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# 博士学位論文

# Uniqueness and stability for double crystals

# (ダブルクリスタルの一意性と安定性)

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## Abstract

We study a mathematical model of small double crystals, that is, two connected regions in $\mathbf{R}^{n+1}$ with prescribed volumes and with surface tension depending on the direction of the each point of the surface. Each double crystal is a critical point of the anisotropic surface energy which is the integral of the surface tension over the surface. We derive the first and the second variation formulas of the energy functional. For $n = 1$ and a certain special energy density function, we classify the double crystals in terms of symmetry and the given areas. Also, we prove that some of the double crystals are unstable, that is they are not local minimizers of the energy.


# 1 Introduction

There was a long-standing conjecture which was called the double bubble conjecture. It says that the standard double bubble provides the least-perimeter way to enclose and separate two given volumes, here the standard double bubble is consisting of three spherical caps meeting along a common circle at 120 degree angles. This conjecture had been believed since about 1870, and was proved in 2002. The existence of the minimizer was proved by F. J. Almgren ([1]) in 1976. (This paper proved, more general case, minimizing surface enclosing $k$ prescribed volumes in $\mathbf{R}^{n+1}$, using geometric measure theory.)In 1993, the double bubble conjecture was proved in the plane by Joel Foisy et al ([3]) advised by Frank Morgan. For higher dimensional case, M. Hutchings ([4]) proved that any minimizer is axially symmetric and he also obtained a bound of the number of connected components of the two regions of a minimizer. Using these results, finally in 2002, the double bubble conjecture was proved by M. Hutchings et al ([5]) in $\mathbf{R}^3$, and a student of Morgan extended it to higher dimensions ([12]).

Double bubbles are a mathematical model of soap bubbles. The energy functional is the total area of the surface. On the other hand, when we think about a mathematical model of anisotropic substance like crystals, we need to consider the energy density function $\gamma : S^n \to \mathbf{R}^+$ depending on the normal direction $N$ of the surface, where $S^n := \{X \in \mathbf{R}^{n+1} \mid \|X\| = 1\}$ is the $n$-dimensional unit sphere in $\mathbf{R}^{n+1}$. $\gamma$ is called an anisotropic energy density function, and its sum (integral) $\mathcal{F} = \int_\Sigma \gamma(N)d\Sigma$ over the surface $\Sigma$ is called an anisotropic (surface) energy. The surface is a constant anisotropic mean curvature (CAMC) surface if it is a critical point of the anisotropic energy for all volume preserving variations. CAMC surfaces are a generalization of CMC (constant mean curvature) surfaces.

In this paper, we extend the double bubble problem to a double crystal (DC) problem, that is, we minimize the anisotropic energy instead of the surface area. The solutions are a mathematical model of multiple crystals.

There were some previous researches relating to the DC problem. Gary R. Lawlor ([10]) determined the energy-minimizer for the case where each energy density function $\gamma_i$ $(i = 0, 1, 2)$ is constant (we consider three surfaces, so we need three kinds of energy density functions). We remark that these $\gamma_i$ are isotropic. His work also means that he gave a new proof of the double bubble conjecture. For $n = 1$, for a norm $\gamma$ in $\mathbf{R}^2$, Frank Morgan et al. ([11]) studied the shapes of the minimizers of the total anisotropic energy $\mathcal{F}$ of curves among curves enclosing prescribed $k$ areas. Especially, they determined the shapes of the all minimizers for the case of $\gamma_i(\nu_1, \nu_2) = |\nu_1| + |\nu_2|$ $(i = 0, 1, 2)((\nu_1, \nu_2) \in S^1)$.

Recall that there is a unique hypersurface that minimizes $\mathcal{F}$ among all closed hypersurfaces enclosing the same volume. This surface is known as the Wulff shape. In this paper, we assume that the Wulff shape is smooth. We will derive the first variation formula for the anisotropic energy $\mathcal{F}$ (Theorem 3.1), and obtain the conditions for a surface $\Sigma$ to be a double crystal (Theorem 3.2). Also, we will obtain the second variation formula for $\mathcal{F}$ (Theorem 3.3) and obtain the condition for a double crystal to be stable. For $n = 1$, we will consider a special energy density function $\gamma := \gamma_1 = \gamma_2 = \gamma_0$ satisfying

$$\gamma(\nu_1, \nu_2) = \left(\nu_1^{2p} + \nu_2^{2p}\right)^{1-\frac{1}{2p}} / \sqrt{\nu_1^{4p-2} + \nu_2^{4p-2}}.$$

We classify the double crystals in terms of symmetry and the given areas. Also, we prove that some of the double crystals are unstable, that is they are not local minimizers of the energy.

We will explain our problem more precisely in §2. In §3.1 , we derive the first and the second variation formulas of the anisotropic surface energy. In §3.2, we study the DC problem in the plane.

In §4, we study the case where the Wulff shape is not necessarily smooth. We try to generalize the research by Morgan et al. ([11]). mentioned above to higher dimensional case. In $\mathbf{R}^3$, for a norm $\gamma$ in $\mathbf{R}^3$, under some additional assumptions, we prove that each minimizer of the anisotropic energy $\mathcal{F}$ among surfaces enclosing prescribed $k$ volumes consists of parts of rescalings of the Wulff shape and $\gamma$-minimizing surfaces (§4.1). Also, again under some additional assumptions, we determine the shapes of the minimizers for the case where $k = 2$ and $\gamma(\nu_1, \nu_2, \nu_3) = |\nu_1| + |\nu_2| + |\nu_3|$ ($(\nu_1, \nu_2, \nu_3) \in S^2$).

## 2 Preliminaries

In this section, first we introduce some fundamental facts about CAMC surfaces (for details, see [6]). Then, we formulate the DC problem.

Let $\gamma : S^n \to \mathbf{R}^+$ be a positive smooth function on the unit sphere $S^n$ in $\mathbf{R}^{n+1}$. We call this function $\gamma$ an anisotropic energy density function. Let $\Sigma$ be an $n$-dimensional oriented compact $C^\infty$ manifold with or without boundary. And let $X : \Sigma \to \mathbf{R}^{n+1}$ be an immersion with Gauss map (unit normal) $N : \Sigma \to S^n$ be its Gauss map. The anisotropic energy of $X$ is defined as

$$\mathcal{F}(X) = \int_\Sigma \gamma(N) d\Sigma,$$

where $d\Sigma$ is the volume form on $\Sigma$ induced by $X$. Any smooth variation $\tilde{X} : \Sigma \times [-\varepsilon_0, \varepsilon_0] \to \mathbf{R}^{n+1}$ ($\varepsilon_0 > 0$) of $X$ can be represented as $\tilde{X}(*, \varepsilon) = X_\epsilon = X + \epsilon(Z + \varphi N) + \mathcal{O}(\epsilon^2)$, where $Z$ is tangent to $X$. The first variation of $\mathcal{F}$ for this variation is (cf. Proof of Proposition 3.1 in [6])

$$\begin{aligned} \delta\mathcal{F} &:= \frac{d}{ds}\Big|_{\varepsilon=0} \mathcal{F}(X_\varepsilon) \\ &= \int_\Sigma \varphi(\mathrm{div}_\Sigma D\gamma - nH\gamma) d\Sigma + \oint_{\partial\Sigma} -\varphi\langle D\gamma, \nu\rangle + \gamma\langle Z, \nu\rangle \, ds, \end{aligned} \tag{1}$$

where $D$ is the gradient on $S^n$, $H$ is the mean curvature of $X$, $\nu$ is the outward pointing unit conormal of $X$ along $\partial\Sigma$, and $ds$ is the $(n-2)$-dimensional volume form of $\partial\Sigma$. $\Lambda := -\mathrm{div}_\Sigma D\gamma + nH\gamma$ is called the anisotropic mean curvature of $X$. $X$ is called a Constant Anisotropic Mean Curvature (CAMC) hypersurface when $\Lambda \equiv$ constant. We remark that $X$ is CAMC if and only if $\delta\mathcal{F} = 0$ for all compactly-supported $(n+1)$-dimensional-volume-preserving variations. For $\gamma \equiv 1$, we get $\Lambda = nH$. It means that CAMC surface is a generalization of CMC surface.

It is known that there is a unique (up to translation in $\mathbf{R}^{n+1}$) minimizer of $\mathcal{F}$ among all closed hypersurfaces enclosing the same volume ([15]), and it is a rescaling of the so-called Wulff shape. The Wulff shape (we denote it by $W$) is a closed convex hypersurface defined by

$$W := \partial \bigcap_{N \in S^n} \{w \in \mathbf{R}^{n+1} | \langle w, N\rangle \le \gamma(N)\}$$

When the $W$ is smooth and strictly convex (that is, all principal curvatures are positive with respect to the inward normal. This condition is equivalent to the condition that $A := D^2\gamma + \gamma \cdot 1$ is positive definite at earh $N \in S^n$, where $D^2\gamma$ is the Hessian of $\gamma$ on $S^n$, and 1 is the identity map on $T_N S^n$. This condition is called the convexity condition), $W$ can be parametrized as an embedding $\Phi : S^n \to W \subset \mathbf{R}^n$:

$$\Phi(N) = D\gamma + \gamma(N)N.$$

The anisotropic mean curvature of $W$ is $n$ with respect to the inward normal.

# 3 Double Crystals problem for smooth Wulff shapes

In this section, we assume that the convexity condition is satisfied.

For later use, we give a new representation of the 1st variation formula:

**Lemma 3.1.** The first variation of $\mathcal{F}$ for the variation $X_\epsilon = X + \epsilon Y + \mathcal{O}(\epsilon^2)$ is

$$\delta\mathcal{F} = -\int_\Sigma \varphi\Lambda \, d\Sigma + \oint_{\partial\Sigma} \langle \Phi, -\varphi\nu + fN\rangle \, ds,$$

where $\varphi := \langle Y, N\rangle$ and $f := \langle Y, \nu\rangle$.

*Proof.* We compute the integrand of the second term of (1).

$$\begin{aligned} -\varphi\langle D\gamma, \nu\rangle + \gamma\langle Z, \nu\rangle &= \langle -\varphi(D\gamma + \gamma N), \nu\rangle + \gamma f \\ &= \langle \Phi, -\varphi\nu\rangle + \langle D\gamma + \gamma N, N\rangle f \\ &= \langle \Phi, -\varphi\nu + fN\rangle. \end{aligned}$$

□

If $n = 1$, curves with constant anisotropic mean curvature are completely determined as follows:

**Lemma 3.2.** Let $n = 1$ and $X : \mathbf{R} \supset I \to \mathbf{R}^2$ be a curve parametrized by arc-length. Then,

$$\Lambda = \kappa/\kappa_W,$$

where $\Lambda$ is the anisotropic mean curvature of $X$, $\kappa$ is the curvature of $X$, and $\kappa_W$ is the curvature of the Wulff shape $W$.

*Proof.* We denote by $\theta$ a point $e^{i\theta}$ in $S^1$. Then, the Wulff shape $W$ is represented by an embedding $\Phi : S^1 \to \mathbf{R}^2$ defined as

$$\Phi(\theta) = \gamma_\theta(\theta)(-\sin\theta, \cos\theta) + \gamma(\theta)(\cos\theta, \sin\theta).$$

Set $X(s) = (x(s), y(s))$. Then, the Gauss map $N$ of $X$ is

$$N(s) = (-y'(s), x'(s)) =: (\cos\theta(s), \sin\theta(s)).$$

Hence, the anisotropic mean curvature $\Lambda$ of $X$ is

$$\Lambda(s) = -\gamma_{\theta s} - \kappa\gamma = -\gamma_{\theta\theta}\theta_s - \kappa\gamma = -\kappa(\gamma_{\theta\theta} + \gamma). \tag{2}$$

On the other hand,

$$\frac{d\Phi}{d\theta} = (\gamma_{\theta\theta} + \gamma)(-\sin\theta, \cos\theta),$$

$$\frac{d^2\Phi}{d\theta^2} = (\gamma_{\theta\theta\theta} + \gamma_\theta)(-\sin\theta, \cos\theta) - (\gamma_{\theta\theta} + \gamma)(\cos\theta, \sin\theta).$$

Hence, by elementary calculations, the curvature $\kappa_W$ of $W$ with respect to the outward pointing unit normal is

$$\kappa_W = \frac{-1}{\gamma_{\theta\theta} + \gamma}. \tag{3}$$

(2) with (3) gives the desired formula. □

**Proposition 3.1.** *If the anisotropic mean curvature $\Lambda$ of a curve $X$ is constant, then either*

1. *$X$ is (a part of) a straight line (when $\Lambda = 0$), or*
2. *$X$ is a part of the Wulff shape up to translation and homothety (when $\Lambda \neq 0$).*

*Proof.* By Lemma 3.2, the curvatureof $X$ is $\kappa = \Lambda\kappa_W$. Hence, by the fundamental theorem for plane curves, we obtain the desired result. □

*Remark* 3.1. For $n \geq 2$, we have great many varieties of CAMC hypersurfaces. For example, [8, §5] gives two parameter family of axisymmetric CAMC surfaces.

Let us explain our problem more precisely. Let $\Sigma_1, \Sigma_2, \Sigma_0$ be three piecewise smooth oriented connected compact hypersurfaces in $\mathbf{R}^{n+1}$ with common boundary $C$ such that $\Sigma_1 \cup \Sigma_0$ (resp. $\Sigma_2 \cup \Sigma_0$) encloses a region $R_1$ (resp. $R_2$) with prescribed volume $V_1$ (resp. $V_2$), and let $\gamma_i$ be energy density functions on $\Sigma_i$. We study the following anisotropic energy of the surface $\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_0$ :

$$\mathcal{F}(\Sigma) := \sum_{i=0}^{2} \int_{\Sigma_i} \gamma_i(N_i) d\Sigma_i, \tag{4}$$

where $N_i : \Sigma_i \to S^n$ is the unit normal vector field along $\Sigma_i$ (we refer to Figure 1 about the directions of $N_i$) and $d\Sigma_i$ is the $n$-dimensional volume form on $\Sigma_i$. The volumes $V_i$ of the region $R_i$ is given by

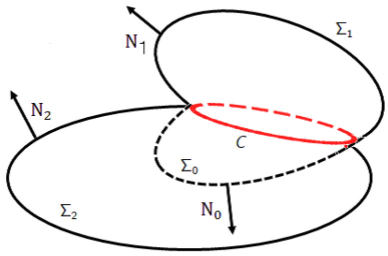


Figure 1: An admissible surface $\Sigma$ in $\mathbf{R}^3$. The red curve $C$ is the common boundary of $\Sigma_1$, $\Sigma_2$ and $\Sigma_0$.We always assume that $\Sigma_0$ is in the middle.

$$\begin{aligned} V_1 &= \frac{1}{n+1}\left\{\int_{\Sigma_1} \langle x_1, N_1\rangle d\Sigma_1 + \int_{\Sigma_0} \langle x_0, N_0\rangle d\Sigma_0\right\}, \\ V_2 &= \frac{1}{n+1}\left\{\int_{\Sigma_2} \langle x_2, N_2\rangle d\Sigma_2 - \int_{\Sigma_0} \langle x_0, N_0\rangle d\Sigma_0\right\}. \end{aligned}$$

Our problem is to study the minimizers of $\mathcal{F}$ among $\Sigma$'s such that $R_1$, $R_2$ have prescribed volumes $V_1$, $V_2$, respectively.

## 3.1 Variation formulas

Throughout this section, $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0$ is such the union of smooth hypersurfaces $\Sigma_0$, $\Sigma_1$, and $\Sigma_2$ with common boundary $C$ as in the last part of §2. We derive the first variation formula for the functional $\mathcal{F}$ defined by (4), and obtain the conditions for critical points.

Let $\tilde{X} : \Sigma \times [-\varepsilon_0, \varepsilon_0] \to \mathbf{R}^{n+1}$ ($\varepsilon_0 > 0$) be a variation of $X : \Sigma \to \mathbf{R}^{n+1}$. $\tilde{X}$ is called an admissible variation if the two volumes $V_1$, $V_2$ are preserved. Such $\tilde{X}$ can be represented as $\tilde{X}(x, \varepsilon) = X_\varepsilon = X + \varepsilon Y + \mathcal{O}(\varepsilon^2)$, and $Y$ is called an admissible variation vector field of $X$. If $Y$ is admissible, then

$$\begin{aligned} \delta V_1 &:= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} V_1(X_\varepsilon) = \int_{\Sigma_1} \langle Y, N_1 \rangle d\Sigma_1 + \int_{\Sigma_0} \langle Y, N_0 \rangle d\Sigma_0 = 0, \quad (5) \\ \delta V_2 &:= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} V_2(X_\varepsilon) = \int_{\Sigma_2} \langle Y, N_2 \rangle d\Sigma_2 - \int_{\Sigma_0} \langle Y, N_0 \rangle d\Sigma_0 = 0. \quad (6) \end{aligned}$$

hold. By a suitable reparametrization of $\tilde{X}$, we may assume that, at each point on $C$, $Y$ is orthogonal to the ($(n-1)$-dimensional) tangent space of $C$. Then, the boundary condition implies the following:

$$Y = \langle Y, N_1 \rangle N_1 + \langle Y, \nu_1 \rangle \nu_1 = \langle Y, N_2 \rangle N_2 + \langle Y, \nu_2 \rangle \nu_2 = \langle Y, N_0 \rangle N_0 + \langle Y, \nu_0 \rangle \nu_0 \quad (7)$$

hold on $C$, where $\nu_i$ is the outward pointing conormal vector for $\Sigma_i$ along $C$.

**Lemma 3.3.** Let $\varphi_i, f_i : \Sigma_i \to \mathbf{R}$ be smooth functions on $\Sigma_i$ satisfying

(i) $\displaystyle\int_{\Sigma_1} \varphi_1 \, d\Sigma_1 + \int_{\Sigma_0} \varphi_0 \, d\Sigma_0 = 0$, $\displaystyle\int_{\Sigma_2} \varphi_2 \, d\Sigma_2 - \int_{\Sigma_0} \varphi_0 \, d\Sigma_0 = 0$, and

(ii) $\varphi_1 N_1 + f_1 \nu_1 = \varphi_2 N_2 + f_1 \nu_2 = \varphi_0 N_0 + f_0 \nu_0$ on $C$.

Then there exists an admissible variation such that the normal (resp. conormal to $C$) component of the variation vector field $Y$ are $\varphi_i N_i$ (resp. $f_i \nu_i$).

*Proof.* We give functions $h_i : \Sigma_i \to \mathbf{R}$ ($i = 1, 2$) such that $\int_{\Sigma_i} h_i d\Sigma_i \neq 0$ holds and each $h_i$ has compact support on the interior of $\Sigma_i$. And we extend each function $h_i$ to 0 on $\Sigma \setminus \Sigma_i$. On the other hand, set

$$Y := \varphi_i N_i + f_i \nu_i \quad \text{on } \Sigma_i, \ i = 0, 1, 2.$$

Then, $Y$ gives a variation vector field of $\Sigma$. Set

$$\begin{aligned} X(s, t_1, t_2) &:= X + sY + t_1 h_1 N_1 + t_2 h_2 N_2, \\ V_i(s, t_1, t_2) &:= V_i(X(s, t_1, t_2)), \quad i = 1, 2. \end{aligned}$$

Set $V_1^0 := V_1(0, 0, 0)$, $V_2^0 := V_2(0, 0, 0)$. Consider the following simultaneous equations.

$$V_1(s, t_1, t_2) = V_1^0, \quad V_2(s, t_1, t_2) = V_2^0.$$

Differentiate $V_1$, $V_2$ at $(s, t_1, t_2) = (0, 0, 0)$ to obtain

$$\frac{\partial V_1}{\partial s}(0,0,0) = \int_{\Sigma_1} \varphi_1 d\Sigma_1 + \int_{\Sigma_0} \varphi_0 d\Sigma_0 = 0,$$

$$\frac{\partial V_2}{\partial s}(0,0,0) = \int_{\Sigma_2} \varphi_2 d\Sigma_2 - \int_{\Sigma_0} \varphi_0 d\Sigma_0 = 0,$$

$$\frac{\partial V_i}{\partial t_j}(0,0,0) = \delta^i_j \int_{\Sigma_i} h_j \, d\Sigma_i \begin{cases} \neq 0, & i = j, \\ = 0, & i \neq j. \end{cases}$$

Therefore, by the implicit function theorem, in there exist a neighborhood $I$ of $s = 0$ and smooth functions $t_1 = t_1(s)$, $t_2 = t_2(s)$ such that $t_1(0) = 0$, $t_2(0) = 0$, $\tilde{V}_1(s) := V_1(s, t_1(s), t_2(s)) = V_1^0$, $\tilde{V}_2(s) := V_1(s, t_1(s), t_2(s)) = V_2^0$ $(s \in I)$. Then,

$$0 = \tilde{V}_i{}'(s) = (V_i)_s + (V_i)_{t_1} t_1'(s) + (V_i)_{t_2} t_2'(s), \quad (i = 1, 2)$$

holds. Hence,

$$\begin{aligned} t_1'(0) &= -\frac{(V_1)_s(0,0,0) + (V_1)_{t_2}(0,0,0)t_2'(0)}{(V_1)_{t_1}(0,0,0)} = 0, \\ t_2'(0) &= -\frac{(V_2)_s(0,0,0) + (V_2)_{t_1}(0,0,0)t_1'(0)}{(V_2)_{t_2}(0,0,0)} = 0. \end{aligned}$$

Consequently,

$$X(s, t_1(s), t_2(s)) = X + sY + t_1(s)h_1N_1 + t_2(s)h_2N_2 = X + sY + \mathcal{O}(s^2)$$

is an admissible variation of $\Sigma$, and so we obtain the desired result. $\square$

Using Lemma 3.1, we immediately obtain the following:

**Theorem 3.1** (First variation formula)**.** For a variation $X_\epsilon = X + \epsilon Y + \mathcal{O}(\epsilon^2)$ of $\Sigma$, the first variation of the anisotropic energy $\mathcal{F}$ is

$$\begin{aligned} \delta\mathcal{F} &:= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{F}(X_\varepsilon) \\ &= \sum_{i=0}^{2} \left[ -\int_{\Sigma_i} \varphi_i \Lambda_i \, d\Sigma_i + (-1)^i \oint_C \langle \Phi_i, -\varphi_i \nu_i + f_i N_i \rangle \, dC \right], \quad (8) \end{aligned}$$

where $\Phi_i = D\gamma_i + \gamma_i N_i$, $\varphi_i = \langle Y, N_i \rangle$, $f_i = \langle Y, \nu_i \rangle$ on $C$, and the orientation of $C$ is chosen so that it is the positive orientation for $\Sigma_1$.

**Definition 3.1.** Each critical point of $\mathcal{F}$ for all admissible variations is called a *double crystal*.

**Theorem 3.2.** A hypersurface $\Sigma$ is a double crystal if and only if there hold:

(i) For $i = 0, 1, 2$, the anisotropic mean curvature $\Lambda_i$ is constant, and $-\Lambda_1 + \Lambda_2 + \Lambda_0 = 0$ holds, and

(ii) at each point $\zeta$ on $C$, $\Phi_0 - \Phi_1 + \Phi_2$ is in the $(n-2)$-dimensional linear subspace determined by the tangent space $T_\zeta C$ of $C$ at $\zeta$.

**Corollary 3.1.** *Assume $\gamma_i \equiv 1$, $i = 0, 1, 2$. Then, $\Sigma$ is a double bubble if and only if*

*(i) For $i = 0, 1, 2$, the mean curvature $H_i$ is constant, and $-H_1 + H_2 + H_0 = 0$ holds, and*

*(ii) at each point on $C$, $N_0 - N_1 + N_2 = 0$.*

*Proof of Theorem 3.2.* Assume that $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is a double crystal. Then, $\Sigma_1$ is a critical point of $\mathcal{F}$ for all admissible variations that fix $\Sigma = \Sigma_0 \cup \Sigma_2$. Hence, $\Lambda_1$ is constant. Similarly, $\Lambda_2$ is constant. Now consider any variation $\Sigma_0(\epsilon)$ of $\Sigma_0$ that fixes $\partial\Sigma_0$. Then, the variation vector field of $\Sigma_0(\epsilon)$ can be extended to an admissible variation vector field of $\Sigma$. In fact, $\Sigma_0(\epsilon)$ can be represented as

$$X_\epsilon = X + \epsilon\varphi_0 N_0 + \mathcal{O}(\epsilon^2),$$

where $\varphi_0 = 0$ on $C$. It is obvious that we can find functions $\varphi_1$, $\varphi_2$, $f_1 = 0$, and $f_2 = 0$ satisfying (i) and (ii) in Lemma 3.3. So, by Lemma 3.3, there exists an admissible variation of $\Sigma$ whose variation vector field is an extension of $Y_0 := \varphi_0 N_0$. We obtain, using Theorem 3.1, (5), and (6),

$$0 = \delta\mathcal{F} = \delta\mathcal{F} + \Lambda_1\delta V_1 + \Lambda_2\delta V_2 = \int_{\Sigma_0} (\Lambda_1 - \Lambda_2 - \Lambda_0)\langle Y_0, N_0\rangle \, d\Sigma_0.$$

Hence, $\Lambda_1 - \Lambda_2 - \Lambda_0 = 0$ holds, which proves the condition 1. Now, assume that the condition 2 does not hold. Then, there exists a non-empty open set $U$ of $C$ such that $(\Phi_0 - \Phi_1 + \Phi_2)(\zeta) \notin T_\zeta C$ for any $\zeta \in U$. Then, we can define a non-zero vector field $\tilde{Y}$ on $C$ with support in $U$ such that $\tilde{Y}$ is orthogonal to $C$ at any $\zeta \in U$ and

$$\oint_C \langle \Sigma_{i=0}^2 (-1)^i \Phi_i, \tilde{Y}\rangle \, dC \neq 0$$

holds. Clearly, $\tilde{Y}$ can be represented as

$$\tilde{Y} = -\varphi_i \nu_i + f_i N_i, \quad i = 0, 1, 2,$$

and $Y := \varphi_i N_i + f_i \nu_i$ can be extended to an admissible variation vector field along $\Sigma$. Here we used Lemma 3.3 again as above. We obtain

$$0 = \delta\mathcal{F} = \delta\mathcal{F} + \Lambda_1 \delta V_1 + \Lambda_2 \delta V_2 = \oint_C \langle \Sigma_{i=0}^2 (-1)^i \Phi_i, \tilde{Y} \rangle \, dC \neq 0,$$

which is a contradiction.

Conversly, assume that the conditions 1 and 2 hold. Then, again by using Theorem 3.1, (5), and (6), for any admissible variation, we have

$$\delta\mathcal{F} = \delta\mathcal{F} + \Lambda_1 \delta V_1 + \Lambda_2 \delta V_2 = 0.$$

Hence, the hypersurface is a double crystal. $\square$

**Definition 3.2.** A double crystal $\Sigma$ is said to be stable if the second variation $\delta^2\mathcal{F}$ is nonnegative for all admissible variations, and otherwise it is said to be unstable.

**Theorem 3.3** (Second variation formula)**.** Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0$ be a double crystal. Then for any admissible variational vector field $Y$, the second variation of the anisotropic energy $\mathcal{F}$ is given by

$$\delta^2\mathcal{F} = \sum_{i=0}^{2} \left[ -\int_{\Sigma_i} \varphi_i L[\varphi_i] d\Sigma_i + (-1)^i \oint_C \varphi_i \langle A_i \nabla \varphi_i - f_i A_i dN_i(\nu_i), \nu_i \rangle \, dC \right], \tag{9}$$

where L is the self-adjoint Jacobi operator

$$L[\varphi_i] := \mathrm{div}(A_i \nabla \varphi_i) + \langle A_i dN_i, dN_i \rangle \varphi_i,$$

$A_i := D^2\gamma_i + \gamma_i \cdot 1$, $\varphi_i = \langle Y, N_i \rangle$, and $f_i = \langle Y, \nu_i \rangle$ on $C$.

*Proof.* The first variation formula (Theorem 3.1) gives

$$\delta\mathcal{F} := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{F}(X_\varepsilon) = \sum_{i=0}^{2} \left[ -\int_{\Sigma_i} \varphi_i \Lambda_i \, d\Sigma_i + (-1)^i \oint_C \langle \Phi_i, \tilde{Y} \rangle \, dC \right],$$

where $\tilde{Y} = -\varphi_i \nu_i + f_i N_i$. Hence, any volume-preserving variation, at a double crystal $\Sigma$, we obtain

$$\delta^2\mathcal{F} = \delta(\delta\mathcal{F} + \Lambda_1 \delta V_1 + \Lambda_2 \delta V_2) = \sum_{i=0}^{2} \left[ -\int_{\Sigma_i} \varphi_i \delta\Lambda_i \, d\Sigma_i + (-1)^i \oint_C \langle \delta\Phi_i, \tilde{Y} \rangle \, dC \right].$$

Note that $\delta\Lambda_i = L[\varphi_i]$ holds (cf. [6]). Also, we compute, on $C$,

$$\begin{aligned}\langle \delta\Phi_i, -\varphi_i\nu_i + f_i N_i\rangle &= -\varphi_i\langle \delta\Phi_i, \nu_i\rangle = -\varphi_i\langle A_i(-\nabla\varphi_i + dN_i(f_i\nu_i)), \nu_i\rangle \\ &= -\varphi_i\langle -A_i\nabla\varphi_i + f_i A_i dN_i(\nu_i), \nu_i\rangle. \end{aligned} \tag{10}$$

□

## 3.2 Double crystals in the plane

In this section, we assume $n = 1$ and apply the above discussion to a certain special energy density function on $S^1$. The Wulff shape corresponding to this energy density function is a smooth square (see Figure. 2). We will discuss the critical points (i.e. double crystals) and their stability.

From Proposition 3.1 and Theorem 3.2, we immediately obtain the following:

**Theorem 3.4.** For $n = 1$, $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is a double crystal if and only if there hold:

(i) Each $\Sigma_i$ is, up to translation, a part of a rescaling of the Wulff shape corresponding to $\gamma_i$.

(ii) $\Phi_0 - \Phi_1 + \Phi_2 = 0$ on the common boundary $C$ ($C$ is a set of two points).

From now on, if we do not say anything special, we assume that the energy density functions $\gamma_i : S^1(\subset \mathbf{R}) \to \mathbf{R}$ are the following special ones.

$$\begin{aligned}\gamma(\nu_1, \nu_2) &:= \gamma_{(p)}(\nu_1, \nu_2) := \gamma_i(\nu_1, \nu_2) \\ &= \left(\nu_1^{2p} + \nu_2^{2p}\right)^{1-\frac{1}{2p}} / \sqrt{\nu_1^{4p-2} + \nu_2^{4p-2}}, \quad i = 0, 1, 2, \end{aligned} \tag{11}$$

where $p$ is any fixed positive integer. Then the Wulff shape is given by

$$\Phi(\theta) := (\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta).$$

### 3.2.1 Classifications of double crystals for a special energy density function

From now on, without loss of generality, we assume $V_1 \leq V_2$. If $\Lambda_i \neq 0$, from (i) in Theorem 3.4, $\Sigma_i$ is represented by

$$X_1(\theta) = -\frac{1}{\Lambda_1}(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta) + (a_1, b_1), \quad \alpha_1 \leq \theta \leq \beta_1, \tag{12}$$

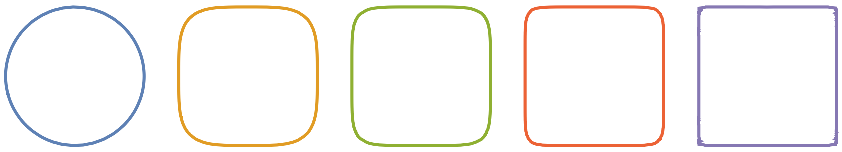

Figure 2: The Wulff shapes $W_{(p)}$ for the energy density $\gamma_{(p)}$ in (11). $W_{(1)}$ is a circle. When $p$ approaches infinity, $W_{(p)}$ converges to a cube.

$$X_2(\theta) = -\frac{1}{\Lambda_2}(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta) + (a_2, b_2), \quad \beta_2 \leq \theta \leq \alpha_2, \tag{13}$$

$$X_0(\theta) = -\frac{1}{\Lambda_0}(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta) + (a_0, b_0), \quad \beta_0 \leq \theta \leq \alpha_0, \tag{14}$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$ correspond to one of the two points in the common boundary $C$, and $\beta_0$, $\beta_1$, $\beta_2$ correspond to the other point in $C$. By the second condition in Theorem 3.4, we have

$$\begin{cases} f(\theta_0)\cos\theta_0 - f(\theta_1)\cos\theta_1 + f(\theta_2)\cos\theta_2 = 0, \\ f(\theta_0)\sin\theta_0 - f(\theta_1)\sin\theta_1 + f(\theta_2)\sin\theta_2 = 0, \end{cases} \quad (\theta_i = \alpha_i, \beta_i), \tag{15}$$

where $f(\theta) = (\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}$.

We can prove the following results about geometry of the double crystals.

**Lemma 3.4.** There are uniquely determined functions $\varphi$, $\psi$ : $\mathbf{S}^1 \to \mathbf{R}$ such that $\theta_2 = \varphi(\theta_1)$ and $\theta_0 = \psi(\theta_1)$ satisfy (15).

*Proof.* We will prove that for fixed $\Phi(\theta_1)$ there are uniquely determined $\Phi(\theta_2)$ and $\Phi(\theta_0)$ satisfying (15). Since we assumed $V_1 \leq V_2$, there are relationships between each $\Phi(\theta_i)$ shown in Figure 3 ($\theta_i = \alpha_i, \beta_i$). We remark that $\Phi(\theta_2)$ and $\Phi(\theta_0)$ trade places with each other depending on $\alpha_i$ and $\beta_i$.

First we fix $\Phi(\alpha_1)$. Let $z$-axis be an axis pointing the same direction of $\Phi(\alpha_1)$ (Figure 4 (a)), and let $w$-axis be an axis orthogonal to the $z$-axis and passing through center point $o$ of the Wulff shape $W$. In addition, let $l$ be a line parallel to the $w$-axis and passing through $\Phi(\alpha_1)$ (Figure 4 (a)).

We remark that the $z$-coordinate of $\Phi(\alpha_2)$ is smaller than that of $\Phi(\alpha_1)$. Otherwise, the condition (15) implies that $\Phi(\alpha_0) \notin W$. On the

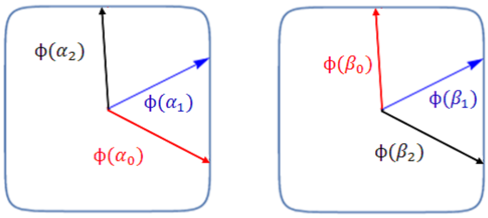


Figure 3: The relationship between each $\Phi(\alpha_i)$ and $\Phi(\beta_i)$.

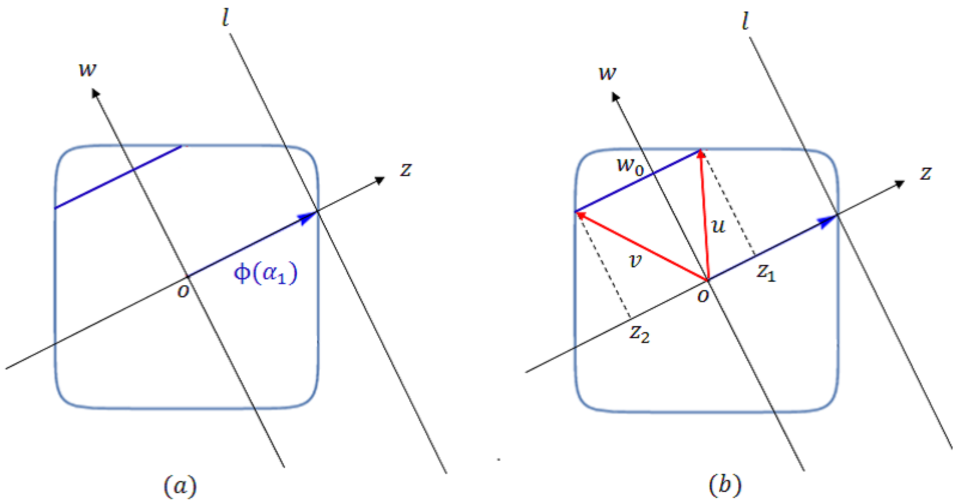


Figure 4:

other hand, by relationship among $\Phi(\alpha_0)$, $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ (Figure 3) can only move in $\{w \geq 0\}$. This means that the movable range of $\Phi(\alpha_2)$ in $W$ is represented as a graph, say $w = f(z)$. Therefore, there are exactly two points $z_1$, $z_2$ in the $z$-axis and one point $w_0$ in the $w$-axis such that $w_0 = f(z_1) = f(z_2)$ and $|\Phi(\alpha_1)| = |z_1 - z_2|$ (Figure 4 (b)). Denote $(z_1, w_0)$ by $u$, and $(z_2, w_0)$ by $v$ (Figure 4 (b)). The vector $\Phi(\alpha_1)$ coincides with vector $u - v$. Set $\Phi(\alpha_2) := u$ and $\Phi(\alpha_0) := -v$, then we obtain the desired result. Similarly, we can prove the statement about $\Phi(\beta_i)$.

□

**Lemma 3.5.** For double crystals, we have the following results about the relationship between $\alpha_i$ and $\beta_i$.

(I) If $\alpha_1 + \beta_1 = 2n_1\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = 2n_i\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).

(II) If $\alpha_1 + \beta_1 = (2n_1 + 1/2)\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = (2n_i + 1/2)\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).

(III) If $\alpha_1 + \beta_1 = (2n_1 + 1)\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = (2n_i + 1)\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).

(IV) If $\alpha_1 + \beta_1 = (2n_1 + 3/2)\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = (2n_i + 3/2)\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).

(V) If $\beta_1 = \alpha_1 + \pi$, then $\beta_0 = \alpha_2 - \pi$ and $\beta_2 = \alpha_0 - \pi$.

*Proof.* By (15),

$$\Phi(\alpha_1) = \Phi(\alpha_0) + \Phi_2(\alpha_2),\ \Phi(\beta_1) = \Phi(\beta_0) + \Phi(\beta_2). \tag{16}$$

For (I)-(IV), by the proof of Lemma 3.4, we obtain the desired result. For (V), by the assumption of (V), $\Phi(\alpha_1) = -\Phi(\beta_1)$. In addition, by (16),

$$\Phi(\alpha_0) + \Phi(\alpha_2) = \Phi(\alpha_1) = -\Phi(\beta_1) = -\Phi(\beta_0) - \Phi(\beta_2).$$

By the proof of Lemma 3.4, we obtain $\Phi(\alpha_0) = -\Phi(\beta_2)$ and $\Phi(\alpha_2) = -\Phi(\beta_0)$. □

Lemma 3.5 gives the following result about about symmetry of double crystals:

**Theorem 3.5.** About the five types of the double crystals in Lemma 3.5, we have the following three types of symmetry (up to translation and homothety) (see Figure 5).

**Type 1** Symmetry with respect to either a horizontal line or a vertical line.

**Type 2** Symmetry with respect to the $\pm\pi/4$ rotation of the horizontal line.

**Type 3** Rotational symmetry with respect to the center point of the smallest cube. In this case, the two bigger Wulff shapes are double size of the smallest one.

Actually, double crystals of Type (I) and (III) have Type 1 symmetry, double crystals of Type (II) and (IV) have Type 2 symmetry, and double crystals of Type (V) have Type 3 symmetry.

*Remark* 3.2. In Type 1 and 2, $\rho$ can be any number bigger than or equal to 1. On the other hand, in Type 3, $\rho$ can take numbers in the interval $[3, 8]$. In fact, in Type 3, two bigger Wulff shapes (black and red shape in Figure 5) are double size of the smallest one (blue shape in Figure 5).

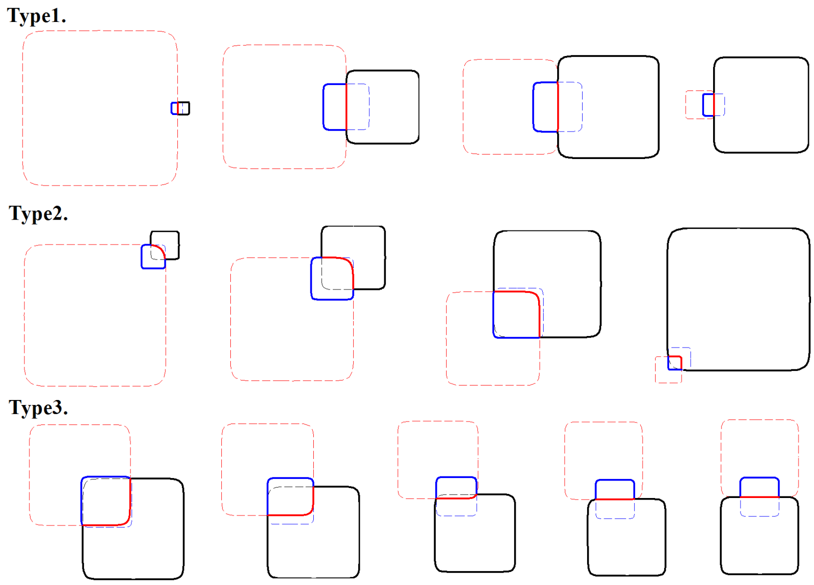


Figure 5: These figures show the three types in Theorem3.5 according to $\rho = V_2/V_1$.

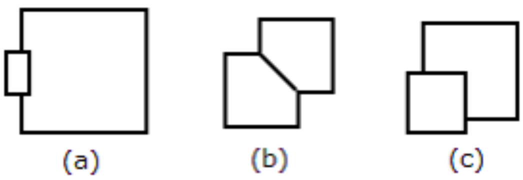


Figure 6:

### 3.2.2 Stability for special energy density function

In this subsection we discuss the stability of the three types of double crystals appeared in Theorem 3.5.

First we give a result about instability of some double crystals which was essentialy proved in [11].

**Lemma 3.6.** Set $\gamma_\infty(\nu_1, \nu_2) = |\nu_1| + |\nu_2|$ $(\nu_1, \nu_2) \in S^1$, and consider an anisotropic surface energy $\mathcal{F}(X) = \displaystyle\int_\Sigma \gamma_\infty(N)\, d\Sigma$. Consider the three types of shapes in Figure 6. Then we can decrease anisotropic the energy of these shapes without changing the enclosed areas.

*Proof.* Note that the anisotropic energy of a horizontal or vertical edge

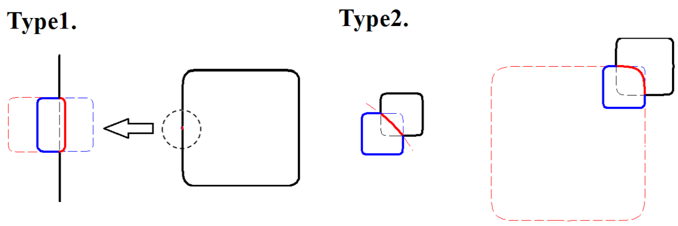


Figure 7: Unstable examples corresponding to Figure 6

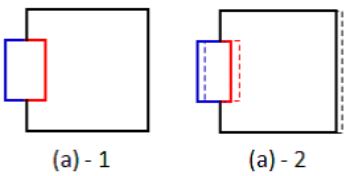


Figure 8: The anisotropic energy of (a)-1 is decreased without changing the enclosed area when it is changed like (a)-2.

is equal to its length, and the anisotropic energy of a diagonal edge is equal to $\sqrt{2}$ times its length. Figures 8 - 11 show how the anisotropic energy is decreased without changing the enclosed area. $\square$

Let us think about the stability of the double crystals for our energy $\gamma = \gamma_{(p)}$ defined in (11). Define two angles $\zeta$ and $\eta$ so that

$$f(\zeta)\cos(\zeta) = \frac{1}{2}, \quad f(\eta)\sin\left(\eta + \frac{\pi}{4}\right) = 2^{\frac{-1-p}{2p}}$$

holds.

Recall that the Wulff shape $W_{(p)}$ for the energy density $\gamma_{(p)}$ in (11) converges to the Wulff shape for $\gamma_\infty$ when $p$ approaches infinity (Figure 2). By Lemma 3.6 and an approximation procedure (Figure 7), we can show the following:

**Proposition 3.2.** *For sufficiently large $p$, we have the following result about the instability of double crystals. Double crystals of type1 in Theorem 3.5 are unstable if $\frac{\pi}{4} < \alpha_0 < \zeta$. Double crystals of type2 are unstable if $\frac{\pi}{4} \leq \alpha_0 < \frac{\pi}{2} - \eta$.*

We apply Theorem 3.3 (second variation formula) to the 2-dimensional

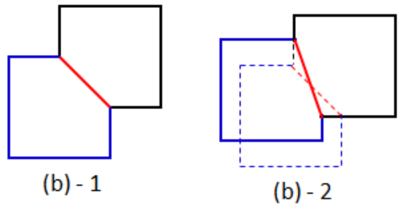


Figure 9: The anisotropic energy of (b)-1 is decreased without changing the enclosed area when it is changed like (b)-2.

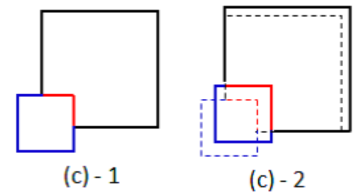


Figure 10: The anisotropic energy of (c)-1 is decreased without changing the enclosed area when it is changed like (c)-2.

case. For admissible variation vector field $Y$, We obtain

$$\delta^2 \mathcal{F} = -\sum_{i=0}^{2} \int_{\Sigma_i} q_i L[q_i] \, d\Sigma_i + [\Lambda_i p_i q_i - A_i(q_i)_t q_i]_a^b \,, \tag{17}$$

where $p_i = \langle Y, v_i \rangle$ and $q_i = \langle Y, N_i \rangle$. We expect that we will be able to prove the following conjecture by using (17).

*Conjecture* 3.1. Except the cases in Proposition 3.2, double crystals of types 1-3 are stable.

Let the Wulff shape be a square. Then the energy minimizing shape is one of the three types in Figure 12 according to $\rho = V_2/V_1$.

We expect that, by using the variational method, we will be able to obtain not only the absolute minimum but also local minimums. It is important to get local minimums because the physical state sometimes takes a local minimum.

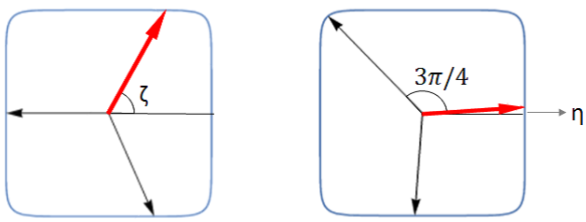


Figure 11: Angle $\xi$ and $\eta$.

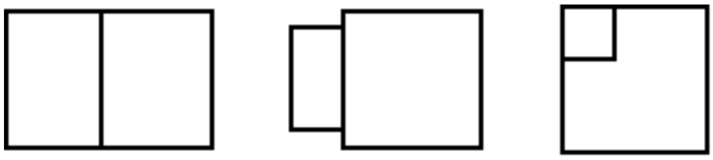

Figure 12: The right side figure is $\rho \leq 2$ and both $R_1$ and $R_2$ are rectangular. The ratio of middle figure is $2 \leq \rho \leq \rho_0 := \frac{43+30\sqrt{2}}{16}$ and $R_1$ is square and $R_2$ is rectangular (this ratio of edge length is $1 : 2$). The ratio of left side figure $\rho \leq \rho_0$, and both $R_1$ and $R_2$ are squares.

# 4 Double crystals problem for general Wulff shapes

In §3, we assumed that the Wulff shape is smooth and strictly convex. However, in general, the Wulff shape is not necessarily smooth. In this section, we discuss double crystals problem that for general Wulff shapes. We try to generalize the research by F. Morgan et al. ([11])

## 4.1 $\gamma$-minimizing surfaces

Let $W$ be the Wulff shape for an energy density function $\gamma : S^2 \to \mathbf{R}^+$. We extend $\gamma$ to a function $\tilde{\gamma} : \mathbf{R}^3 \to \mathbf{R}$ on $\mathbf{R}^3$ as follows:

$$\begin{cases} \tilde{\gamma}(0) = 0 & x = 0 \\ \tilde{\gamma}(x) = |x| \cdot \gamma(\frac{x}{|x|}) & x \neq 0 \end{cases}.$$

We assume that $\tilde{\gamma}$ satisfies

$$\tilde{\gamma}(u+v) \leq \tilde{\gamma}(u) + \tilde{\gamma}(v) \quad ({}^\forall u, v \in \mathbf{R}^3),$$

that is, $\tilde{\gamma}$ is a *norm*.

**Definition 4.1.** Let $\Gamma$ be a piecewise smooth simple closed plane curve in $\mathbf{R}^3$. Denote by $S(\Gamma)$ the set of all compact connected oriented piecewise smooth surfaces in $\mathbf{R}^3$ bounded by $\Gamma$. $\Sigma_0 \in \mathcal{S}(\Gamma)$ is called a $\gamma$-minimizing surface if $\mathcal{F}(\Sigma_0) = \min\{\mathcal{F}(\Sigma) \mid \Sigma \in \mathcal{S}(\Gamma)\}$.

For a plane $H$ in $\mathbf{R}^3$. Denote by $H^+$, $H^-$ the two closed half spaces bounded by $H$. For subset $A \subset \mathbf{R}^3$, $H$ is the *support plane* of $A$ at $x \in A$ if $x \in A \cap H$, and either $A \subseteq H^+$ or $A \subseteq H^-$ is satisfied.

**Definition 4.2.** All vectors that are normal to the supporting plane at $w \in W$ are called *exterior normal vectors* at $w$.

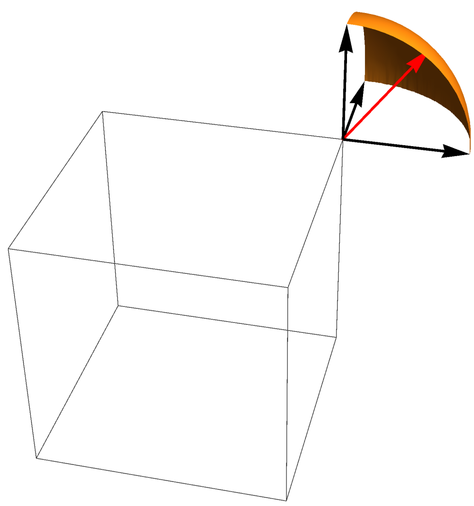

Figure 13: For the Wulff shape is a cube, we draw exterior normal vectors (red vector). And this movable range is a part of a sphere.

*Remark* 4.1. If the Wulff shape $W$ is of $C^1$, for any $w \in W$, the exterior normal vector at $w$ is unique. If $W$ is piecewise smooth and convex, for the point that is not smooth, the exterior normal vector is not unique (Figure 13).

It is easy to prove the following:

**Lemma 4.1.** Let $w$ be a point on the Wulff shape $W$. Then, $\langle w, \nu \rangle = \gamma(\nu)$ if and only if $\nu$ is an exterior normal vector to $W$ at $w$.

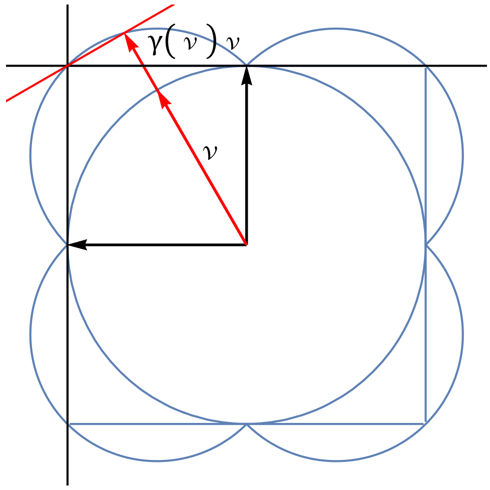


Figure 14: Wulff shape and frank shape

**Proposition 4.1.** *Let $\Gamma$ be a piecewise smooth simple closed plane curve in $\mathbf{R}^3$. Let $\Pi(\Gamma)$ be a plane including $\Gamma$, and let $w$ be a point on the Wulff shape $W$ such that the normal vector of $\Pi(\Gamma)$ is one of the exterior normal vectors at $w$. Then oriented surface $X \in \mathcal{S}(\Gamma)$ is a $\gamma$-minimizing surface if and only if all normal vectors of $X$ are exterior normal vectors to $W$ at $w$.*

*Remark* 4.2. If the Wulff shape is of $C^1$, then all $\gamma$-minimizing surfaces are planer.

*Proof.*

$$\mathcal{F}(X) = \int_{\Sigma} \gamma(\nu) d\Sigma \geq \int_{\Sigma} \langle w, \nu \rangle d\Sigma.$$

By Lemma4.1, the equality holds if and only if $\nu$ is an exterior normal vector to $W$ at $w$.

□

**Proposition 4.2.** *Let $V > 0$ and let $\Gamma$ be a piecewise smooth simple plane curve in $\mathbf{R}^3$. We assume that $\Gamma$ is the curve of a section of a rescaling of the Wulff shape $W$ cut by a plane or $\Gamma$ consists of line segments each of which is parallel to an edge of $W$. Let $\Sigma \in \mathcal{S}(\Gamma)$ such that $\Sigma$ and $\Gamma$ enclose volume $V$. $\Sigma$ is either a $\gamma$-minimizing surface or a rescaling of $W$.*

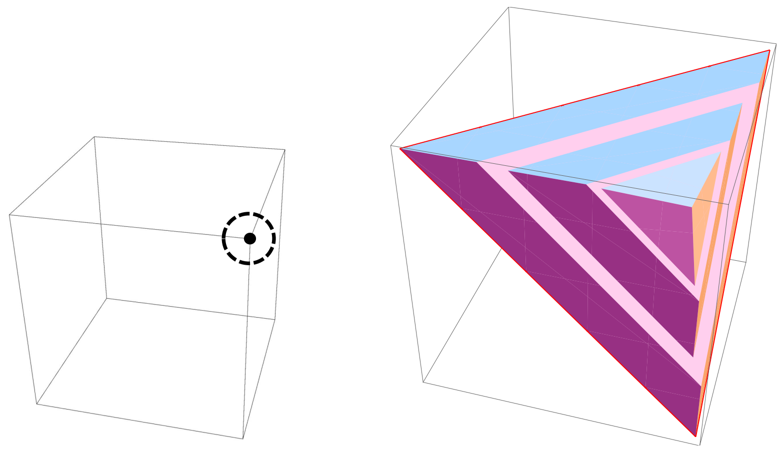

Figure 15: The left figure is the Wulff shape $W$. The right figure is a $\gamma$-minimizing surface whose boundary is consists of the three red segments. All normal vectors of this shape include exterior normal vectors at a point of $W$.

*Proof.* We suppose that $\Pi(\Gamma)$ is a plane including $\Gamma$, and $w$ is a point on $W$ such that the normal vector of $\Pi(\Gamma)$ is one of the vector into exterior normal vectors at $w$. If $W$ is of $C^1$ at $w$, there is a portion of a scaling of $W$ enclosing volume $V$. If $W$ is not of $C^1$ at $w$, let $V_0$ be volume of the smaller side of $W$ cut by $\Pi(\Gamma)$. If $V \geq V_0$, there is a portion of some scaling of $W$ enclosing $V$. If $V < V_0$, by Proposition 4.1 there is $\gamma$-minimizing surface enclosing $V$. $\square$

## 4.2 Existence of Wulff cluster

**Theorem 4.1.** Let $\gamma$ be an anisotropic energy density function satisfying $\gamma(\nu) = \gamma(-\nu)$. Given the following prescribed volumes $V_1, V_2, \ldots, V_m > 0$, let $\Omega$ be the set of all $\mathcal{S} = \bigcup_{i=1}^{N} S_i$ ($N$ can be any integer greater than $m$) satisfying the following (i)-(iii).

(i) $S_1, \ldots S_N$ are oriented compact connected piecewise smooth surfaces with or without boundary.

(ii) If $\partial S_i \neq \emptyset$, then either $\partial S_i$ is the boundary of a rescaling of a section of $W$ cut by a plane, or $\partial S_i$ consists of a finitely many line segments each of which is parallel to an edge of $W$.

(iii) $\mathcal{S} = \bigcup_{i=1}^{N} S_i$ divides $\mathbf{R}^3$ into $(m+1)$ connected regions $R_0, R_1, \ldots, R_m$, where $R_0$ is unbounded and each $R_i$ is bounded and has volume $V_i$ $(i = 1, \ldots, m)$.

We assume that there is $\mathcal{S} \in \Omega$ such that $\mathcal{S}$ minimizes the anisotropic energy in $\Omega$. Then each $S_i$ is either a $\gamma$-minimizing surface for $\partial S_i$ or a portion a rescaling of the Wulff shape for $\gamma$.

*Remark* 4.3.

1. Each $\gamma$-minimizing surface can be replaced by a finite number of plane regions that are parallel to the surfaces of the Wulff shape without changing the energy. Then, the boundary curve also can be replaced by line segments each of which is parallel to an edge of the Wulff shape without changing the energy.

2. When $\mathcal{S} \in \Omega$ minimizes the anisotropic energy in $\Omega$, we call $\mathcal{S}$ $\gamma$-minimizing cluster.

*Proof.* Let $D_0$ be a closed domain in $\mathbf{R}^2$, and Let $f : D_0 \to \mathbf{R}^3$ be a piecewise smooth surface, where $f(D_0)$ does not have selfintersection and minimizes the surface energy. Let $D \subseteq D_0$ $(\partial D \neq \phi)$. We suppose that $f_0 := f|_D$ is neither a $\gamma$-minimizing surface for $\partial S_i$ nor any portion of the boundary of any rescaling of $W$. By Proposition 4.2, there is a piecewise smooth surface $f_1 : D \to \mathbf{R}^3$ such that $f_1(D)$ does not have selfintersection, $f_1(\partial D) = f_0(\partial D)$, Vol($f_1(D)$)=Vol($f_0(D)$), and $f_1(D)$ is either a $\gamma$-minimizing surface for $\partial S_i$ or a portion of a rescaling of $W$. We consider $f_t := (1-t)f_0 + tf_1$ $(t \in [0,1])$, then $f_t$ satisfies for small $t$,

$$\begin{aligned} f_t(\partial D) = f_1(\partial D) = f_0(\partial D), \\ \mathrm{Vol}(f_t(D)) = \mathrm{Vol}(f_1(D)) = \mathrm{Vol}(f_0(D)), \\ \mathcal{F}(f_t(D)) < \mathcal{F}(f_0(D)). \end{aligned} \tag{18}$$

We will prove that there is small $t$ such that $f_t : D \to \mathbf{R}^3$ is an injection. Let $p_1, p_2 \in D$ and $p_1 \neq p_2$. First claim that there is $\delta > 0$ such that

$$|f_1(p_1) - f_1(p_2)| \geq \delta |p_1 - p_2|. \tag{19}$$

If $f_1(D)$ is a $\gamma$-minimizing surface, then $f_1$ is a graph. We denote by $(p, \varphi(p))$ $f_1(D)$, then

$$\begin{aligned} |f_1(p_1) - f_1(p_2)| &= |(p_1, \varphi(p_1)) - (p_2, \varphi(p_2)) \\ &= \sqrt{|p_1 - p_2|^2 + |\varphi_1(p_1) - \varphi_2(p_2)|^2}. \end{aligned}$$

Hence,

$$\left(\frac{|f_1(p_1)-f_1(p_2)|}{|p_1-p_2|}\right)^2 = 1+\frac{(\varphi_1(p_1)-\varphi_2(p_2))^2}{(p_1-p_2)^2} \geq 1.$$

If $f_1(D)$ is a portion of a rescaling of $W$, then since $W$ is convex, we obtain the desired result.

On the other hand,

$$\begin{aligned} |f_t(p_1)-f_t(p_2)| &= |(1-t)f_0(p_1)+tf_1(p_1)-(1-t)f_0(p_2)-tf_1(p_2)| \\ &= |(1-t)\{f_0(p_1)-f_0(p_2)\}+t\{f_1(p_1)-f_1(p_2)\}| \\ &\geq -(1-t)|f_0(p_1)-f_0(p_2)|+t|f(p_1)-f(p_2)|. \quad (20) \end{aligned}$$

Since $D \subset \mathbf{R}^2$ is closed, we get $R := \max_{p\in D}|f_0(p)| < +\infty$. For $t > 0$ satisfying

$$t > \frac{2R}{\delta|p_1-p_2|+2R},$$

by using (19), we get

$$f_1(p_1) \neq f_1(p_2),$$

which combined with (18) contradicts the minimality of the energy. □

## 4.3 Double crystals for a cubic Wulff shape

Set $\gamma_\infty(\nu) = |\nu_1|+|\nu_2|+|\nu_3|$ on $\mathbf{R}^3$ where $\nu = (\nu_1,\nu_2,\nu_3)$. For this anisotropic energy density function $\gamma_\infty$, the Wulff shape $W$ is a cube. In this subsection, for given $V_1, V_2 > 0$, a double cluster is a union of oriented compact piecewise smooth surfaces enclosing two regions with volume $V_1, V_2$. If a double cluster is a minimizer of the anisotropic enerfy $\mathcal{F}$, then it is called a $\gamma$-minimizing double cluster.

There are some kinds of double cluster in Figure 16. In this subsection, we firstly decide one type in Figure 16. We secondly consider these combination. We remark that Type 3 of Figure 16 are excepted from the candidates. Because we decrease the surface energy connecting domain A and B.

**Lemma 4.2.** For a $\gamma_\infty$-minimizing double cluster with connected exterior, it is possible to decrease the volume of either of $R_1$ and $R_2$ and decrease surface energy $\mathcal{F}$.

*Proof.* By Remark 4.3. 1, we transform the double cluster into three types plane regions. We consider one connected domain $R_1$ looks like

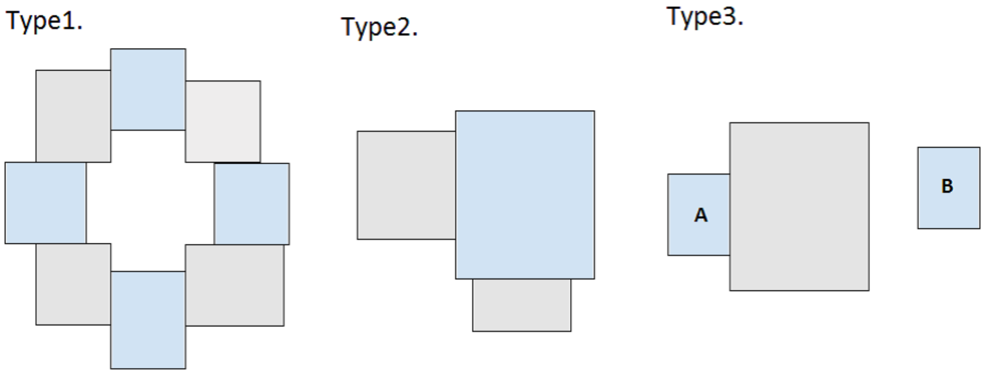


Figure 16: Double clusters.

Figure 17, and some corners of the top or bottom planes. We notice one corner that occupy one over eight of the sphere $S^2$. Denote by $S$ this region. If $R_2$ include three regions of adjacent to $S$, then a part this corner transfers volume from $R_1$ and $R_2$ (Figure 18). Next we scaling returns $R_2$ its original volume, decreases the volume of $R_1$. We finally reduce the surface energy. If $R_2$ does not include three regions of adjacent to $S$, then $R_1$ is adjacent to exterior $R_0$. Since the exterior of double cluster is connected, $R_1$ can be varied so that the both the volume and the energy decrease. $\square$

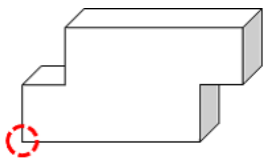

Figure 17: One connected domain. Red broken circle shows one corner of the bottom plane.

**Lemma 4.3.** The energy of a $\gamma_\infty$-minimizing double cluster is a strictly increasing function of each of the two volumes.

*Proof.* For given two volumes $V_1$ and $V_2$, assume that a double cluster $D$ minimizes the anisotropic energy among all double clusters enclosing $V_1$ and any $V \geq V_2$. Denote by $R_1$ the region of $D$ with volume $V_1$, and $R$ the other region of $D$. Let $\tilde{V}$ be the volume of $R$. Denote by $R_0$ the

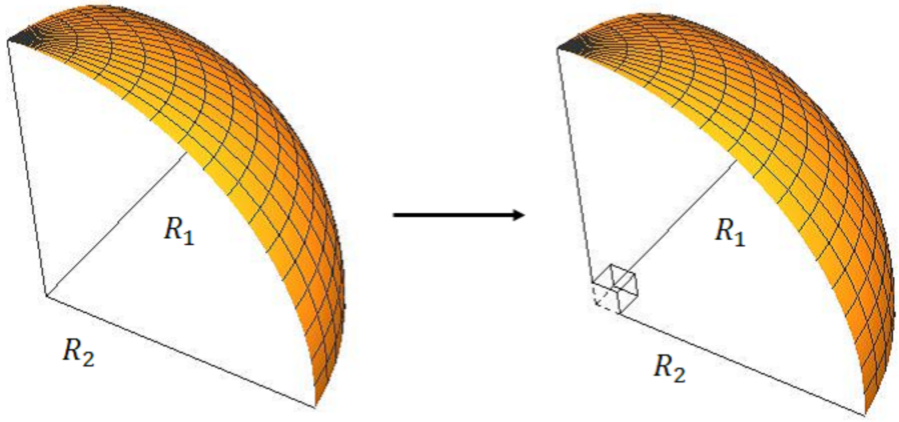


Figure 18: Adjusting the corners of a region, followed by scaling down, decreases one volume and decreases anisotropic energy.

exterior of $D$. $R_0$ is decomposed as $R_0 = R_\infty \cup R_B$, where $R_\infty$ is the unbounded connected component of $R_0$, and $R_B := R_0 - R_\infty$. Denote by $V_B$ the volume of $R_B$. First we show that the exterior of $D$ is connected, that is $R_B = \emptyset$. If not, set $\hat{R} := R \cup R_\infty$. Then, it is clear that $R_1 \cup \hat{R}$ is a $\gamma_\infty$-minimizing double cluster for volumes $V_1$, $\tilde{V} + V_B$. Then, by Lemma 4.2, it is possible to decrease the energy. This contradicts the assumption that $D$ is $\gamma_\infty$-minimizing.

Next, we prove that $\tilde{V} = V_2$. If not, $\tilde{V} > V_2$. Again, by using Lemma 4.2, we can decrease the volume of $R$ and at the same time decrease the energy, which contradicts again the $\gamma_\infty$-minimality. Therefore, $D$ has connected exterior and is a $\gamma_\infty$-minimizing double cluster with volumes $V_1$ and $V_2$. Hence, by Lemma 4.2, we obtain the desired result.

□

**Proposition 4.3.** *A $\gamma_\infty$-minimizing double cluster has connected exterior.*

*Proof.* We assume that a $\gamma_\infty$-minimizing double cluster does not have connected exterior. This contradicts Lemma 4.3. □

Hence, Type2 of Figure 16 is not a $\gamma_\infty$-minimizer. We will prove that a $\gamma_\infty$-minimizing double cluster has one domain for one volume, thereby we prepare some words. A *subregion* is a connected compact component of one of the regions $R_1, R_2$. A *simple subregion* shares exactly one boundary surface with the exterior.

**Lemma 4.4.** A $\gamma_\infty$-minimizing double cluster has at least two simple subregion.

*Proof.* First we show that there are at least two regions with exterior boundary. Assume that there is only one such region $R_1$. Then, by moving the other region $R_2$ to the outside of the cluster and replacing $R_1$ to $R_1 \cup R_2$, we get a contradiction be using Lemma 4.3. Hence, there are at least two subregions with an exterior boundary. If two of these regions are simple, we have the desired result. Otherwise, there exists a subregion with at least two exterior boundary surfaces. This subregion divides the remainder of the double cluster into at least two pieces $D_1$ and $D_2$. By induction, one area that each of $D_1$ and $D_2$ contains a simple subregion.

□

**Theorem 4.2.** We use the same notations as in Theorem 4.1. Let $m = 2$. We assume that $\partial S_i$ is the boundary of a section of a rescaling of the Wulff shape $W$ cut by a plane or $\partial S_i$ consists of line segments each of which is parallel to an edge of $W$. Set $\rho = \frac{V_1}{V_2}$. Then , $\gamma_\infty$-minimizing double clusters are classified into the following three classes by the ratio $\rho$, where $\mathrm{Vol}(R_i) = V_i$ and $a, b, c$ and $x, y, z$ are length (Figure 19). (The larger region is denoted by $R_1$ and the smaller region is denoted by $R_2$.)

1. $\rho \leq 2$; both $R_1$ and $R_2$ are cuboid. $y = z = \sqrt[3]{2(V_1 + V_2)/3}$.
2. $2 \leq \rho \leq \rho_0 :\fallingdotseq 11.2\cdots$; $R_1$ is a cube, and $R_2$ is a cuboid ($a : b : c = 1 : 2 : 2$).
3. $\rho \leq \rho_0$; both $R_1$ and $R_2$ are cubes.

*Proof.* By using the above observation, we can show that $\gamma_\infty$-minimizing double clusters are classified into the three types shown in Figure 19. We fix volumes $V_1$, $V_2$. Let $E_i$ be the minimum of the surface energy of type $i$, and let $A_i$ be the surface area of the energy minimizer of type $i$. For energy density function $\gamma_\infty$, since the surface energy is equal to the surface area, we can calculate these surface area. For type1,

$$A_1 = 2xy + 2ab + 2xz + 2az + 3yz.$$

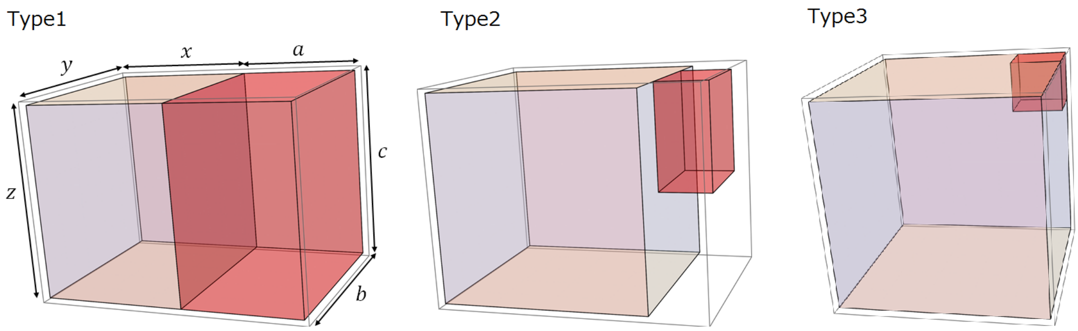


Figure 19: Double clusters

Using $V_1$ and $V_2$, we obtain

$$\begin{aligned} A_1 &= \frac{2(V_1+V_2)}{z} + \frac{2(V_1+V_2)}{y} + 3yz \\ &\geq 3\sqrt[3]{\frac{2(V_1+V_2)}{z} \cdot \frac{2(V_1+V_2)}{y} \cdot 3yz} \\ &= 3\sqrt[3]{12(V_1+V_2)^2}. \end{aligned}$$

Hence, we get

$$E_1 = 3\sqrt[3]{12(V_1+V_2)^2}.$$

We use the same method to get the surface energy $E_2$ (resp. $E_3$) of type2 (resp. type3).

$$E_2 = 3(2\sqrt[3]{(V_1)^2} + \sqrt[3]{(2V_2)^2}),\ E_3 = 3(2\sqrt[3]{(V_1+V_2)^2} + \sqrt[3]{(V_2)^2}).$$

For type1 and type2, since $E_1 \geq E_2$,

$$3\sqrt[3]{12(V_1+V_2)^2} \geq 3(2\sqrt[3]{(V_1)^2} + \sqrt[3]{(2V_2)^2}).$$

Since $\rho \geq 1$, we get $\rho \geq 2$. For type2 and type3, since $E_2 \geq E_3$,

$$2(\rho V_2)^{2/3}(2V_2)^{2/3} \geq 2\{(\rho+1)V_2\}^{2/3} + V_2^{2/3}.$$

hence, we obtain

$$12\left(2^{2/3}-1\right)\rho^{4/3} - 16\rho + 6\left(2^{2/3}-1\right)^2\rho^{2/3} + \left(2^{2/3}-1\right)^3 - 8 \geq 0. \quad (21)$$

Since $\rho \geq 1$, we get $\rho \geq 11.2\cdots$. $\square$

# 5 Appendix

### 1. Wulff shape corresponding to a special energy density function

The case where a energy density function is

$$\gamma_{(p)}(N_1, N_2) := \gamma(N_1, N_2) = \left(N_1^{2p} + N_2^{2p}\right)^{1-\frac{1}{2p}} / \sqrt{N_1^{4p-2} + N_2^{4p-2}},$$

then the Wulff shape corresponding to this $\gamma$ is

$$W_{(p)} : \Phi(\theta) := (\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta).$$

Now, we will proof $W_{(\infty)}$ is a square. If $|\cos\theta| \geq |\sin\theta|$, then,

$$(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}} \geq (\cos^{2p}\theta)^{-\frac{1}{2p}} = \frac{1}{|\cos\theta|}.$$

On the other hand,

$$(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}} \leq (2\cos^{2p}\theta)^{-\frac{1}{2p}} = 2^{-\frac{1}{2p}}(\cos^{2p}\theta)^{-\frac{1}{2p}}. \tag{22}$$

If $p$ is close to $\infty$, then (22) is close to $1/|\cos\theta|$, so we obtain

$$W_{(\infty)} : \frac{1}{|\cos\theta|}(\cos\theta, \sin\theta).$$

If $|\cos\theta| \leq |\sin\theta|$ then,

$$W_{(\infty)} : \frac{1}{|\sin\theta|}(\cos\theta, \sin\theta).$$

We have gotten the desired result.

### 2. Details about the proof of Lemma 3.6

We want to vary (b)-1 in Figure 20 to (b)-2, so that the two shaded regions in (b)-3 have the same area. Then, the volumes $V_1$, $V_2$ are preserved. This is achieved by choosing the lengths $x$ and $y$ so that the two green lines are parallel, that is, $x$ and $y$ satisfy $xy + (x-y)a = 0$ (length $a$ is fixed). Then, the anisotropic energy $\mathcal{F}$ of the new curves minus that of the original curves is

$$x - y = -\frac{x^2}{a-x}. \tag{23}$$

If $0 < x < a$, then (23) is negative. Hence, the anisotropic energy $\mathcal{F}$ is decreased without changing the enclosing area.

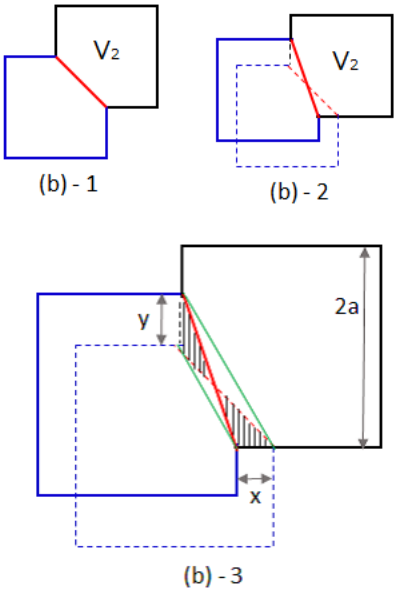


Figure 20: The anisotropic energy of (b)-1 is decreased without changing the enclosed areas when it is changed like (b)-2.

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