuous, let $U \subset \mathbb{S}^1$ be an open subset. For any n , $\iota_n^{-1}(f^{-1}(U)) = f_n^{-1}(U)$ is open in D_n . Hence $f^{-1}(U)$ is open. So f is continuous.

- (2) To prove that ψ is injective, we assume that $\psi(\varprojlim_n f_n) = 0$. Then $f(x_n) = f_n(x_n) = 0$ for all $x_n \in D_n$. This means that $f_n = 0$ for any n . Consequently, $\varinjlim_n f_n = 0$, and hence ψ is injective.

To show that ψ is surjective, let $f : \varinjlim_n D_n \rightarrow \mathbb{S}^1$ be a continuous map. Set $f_n = f \circ \iota_n$. Clearly, f_n is continuous. So $f_n \in \widehat{D}_n$. Moreover, for all $n' \geq n$, $f_n = f \circ \iota_n = f \circ \iota_{n'} \circ \iota_{n,n'} = f_{n'} \circ \iota_{n,n'}$. That is, $\{f_n\}_n$ forms a projective limit. Obviously, $\psi(\varprojlim_n f_n) = f$.

- (3) To prove that ψ is continuous, it suffices to show that for open subsets of $\varinjlim_n \widehat{D}_n$ in the form $W(K, V)$, $\psi^{-1}(W(K, V))$ is open in $\varprojlim_n \widehat{D}_n$, where K is a compact subset of $\varinjlim_n D_n$ and V is an open subset of \mathbb{S}^1 . By our assumptions, for any n' , $\iota_{n'}^{-1}(D_n)$ is closed. Hence $D_n \subset \varinjlim_n D_n$ are closed. So $K_n := K \cap D_n$ are compact. Since $\psi^{-1}(W(K, V)) = \varprojlim_n W(K_n, V)$, it suffices to show that $\varprojlim_n W(K_n, V)$ is open. This is a consequence of our assumptions. Indeed, since our system is compact oriented, there exists a certain n_0 such that $K = \varinjlim_n K_n \subset D_{n_0}$. Hence, $K_n = K_{n_0} = K$ for all $n \geq n_0$. $\varprojlim_n W(K_n, V) = \pi_{n_0}^{-1}(W(K_{n_0}, V))$. So, $\varprojlim_n W(K_n, V)$ is open.

- (4) To prove that ψ is open, let U be an open subset of $\varinjlim_n \widehat{D}_n$ such that $U = \pi_n^{-1}(W(K_n, V))$ for a compact subset K_n of P_n for some n , where $\pi_n : \varinjlim_n \widehat{D}_n \rightarrow \widehat{D}_n$. Then $W(K_n, V)$ is open in \widehat{D}_n . Note that we have $\psi(U) = W(K_n, V)$. So ψ is open. This proves the lemma. □

Clearly, our proposition is a direct consequence of the lemma above. Hence, the proposition is proven. □

5.3 Completeness and Compact Orientedness of Arithmetic Adelic Groups

For basis of complete topological groups, please refer to [2] and [3].

In the sequel, we use simply \mathbb{A} to denote arithmetic adelic groups $\mathbb{A}_X^{\text{fin}}$, or $\mathbb{A}_{X,01}$, or $\mathbb{A}_{X,02}$ or \mathbb{A}_X^∞ . Similarly, we use $\mathbb{A}(D)$ to denote arithmetic adelic groups $\mathbb{A}_{X,12}(D)$, or $\mathbb{A}_{X,1}(D)$, or $\mathbb{A}_{X,2}(D)$ or $\mathbb{A}_{X,1}^\infty(D)$. When \mathbb{A} and/or $\mathbb{A}(D)$ represent what will be clear according to the text involved below.

Proposition 5.5 ([15]). *The subgroups $\mathbb{A}(D)$ of \mathbb{A} are complete and hence closed.*

Proof. Since $\mathbb{A}(D)/\mathbb{A}(E)$ are complete Hausdorff locally compact, as a projective limit of complete spaces, $\mathbb{A}(D) = \varprojlim_{E:D \supset E} \mathbb{A}(D)/\mathbb{A}(E)$ is complete. It is also closed since \mathbb{A} is Hausdorff. \square

Lemma 5.6 ([15]). *Let $\{\mathbb{A}(D_n)\}_n$ be a strictly increasing sequence and $\{a_n\}_n$ be a sequence of elements of \mathbb{A} . Assume that $a_n \in \mathbb{A}(D_n) - \mathbb{A}(D_{n-1})$ for all $n \geq 1$. Then there exists an open subset U of $\varinjlim_n \mathbb{A}(D_n)$ such that $a_1, \dots, a_n, \dots \notin U$ and $a_{m+1}, \dots, a_n, \dots \notin U + \mathbb{A}(D_m)$ for all $m < n$.*

Proof. We separate the finite and infinite adeles.

[Finite Adeles] Since $\mathbb{A}(D_1)/\mathbb{A}(D_0)$ is Hausdorff, there exists an open, and hence closed, subgroup $U_1 \subset \mathbb{A}(D_1)$ such that $a_1 \notin U_1$ and $U_1 \supset \mathbb{A}(D_0)$. Since $\mathbb{A}(D_1)$ is complete and U_1 is closed in $\mathbb{A}(D_1)$, U_1 is complete as well. Now, viewing in $\mathbb{A}(D_2)$, since $\mathbb{A}(D_2)$ is Hausdorff, U_1 is a complete subgroup, so U_1 is closed in $\mathbb{A}(D_2)$. Hence $\mathbb{A}(D_2)/U_1$ is Hausdorff too. Therefore, there exists an open and hence closed subgroup $V_{2,0}$ of $\mathbb{A}(D_2)$ such that $a_1, a_2 \notin V_{2,0}$ and $V_{2,0} \supset U_1$. In addition, $\mathbb{A}(D_2)/\mathbb{A}(D_1)$ is Hausdorff, there exists an open subgroup $V_{2,1}$ such that $a_2 \notin V_{2,1}$ and $V_{2,1} \supset \mathbb{A}(D_2)$. Consequently, if we set $U_2 = V_{2,0} \cap V_{2,1}$, U_2 is an open hence closed subgroup of $\mathbb{A}(D_2)$ such that $a_1, a_2 \notin U_2$, $a_2 \notin U_2 + \mathbb{A}(D_1)$ and $U_2 \supset U_1$. So, inductively, we may assume that there exists an increasing sequence of open subgroups U_1, \dots, U_{n-1} satisfying the properties required. In particular, the following quotient groups

$$\mathbb{A}(D_n)/U_{n-1} + \mathbb{A}(D_0) (= \mathbb{A}(D_n)/U_{n-1}), \dots, \mathbb{A}(D_n)/U_{n-1} + \mathbb{A}(D_{n-1}) (= \mathbb{A}(D_{n-1}))$$

are Hausdorff. Hence there are open subgroups $V_{n,m}$, $0 \leq m \leq n-1$ of $\mathbb{A}(D_n)$ such that $a_{m+1}, \dots, a_n \notin V_{n,m}$ and $V_{n,m} \supset U_{n-1} + \mathbb{A}(D_m)$. Define $U_n := \bigcap_{m=1}^{n-1} V_{n,m}$. Then U_n is an open subgroup of $\mathbb{A}(D_n)$ satisfying $a_1, \dots, a_n \notin U_n$, $a_{m+1}, \dots, a_n \notin U_n + \mathbb{A}(D_m)$, $1 \leq m \leq n-1$ and $U_n \supset U_{n-1}$. Accordingly, if we let $U = \varinjlim_n U_n$, by definition, U is an open subgroup of $\varinjlim_n \mathbb{A}(D_n)$, and from our construction, $a_1, \dots, a_n, \dots \notin U$ and $a_{m+1}, \dots, a_n, \dots \notin U + \mathbb{A}(D_m)$, $m \geq 1$.

[Infinite Adeles] Since $\mathbb{A}(D_1)$ is Hausdorff, there exists an open subset U_1 of $\mathbb{A}(D_1)$ such that $a_1 \notin \mathbb{A}(D_1)$. Moreover, since $\mathbb{A}(D_2) \simeq \mathbb{A}(D_2)/\mathbb{A}(D_1) \oplus \mathbb{A}(D_1)$ and $\mathbb{A}(D_2)/\mathbb{A}(D_1)$ is Hausdorff, there exists an open subset U_2 of $\mathbb{A}(D_2)$ such that $a_1, a_2 \notin U_2$ and $U_2 \cap \mathbb{A}(D_1) = U_1$. In particular, $a_2 \notin U_2 + \mathbb{A}(D_1)$. Similarly, as above, with an inductive process, based on the fact that $\mathbb{A}(D_n) \simeq \mathbb{A}(D_n)/\mathbb{A}(D_{n-1}) \oplus \mathbb{A}(D_{n-1})$ and $\mathbb{A}(D_n)/\mathbb{A}(D_{n-1})$ is Hausdorff, there exists an open subset U_n of $\mathbb{A}(D_n)$ such that $a_1, \dots, a_n \notin U_n$ and $U_n \cap \mathbb{A}(D_n) = U_{n-1}$. Consequently, $a_{m+1}, \dots, a_n \notin U_n + \mathbb{A}(D_m)$, $1 \leq m \leq n-1$. In this way, we obtain an infinite increasing sequence of open subsets U_n . Let $U = \varinjlim_n U_n$.

Then we have $a_1, \dots, a_n, \dots \notin U$ and $a_{m+1}, \dots, a_n, \dots \notin U + \mathbb{A}(D_m)$, $m \geq 1$. This proves the lemma. \square

Proposition 5.7 ([15]). \mathbb{A} is complete.

Proof. Let $\{a_n\}_n$ be a Cauchy sequence of \mathbb{A} . We will show that there exists a divisor D such that $\{a_n\}_n \subset \mathbb{A}(D)$. Assume that, on the contrary, for all divisors D , $\{a_n\}_n \not\subset \mathbb{A}(D)$. Then there exists a subsequence $\{a_{k_n}\}_n$ of $\{a_n\}_n$, a (strictly increasing) sequence $\{D_n\}_n$, and an open neighborhood U of 0 in $\varinjlim_n \mathbb{A}(D_n)$ such that

- (i) $a_{k_n} \in \mathbb{A}(D_n) - \mathbb{A}(D_{n-1})$ ($n \geq 2$),
- (ii) $a_{k_1}, \dots, a_{k_n}, \dots \notin U$,
- (iii) $a_{k_{i+1}}, \dots, a_{k_n}, \dots \notin U + \mathbb{A}(D_i)$ ($i \geq 1$).

Since $\{a_{k_n}\}_n$ is not a Cauchy sequence of $\varinjlim_n \mathbb{A}(D_n)$, we get a contradiction. Therefore, there exists a divisor D such that $\{a_n\}_n \subset \mathbb{A}(D)$. But $\mathbb{A}(D)$ is complete, the Cauchy sequence $\{a_n\}_n$ is convergent in $\mathbb{A}(D)$. \square

Proposition 5.8 ([15]). \mathbb{A} is compact oriented.

Proof. Assume that for all divisors D , $K \not\subset \mathbb{A}(D)$. Then there exists a sequence $\{a_n\}_n$ in K , a (strictly increasing) sequence $\{D_n\}_n$, and an open neighborhood U of 0 in $\varinjlim_n \mathbb{A}(D_n)$ such that

- (i) $a_n \in \mathbb{A}(D_n) - \mathbb{A}(D_{n-1})$ ($n \geq 2$),
- (ii) $a_1, \dots, a_n, \dots \notin U$,
- (iii) $a_{i+1}, \dots, a_n, \dots \notin U + \mathbb{A}(D_i)$ ($i \geq 1$).

$\{U + \mathbb{A}(D_n)\}_n$ is an open covering of $K \cap \varinjlim_n \mathbb{A}(D_n)$ and admits no finite sub-covering. On the other hand, $K \cap \varinjlim_n \mathbb{A}(D_n)$ is compact, a contradiction. Indeed, since $\varinjlim_n \mathbb{A}(D_n)$ is complete and \mathbb{A} is Hausdorff, $\varinjlim_n \mathbb{A}(D_n)$ is closed in \mathbb{A} and hence $K \cap \varinjlim_n \mathbb{A}(D_n)$ is compact. This completes the proof. \square

5.4 Double Dual of Arithmetic Adelic Rings

Proposition 5.9 ([15]). As topological groups, we have the following isomorphisms.

$$(i) \quad \widehat{\mathbb{A}} \simeq \varprojlim_D \varinjlim_{E:E \leq D} (\mathbb{A}(D)/\mathbb{A}(E))^\wedge,$$

$$(ii) \quad \widehat{\widehat{\mathbb{A}}} \simeq \mathbb{A}.$$

Proof. (i)

$$\begin{aligned} \widehat{\mathbb{A}} &\simeq \left(\varinjlim_D \varprojlim_{E:E \leq D} \mathbb{A}(D)/\mathbb{A}(E) \right)^\wedge \\ &\simeq \varprojlim_D \left(\varinjlim_{E:E \leq D} \mathbb{A}(D)/\mathbb{A}(E) \right)^\wedge \\ &\simeq \varprojlim_D \varinjlim_{E:E \leq D} (\mathbb{A}(D)/\mathbb{A}(E))^\wedge. \end{aligned}$$

(ii) Remarking that $(\mathbb{A}(D)/\mathbb{A}(E))^\widehat{\sim} \simeq \mathbb{A}(D)/\mathbb{A}(E)$ since $\mathbb{A}(D)/\mathbb{A}(E)$ are Hausdorff locally compact groups,

$$\begin{aligned} \widehat{\mathbb{A}} &\simeq (\varprojlim_D \varinjlim_{E:E \leq D} (\mathbb{A}(D)/\mathbb{A}(E))^\widehat{\sim})^\widehat{\sim} \\ &\simeq \varinjlim_D (\varprojlim_{E:E \leq D} (\mathbb{A}(D)/\mathbb{A}(E))^\widehat{\sim})^\widehat{\sim} \\ &\simeq \varinjlim_D \varinjlim_{E:E \leq D} (\mathbb{A}(D)/\mathbb{A}(E))^\widehat{\sim} \\ &\simeq \varinjlim_D \varinjlim_{E:E \leq D} \mathbb{A}(D)/\mathbb{A}(E) = \mathbb{A}. \end{aligned}$$

□

5.5 Continuity of Scalar Product

Proposition 5.10 ([15]). *For a fixed element a of $\mathbb{A}_X^{\text{fin}}$, resp. of \mathbb{A}_X^∞ , the induced scalar product map: $\phi_a^{\text{fin}} : \mathbb{A}_X^{\text{fin}} \xrightarrow{\times a} \mathbb{A}_X^{\text{fin}}$, resp. $\phi_a^\infty : \mathbb{A}_X^\infty \xrightarrow{\times a} \mathbb{A}_X^\infty$ is continuous.*

Proof. If $a = 0$, there is nothing to prove. Assume, from now on, that

$$0 \neq a = (a_C) = \left(\sum_{i_C=i_{C,0}}^{\infty} h_C(a_{i_C}) \pi_C^{i_C} \right) \in \mathbb{A}_X^{\text{fin}}.$$

Here, for each C , we assume that $a_{i_{C,0}} \neq 0$. To prove that ϕ_a^{fin} is continuous, it suffices to show that for an open subgroup

$$U = (U_C) = \left(\sum_{j_C=-\infty}^{r_C-1} h_C(\mathbb{A}_{C,1}(D_{j_C})) \pi_C^{j_C} + \sum_{j_C=r_C}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{j_C} \right) \cap \mathbb{A}_X^{\text{fin}},$$

as an open neighborhood of 0, its inverse image $(\phi_a^{\text{fin}})^{-1}(U)$ contains an open subgroup. For later use, set $I_C := r_C - i_{C,0}$.

Let

$$b = (b_C) = \left(\sum_{k_C=-\infty}^{\infty} h_C(b_{k_C}) \pi_C^{k_C} \right) \in (\phi_a^{\text{fin}})^{-1}(U) \subset \mathbb{A}_X^{\text{fin}}.$$

Then, for each fixed C ,

$$a_C b_C = \sum_{l_C=-\infty}^{\infty} \left(\sum_{i_C=i_{C,0}}^{\infty} h_C(a_{i_C}) h_C(b_{l_C-i_C}) \right) \pi_C^{l_C}.$$

Recall that h_C is the Madunts-Zhukov lifting map

$$h_C : \mathbb{A}_{C,01} \simeq \left(\prod'_{x:x \in C} \mathcal{O}_{C,x} \right) / (\pi_C \prod'_{x:x \in C} \mathcal{O}_{C,x}) \xrightarrow{\text{lifting}} \prod'_{x:x \in C} \mathcal{O}_{C,x}.$$

Thus if $b_{k_C} \in \mathbb{A}_{C,01}$, we always have

$$h_C(a_{i_C}) h_C(b_{l_C-i_C}) \in \sum_{m_C=0}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{m_C}.$$

Moreover, if we write, as we can, $a_{i_C} \in \mathbb{A}_{C,1}(F_{i_C})$, $b_{k_C} \in \mathbb{A}_{C,1}(E_{k_C})$ for some divisors F_{i_C} and E_{k_C} , we have

$$h_C(a_{i_C})h_C(b_{l_C-i_C}) \in \sum_{m_C=0}^{\infty} h_C(\mathbb{A}_{C,1}(F_{i_C} + E_{l_C-i_C})) \pi_C^{m_C}.$$

Now write

$$b_C = \left(\sum_{k_C=-\infty}^{I_C-1} + \sum_{k_C=I_C}^{\infty} \right) h_C(b_{k_C}) \pi_C^{k_C}.$$

We will construct the required open subgroup according to the range of the degree index k_C .

(i) If

$$b_C \in \left(\sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{k_C} \right) \cap \left(\prod'_{x \in C} k(X)_{C,x} \right),$$

we have $a_C b_C \in U_C$.

(ii) To extend the range including to also the degree $I_C - 1$, choose a divisor E_{I_C-1} such that

$$h_C(\mathbb{A}_{C,1}(F_{i_{C,0}} + E_{I_C-1})) \subset h_C(\mathbb{A}_{C,1}(D_{r_{C-1}})).$$

Then if we choose

$$b_C \in \left(h_C(\mathbb{A}_{C,1}(E_{I_C-1})) \pi_C^{I_C-1} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{k_C} \right) \cap \left(\prod'_{x \in C} k(X)_{C,x} \right),$$

we also have $a_C b_C \in U_C$.

(iii) Similarly, to extend the range including the degree $I_C - 2$, choose a divisor E_{I_C-2} such that

$$\begin{aligned} h_C(\mathbb{A}_{C,1}(F_{i_{C,0}} + E_{I_C-2})) &\subset h_C(\mathbb{A}_{C,1}(D_{r_{C-2}})) \cap h_C(\mathbb{A}_{C,1}(D_{r_{C-1}})) \\ h_C(\mathbb{A}_{C,1}(F_{i_{C,0}+1} + E_{I_C-2})) &\subset h_C(\mathbb{A}_{C,1}(D_{r_{C-1}})). \end{aligned}$$

Then, if we choose

$$b_C \in \left(\sum_{k_C=I_C-2}^{I_C-1} h_C(\mathbb{A}_{C,1}(E_{k_C})) \pi_C^{k_C} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{k_C} \right) \cap \left(\prod'_{x \in C} k(X)_{C,x} \right),$$

then we have $a_C b_C \in U_C$.

Continuing this process repeatedly, we obtain divisors E_{k_C} 's such that, for

$$b_C \in V_C := \left(\sum_{k_C=-\infty}^{I_C-1} h_C(\mathbb{A}_{C,1}(E_{k_C})) \pi_C^{k_C} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{k_C} \right) \cap \left(\prod'_{x \in C} k(X)_{C,x} \right),$$

we have $a_C b_C \in U_C$.

Since, for all but finitely many C , $r_C \leq 0$ and $i_{C,0} \geq 0$, or better, $I_C \leq 0$. Therefore, from above discussions, we conclude that $\prod_C V_C \cap \mathbb{A}_X^{\text{fin}}$ is an open subgroup of $\mathbb{A}_X^{\text{fin}}$ and $a\left(\prod_C V_C \cap \mathbb{A}_X^{\text{fin}}\right) \subset U$.

In particular, ϕ_a^{fin} is continuous.

A similar proof works for ϕ_a^{∞} . □

5.6 Continuity of Residue Maps

Fix a non-zero rational differential ω on X . Then for an element a of $\mathbb{A}_X^{\text{fin}}$, resp. \mathbb{A}_X^∞ , induced from the natural residue pairing $\langle \cdot, \cdot \rangle_\omega$, we get a natural map $\phi_a^{\text{fin}} := \langle a, \cdot \rangle_\omega : \mathbb{A}_X^{\text{fin}} \rightarrow \mathbb{R}/\mathbb{Z}$, resp. $\phi_a^\infty := \langle a, \cdot \rangle_\omega : \mathbb{A}_X^\infty \rightarrow \mathbb{R}/\mathbb{Z}$.

Lemma 5.11 ([15]). *Let a be a fixed element in $\mathbb{A}_X^{\text{fin}}$, resp. \mathbb{A}_X^∞ . Then $\phi_a^\infty := \langle a, \cdot \rangle_\omega : \mathbb{A}_X^\infty \rightarrow \mathbb{R}/\mathbb{Z}$ is continuous. In particular, the residue map on arithmetic adèles \mathbb{A}_X^{ar} is continuous.*

Proof. We prove only for ϕ_a^{fin} , as a similar proof works ϕ_a^∞ . Write $\mathbb{A}_X^{\text{fin}} = \prod'_F F$ where the product ranges over 2-dimensional local fields. And, for each 2-dimensional local field F , fix an element t_F of F such that for equal characteristic field F , t_F is a uniformizer of F , while for mixed characteristic field F , t_F is a lift of a uniformizer of its residue field. Since the scalar product is continuous, to prove the continuity of $\langle a, \cdot \rangle_\omega$, it suffices to show that the residue map $\text{Res} : \mathbb{A}_X^{\text{fin}} \rightarrow \mathbb{R}/\mathbb{Z}$, $(x_F) \mapsto \sum_F \text{res}_F(x_F dt_F)$ is continuous. (Note that, by the definition of $\mathbb{A}_X^{\text{fin}}$, the above summation is a finite sum.) Since the open subgroup

$$\left(\sum_{i_C=-\infty}^{-1} h_C(\mathbb{A}_{C,1}(0)) \pi_C^{i_C} + \sum_{i_C=0}^{\infty} h_C(\mathbb{A}_{C,01}) \pi_C^{i_C} \right) \cap \mathbb{A}_X^{\text{fin}}$$

is contained, the kernel of the residue map is an open subgroup. This proves the lemma. \square

5.7 Self-Duality of Arithmetic Adelic Rings

We will treat both $\mathbb{A}_X^{\text{fin}}$ and \mathbb{A}_X^∞ simultaneously. So as before, we use \mathbb{A} to represent them.

Recall that, for a fixed $a \in \mathbb{A}$, the map $\langle a, \cdot \rangle_\omega : \mathbb{A} \rightarrow \mathbb{S}^1$ is continuous. Accordingly, we define a map $\varphi : \mathbb{A} \rightarrow \widehat{\mathbb{A}}$, $a \mapsto \varphi_a := \langle a, \cdot \rangle_\omega$.

Proposition 5.12 ([15]). *For the map $\varphi : \mathbb{A} \rightarrow \widehat{\mathbb{A}}$, $a \mapsto \varphi_a := \langle a, \cdot \rangle_\omega$, we have the follows.*

- (1) φ is continuous.
- (2) φ is injective.
- (3) The image of φ is dense.
- (4) φ is open.

Proof. (1) For an open subset $W(K, V)$ of $\widehat{\mathbb{A}}$, where K is a compact subset of \mathbb{A} and V is an open subset of \mathbb{S}^1 , let $U := \varphi^{-1}(W(K, V))$. Since $\widehat{\mathbb{A}} = \varprojlim_D \varinjlim_{E:E \leq D} (\mathbb{A}(D)/\mathbb{A}(E))^\wedge$, we may write

$$\chi_0 := \langle 1, \cdot \rangle_\omega \text{ as } \varprojlim_D \varinjlim_{E:E \leq D} \chi_{D/E} \text{ with } \chi_{D/E} \in (\mathbb{A}(D)/\mathbb{A}(E))^\wedge. \text{ Accordingly, write}$$

$$\begin{aligned} A_{D/E} &:= \mathbb{A}(D)/\mathbb{A}(E), \\ K_{D/E} &:= K \cap \mathbb{A}(D)/K \cap \mathbb{A}(E), \\ U_{D/E} &:= \{a_{D/E} \in A_{D/E} \mid \chi_{D/E}(a_{D/E} K_{D/E}) \subset V \text{ (} V : \text{open)}\}. \end{aligned}$$

Since, for a fixed divisor D , $\mathbb{A}(D)$ is closed in \mathbb{A} , $K \cap \mathbb{A}(D)$ is a compact subset. So, for $E \leq D$, $K_{D/E}$ is compact in $A_{D/E}$. Consequently, from the non-degeneracy of $\chi_{D/E}$ on locally compact spaces, $U_{D/E}$ is an open subset of \mathbb{A} , and $U := \varinjlim_D \varprojlim_{E:E \leq D} U_{D/E}$. We claim that U is open. Indeed, since \mathbb{A} is compact oriented, for compact K , there exists a divisor D_1 such that

$K \subset \mathbb{A}(D_1)$. On the other hand, since χ_0 is continuous, there exists a divisor D_2 such that $\mathbb{A}(D_1 + D_2) \subset \text{Ker}(\chi_0)$. Hence $U \supset \mathbb{A}(D_2)$. Thus, for a fixed D , with respect to sufficiently small $E \leq D$, we have $U_{D/E} = A_{D/E}$. This verifies that U is open, and hence proves (1), since the topology of $\widehat{\mathbb{A}}$ is generated by the open subsets of the form $W(K, V)$.

(2) This is a direct consequence of the non-degeneracy of the residue pairing. So we have (2).

(3) To prove (3), we use the fact that $\psi : \mathbb{A} \simeq \widehat{\mathbb{A}}$, where, for $a \in \mathbb{A}$, ψ_a is given by $\psi_a : \widehat{\mathbb{A}} \rightarrow \mathbb{S}^1, \chi \mapsto \chi(a)$. Thus to show that the image of φ is dense, it suffices to show that the annihilator subgroup $\text{Ann}(\text{Im}(\varphi))$ of $\text{Im}(\varphi)$ is zero. Let then $x \in \text{Ann}(\text{Im}(\varphi))$ be an annihilator of $\text{Im}(\varphi)$. Then, by definition, $\{0\} = \psi_x(\{\varphi_a \mid a \in \mathbb{A}\}) = \{\varphi_a(x) \mid a \in \mathbb{A}\}$. That is to say, $\langle a, x \rangle_\omega = 0$ for all $a \in \mathbb{A}$. But the residue pairing is non-degenerate. So, $x = 0$.

(4) This is the dual of (2). Indeed, let $U \subset \mathbb{A}$ be an open subset of \mathbb{A} . Then $U \cap \mathbb{A}(D)$ is open in $\mathbb{A}(D)$. Write

$$\begin{aligned} U_{D/E} &:= U \cap \mathbb{A}(D) / U \cap \mathbb{A}(E) \\ K_{D/E} &:= \{a_{D/E} \in A_{D/E} \mid \chi_{D/E}(a_{D/E} U_{D/E}) \subset V \text{ (} V : \text{open)}\} \end{aligned}$$

Since $\mathbb{A}(D)$ is closed, $U_{D/E}$ is open in $A_{D/E}$. This, together with the fact that $\chi_{D/E}$ is non-degenerate on its locally compact base space, implies that $K_{D/E}$ is a compact subset. Let $K := \varinjlim_D \varprojlim_{E: E \leq D} K_{D/E}$. Since U is open, there exists a divisor E such that $\mathbb{A}(E) \subset U$. This implies that there exists a divisor D such that $K \subset \mathbb{A}(D)$. Otherwise, assume that, for any D , we have $K \not\subset \mathbb{A}(D)$. Then, there exists an element $k \in K$ such that $k \notin \mathbb{A}((\omega) - E)$. Hence we have $\chi(k\mathbb{A}(E)) \neq \{0\}$, a contradiction. This then completes the proof of (4). \square

Theorem 5.13 (Sugahara-Weng [15]). *Let X be an arithmetic surface. Then, as topological groups, we have the following canonical isomorphisms.*

- (i) $\widehat{\mathbb{A}}_X^{\text{fin}} \simeq \mathbb{A}_X^{\text{fin}}$.
- (ii) $\widehat{\mathbb{A}}_X^{\infty} \simeq \mathbb{A}_X^{\infty}$.
- (iii) $\widehat{\mathbb{A}}_X^{\text{ar}} \simeq \mathbb{A}_X^{\text{ar}}$.

Proof. We have an injective continuous open morphism $\varphi : \mathbb{A} \rightarrow \widehat{\mathbb{A}}$. So it suffices to show that φ is surjective. But this is a direct consequence of the fact that φ is dense, since both \mathbb{A} and $\widehat{\mathbb{A}}$ are complete and Hausdorff. This proves the theorem. \square

6 Duality Theorem for Arithmetic Curves

Proposition 6.1 (Tate ([16], Lem 4.1.5)). *For any element $x \in F$, residue formula*

$$\sum_{v \in S} \lambda_v(\mathrm{Tr}_{F_v/\mathbb{Q}_v}(x)) = 0$$

holds.

Definition 6.2 (Tate ([16], § 4.1)). We define a pairing $\langle \cdot, \cdot \rangle$ for an adelic ring \mathbb{A}_F

$$\langle \cdot, \cdot \rangle : \mathbb{A}_F \times \mathbb{A}_F \rightarrow \mathbb{R}/\mathbb{Z}; ((x_v), (y_v)) \mapsto \sum_{v \in S} \lambda_v(\mathrm{Tr}_{F_v/\mathbb{Q}_v}(x_v y_v)).$$

Theorem 6.3 (Tate ([16], Theorems 4.1.1, 4.1.4)). *For the above pairing $\langle \cdot, \cdot \rangle$ of an adelic ring \mathbb{A}_F , we have the follows.*

- (i) $\langle \cdot, \cdot \rangle$ is perfect.
- (ii) (Self-duality) $\widehat{\mathbb{A}_F} = \mathbb{A}_F$, where \widehat{X} denote the Pontryagin duality of X .
- (iii) $F^\perp = F$.
- (iv) (Weng ([18], §1.3)) For any divisor D on Y , $\mathbb{A}_{Y,1}(D)^\perp = \mathbb{A}_{Y,1}(K_F - D)$, where K_F denote codifferent of F on \mathbb{Q} .

Theorem 6.4 (Weng ([18], Prop. 3)). *Let D be an Arakelov divisor on Y , we have the following canonical isomorphism as topological groups:*

$$H_{\mathrm{ar}}^1(\widehat{Y}, D) \simeq H_{\mathrm{ar}}^0(Y, K_F - D).$$

Since $H_{\mathrm{ar}}^0(Y, D)$ is discrete and $H_{\mathrm{ar}}^1(Y, D)$ is compact, using Fourier analysis for locally compact groups, we obtain their arithmetic counts $h_{\mathrm{ar}}^0(Y, D)$ and $h_{\mathrm{ar}}^1(Y, D)$ (See [18], Defs 2, 3).

Theorem 6.5 (Weng ([18], Theorem 2)). *For any divisor D on Y , we have the follows.*

- (i) (Arithmetic duality)

$$h_{\mathrm{ar}}^1(Y, K_F - D) = h_{\mathrm{ar}}^0(Y, D).$$

- (ii) (Arithmetic Riemann-Roch theorem)

$$h_{\mathrm{ar}}^0(Y, D) - h_{\mathrm{ar}}^1(Y, D) = \deg(D) - \frac{1}{2} \log |\Delta_F|,$$

where Δ_F denote the discriminant of F .

7 Duality Theorem for Arithmetic Surfaces

Theorem 7.1 (Sugahara-Weng [15]). *Let X be an arithmetic surface. Fix a rational differential $0 \neq \omega \in \Omega_{k(X)/F}$, and denote by $\langle \cdot, \cdot \rangle_\omega$ the natural residue pairing on the arithmetic adelic ring \mathbb{A}_X^{ar} induced by ω . Then the following holds.*

(i) For a divisor D on X , $(\mathbb{A}_{X,12}^{\text{ar}}(D))^\perp = \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)$.

(ii) $(\mathbb{A}_{X,01}^{\text{ar}})^\perp = \mathbb{A}_{X,01}^{\text{ar}}$.

(iii) $(\mathbb{A}_{X,02}^{\text{ar}})^\perp = \mathbb{A}_{X,02}^{\text{ar}}$.

Proof. (i) To prove $(\mathbb{A}_{X,12}^{\text{ar}}(D))^\perp = \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)$, we go as follows. Set $\tilde{\pi} : X \rightarrow Y \rightarrow \text{Spec } \mathbb{Z}$. Then, for an open subset U of X , the dualizing sheaf $\omega_{\tilde{\pi}}$ of $\tilde{\pi}$ can be written as

$$\omega_{\tilde{\pi}}(U) = \{\omega \in \Omega_{k(X)/\mathbb{Q}} \mid \text{Res}_{C,x}(f\omega) = 0 \ (x \in C \subset U), f \in \mathcal{O}_{X,C}\}$$

(See e.g., [8], Theorem 5.7). By a similar argument as in the proof of this result, we have, for a fixed irreducible curve C_0 ,

$$\omega_{\pi,C_0} = \{\omega \in \Omega_{k(X)/\mathbb{Q}} \mid \text{Res}_{C_0,x}(f\omega) = 0 \ (x \in C_0, f \in \mathcal{O}_{X,C_0})\}.$$

This is just the set of differentials ω satisfying $\text{ord}_{C_0}((\omega)) \geq 0$. Moreover, we also have, a fixed pair $x \in C_0$,

$$\omega_{\pi,C_0} \otimes_{\mathcal{O}_{C_0}} \mathcal{O}_{C_0,x_0} = \{\omega \in \Omega_{k(X)_{C_0,x_0}/\mathbb{Q}_{\tilde{\pi}(x_0)}} \mid \text{Res}_{C_0,x_0}(f\omega) = 0 \ (f \in \mathcal{O}_{C_0,x_0})\}.$$

These results implies that for a fixed pair (x_0, C_0) the following conditions (1), (2) are equivalent.

(1) For any $f \in \mathcal{O}_{X,C_0}$, $\text{Res}_{C_0,x_0}(f\omega) = 0$.

(2) $\text{ord}_{C_0}((\omega)) \geq 0$.

By a similar argument, we have that for a fixed closed point P_0 on X_F the following conditions (3), (4) are equivalent.

(3) For any $f \in \mathcal{O}_{X_F,P_0}$, $\text{Res}_{P_0}(f\omega) = 0$.

(4) $\text{ord}_{P_0}((\omega)) \geq 0$.

Given these facts, we conclude $(\mathbb{A}_{X,12}^{\text{ar}}(D))^\perp = \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)$.

(ii) To show $\mathbb{A}_{X,01}^{\text{ar}} \subset (\mathbb{A}_{X,01}^{\text{ar}})^\perp$, we use Theorem 4.6. By definition, elements of $\mathbb{A}_{X,01}^{\text{ar}}$ are independent of closed points $x \in X$. So we may write components of $f = (f_{C,x}) \times (f_P) \in \mathbb{A}_{X,01}^{\text{ar}}$ as $f_{C,x} = f_C$ and $f_P = f_{E_P}$, which are independent of closed points of X . For $f, g \in \mathbb{A}_{X,01}^{\text{ar}}$, we have then

$$\langle f, g \rangle_\omega = \sum_{V:\text{vertical}} \sum_{x:x \in V} \text{Res}_{V,x}(f_V g_V \omega) + \sum_{P \in X_F} \left(\sum_{x:x \in E_P} \text{Res}_{E_P,x}(f_{E_P} g_{E_P} \omega) + \text{Res}_P(f_{E_P} g_{E_P} \omega) \right) = 0.$$

Now, for vertical curves V , by Theorem 4.6(iii), we have

$$\sum_{x:x \in V} \text{Res}_{V,x}(f_V g_V \omega) = 0.$$

Furthermore, by Theorem 4.6(ii), we have, for closed points $P \in X_F$,

$$\sum_{x: x \in E_P} \text{Res}_{E_P, x}(f_{E_P} g_{E_P} \omega) + \text{Res}_P(f_{E_P} g_{E_P} \omega) = 0.$$

This then proves that $\mathbb{A}_{X,01}^{\text{ar}} \subset (\mathbb{A}_{X,01}^{\text{ar}})^{\perp}$.

Next we prove that $(\mathbb{A}_{X,01}^{\text{ar}})^{\perp} \subset \mathbb{A}_{X,01}^{\text{ar}}$. For this purpose, we use the ind-pro structure of \mathbb{A}_X^{ar} . For divisors D and vertical curves C , using (i), we obtain perfect pairings

$$\mathbb{A}_{X,12}^{\text{ar}}(D)/\mathbb{A}_{X,12}^{\text{ar}}(D-C) \times \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D + C)/\mathbb{A}_{X,12}^{\text{ar}}((\omega) - D) \rightarrow \mathbb{R}/\mathbb{Z} \quad (1)$$

induced by the residue pairing $\langle \cdot, \cdot \rangle_{\omega}$. By above result $\mathbb{A}_{X,01}^{\text{ar}} \subset (\mathbb{A}_{X,01}^{\text{ar}})^{\perp}$, we see that the perfect residue pairing $\langle \cdot, \cdot \rangle_{\omega}$ annihilates $\mathbb{A}_{X,01}^{\text{ar}} \times \mathbb{A}_{X,01}^{\text{ar}}$, hence the above perfect pairing (1) annihilates $\mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(D-C) \times \mathbb{A}_{X,1}^{\text{ar}}((\omega) - D + C)/\mathbb{A}_{X,1}^{\text{ar}}((\omega) - D)$. Note that we have isomorphisms

$$\begin{cases} \mathbb{A}_{X,12}^{\text{ar}}(D)/\mathbb{A}_{X,12}^{\text{ar}}(D-C) \simeq \mathbb{A}_{C,01}, \\ \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(D-C) \simeq \mathbb{A}_{C,0} = k(C), \\ \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D + C)/\mathbb{A}_{X,12}^{\text{ar}}((\omega) - D) \simeq \mathbb{A}_{C,01}, \\ \mathbb{A}_{X,1}^{\text{ar}}((\omega) - D + C)/\mathbb{A}_{X,1}^{\text{ar}}((\omega) - D) \simeq \mathbb{A}_{C,0} = k(C). \end{cases}$$

Hence the above pairing (1) can be regarded as a perfect pairing $\mathbb{A}_{C,01} \times \mathbb{A}_{C,01} (\rightarrow \mathbb{F}_p) \rightarrow \mathbb{R}/\mathbb{Z}$ on the vertical curve C/\mathbb{F}_p ($\{p\} = \tilde{\pi}(C)$), which annihilates $\mathbb{A}_{C,0} \times \mathbb{A}_{C,0}$. In particular, for some non-zero $\omega_C \in \Omega_{k(C)/\mathbb{F}_p}$ and a certain $(a_x) \in \mathbb{A}_{C,01}$, we can write this perfect pairing as a residue pairing

$$\langle \cdot, \cdot \rangle_{\omega_C, (a_x)} : \mathbb{A}_{C,01} \times \mathbb{A}_{C,01} \rightarrow \mathbb{R}/\mathbb{Z}; \quad ((f_x), (g_x)) \mapsto \sum_x \text{Res}_x(f_x g_x a_x \omega_C). \quad (2)$$

It is well-known that for a perfect pairing of $\mathbb{A}_{C,01}$ annihilating $\mathbb{A}_{C,0} \times \mathbb{A}_{C,0}$, we have $\mathbb{A}_{C,0}^{\perp} = \mathbb{A}_{C,0}$ (See e.g., [5], §4). Therefore, if we set $G(D) = (\mathbb{A}_{X,01}^{\text{ar}})^{\perp} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)$, we have that

$$\begin{aligned} G(D)/G(D-C) &= (\mathbb{A}_{X,1}^{\text{ar}}((\omega) - D + C)/\mathbb{A}_{X,1}^{\text{ar}}((\omega) - D))^{\perp} \\ &= \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(D-C) \end{aligned} \quad (3)$$

with respect to the perfect pairing (1) and for a certain $(a_x) \in \mathbb{A}_{C,0}$ used in the residue pairing (2). When necessary, with a possible modification on ω_C , without loss of generality, we may and will assume that $(a_x) = 1$, and write $\langle \cdot, \cdot \rangle_{\omega_C, (a_x)}$ simply as $\langle \cdot, \cdot \rangle_{\omega_C}$. In parallel, when C is horizontal, by a similar argument as above, we get the same conclusion. Consequently, using (3), we have, for any irreducible curves C_1, C_2 , the following commutative diagram with exact columns

$$\begin{array}{ccc} \mathbb{A}_{X,1}^{\text{ar}}(D-C_1)/\mathbb{A}_{X,1}^{\text{ar}}(D-C_1-C_2) & \xlongequal{\quad} & G(D-C_1)/G(D-C_1-C_2) \\ \downarrow & & \downarrow \\ \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(D-C_1-C_2) & \longrightarrow & G(D)/G(D-C_1-C_2) \\ \downarrow & & \downarrow \\ \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(D-C_1) & \xlongequal{\quad} & G(D)/G(D-C_1). \end{array}$$

This then proves that the horizontal map in the middle is surjective. Furthermore, the horizontal map in the middle is injective, since, from above, $\mathbb{A}_{X,01}^{\text{ar}} \subset (\mathbb{A}_{X,01}^{\text{ar}})^{\perp}$. Therefore, we have

$$G(D)/G(D - C_1 - C_2) = \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(D - C_1 - C_2).$$

Repeating a similar argument as above, we have, for any divisors $D \geq E$

$$G(D)/G(E) = \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(E). \quad (4)$$

Since we have

$$\mathbb{A}_X^{\text{ar}} = \lim_{\substack{\longrightarrow \\ D}} \lim_{\substack{\longleftarrow \\ E: E \leq D}} \mathbb{A}_{X,12}^{\text{ar}}(D)/\mathbb{A}_{X,12}^{\text{ar}}(E),$$

using (4), we conclude that

$$\begin{aligned} (\mathbb{A}_{X,01}^{\text{ar}})^\perp &= \lim_{\substack{\longrightarrow \\ D}} \lim_{\substack{\longleftarrow \\ E: E \leq D}} G(D)/G(E) \\ &= \lim_{\substack{\longrightarrow \\ D}} \lim_{\substack{\longleftarrow \\ E: E \leq D}} \mathbb{A}_{X,1}^{\text{ar}}(D)/\mathbb{A}_{X,1}^{\text{ar}}(E) = \mathbb{A}_{X,01}^{\text{ar}}. \end{aligned}$$

This proves that $(\mathbb{A}_{X,01}^{\text{ar}})^\perp = \mathbb{A}_{X,01}^{\text{ar}}$.

- (iii) Next we prove $\mathbb{A}_{X,02}^{\text{ar}} \subset (\mathbb{A}_{X,02}^{\text{ar}})^\perp$. By definition, since elements of $\mathbb{A}_{X,02}^{\text{ar}}$ are independent of irreducible curves C of X and closed points $P \in X_F$, we may write components of $f = (f_{C,x}) \times (f_P) \in \mathbb{A}_{X,02}^{\text{ar}}$ as $f_{C,x} = f_x$ and $f_P = f_\infty$, which are independent of irreducible curves C and closed points P . Hence, for $f, g \in \mathbb{A}_{X,02}^{\text{ar}}$, we have

$$\langle f, g \rangle_\omega = \sum_x \sum_{C: x \in C} \text{Res}_{C,x}(f_x g_x \omega) + \sum_P \text{Res}_P(f_\infty g_\infty \omega) = 0.$$

Now, by Theorem 4.6(i), we have, for any closed point x ,

$$\sum_{C: x \in C} \text{Res}_{C,x}(f_x g_x \omega) = 0.$$

Furthermore,

$$\sum_P \text{Res}_P(f_\infty g_\infty \omega) = 0$$

since we can use a standard residue formula for a curve X_F/F (See [14], §II. 7, Prop. 6).

Therefore, $\mathbb{A}_{X,02}^{\text{ar}} \subset (\mathbb{A}_{X,02}^{\text{ar}})^\perp$.

Finally, we show that $(\mathbb{A}_{X,02}^{\text{ar}})^\perp \subset \mathbb{A}_{X,02}^{\text{ar}}$. Fix a vertical curve C . Then, for any divisor D on X , we have

$$\begin{cases} \mathbb{A}_{X,2}^{\text{ar}}(D)/\mathbb{A}_{X,2}^{\text{ar}}(D - C) \simeq \mathbb{A}_{C,1}(D|_C), \\ \mathbb{A}_{X,2}^{\text{ar}}((\omega) - D + C)/\mathbb{A}_{X,2}^{\text{ar}}((\omega) - D) \simeq \mathbb{A}_{C,1}((\omega'_C) - D|_C), \end{cases}$$

for a certain $\omega'_C \in \Omega_{k(C)/\mathbb{F}_p}$ satisfying $(\omega'_C) = ((\omega) + C)|_C$ by adjunction formula (see [6], Theorem 3.6). We claim that $(\omega_C) = (\omega'_C)$. Indeed, since $\mathbb{A}_{C,1}(D|_C)^\perp = \mathbb{A}_{C,1}((\omega_C) - D|_C)$ and $\mathbb{A}_{X,02}^{\text{ar}} \subset (\mathbb{A}_{X,02}^{\text{ar}})^\perp$, $\mathbb{A}_{C,1}((\omega'_C) - D|_C) \subset \mathbb{A}_{C,1}((\omega_C) - D|_C)$. This implies that $(\omega_C) \geq (\omega'_C)$ and hence $(\omega_C) = (\omega'_C)$, because there is no $f \in k(C)$ such that $(f) > 0$. Thus, for a residue pairing (2), we have $(\mathbb{A}_{C,1}((\omega'_C) - D|_C))^\perp = \mathbb{A}_{C,1}(D|_C)$. By a similar argument as a proof of (4), we have that for any integers $n \geq m \in \mathbb{Z}$,

$$H(nC)/H(mC) = \mathbb{A}_{X,2}^{\text{ar}}(nC)/\mathbb{A}_{X,2}^{\text{ar}}(mC), \quad (5)$$

where we set $H(D) = (\mathbb{A}_{X,02}^{\text{ar}})^{\perp} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)$. Since we have

$$\mathbb{A}_X^{\text{ar}} = \lim_{\substack{\longrightarrow \\ D}} \lim_{\substack{\longleftarrow \\ E: E \leq D}} \mathbb{A}_{X,12}^{\text{ar}}(D) / \mathbb{A}_{X,12}^{\text{ar}}(E),$$

using (5), we get

$$\begin{aligned} (\mathbb{A}_{X,02}^{\text{ar}})^{\perp} &= \lim_{\substack{\longrightarrow \\ D}} \lim_{\substack{\longleftarrow \\ E: E \leq D}} H(D) / H(E) \\ &\supset \lim_{\substack{\longrightarrow \\ n}} \lim_{\substack{\longleftarrow \\ m \leq n}} H(nC) / H(mC) \\ &= \lim_{\substack{\longrightarrow \\ n}} \lim_{\substack{\longleftarrow \\ m \leq n}} \mathbb{A}_{X,2}^{\text{ar}}(nC) / \mathbb{A}_{X,2}^{\text{ar}}(mC) \\ &= \prod'_{x: x \in C} k(X)_x. \end{aligned}$$

Hence for any $x \in C$, a (x, C) -component of $(\mathbb{A}_{X,02}^{\text{ar}})^{\perp}$ is an element of $k(X)_x$ (*). To prove the rest, we take a classical approach with a use of Chinese remainder theorem, using an idea in the proof of Prop. 1 of [12]. To be more precise, fix $x_0 \in X$ and a hyperplane H on X satisfying that $x_0 \in H$. Setting $V = \text{Spec } \mathcal{O}_{X,x_0} - H$, V is affine. We claim that for any family of prime divisors D_j on V such that $D_i \cap D_j = \emptyset$ if $i \neq j$, and any rational functions f_0, f_1, \dots, f_n on V , and any fixed divisor D supported on D_i 's, there exists a rational function g such that

$$\begin{cases} \text{ord}_{D_i}(f_i - g) \geq \text{ord}_{D_i}(D) & (i = 1, \dots, n) \\ \text{ord}_{D_i}(g) \geq \text{ord}_{D_i}(D) & (i \neq 1, \dots, n). \end{cases}$$

Indeed, by clearing the common denominators for f_i 's (with a modification of f_i 's if necessary), we may assume that f_i 's are all regular. Then by applying the Chinese remainder theorem to the fractional ideals $\mathcal{P}_i^{\text{ord}_{D_i}(D)}$, $i = 0, \dots, n$, and $\bigcap_{i \notin \{0, 1, \dots, n\}} \mathcal{P}_i^{\text{ord}_{D_i}(D)}$, where \mathcal{P}_i are the prime ideals associated to the prime divisors D_i , we see the existence of such a g . Associated to $f \in (\mathbb{A}_{X,02}^{\text{ar}})^{\perp}$, form a new adèle $f' \in \mathbb{A}_{X,012}^{\text{ar}}$ by setting $f'_{C,x} = f_{C,x} - f_{H,x}$ where H is a fixed vertical hyperplane through x . By (*), the definition of $f'_{C,x}$ is well-defined. Then

$$\begin{aligned} \sum_{C: x \in C} \text{Res}_{C,x}(f'_{C,x} g_x \omega) &= \sum_{C: x \in C} \text{Res}_{C,x}(f_{C,x} g_x \omega) - \sum_{C: x \in C} \text{Res}_{C,x}(f'_{H,x} g_x \omega) \\ &= 0 - 0 = 0. \end{aligned}$$

Now, applying the above existence to obtain a g satisfying that for any fixed rational function f_0 and any fixed curve $C_0 \ni x$, we have

$$\begin{cases} \text{ord}_C(f'_{C,x}) + \text{ord}_C(f_0 - g) + \text{ord}_C((\omega)) \geq 0 & (C = C_0) \\ \text{ord}_C(f'_{C,x}) + \text{ord}_C(g) + \text{ord}_C((\omega)) \geq 0 & (C \neq C_0, H). \end{cases}$$

Consequently, by the definition of the residue map, with $f'_{H,x} = 0$ in mind, we get, for any $f_0 \in k(X)$ and the corresponding g just chosen,

$$0 = \sum_{C: x \in C} \text{Res}_{C,x}(f'_{C,x} g \omega) = \text{Res}_{C_0,x}(f'_{C_0,x} g \omega) = \text{Res}_{C_0,x}(f'_{C_0,x} f_0 \omega).$$

Since the last quantity is always zero for all f_0 , this then implies that $f'_{C_0,x} = 0$, namely, $f_{C_0,x} = f_{H,x_0}$. To end the proof of (iii), we still need to show that $f_P = f_{P_0}$ for a fixed $P_0 \in X_F$ and all $P \in X_F$. But this is amount to a use of a similar argument just said again, based on Chinese remainder theorem (See [5], §4). Thus, if $f = (f_{C,x}) \times (f_P) \in (\mathbb{A}_{X,02}^{\text{ar}})^{\perp}$, then $f_{C,x} = f_{C_0,x}$ and $f_P = f_{P_0}$ for fixed C_0 and P_0 . Therefore $f \in \mathbb{A}_{X,02}^{\text{ar}}$. That is to say, $(\mathbb{A}_{X,02}^{\text{ar}})^{\perp} = \mathbb{A}_{X,02}^{\text{ar}}$. This proves (iii). □

Theorem 7.2 (Duality theorem, Sugahara-Weng [15]). *Let $0 \neq \omega \in \Omega_{k(X)/F}$. For any $i \in \{0, 1, 2\}$, we have isomorphisms as topological groups*

$$H_{\text{ar}}^i(\widehat{X, D}) \simeq H_{\text{ar}}^{2-i}((\omega) - D).$$

Proof.

(1) Duality between H_{ar}^0 and H_{ar}^2

$$\begin{aligned} H_{\text{ar}}^0(\widehat{X, D}) &= (\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D))^\wedge \\ &\simeq \mathbb{A}_X^{\text{ar}} / (\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D))^\perp \\ &= \mathbb{A}_X^{\text{ar}} / (\mathbb{A}_{X,01}^{\text{ar}})^\perp + (\mathbb{A}_{X,02}^{\text{ar}})^\perp + (\mathbb{A}_{X,12}^{\text{ar}}(D))^\perp \\ &= \mathbb{A}_X^{\text{ar}} / \mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D) = H_{\text{ar}}^2(X, (\omega) - D), \end{aligned}$$

(2) Duality among H_{ar}^1

$$\begin{aligned} H_{\text{ar}}^1(\widehat{X, D}) &\simeq \left(\frac{\mathbb{A}_{X,02}^{\text{ar}} \cap (\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D))}{\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)} \right)^\wedge \\ &\simeq \frac{(\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}})^\perp \cap (\mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D))^\perp}{(\mathbb{A}_{X,02}^{\text{ar}})^\perp + (\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D))^\perp} \\ &= \frac{(\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}) \cap (\mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D))}{\mathbb{A}_{X,02}^{\text{ar}} + (\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D))} \\ &\simeq \frac{(\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}) \cap \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)}{(\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}}) + (\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D))} \simeq H_{\text{ar}}^1(X, (\omega) - D). \end{aligned}$$

□

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