Kaneko-Zagier type differential equation for Jacobi forms

喜友名, 朝也

https://doi.org/10.15017/1500516

出版情報：九州大学, 2014, 博士（数理学）, 課程博士
バージョン：

Kaneko-Zagier type differential equation for Jacobi forms

Tomoya Kiyuna

Doctoral thesis

Graduate School of Mathematics
Kyushu University

Acknowledgments

I would like to express sincere gratitude to my supervisor Professor Masanobu Kaneko for warm encouragement and valuable comments. I also would like to thank my father Asao Kiyuna, my mother Keiko Kiyuna, and my older sisters Ayaka Kiyuna and Yuka Kiyuna for encouraging and supporting me during my research in graduate school, without which my research would not be successful.

Contents

1 Introduction **3**

2 Modular forms of one variable **5**
- 2.1 Definition of modular forms 5
- 2.2 Examples of modular forms and quasimodular forms 6

3 The K-Z equation ($\sharp_k$) for modular forms **10**
- 3.1 A method for deriving ($\sharp_k$) 10
- 3.2 Existing results on ($\sharp_k$), especially modular and quasimodular solutions 10

4 Jacobi forms with respect to $SL_2(\mathbb{Z})$ **15**

5 The equation ($\flat_{k,m}$) for Jacobi forms **20**
- 5.1 The case of $m = 1$ 20
  - 5.1.1 A method for deriving ($\flat_k$) and properties of ($\flat_k$) 20
  - 5.1.2 Jacobi form solutions 27
  - 5.1.3 Key proposition 31
  - 5.1.4 Proof of Theorem 5.8 34
- 5.2 The case of general index m 35
- 5.3 A connection with ($\flat_{k,m}$), ($\sharp_k$) and the heat equation 36

1 Introduction

In this paper, we study the following fourth order linear partial differential equation related to Jacobi forms of weight k and index m:

$$\begin{aligned}\phi^{[4]}(\tau,z) - 8m\phi^{[2](1)}(\tau,z) + \frac{(2k+1)m}{3}E_2(\tau)\phi^{[2]}(\tau,z)\\ + 16m^2\phi^{(2)}(\tau,z) - \frac{4(2k+1)m^2}{3}E_2(\tau)\phi'(\tau,z)\\ + \frac{(2k-1)(2k+1)m^2}{3}E_2'(\tau)\phi(\tau,z) = 0,\end{aligned} \tag{$\flat_{k,m}$}$$

where k is a rational number, m a natural number, τ a variable in the upper half plane $\mathfrak{H}$, z a variable in the complex plane $\mathbb{C}$, $E_2(\tau)$ the Eisenstein series of weight 2 with respect to the full modular group $SL_2(\mathbb{Z})$, the symbol $'$ the differential operator $(2\pi\sqrt{-1})^{-1}d/d\tau$, $\phi^{(n)} := (2\pi\sqrt{-1})^{-n}\partial^n\phi/\partial\tau^n$ and $\phi^{[n]} := (2\pi\sqrt{-1})^{-n}\partial^n\phi/\partial z^n$ for function $\phi = \phi(\tau,z)$ of two variables. If $m = 1$, we write ($\flat_k$) simply for ($\flat_{k,1}$). This equation ($\flat_{k,m}$) is a Jacobi form analog of the following second order linear ordinary differential equation related to elliptic modular forms:

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0. \tag{$\sharp_k$}$$

The equation ($\sharp_k$) was derived by Kaneko and Zagier in [KZ98]. We call the equation ($\sharp_k$) the Kaneko-Zagier differential equation (the K-Z equation, for short).

The K-Z equation ($\sharp_k$) is closely connected with several mathematical areas, for example, elliptic curves [KZ98], modular forms of one variable [KZ98], [KK03], [Gu15] and the conformal field theory [KNS13]. Among them, we mention in particular a recent work of Guerzhoy [Gu15] who proved that mixed mock modular forms [DMZ, §7.3] arise as solutions of the K-Z equation ($\sharp_k$). It is possible that our equation ($\flat_{k,m}$) also admits a new type of Jacobi form as a solution.

This paper is organized as follows. In Section 2, we review the theory of modular forms of one variable. We outline a method for deriving the K-Z equation ($\sharp_k$) (employing the Ramanujan-Serre derivative) and summarize existing results on ($\sharp_k$) in Section 3. In Section 4, we review the theory of Jacobi forms presented in [EZ85]. The equation ($\flat_{k,m}$) is treated in Section 5. Because Section 5 is a main part of the paper, we explain the contents of the section in detail. In Subsection 5.1, we consider the equation ($\flat_k$) for Jacobi forms of index 1. After deriving the equation ($\flat_k$) (employing the modified heat operator) and showing several properties of ($\flat_k$), we explicitly define a series of Jacobi forms ϕ_k of weights $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 4 \pmod 6$ and prove that ϕ_k is a solution of ($\flat_k$). In Subsection 5.2, we consider the equation ($\flat_{k,m}$) for Jacobi forms of general index m. We discuss a connection with ($\flat_{k,m}$), ($\sharp_k$) and the classical heat equation in Subsection 5.3.

Notation

For a commutative ring R with an identity element, we denote by $\mathrm{Mat}_n(R)$ the set of square matrices of size n with entries in R. Throughout the paper, the notations τ and z mean variables in $\mathfrak{H}$ and $\mathbb{C}$, respectively. Also, we use the notations $\mathbf{e}(w) := \exp(2\pi\sqrt{-1}w)$ for variable w, $q := \mathbf{e}(\tau)$, $\zeta := \mathbf{e}(z)$, $\,' := (2\pi\sqrt{-1})^{-1}\partial/\partial\tau$, $\phi^{(n)} := (2\pi\sqrt{-1})^{-n}\partial^n\phi/\partial\tau^n$ and $\phi^{[n]} := (2\pi\sqrt{-1})^{-n}\partial^n\phi/\partial z^n$ for function $\phi = \phi(\tau, z)$ of two variables. In addition, for any $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, we define $\log z$ and z^α as $\log z := \log|z| + \sqrt{-1}\arg z$ with $-\pi < \arg z \leq \pi$ and $z^\alpha := \exp(\alpha \log z)$, respectively.

2 Modular forms of one variable

In this section, we review the theory of modular forms of one variable.

2.1 Definition of modular forms

First, we present definitions. Let $SL_2(\mathbb{Z})$ be the full modular group and $\mathfrak{H}$ the upper half plane:

$$SL_2(\mathbb{Z}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{Z}) \mid ad - bc = 1 \right\},$$
$$\mathfrak{H} := \left\{ \tau = x + y\sqrt{-1} \in \mathbb{C} \mid y > 0 \right\}.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}$ as $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) \mapsto (a\tau + b)/(c\tau + d)$. For $N \in \mathbb{N}$, we define the subgroups $\Gamma_0(N)$, $\Gamma_0^0(N)$ and $\Gamma(N)$ of $SL_2(\mathbb{Z})$ as

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \,|\, c \equiv 0 \,(\mathrm{mod}\ N) \right\},$$
$$\Gamma_0^0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \,|\, b \equiv 0 \,(\mathrm{mod}\ N) \right\},$$
$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^0(N) \,|\, a \equiv d \equiv 1 \,(\mathrm{mod}\ N) \right\}.$$

A subgroup Γ of $SL_2(\mathbb{Z})$ is a congruence subgroup of level N if $\Gamma(N) \subset \Gamma$. For example, the sets $SL_2(\mathbb{Z})$, $\Gamma_0(N)$, $\Gamma_0^0(N)$ and $\Gamma(N)$ are congruence subgroups of level N. Especially, the set $\Gamma(N)$ is called the principal congruence subgroup of level N.

Let Γ be a congruence subgroup of level N and χ be a character on Γ. A holomorphic function f is a modular form (or an elliptic modular form) of weight k with character χ with respect to Γ if it satisfies the following two conditions

- $f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)(c\tau + d)^k f(\tau)$ (for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathfrak{H}$),
- $f(\tau)$ is holomorphic at all cusps of Γ.

When $-1_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma$, because f satisfies $f(\tau) = \chi(-1_2)(-1)^k f(\tau)$, we see that f is identically zero if $\chi(-1_2) \neq (-1)^k$. We denote by $M_k(\Gamma, \chi)$ the $\mathbb{C}$-vector space of modular forms of weight k with character χ with respect to Γ. It is well-known that $\dim_{\mathbb{C}} M_k(\Gamma, \chi) < \infty$. Then, we denote by $S_k(\Gamma, \chi)$ the subspace of $M_k(\Gamma, \chi)$ consisting functions which vanish at all cusps of Γ. An element in $S_k(\Gamma, \chi)$ is called a cusp form (or an elliptic cusp form). If χ is the trivial character, we write $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) simply for $M_k(\Gamma, \chi)$ (resp. $S_k(\Gamma, \chi)$). Moreover, if $\Gamma = SL_2(\mathbb{Z})$, we denote $M_k(SL_2(\mathbb{Z}))$(resp. $S_k(SL_2(\mathbb{Z}))$) by M_k (resp. S_k).

2.2 Examples of modular forms and quasimodular forms

Let us give examples of modular forms, modular functions and quasimodular forms with respect to several congruence subgroups. These functions will be used later (see Subsection 3.2).

First, we consider the case of level 1. Let $E_k(\tau)$ be the normalized Eisenstein series of weight k ($\in 2\mathbb{N}$) with respect to $SL_2(\mathbb{Z})$:

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\Big(\sum_{0<d|n} d^{k-1}\Big)q^n,$$

where B_k is the kth Bernoulli number defined as the following generating function

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

It is well-known that $0 \neq E_k \in M_k$ for even integer $k \geq 4$ and E_2 satisfies

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 E_2(\tau) + \frac{6c(c\tau+d)}{\pi\sqrt{-1}} \tag{1}$$

$$\text{(for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \text{ and } \tau \in \mathfrak{H}).$$

A homogeneous element in $\mathbb{C}[E_2, E_4, E_6]$ of degree k is called a quasimodular form of weight k with respect to $SL_2(\mathbb{Z})$. Here, we consider that the generators E_2, E_4 and E_6 have degrees $2, 4$ and 6, respectively. The Eisenstein series E_2 is a quasimodular form of weight 2 with respect to $SL_2(\mathbb{Z})$. Let

$$\Delta(\tau) := \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots$$

be the discriminant function. The function $\Delta(\tau)$ is a cusp form of weight 12. A meromorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a modular function with respect to a congruence subgroup Γ if it satisfies the following two conditions

- $f\big(\frac{a\tau+b}{c\tau+d}\big) = f(\tau)$ (for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathfrak{H}$),
- $f(\tau)$ is meromorphic at all cusps of Γ.

For example, the modular invariant

$$j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + \cdots$$

is a modular function with respect to $SL_2(\mathbb{Z})$.

Next, we consider the case of level 2. We define modular forms $E_2^{(2)}(\tau)$ and $\Delta_4^{(2)}(\tau)$ and the modular function $j^{(2)}(\tau)$ with respect to $\Gamma_0(2)$ as follows:

$$\begin{aligned} E_2^{(2)}(\tau) &:= 2E_2(2\tau) - E_2(\tau) \\ &\qquad = 1 + 24 \sum_{n=1}^{\infty} \Big(\sum_{\substack{0<d|n \\ d:\text{odd}}} d \Big) q^n \in M_2(\Gamma_0(2)), \\ \Delta_4^{(2)}(\tau) &:= \frac{\eta(2\tau)^{16}}{\eta(\tau)^8} = \sum_{n=1}^{\infty} \Big(\sum_{\substack{0<d|n \\ d:\text{odd}}} (n/d)^3 \Big) q^n \in M_4(\Gamma_0(2)), \\ j^{(2)}(\tau) &:= \frac{E_2^{(2)}(\tau)^2}{\Delta_4^{(2)}(\tau)} = \frac{1}{q} + 40 + 276q - 2048q^2 + \cdots . \end{aligned}$$

Here

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta function.

Let us consider the case of level 3. We define modular forms $E_1^{(3)}(\tau)$ and $\Delta_3^{(3)}(\tau)$ and the modular function $j^{(3)}(\tau)$ with respect to $\Gamma_0(3)$ as follows:

$$\begin{aligned} E_1^{(3)}(\tau) &:= 1 + 6 \sum_{n=1}^{\infty} \Big(\sum_{0<d|n} \Big(\frac{d}{3}\Big) \Big) q^n = 1 + 6q + 6q^3 + 6q^4 + \cdots \\ &\qquad \in M_1(\Gamma_0(3), \Big(\frac{d}{3}\Big)), \\ \Delta_3^{(3)}(\tau) &:= \frac{\eta(3\tau)^9}{\eta(\tau)^3} = \sum_{n=1}^{\infty} \Big(\sum_{0<d|n} \Big(\frac{d}{3}\Big) (n/d)^2 \Big) q^n \\ &\qquad = q + 3q^2 + 9q^3 + 13q^4 + \cdots \in M_3(\Gamma_0(3), \Big(\frac{d}{3}\Big)), \\ j^{(3)}(\tau) &:= \frac{E_1^{(3)}(\tau)^3}{\Delta_3^{(3)}(\tau)} = \frac{1}{q} + 15 + 54q - \cdots , \end{aligned}$$

where $(\frac{d}{3})$ is the Legendre symbol. (Note that $\Gamma_0(3) \ni \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (\frac{d}{3}) \in \{\pm 1\}$ is a character on $\Gamma_0(3)$.)

Finally, we consider the case of level 4. In the case, we have to treat modular forms of half-integral weight. In order to describe modular forms of half-integral weight, we need the automorphic factor $J(\gamma,\tau)$ on $\Gamma_0(4)$:

$$J(\gamma,\tau) := \frac{\theta(\frac{a\tau+b}{c\tau+d})}{\theta(\tau)} = \epsilon_{c,d}\sqrt{c\tau+d},$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, $\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$ and

$$\epsilon_{c,d} := \left(\frac{c}{d}\right) \times \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ -i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

(The symbol $(\frac{c}{d})$ is the Kronecker symbol.) For $k \in \mathbb{N}$, a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a modular form of weight $k-1/2$ with respect to a congruence subgroup $\Gamma \subset \Gamma_0(4)$ if it satisfies the following two conditions

- $f\left(\frac{a\tau+b}{c\tau+d}\right) = J(\gamma,\tau)^{2k-1} f(\tau) \quad$ (for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathfrak{H}$),
- $f(\tau)$ is holomorphic at all cusps of Γ.

We denote by $M_{k-1/2}(\Gamma)$ the $\mathbb{C}$-vector space of modular forms of weight $k-1/2$ with respect to Γ. The theta series $\theta(\tau)$ is a modular form of weight $1/2$ with respect to $\Gamma_0(4)$. Further, we define the Kohnen plus space $M^+_{k-1/2}$ [Ko80], which is closely connected to the space of Jacobi forms of index 1 (see Theorem 4.8):

$$M^+_{k-\frac{1}{2}} := \left\{ f(\tau) = \sum_{n\geq 0} c(n)q^n \in M_{k-\frac{1}{2}}(\Gamma_0(4)) \;\middle|\; c(n) = 0 \text{ unless } (-1)^{k-1}n \equiv 0 \text{ or } 1 \pmod 4 \right\}.$$

We end this subsection by defining modular forms $E_2^{(4)}(\tau)$ and $\Delta_2^{(4)}(\tau)$ and the modular function $j^{(4)}(\tau)$ with respect to $\Gamma_0(4)$, which appear in Theorem

3.9:

$$
\begin{aligned}
E_2^{(4)}(\tau) &:= \frac{1}{3}(4E_2(4\tau) - E_2(\tau)) \\
&= 1 + 8q + 24q^2 + 32q^3 + \cdots \in M_2(\Gamma_0(4)), \\
\Delta_2^{(4)}(\tau) &:= \frac{\eta(4\tau)^8}{\eta(2\tau)^4} = \sum_{\substack{n=1 \\ n:\mathrm{odd}}}^{\infty} \Big(\sum_{0<d|n} d \Big) q^n \\
&= q + 4q^3 + 6q^5 + 8q^7 + \cdots \in M_2(\Gamma_0(4)), \\
j^{(4)}(\tau) &:= \frac{E_2^{(4)}(\tau)}{\Delta_2^{(4)}(\tau)} = \frac{1}{q} + 8 + 20q - 62q^3 + \cdots .
\end{aligned}
$$

Remark 2.1. The relations

$$
\begin{aligned}
E_2^{(4)}(\tau)^{\frac{1}{4}} &= \theta(\tau) = 1 + 2q + 2q^4 + \cdots , \\
\Delta_2^{(4)}(\tau)^{\frac{1}{4}} &= \frac{1}{2} \sum_{n \in \mathbb{Z}} q^{(n+\frac{1}{2})^2} = q^{\frac{1}{4}} + q^{\frac{9}{4}} + q^{\frac{25}{4}} + \cdots \in M_{\frac{1}{2}}(\Gamma_0^0(4)),
\end{aligned}
$$

hold.

3 The K-Z equation $(\sharp_k)$ for modular forms

In this section, we consider the K-Z equation $(\sharp_k)$. We omit the proofs of theorems in the section. (For further details, see [KZ98] and [KK03].)

3.1 A method for deriving $(\sharp_k)$

As mentioned in Section 1, the K-Z equation $(\sharp_k)$ for modular forms was derived by Kaneko and Zagier in [KZ98]. Let us review their method for deriving the equation $(\sharp_k)$. First, we define the Ramanujan-Serre derivative

$$\vartheta_k := \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau} - \frac{k}{12}E_2 : M_k \longrightarrow M_{k+2}.$$

(We may drop the subscript k when the weight k is clear from the context.) For $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 0$ or $4 \pmod 6$, we can consider the endomorphism

$$\varphi_k := \frac{1}{E_4}\vartheta_{k+2} \circ \vartheta_k : M_k \longrightarrow M_k,$$

because $\dim_{\mathbb{C}} M_k = \dim_{\mathbb{C}} M_{k+4}$ and thus $M_{k+4} = E_4 {\cdot} M_k$. Looking at the constant term of $\varphi_k(f)$, we see that the value $k(k+2)/144$ is an eigenvalue of φ_k. Using $E_2' = (E_2^2 - E_4)/12$, we find that the equation

$$\varphi_k(f)(\tau) = \frac{k(k+2)}{144}f(\tau)$$

is equivalent to the K-Z equation $(\sharp_k)$.

Remark 3.1. In general, a solution of $(\sharp_k)$ is not necessarily a modular form.

3.2 Existing results on $(\sharp_k)$, especially modular and quasi-modular solutions

In this subsection, we summarize existing results concerning the K-Z equation $(\sharp_k)$. A study on the K-Z equation $(\sharp_k)$ has originated with Kaneko and Zagier [KZ98]. They found that the polynomial

$$\mathrm{ss}_p(X) := \prod_{\substack{E/\overline{\mathbb{F}}_p \\ \text{supersingular}}} (X - j(E)) \in \mathbb{F}_p[X],$$

where $j(E)$ is the j-invariant of an elliptic curve $E/\overline{\mathbb{F}}_p$, is closely connected to modular solutions of $(\sharp_k)$.

Theorem 3.2. ([KZ98, Theorem 1]) *Let $p \geq 5$ be a prime number. Define integers $\gamma \in \mathbb{Z}_{\geq 0}$, $\delta \in \{0,1,2\}$ and $\epsilon \in \{0,1\}$ through the unique expression $p-1 = 12\gamma + 4\delta + 6\epsilon$. Take $f_{p-1} \in M_{p-1}$ such that f_{p-1} is a solution of $(\sharp_{p-1})$ and the constant term of the Fourier expansion of f_{p-1} is equal to $(-1)^\gamma\binom{\frac{p}{6}-1}{\gamma}$. Define $F_{p-1}(X) \in \mathbb{Q}[X]$ through the unique expression $f_{p-1} = \Delta^\gamma E_4^\delta E_6^\epsilon F_{p-1}(j)$. Then we have*

$$\mathrm{ss}_p(X) \equiv X^\delta (X-1728)^\epsilon F_{p-1}(X) \pmod{p}.$$

The reason why $F_{p-1}(X) \in \mathbb{Q}[X]$ is that the Fourier expansion of $j(\tau)$ has integral coefficients and $F_{p-1}(j)$ is the unique polynomial solution of the following Gauss hypergeometric differential equation:

$$j(j-1728)F''_{p-1} + \{(1-\nu_1)j + (1-\nu_0)(j-1728)\}F'_{p-1} + \gamma(\gamma-\nu_\infty)F_{p-1} = 0, \quad (2)$$

where $\nu_0 := (1-2\delta)/3, \nu_1 := (1-2\epsilon)/2$ and $\nu_\infty := p/6$. (The equation (2) is obtained by transforming $(\sharp_k)$ in terms of j.)

Next, let us consider modular and quasimodular solutions of the equation $(\sharp_k)$. Kaneko and Koike gave the explicit modular and quasimodular solutions of the equation $(\sharp_k)$ for $k \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Q}_{\geq 0}$ with $k \equiv 1/2 \pmod 3$ in [KK03]. Recall the definition of the Gauss hypergeometric series

$${}_2F_1(a,b,c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where

$$(a)_n := \begin{cases} 1 & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1) & \text{if } n \geq 1. \end{cases}$$

First, we consider the cases of all $k \in \mathbb{Z}_{\geq 0}$ except $k \equiv 5 \pmod 6$.

Theorem 3.3. ([KK03, Theorem 1 (i) (ii) (iii)]) (i) *For each $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 0$ or $4 \pmod{12}$, the modular form*

$$\begin{aligned} &E_4(\tau)^{\frac{k}{4}} {}_2F_1\Big(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\Big) \\ &= \sum_{0 \leq i \leq \frac{k}{12}} \frac{(-\frac{k}{12})_i (-\frac{k-4}{12})_i}{(-\frac{k-5}{6})_i\, i!} 1728^i \Delta(\tau)^i E_4(\tau)^{\frac{k}{4}-3i} \\ &= 1 + O(q) \in M_k \end{aligned}$$

is a solution of $(\sharp_k)$.

(ii) *For each* $k \in \mathbb{Z}_{\geq 0}$ *with* $k \equiv 6$ *or* $10 \pmod{12}$, *the modular form*

$$\begin{aligned}
&E_4(\tau)^{\frac{k-6}{4}} E_6(\tau) {}_2F_1\Big(-\frac{k-6}{12}, -\frac{k-10}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\Big) \\
&\qquad = E_6(\tau) \sum_{0 \leq i \leq \frac{k-6}{12}} \frac{(-\frac{k-6}{12})_i (-\frac{k-10}{12})_i}{(-\frac{k-5}{6})_i i!} 1728^i \Delta(\tau)^i E_4(\tau)^{\frac{k-6}{4}-3i} \\
&\qquad = 1 + O(q) \in M_k
\end{aligned}$$

is a solution of $(\sharp_k)$.

(iii) *For each* $k \in \mathbb{Z}_{\geq 0}$ *with* $k \equiv 2 \pmod 6$, *the modular forms*

$$\begin{aligned}
&E_2^{(2)}(\tau)^{\frac{k}{2}} {}_2F_1\Big(-\frac{k}{4}, -\frac{k-2}{4}, -\frac{k-5}{6}; \frac{64}{j^{(2)}(\tau)}\Big) \\
&\qquad = \sum_{0 \leq i \leq \frac{k}{4}} \frac{(-\frac{k}{4})_i (-\frac{k-2}{4})_i}{(-\frac{k-5}{6})_i i!} 64^i \Delta_4^{(2)}(\tau)^i E_2^{(2)}(\tau)^{\frac{k}{2}-2i} \\
&\qquad = 1 + O(q) \in M_k(\Gamma_0(2))
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_4^{(2)}(\tau)^{\frac{k+1}{6}} E_2^{(2)}(\tau)^{\frac{k-2}{6}} {}_2F_1\Big(-\frac{k-2}{12}, -\frac{k-8}{12}, -\frac{k+7}{6}; \frac{64}{j^{(2)}(\tau)}\Big) \\
&\qquad = \sum_{0 \leq i \leq \frac{k-2}{12}} \frac{(-\frac{k-2}{12})_i (-\frac{k-8}{12})_i}{(\frac{k+7}{6})_i i!} 64^i \Delta_4^{(2)}(\tau)^{\frac{k+1}{6}+i} E_2^{(2)}(\tau)^{\frac{k-2}{6}-2i} \\
&\qquad = q^{\frac{k+1}{6}} + O(q^{\frac{k+7}{6}}) \in M_k(\Gamma(2))
\end{aligned}$$

are solutions of $(\sharp_k)$.

(iv) *For each* $k \in \mathbb{Z}_{\geq 0}$ *with* $k \equiv 1$ *or* $3 \pmod 6$, *the modular forms*

$$\begin{aligned}
&E_1^{(3)}(\tau)^k {}_2F_1\Big(-\frac{k}{3}, -\frac{k-1}{3}, -\frac{k-5}{6}; \frac{27}{j^{(3)}(\tau)}\Big) \\
&\qquad = \sum_{0 \leq i \leq \frac{k}{3}} \frac{(-\frac{k}{3})_i (-\frac{k-1}{3})_i}{(-\frac{k-5}{6})_i i!} 27^i \Delta_3^{(3)}(\tau)^i E_1^{(3)}(\tau)^{k-3i} \\
&\qquad = 1 + O(q) \in M_k(\Gamma_0(3), \Big(\frac{d}{3}\Big))
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_3^{(3)}(\tau)^{\frac{k+1}{6}} E_1^{(3)}(\tau)^{\frac{k-1}{2}} {}_2F_1\Big(-\frac{k-1}{6}, -\frac{k-3}{6}, -\frac{k+7}{6}; \frac{27}{j^{(3)}(\tau)}\Big) \\
&\qquad = \sum_{0\le i\le \frac{k-1}{6}} \frac{(-\frac{k-1}{6})_i(-\frac{k-3}{6})_i}{(\frac{k+7}{6})_i i!} 27^i \Delta_3^{(3)}(\tau)^{\frac{k+1}{6}+i} E_1^{(3)}(\tau)^{\frac{k-1}{2}-3i} \\
&\qquad = q + O(q^{\frac{k+7}{6}}) \in M_k(\Gamma(3))
\end{aligned}$$

are solutions of $(\sharp_k)$.

Remark 3.4. The cases (i) and (ii) are contained in [KZ98, Theorem 5].

Then, let us consider the case of $k \in \mathbb{Z}_{\ge 0}$ with $k \equiv 5 \pmod 6$. In the case, the equation $(\sharp_k)$ has quasimodular solutions of weight $k+1$. To describe this, we need the following polynomials:

$$\begin{aligned}
&A_0(X) := 1,\ B_0(X) := 0,\ A_1(X) := X,\ B_1(X) := 1, \\
&A_{n+1}(X) := XA_n(X) + \rho_n A_{n-1}(X) \qquad (n \ge 1), \\
&B_{n+1}(X) := XB_n(X) + \rho_n B_{n-1}(X) \qquad (n \ge 1),
\end{aligned}$$

where

$$\rho_n := 12\frac{(6n+1)(6n+5)}{n(n+1)} \quad (n \ge 1).$$

The first few examples of the polynomials and ρ_n are

$$\begin{aligned}
&A_2(X) = X^2 + 462,\ B_2(X) = X, \\
&A_3(X) = X^3 + 904X,\ B_3(X) = X^2 + 442, \\
&A_4(X) = X^4 + 1341X^2 + 201894,\ B_4(X) = X^3 + 879X, \\
&\rho_1 = 462,\ \rho_2 = 442,\ \rho_3 = 437.
\end{aligned}$$

Remark 3.5. By the definitions of the polynomials and the induction on $n \ge 0$, we find that $\deg A_n(X) = n$ (resp. $\deg B_n(X) = n-1$) and $A_n(X)$ is even or odd (resp. $B_n(X)$ is odd or even) according to whether n is even or odd.

Theorem 3.6. ([KK03, Theorem 2]) *For each* $k \in \mathbb{Z}_{\ge 0}$ *with* $k \equiv 5 \pmod 6$, *the quasimodular form*

$$\sqrt{\Delta(\tau)}^{\frac{k-5}{6}} A_{\frac{k-5}{6}}\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right)\frac{E_4'(\tau)}{240} - \sqrt{\Delta(\tau)}^{\frac{k-5}{6}+1} B_{\frac{k-5}{6}}\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right).$$

of weight $k+1$ *is a solution of* $(\sharp_k)$.

Remark 3.7. Noting Remark 3.5 and $E_4' = (E_2E_4 - E_6)/3$, we see that $\sqrt{\Delta}$ in the definition of the above equation cancels out and therefore the above function is an element in $\mathbb{Q}[E_2, E_4, E_6]$.

From Theorem 3.3 and Theorem 3.6, we obtain the following theorem:

Theorem 3.8. *For all $k \in \mathbb{Z}_{\geq 0}$, there exist modular or quasimodular solutions of $(\sharp_k)$.*

Finally, we consider the case of $k \in \mathbb{Q}_{\geq 0}$ with $k \equiv \frac{1}{2} \pmod 3$.

Theorem 3.9. ([KK03, Theorem 1 (iv)]) *For each $k \in \mathbb{Q}_{\geq 0}$ with $k \equiv \frac{1}{2}$ $\pmod 3$, the modular forms*

$$\begin{aligned}
E_2^{(4)}(\tau)^{\frac{k}{2}} {}_2F_1\Big(-\frac{2k-1}{6}, -\frac{k}{2}, -\frac{k-5}{6}; \frac{16}{j^{(4)}(\tau)}\Big) \\
&= \sum_{0 \leq i \leq \frac{2k-1}{6}} \frac{(-\frac{2k-1}{6})_i(-\frac{k}{2})_i}{(-\frac{k-5}{6})_i i!} 16^i \Delta_2^{(4)}(\tau)^i E_2^{(4)}(\tau)^{\frac{k}{2}-i} \\
&= 1 + O(q) \in M_k(\Gamma_0(4))
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2^{(4)}(\tau)^{\frac{k+1}{6}} E_2^{(4)}(\tau)^{\frac{2k-1}{6}} {}_2F_1\Big(-\frac{2k-1}{6}, -\frac{k-2}{6}, \frac{k+7}{6}; \frac{16}{j^{(4)}(\tau)}\Big) \\
&= \sum_{0 \leq i \leq \frac{2k-1}{6}} \frac{(-\frac{2k-1}{6})_i(-\frac{k-2}{6})_i}{(\frac{k+7}{6})_i i!} 16^i \Delta_2^{(4)}(\tau)^{\frac{k+1}{6}+i} E_2^{(4)}(\tau)^{\frac{2k-1}{6}-i} \\
&= q^{\frac{k+1}{6}} + O(q^{\frac{k+7}{6}}) \in M_k(\Gamma_0^0(4))
\end{aligned}$$

are solutions of $(\sharp_k)$.

Using Theorem 3.8 and Theorem 3.9, we can construct solutions of $(\flat_{k,m})$. (See Subsection 5.3.)

4 Jacobi forms with respect to $SL_2(\mathbb{Z})$

Here, we summarize the theory of Jacobi forms. (For further details, see [EZ85].)

First, we define Jacobi forms. A holomorphic function $\phi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ is a Jacobi form of weight k ($\in \mathbb{Z}$) and index m ($\in \mathbb{Z}$) with respect to $SL_2(\mathbb{Z})$ if it satisfies the following conditions

- $\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \mathbf{e}\left(\frac{cmz^2}{c\tau+d}\right)\phi(\tau, z)$

$$\text{(for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \text{ and } (\tau, z) \in \mathfrak{H} \times \mathbb{C}),$$

- $\phi(\tau, z + \lambda\tau + \mu) = \mathbf{e}(-\lambda^2 m\tau - 2\lambda m z)\phi(\tau, z)$

$$\text{(for all } (\lambda, \mu) \in \mathbb{Z}^2 \text{ and } (\tau, z) \in \mathfrak{H} \times \mathbb{C}),$$

- $\phi(\tau, z)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ 4mn \geq r^2}} c(n, r) q^n \zeta^r.$$

We denote by $J_{k,m}$ the $\mathbb{C}$-vector space of Jacobi forms of weight k and index m with respect to $SL_2(\mathbb{Z})$. If $m = 1$, we write J_k for $J_{k,1}$. A Jacobi form $\phi \in J_{k,m}$ that satisfies the stronger condition $c(n, r) = 0$ if $4mn = r^2$ is called a Jacobi cusp form.

Theorem 4.1. ([EZ85, Theorem 1.1, Corollary of Theorem 9.2]) *The space $J_{k,m}$ is finite dimensional. Further, if $k \geq m \geq 1$, then we have*

$$\dim_{\mathbb{C}} J_{k,m} = \begin{cases} \displaystyle\sum_{\nu=0}^{m} \dim_{\mathbb{C}} M_{k+2\nu} - \sum_{\nu=0}^{m} \left\lceil \frac{\nu^2}{4m} \right\rceil & \text{if } k \text{ is even,} \\[2ex] \displaystyle\sum_{\nu=1}^{m-1} \dim_{\mathbb{C}} M_{k+2\nu-1} - \sum_{\nu=1}^{m-1} \left\lceil \frac{\nu^2}{4m} \right\rceil & \text{if } k \text{ is odd,} \end{cases} \tag{3}$$

where $\lceil x \rceil$ is the smallest integer $n \geq x$.

A basic example of Jacobi forms is the Jacobi-Eisenstein series $E_{k,m}(\tau, z)$ of weight $k \geq 4$ and index $m \geq 1$:

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-k} \sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(\lambda^2 m \frac{a\tau + b}{c\tau + d} + 2\lambda m \frac{z}{c\tau + d} - \frac{cmz^2}{c\tau + d}\Big).$$

To describe the Fourier expansion of $E_{k,m}(\tau, z)$, we define

$$\alpha_{k,m} := (-1)^{\frac{k}{2}} \frac{2^{2-k}\pi^{k-\frac{1}{2}}}{m^{k-1}\Gamma\big(k - \frac{1}{2}\big)\zeta(k-1)},$$

where $\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann zeta function.

Theorem 4.2. ([EZ85, Theorem 2.1]) *The Fourier expansion of $E_{k,m}(\tau, z)$ is given by*

$$E_{k,m}(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn \geq r^2}} e_{k,m}(n, r) q^n \zeta^r.$$

Here

$$e_{k,m}(n, r) := \begin{cases} 1 & \text{if } 4mn = r^2 \text{ and } r \equiv 0 \pmod{2m}, \\ 0 & \text{if } 4mn = r^2 \text{ and } r \not\equiv 0 \pmod{2m}, \\ \alpha_{k,m}(4mn - r^2)^{k-\frac{3}{2}} \displaystyle\sum_{a=1}^{\infty} \frac{N_a(m, r, n)}{a^{k-1}} & \text{if } 4mn > r^2, \end{cases}$$

where $N_a(m, r, n)$ represents the order of the set $\{\lambda \pmod{a} \mid m\lambda^2 + r\lambda + n \equiv 0 \pmod{a}\}$.

Remark 4.3. Using Cohen's function (cf. [C75] and [Z77, Proposition 3]), we can compute the Fourier coefficients $e_{k,m}(n, r)$ of $E_{k,m}(\tau, z)$ more explicitly.

For example,

$$\begin{aligned} E_{4,1}(\tau, z) &= 1 + (\zeta^{-2} + 56\zeta^{-1} + 126 + 56\zeta + \zeta^2)q \\ &\quad + (126\zeta^{-2} + 576\zeta^{-1} + 756 + 576\zeta + 126\zeta^2)q^2 + \cdots, \\ E_{6,1}(\tau, z) &= 1 + (\zeta^{-2} - 88\zeta^{-1} - 330 - 88\zeta + \zeta^2)q \\ &\quad + (-330\zeta^{-2} - 4224\zeta^{-1} - 7524 - 4224\zeta - 330\zeta^2)q^2 + \cdots. \end{aligned}$$

They are important, because they are generators of the space $J_{*,1} := \bigoplus_k J_k$ of Jacobi forms of index 1 that is a free module of rank 2 over the ring $M_* := \bigoplus_k M_k$:

Theorem 4.4. ([EZ85, Theorem 3.5]) (i) We have

$$J_{*,1} = M_* E_{4,1} \oplus M_* E_{6,1}.$$

(ii) We have the isomorphism

$$J_k \cong M_k \oplus S_{k+2}.$$

Next, let us describe theta expansions of Jacobi forms. The following lemma about Fourier coefficients of Jacobi forms plays an important role in theta expansions:

Lemma 4.5. ([EZ85, Theorem 2.2]) *Let $\phi = \sum c(n,r)q^n\zeta^r$ be a Jacobi form of weight k and index m with respect to $SL_2(\mathbb{Z})$. If $r \equiv r' \pmod{2m}$ and $4mn - r^2 = 4mn' - r'^2$, then $c(n,r) = c(n',r')$.*

By Lemma 4.5, for all $\mu \in \mathbb{Z}/2m\mathbb{Z}$ and $N \in \mathbb{Z}_{\geq 0}$,

$$c_\mu(N) := \begin{cases} c(\frac{N+r^2}{4m}, r) & \text{if } N \equiv -r^2 \pmod{4m} \text{ with } r \equiv \mu \pmod{2m}, \\ 0 & \text{otherwise} \end{cases}$$

is well-defined. Here $r \equiv \mu \pmod{2m}$ represents $r \in \mu$. Then we have

$$\begin{aligned} \phi(\tau,z) &= \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn\geq r^2}} c(n,r)q^n\zeta^r \\ &= \sum_{r\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \\ 4mn\geq r^2}} c(n,r)q^n\zeta^r \\ &= \sum_{\mu \pmod{2m}} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \pmod{2m}}} \sum_{\substack{n\in\mathbb{Z} \\ 4mn\geq r^2}} c(n,r)q^n\zeta^r \\ &= \sum_{\mu \pmod{2m}} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \pmod{2m}}} \sum_{N\geq 0} c_\mu(N) q^{\frac{N+r^2}{4m}} \zeta^r \\ &= \sum_{\mu \pmod{2m}} h_\mu(\tau)\theta_{m,\mu}(\tau,z), \end{aligned} \tag{4}$$

where

$$h_\mu(\tau) := \sum_{N\geq 0} c_\mu(N) q^{\frac{N}{4m}},$$

and

$$\theta_{m,\mu}(\tau,z) := \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu \pmod{2m}}} q^{\frac{r^2}{4m}}\zeta^r.$$

We call the expansion (4) the theta expansion of $\phi(\tau,z)$. Note that $\theta_{m,\mu}(\tau,z)$ are independent of $\phi(\tau,z)$.

As an application of theta expansions of Jacobi forms, we mention Theorem 4.7. First, we prove the transformation laws of $\theta_{m,\mu}(\tau,z)$ and $h_\mu(\tau)$:

Proposition 4.6. *Under the above notations, we have*

(i) $\theta_{m,\mu}(\tau+1,z) = \mathbf{e}\Big(\dfrac{\mu^2}{4m}\Big)\theta_{m,\mu}(\tau,z)$,

(ii) $\theta_{m,\mu}\Big(-\dfrac{1}{\tau},\dfrac{z}{\tau}\Big) = \Big(\dfrac{\tau}{2m\sqrt{-1}}\Big)^{\frac{1}{2}}\mathbf{e}\Big(\dfrac{mz^2}{\tau}\Big)\displaystyle\sum_{\nu \pmod{2m}} \mathbf{e}\Big(-\dfrac{\mu\nu}{2m}\Big)\theta_{m,\nu}(\tau,z)$,

(iii) $h_\mu(\tau+1) = \mathbf{e}\Big(-\dfrac{\mu^2}{4m}\Big)h_\mu(\tau)$,

(iv) $h_\mu\Big(-\dfrac{1}{\tau}\Big) = \dfrac{\tau^k}{\Big(\frac{2m\tau}{\sqrt{-1}}\Big)^{\frac{1}{2}}}\displaystyle\sum_{\nu \pmod{2m}} \mathbf{e}\Big(\dfrac{\mu\nu}{2m}\Big)h_\nu(\tau)$,

where $\mathbf{e}(\frac{\xi}{\cdot}) := \mathbf{e}(\frac{x}{\cdot})$ *for* $\xi = x \pmod{*} \in \mathbb{Z}/*\mathbb{Z}$.

Proof. The assertion (i) is easy from the definition of $\theta_{m,\mu}(\tau,z)$. Using the Poisson summation formula, we obtain the assertion (ii). From (i) (resp. (ii)) and the invariance of ϕ under $(\tau,z)\mapsto(\tau+1,z)$ (resp. the transformation law of ϕ under $(\tau,z)\mapsto(-1/\tau,z/\tau)$), we obtain the assertion (iii) (resp. (iv)). This completes the proof of Proposition 4.6. □

Theorem 4.7. ([EZ85, Theorem 5.1]) *The theta expansion* (4) *gives an isomorphism between* $J_{k,m}$ *and the* $\mathbb{C}$*-vector space of vector-valued modular forms* $(h_\mu)_{\mu \pmod{2m}}$ *with respect to* $SL_2(\mathbb{Z})$ *satisfying Proposition* 4.6 (iii) *and* (iv) *and bounded as* $y = \mathrm{Im}(\tau)\to\infty$.

Proof. Because the group $SL_2(\mathbb{Z})$ is generated by the matrices $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, we arrive at the claim from Proposition 4.6. □

Further, when k is an even integer and $m = 1$, using Theorem 4.7, we obtain the following theorem:

Theorem 4.8. ([EZ85, Theorem 5.4]) *Let k be an even integer. Then we have the isomorphism*

$$\begin{array}{ccc} M^{+}_{k-\frac{1}{2}} & \cong & J_k \\ \cup & & \cup \\ \displaystyle\sum_{\substack{N \geq 0 \\ N \equiv 0 \,\mathrm{or}\, 3 \pmod 4}} c(N) q^N & \mapsto & \displaystyle\sum_{\substack{n, r \in \mathbb{Z} \\ 4n \geq r^2}} c(4n - r^2) q^n \zeta^r, \end{array}$$

where $M^{+}_{k-1/2}$ is the Kohnen plus space appearing in subsection 2.2.

Remark 4.9. The above isomorphism is compatible with the Peterson scalar products, with the actions of Hecke operators, and with the structures of $M_{2*-1/2} := \bigoplus_{k:\mathrm{even}} M_{k-1/2}$ and $J_{*,1}$ as modules over M_*.

5 The equation $(\flat_{k,m})$ for Jacobi forms

In this section, we study the equation $(\flat_{k,m})$ which is a Jacobi form analog of the K-Z equation $(\sharp_k)$. Some of the results given in this section are based on [Ki14].

5.1 The case of $m = 1$

In this subsection, we consider the equation

$$\begin{aligned}\phi^{[4]}(\tau,z) - 8\phi^{[2](1)}(\tau,z) + \frac{2k+1}{3}E_2(\tau)\phi^{[2]}(\tau,z) \\ + 16\phi^{(2)}(\tau,z) - \frac{4(2k+1)}{3}E_2(\tau)\phi'(\tau,z) \\ + \frac{(2k-1)(2k+1)}{3}E_2'(\tau)\phi(\tau,z) = 0. \end{aligned} \tag{$\flat_k$}$$

5.1.1 A method for deriving $(\flat_k)$ and properties of $(\flat_k)$

The equation $(\flat_k)$ for Jacobi forms of index 1 is derived in a similar manner to the K-Z equation $(\sharp_k)$. In order to derive the equation $(\flat_k)$, we define the heat operator L and the modified heat operator ∂_k [R09]:

$$\begin{aligned} L &:= \frac{1}{(2\pi\sqrt{-1})^2}\Big(8\pi\sqrt{-1}\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\Big), \\ \partial_k &:= L - \frac{2k-1}{6}E_2 : J_k \longrightarrow J_{k+2}. \end{aligned}$$

(We shall often drop the subscript k as with ϑ_k.) Note that for $f \in M_k$ and $\phi \in J_l$, the Leibniz rule

$$\partial_{k+l}(f\phi) = 4\vartheta_k(f)\phi + f\partial_l(\phi)$$

holds (note the factor 4 on the right). Combining this rule with $\Delta' = \Delta E_2$ (i.e. $\vartheta(\Delta) = 0$), we have

$$\partial_{12+l}(\Delta\phi) = \Delta\partial_l(\phi).$$

These relations will be used later.

Let us derive the equation $(\flat_k)$ for Jacobi forms of index 1. From Theorem 4.4 (ii) and the well-known dimension formulas for spaces of modular forms and cusp forms, for even integer $k \geq 4$, we have

$$\dim_{\mathbb{C}} J_k = \begin{cases} 2\gamma & \text{if } k \equiv 0 \pmod{12}, \\ \\ 2\gamma + 1 & \text{if } k \equiv 4,\ 6 \text{ or } 8 \pmod{12}, \\ \\ 2\gamma + 2 & \text{if } k \equiv 2 \text{ or } 10 \pmod{12}, \end{cases}$$

where $\gamma \in \mathbb{Z}_{\geq 0}$ is defined through the unique expression $k = 12\gamma + 4\delta + 6\epsilon$ ($\delta \in \{0, 1, 2\}, \epsilon \in \{0, 1\}$). Using this dimension formula, we can easily check that $\dim_{\mathbb{C}} J_k = \dim_{\mathbb{C}} J_{k+4}$ and thus $J_{k+4} = E_4 \cdot J_k$ if $k \equiv 4 \pmod 6$. Therefore, for $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 4 \pmod 6$, we can define the endomorphism

$$\varpi_k := \frac{1}{E_4} \partial_{k+2} \circ \partial_k : J_k \longrightarrow J_k.$$

(We may use Theorem 4.1 to check the well-definedness of ϖ_k.) Because the constant term of $\varpi_k(\phi)$ is $\kappa_k := (2k-1)(2k+3)/36$ times the constant term of ϕ, the map ϖ_k preserves the codimension 1 subspace of Jacobi cusp forms (cf. [EZ85, a paragraph before Theorem 2.3]). It follows that the value κ_k is an eigenvalue of ϖ_k. Then, using $E_2' = (E_2^2 - E_4)/12$, we find that the equation

$$\varpi_k(\phi)(\tau, z) = \kappa_k \phi(\tau, z)$$

is equivalent to the equation $(\flat_k)$.

Remark 5.1. In general, a solution of $(\flat_k)$ is not necessarily a Jacobi form of index 1.

Although there exist other eigenvalues of ϖ_k in general, the following proposition suggests that the value κ_k is basic among the eigenvalues of ϖ_k.

Proposition 5.2. *For each $l \in \mathbb{Z}_{\geq 0}$ with $l \equiv 4 \pmod 6$, choose an eigenvector $\phi_l \in J_l$ with eigenvalue κ_l. Then the form $\Delta^i \phi_{k-12i} \in J_k$ is an eigenvector of ϖ_k with eigenvalue κ_{k-12i} for each i $(0 \leq i \leq \gamma)$.*

Proof. Since $\partial(\Delta^i\phi) = \Delta^i\partial(\phi)$, we have

$$\begin{aligned}\varpi_k(\Delta^i\phi_{k-12i}) &= \frac{1}{E_4}\Delta^i\partial^2(\phi_{k-12i})\\ &= \Delta^i\varpi_{k-12i}(\phi_{k-12i})\\ &= \kappa_{k-12i}\Delta^i\phi_{k-12i}.\end{aligned}$$

This completes the proof of Proposition 5.2. $\square$

We end this subsection by proving two propositions on $(\flat_k)$. The first proposition represents that the space of solutions of $(\flat_k)$ is closed under the actions of $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$:

Proposition 5.3. *If $\phi(\tau, z)$ is a solution of $(\flat_k)$, then the two functions*

$$(c\tau+d)^{-k}\mathbf{e}\Big(\frac{-cz^2}{c\tau+d}\Big)\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) \text{ and } \mathbf{e}(\lambda^2\tau+2\lambda z)\phi(\tau, z+\lambda\tau+\mu)$$

are also solutions of $(\flat_k)$ for all $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $(\lambda,\mu) \in \mathbb{Z}^2$.

Proof. Define

$$\begin{aligned}\psi(\tau, z) &:= (c\tau+d)^{-k}\mathbf{e}\Big(\frac{-cz^2}{c\tau+d}\Big)\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big),\\ \chi(\tau, z) &:= \mathbf{e}(\lambda^2\tau+2\lambda z)\phi(\tau, z+\lambda\tau+\mu).\end{aligned}$$

By direct calculations, we have

$$
\begin{aligned}
&\psi^{[4]}(\tau,z) - 8\psi^{[2](1)}(\tau,z) + \frac{2k+1}{3}E_2(\tau)\psi^{[2]}(\tau,z) + 16\psi^{(2)}(\tau,z)\\
&\qquad - \frac{4(2k+1)}{3}E_2(\tau)\psi'(\tau,z) + \frac{(2k-1)(2k+1)}{3}E_2'(\tau)\psi(\tau,z)\\
&= (c\tau+d)^{-k-4}\mathbf{e}\Big(-\frac{cz^2}{c\tau+d}\Big)\\
&\quad \times \Big\{\phi^{[4]}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) - 8\phi^{[2](1)}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\\
&\qquad\qquad + \frac{2k+1}{3}E_2\Big(\frac{a\tau+b}{c\tau+d}\Big)\phi^{[2]}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\\
&\qquad\qquad + 16\phi^{(2)}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\\
&\qquad\qquad - \frac{4(2k+1)}{3}E_2\Big(\frac{a\tau+b}{c\tau+d}\Big)\phi'\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\\
&\qquad\qquad + \frac{(2k-1)(2k+1)}{3}E_2'\Big(\frac{a\tau+b}{c\tau+d}\Big)\phi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\Big\}\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\chi^{[4]}(\tau,z) - 8\chi^{[2](1)}(\tau,z) + \frac{2k+1}{3}E_2(\tau)\chi^{[2]}(\tau,z) + 16\chi^{(2)}(\tau,z)\\
&\qquad - \frac{4(2k+1)}{3}E_2(\tau)\chi'(\tau,z) + \frac{(2k-1)(2k+1)}{3}E_2'(\tau)\chi(\tau,z)\\
&= \mathbf{e}(\lambda^2\tau + 2\lambda z)\\
&\quad \times \Big\{\phi^{[4]}(\tau,z+\lambda\tau+\mu) - 8\phi^{[2](1)}(\tau,z+\lambda\tau+\mu)\\
&\qquad\qquad + \frac{2k+1}{3}E_2(\tau)\phi^{[2]}(\tau,z+\lambda\tau+\mu)\\
&\qquad\qquad + 16\phi^{(2)}(\tau,z+\lambda\tau+\mu)\\
&\qquad\qquad - \frac{4(2k+1)}{3}E_2(\tau)\phi'(\tau,z+\lambda\tau+\mu)\\
&\qquad\qquad + \frac{(2k-1)(2k+1)}{3}E_2'(\tau)\phi(\tau,z+\lambda\tau+\mu)\Big\}\\
&= 0.
\end{aligned}
$$

Therefore, the two functions $\psi(\tau,z)$ and $\chi(\tau,z)$ are also solutions of $(\flat_k)$. $\square$

The second proposition describes a characterization of the equation $(\flat_k)$. Let us consider the following differential equation for holomorphic functions $\phi(\tau,z)$ on $\mathfrak{H}\times\mathbb{C}$:

$$\begin{aligned}\phi^{[4]}(\tau,z)+A_1(\tau)\phi^{[2](1)}(\tau,z)+A_2(\tau)\phi^{[2]}(\tau,z)\\ +A_3(\tau)\phi^{(2)}(\tau,z)+A_4(\tau)\phi'(\tau,z)+A_5(\tau)\phi(\tau,z)=0,\end{aligned}\tag{5}$$

where $A_i(\tau)$ $(1\le i\le 5)$ are holomorphic functions on $\mathfrak{H}$. Further, fixing an integer k, we impose the following condition on the equation (5):

Condition 5.4. (i) *If $\phi(\tau,z)$ is a solution of* (5)*, then the two functions*

$$(c\tau+d)^{-k}\mathbf{e}\Big(\frac{-cz^2}{c\tau+d}\Big)\phi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\quad\textit{and}\quad \mathbf{e}(\lambda^2\tau+2\lambda z)\phi(\tau,z+\lambda\tau+\mu)$$

are also solutions of (5) *for all* $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$ *and* $(\lambda,\mu)\in\mathbb{Z}^2$.
(ii) *The functions $A_i(\tau)$ $(1\le i\le 5)$ are bounded when $y=\mathrm{Im}(\tau)\to\infty$.*

Proposition 5.5. *The differential equation* (5) *satisfying Condition* 5.4 *is essentially the equation $(\flat_k)$. More precisely, if a function ϕ is a solution of the equation* (5) *satisfying Condition* 5.4*, then the function $\Delta^\beta\phi$ for a suitable constant β is a solution of the equation $(\flat_{12\beta+k})$.*

Proof. First, define $\chi(\tau,z):=\mathbf{e}(\lambda^2\tau+2\lambda z)\phi(\tau,z+\lambda\tau+\mu)$. From Condition 5.4 (i), we have

$$\begin{aligned}0&=\mathbf{e}(-\lambda^2\tau-2\lambda z)\Big\{\chi^{[4]}(\tau,z)+A_1(\tau)\chi^{[2](1)}(\tau,z)+A_2(\tau)\chi^{[2]}(\tau,z)\\&\qquad\qquad+A_3(\tau)\chi^{(2)}(\tau,z)+A_4(\tau)\chi'(\tau,z)+A_5(\tau)\chi(\tau,z)\Big\}\\&=\phi^{[4]}(\tau,z+\lambda\tau+\mu)+A_1(\tau)\phi^{[2](1)}(\tau,z+\lambda\tau+\mu)\\&\quad+(24\lambda^2+5\lambda^2A_1(\tau)+A_2(\tau)+\lambda^2A_3(\tau))\phi^{[2]}(\tau,z+\lambda\tau+\mu)\\&\quad+A_3(\tau)\phi^{(2)}(\tau,z+\lambda\tau+\mu)\\&\quad+(4\lambda^2A_1(\tau)+2\lambda^2A_3(\tau)+A_4(\tau))\phi'(\tau,z+\lambda\tau+\mu)\\&\quad+(16\lambda^4+4\lambda^4A_1(\tau)+4\lambda^2A_2(\tau)+\lambda^4A_3(\tau)\\&\qquad\qquad+\lambda^2A_4(\tau)+A_5(\tau))\phi(\tau,z+\lambda\tau+\mu)\\&\quad+(8\lambda+\lambda A_1(\tau))\phi^{[3]}(\tau,z+\lambda\tau+\mu)\\&\quad+(32\lambda^3+8\lambda^3A_1(\tau)+4\lambda A_2(\tau)+2\lambda^3A_3(\tau)\\&\qquad\qquad+\lambda A_4(\tau))\phi^{[1]}(\tau,z+\lambda\tau+\mu)\\&\quad+(4\lambda A_1(\tau)+2\lambda A_3(\tau))\phi^{[1](1)}(\tau,z+\lambda\tau+\mu).\end{aligned}$$

Comparing this equation with

$$\begin{aligned}&\phi^{[4]}(\tau, z+\lambda\tau+\mu)+A_1(\tau)\phi^{[2](1)}(\tau, z+\lambda\tau+\mu)\\&+A_2(\tau)\phi^{[2]}(\tau, z+\lambda\tau+\mu)+A_3(\tau)\phi^{(2)}(\tau, z+\lambda\tau+\mu)\\&+A_4(\tau)\phi'(\tau, z+\lambda\tau+\mu)+A_5(\tau)\phi(\tau, z+\lambda\tau+\mu)=0,\end{aligned}$$

we have

$$\begin{aligned}A_1(\tau)&=-8,\\A_3(\tau)&=16,\\A_4(\tau)&=-4A_2(\tau).\end{aligned}$$

Therefore, the equation (5) becomes

$$\begin{aligned}&\phi^{[4]}(\tau, z)-8\phi^{[2](1)}(\tau, z)+A_2(\tau)\phi^{[2]}(\tau, z)\\&\qquad+16\phi^{(2)}(\tau, z)-4A_2(\tau)\phi'(\tau, z)+A_5(\tau)\phi(\tau, z)=0.\end{aligned}\tag{6}$$

Then, define $\psi(\tau,z) := (c\tau+d)^{-k}\mathbf{e}\left(\frac{-cz^2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$. From Condition 5.4 (i), we have

$$\begin{aligned}0&=(c\tau+d)^{k+4}\mathbf{e}\left(\frac{cz^2}{c\tau+d}\right)\Big\{\psi^{[4]}(\tau, z)-8\psi^{[2](1)}(\tau, z)+A_2(\tau)\psi^{[2]}(\tau, z)\\&\qquad\qquad+16\psi^{(2)}(\tau, z)-4A_2(\tau)\psi'(\tau, z)+A_5(\tau)\psi(\tau, z)\Big\}\\&=\phi^{[4]}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)-8\phi^{[2](1)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)\\&\quad+\left\{(c\tau+d)^2A_2(\tau)+\frac{2(2k+1)}{\pi\sqrt{-1}}c(c\tau+d)\right\}\phi^{[2]}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)\\&\quad+16\phi^{(2)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)\\&\quad-4\left\{(c\tau+d)^2A_2(\tau)+\frac{2(2k+1)}{\pi\sqrt{-1}}c(c\tau+d)\right\}\phi'\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)\\&\quad+\left\{(c\tau+d)^4A_5(\tau)+\frac{2k-1}{\pi\sqrt{-1}}c(c\tau+d)^3A_2(\tau)\right.\\&\qquad\qquad\left.+\frac{(2k-1)(2k+1)}{(\pi\sqrt{-1})^2}c^2(c\tau+d)^2\right\}\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right).\end{aligned}$$

Comparing this equation with

$$\begin{aligned}\phi^{[4]}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)-8\phi^{[2](1)}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)&\\
+A_2\Big(\frac{a\tau+b}{c\tau+d}\Big)\phi^{[2]}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)&\\
+16\phi^{(2)}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)&\\
-4A_2\Big(\frac{a\tau+b}{c\tau+d}\Big)\phi'\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)&\\
+A_5\Big(\frac{a\tau+b}{c\tau+d}\Big)\phi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)&=0,\end{aligned}$$

we have

$$A_2\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^2A_2(\tau)+\frac{2(2k+1)}{\pi\sqrt{-1}}c(c\tau+d), \tag{7}$$

$$\begin{aligned}A_5\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^4A_5(\tau)+\frac{2k-1}{\pi\sqrt{-1}}c(c\tau+d)^3A_2(\tau)&\\
+\frac{(2k-1)(2k+1)}{(\pi\sqrt{-1})^2}c^2(c\tau+d)^2.&\end{aligned} \tag{8}$$

Define $A(\tau):=A_2(\tau)-(2k+1)E_2(\tau)/3$. From (7) and the transformation formula (1) of $E_2(\tau)$, we have

$$A\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^2A(\tau). \tag{9}$$

Next, define $B(\tau):=A_5(\tau)-(2k-1)A_2'(\tau)$. From (8) and

$$A_2'\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^4A_2'(\tau)+\frac{c(c\tau+d)^3}{\pi\sqrt{-1}}A_2(\tau)+\frac{2k+1}{(\pi\sqrt{-1})^2}c^2(c\tau+d)^2,$$

we have

$$B\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^4B(\tau). \tag{10}$$

Hence, from (9), (10) and Condition 5.4 (ii), we have

$$A(\tau)\in M_2=\{0\},\text{ i.e. }A_2(\tau)=\frac{2k+1}{3}E_2(\tau),$$

$$B(\tau)\in M_4=\mathbb{C}\cdot E_4,\text{ i.e. }A_5(\tau)=\frac{(2k-1)(2k+1)}{3}E_2'(\tau)+\alpha E_4(\tau)$$

with some constant $\alpha \in \mathbb{C}$, and therefore the equation (6) becomes

$$\begin{aligned}\phi^{[4]}(\tau,z) - 8\phi^{[2](1)}(\tau,z) + \frac{2k+1}{3}E_2(\tau)\phi^{[2]}(\tau,z) \\ + 16\phi^{(2)}(\tau,z) - \frac{4(2k+1)}{3}E_2(\tau)\phi'(\tau,z) \\ + \left\{\frac{(2k-1)(2k+1)}{3}E_2'(\tau) + \alpha E_4(\tau)\right\}\phi(\tau,z) = 0. \qquad (11)\end{aligned}$$

Let β be a solution of

$$16\beta^2 + \frac{4(2k+1)}{3}\beta + \alpha = 0.$$

We see by a direct calculation that if ϕ is a solution of (11), then $\Delta^\beta\phi$ is a solution of $(\flat_{12\beta+k})$. Therefore, without loss of generality, we may assume that $\alpha = 0$. □

5.1.2 Jacobi form solutions

In this subsection, we present the main result of the paper (Theorem 5.8). First, as preparation, we define the polynomials $P_n(X)$, $Q_n(X)$, $R_n(X)$ and $S_n(X)$ $(n \geq 0)$ as follows:

$$\begin{aligned}&P_0(X) := 1,\ Q_0(X) := 0,\ R_0(X) := 1,\ S_0(X) := 1, \\ &P_{n+1}(X) := (X-1728)R_n(X) + \lambda_{2n+1}P_n(X) && (n \geq 0), \\ &Q_{n+1}(X) := S_n(X) + \lambda_{2n+1}Q_n(X) && (n \geq 0), \\ &R_{n+1}(X) := P_{n+1}(X) + \lambda_{2n+2}R_n(X) && (n \geq 0), \\ &S_{n+1}(X) := (X-1728)Q_{n+1}(X) + \lambda_{2n+2}S_n(X) && (n \geq 0),\end{aligned}$$

where

$$\lambda_n := \begin{cases} \dfrac{5472}{7} & \text{if } n = 1, \\ 48\dfrac{(12n-1)(12n+7)}{(4n-1)(4n+3)} & \text{if } n \geq 2. \end{cases}$$

The first few examples of the polynomials and λ_n are

$$
\begin{aligned}
&P_1(X) = X - \frac{6624}{7},\ Q_1(X) = 1,\ R_1(X) = X - \frac{5520}{11},\ S_1(X) = X - \frac{98832}{77},\\
&P_2(X) = X^2 - 1792X + \frac{4981248}{11},\ Q_2(X) = X - \frac{5920}{7},\\
&R_2(X) = X^2 - \frac{25776}{19}X + \frac{48982272}{209},\ S_2(X) = X^2 - \frac{284400}{133}X + \frac{120043776}{133},\\
&P_3(X) = X^3 - \frac{60960}{23}X^2 + \frac{8655213312}{4807}X - \frac{43547369472}{209},\\
&Q_3(X) = X^2 - \frac{274368}{161}X + \frac{1637722368}{3059},\\
&R_3(X) = X^3 - \frac{19952}{9}X^2 + \frac{306732544}{253}X - \frac{66933919744}{627},\\
&S_3(X) = X^3 - \frac{188912}{63}X^2 + \frac{58720768}{23}X - \frac{699840458752}{1311},\\
&\lambda_2 = \frac{34224}{77},\ \lambda_3 = \frac{4816}{11},\ \lambda_4 = \frac{8272}{19},\ \lambda_5 = \frac{189744}{437},\ \lambda_6 = \frac{89744}{207}.
\end{aligned}
$$

The sequence $\{\lambda_n\}_{n\geq 1}$ is related to the hypergeometric series ${}_2F_1$:

Proposition 5.6. *We have*

$$
\frac{{}_2F_1(\frac{31}{24}, \frac{19}{24}, \frac{7}{4}; \frac{1728}{j})}{{}_2F_1(\frac{7}{24}, \frac{19}{24}, \frac{7}{4}; \frac{1728}{j})} = \cfrac{1}{1 - \cfrac{\lambda_1 j^{-1}}{1 - \cfrac{\lambda_2 j^{-1}}{1 - \cfrac{\lambda_3 j^{-1}}{1 - \cdots}}}}.
$$

Proof. For simplicity of notation, we use $F := {}_2F_1$. Let us define the sequence $\{a_n\}_{n\geq 1}$ as follows:

$$
\begin{aligned}
a_{2n-1} &:= -\frac{(24n-1)(24n+7)}{36(8n-1)(8n+3)},\\
a_{2n} &:= -\frac{(24n+11)(24n+19)}{36(8n+3)(8n+7)}.
\end{aligned}
$$

By a direct calculation, we have

$$
\begin{aligned}
\frac{F(\frac{31}{24},\frac{19}{24},\frac{7}{4};x)}{F(\frac{7}{24},\frac{19}{24},\frac{7}{4};x)} &= \frac{F(\frac{31}{24},\frac{19}{24},\frac{7}{4};x)}{F(\frac{31}{24},\frac{19}{24},\frac{7}{4};x)-\frac{19x}{42}F(\frac{7}{24}+1,\frac{19}{24}+1,\frac{7}{4}+1;x)} \\
&= \cfrac{1}{1-\cfrac{\frac{19x}{42}}{\cfrac{F(\frac{31}{24},\frac{19}{24},\frac{7}{4};x)}{F(\frac{31}{24},\frac{19}{24}+1,\frac{7}{4}+1;x)}}} \\
&= \cfrac{1}{1-\cfrac{\frac{19x}{42}}{1+\cfrac{a_1x}{1+\cfrac{a_2x}{1+\cdots}}}}, \qquad (12)
\end{aligned}
$$

where we use Gauss's contiguous relation

$$
F(a+1,b,c;x)-F(a,b,c;x)=\frac{bx}{c}F(a+1,b+1,c+1;x)
$$

[Ga1812, the equation (18) in §11] at the first equality and Gauss's formula for the continued fraction expansion of a quotient of Gauss hypergeometric series

$$
\frac{F(a,b,c;x)}{F(a,b+1,c+1;x)}=1+\cfrac{d_1x}{1+\cfrac{d_2x}{1+\cfrac{d_3x}{1+\cdots}}},
$$

$$
d_{2k-1}:=-\frac{(a+k-1)(c-b+k-1)}{(c+2k-2)(c+2k-1)}, \quad d_{2k}:=-\frac{(b+k)(c-a+k)}{(c+2k-1)(c+2k)}
$$

[Ga1812, the equation (25) in §12] at the third equality. Substituting $x=1728/j$ into the equation (12), we obtain the claim. $\square$

From the definitions of the polynomials, we find that $\deg P_n(X)=\deg R_n(X)=\deg S_n(X)=n$ $(n\geq 0)$ and $\deg Q_n(X)=n-1$ $(n\geq 1)$, and the polynomials satisfy the three-term recursions

$$\begin{aligned}
P_{n+1}(X) &= (X-a_n)P_n(X) - b_nP_{n-1}(X) \ (n \geq 1),\\
Q_{n+1}(X) &= (X-a_n)Q_n(X) - b_nQ_{n-1}(X) \ (n \geq 1),\\
R_{n+1}(X) &= (X-c_n)R_n(X) - d_nR_{n-1}(X) \ (n \geq 1),\\
S_{n+1}(X) &= (X-c_n)S_n(X) - d_nS_{n-1}(X) \ (n \geq 1),
\end{aligned}$$

where a_n, b_n, c_n, d_n $(n \geq 1)$ are defined as

$$a_n := 1728 - \lambda_{2n} - \lambda_{2n+1} = 96\frac{576n^2+432n-83}{(8n-1)(8n+7)},$$

$$b_n := \lambda_{2n-1}\lambda_{2n} = \begin{cases} \dfrac{187273728}{539} & \text{if } n = 1,\\ \\ 2304\dfrac{(24n-13)(24n-5)(24n-1)(24n+7)}{(8n-5)(8n-1)^2(8n+3)} & \text{if } n \geq 2, \end{cases}$$

$$c_n := 1728 - \lambda_{2n+1} - \lambda_{2n+2} = 96\frac{576n^2+1008n+277}{(8n+3)(8n+11)},$$

$$d_n := \lambda_{2n}\lambda_{2n+1} = 2304\frac{(24n-1)(24n+7)(24n+11)(24n+19)}{(8n-1)(8n+3)^2(8n+7)}.$$

Remark 5.7. We may adopt these recursions as definitions of the polynomials, because polynomials are uniquely determined by the above recursions with initial conditions $P_0(X) = R_0(X) = S_0(X) = 1, Q_0(X) = 0, P_1(X) = X - 6624/7$, $Q_1(X) = 1, R_1(X) = X - 5520/11$ and $S_1(X) = X - 98832/77$.

Let us describe the main result of the paper. For each $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 4 \pmod 6$, we define the Jacobi form $\phi_k \in J_k$ as follows:

$$\phi_k := \begin{cases} \Delta^n P_n(j)E_{4,1} + \dfrac{7}{11}\Delta^{n-1}E_4E_6Q_n(j)E_{6,1} & \text{if } k = 4+12n \ (n \in \mathbb{Z}_{\geq 0}),\\ \\ \Delta^n E_6R_n(j)E_{4,1} + \dfrac{7}{11}\Delta^n E_4S_n(j)E_{6,1} & \text{if } k = 10+12n \ (n \in \mathbb{Z}_{\geq 0}). \end{cases}$$

Theorem 5.8. *The Jacobi form ϕ_k is a solution of the equation $(\flat_k)$.*

Theorem 5.8 is proved in Subsection 5.1.4. We end this subsection by giving the first few examples of ϕ_k:

$$
\begin{aligned}
\phi_4 &= E_{4,1}, \\
\phi_{10} &= E_6 E_{4,1} + \frac{7}{11} E_4 E_{6,1}, \\
\phi_{16} &= \Delta\Big(j - \frac{6624}{7}\Big) E_{4,1} + \frac{7}{11} E_4 E_6 E_{6,1} = \frac{19E_4^3 + 23E_6^2}{42} E_{4,1} + \frac{7}{11} E_4 E_6 E_{6,1}, \\
\phi_{22} &= \Delta E_6 \Big(j - \frac{5520}{11}\Big) E_{4,1} + \frac{7}{11} \Delta E_4 \Big(j - \frac{98832}{77}\Big) E_{6,1} \\
&= \frac{281E_4^3 + 115E_6^2}{396} E_6 E_{4,1} + \frac{713E_4^3 + 2059E_6^2}{4356} E_4 E_{6,1}, \\
\phi_{28} &= \Delta^2 \Big(j^2 - 1792j + \frac{4981248}{11}\Big) E_{4,1} + \frac{7}{11} \Delta E_4 E_6 \Big(j - \frac{5920}{7}\Big) E_{6,1} \\
&= \frac{817E_4^6 + 5230E_4^3E_6^2 + 1081E_6^4}{7128} E_{4,1} + \frac{193E_4^3 + 185E_6^2}{594} E_4 E_6 E_{6,1}.
\end{aligned}
$$

5.1.3 Key proposition

In order to prove Theorem 5.8, we need a proposition, namely, Proposition 5.11 given below. In this subsection, we prove the proposition. First, we define Rankin-Cohen brackets for modular forms and Jacobi forms [I12, p.60]. For $f \in M_k$ and $\phi \in J_l$, the Rankin-Cohen bracket $[f, \phi] \in J_{k+l+2}$ is defined as

$$
\begin{aligned}
[f, \phi] &:= kfL(\phi) - 2(2l-1)f'\phi \\
&= kf\partial_l(\phi) - 2(2l-1)\vartheta_k(f)\phi.
\end{aligned}
$$

Lemma 5.9. *Let ϕ_k be a solution of $(\flat_k)$. Then we have*

(i) $\partial_{k+6}([E_4, \phi_k]) = \dfrac{2k-9}{9}[E_6, \phi_k]$,

(ii) $\partial_{k+8}([E_6, \phi_k]) = \dfrac{2k-13}{4}E_4[E_4, \phi_k]$,

(iii) *the function* $[E_4, \phi_k]/\Delta$ *is a solution of* $(\flat_{k-6})$.

(*Here, in the brackets, ϕ_k is regarded as being weight k and index* 1.)

Proof. Note that ϕ_k satisfies the relation

$$\partial^2(\phi_k) = \kappa_k E_4 \phi_k, \tag{13}$$

where $\kappa_k = (2k-1)(2k+3)/36$. Using $\vartheta(E_4) = -E_6/3$ and $\vartheta(E_6) = -E_4^2/2$, we obtain the assertions (i) and (ii). Since $\partial(\phi/\Delta) = \partial(\phi)/\Delta$, the assertion (iii) is easily obtained from (i) and (ii). $\square$

Lemma 5.10. *Assume that ψ_k and ψ_{k-6} are solutions of $(\flat_k)$ and $(\flat_{k-6})$, respectively. Define $\psi_{k+6} := E_6\psi_k + \Delta\psi_{k-6}$. Then the function ψ_{k+6} is a solution of $(\flat_{k+6})$ if and only if the relation*

$$[E_4, \psi_k] = -\frac{4(2k+1)}{3}\Delta\psi_{k-6}$$

holds.

Proof. Using

$$\begin{aligned}
\partial(E_6\psi_k) &= 4\vartheta(E_6)\psi_k + E_6\partial(\psi_k)\\
&= \frac{1}{6}[E_6, \psi_k] - \frac{2k+11}{6}E_4^2\psi_k,\\
\partial(E_4^2\psi_k) &= 4\vartheta(E_4^2)\psi_k + E_4^2\partial(\psi_k)\\
&= \frac{E_4}{4}[E_4, \psi_k] - \frac{2k+15}{6}E_4E_6\psi_k
\end{aligned}$$

and Lemma 5.9 (ii), we have

$$\begin{aligned}
\partial^2(\psi_{k+6}) &= \partial^2(E_6\psi_k) + \partial^2(\Delta\psi_{k-6})\\
&= \partial\Big(\frac{1}{6}[E_6, \psi_k] - \frac{2k+11}{6}E_4^2\psi_k\Big) + \Delta\partial^2(\psi_{k-6})\\
&= \frac{1}{6}\partial([E_6, \psi_k]) - \frac{2k+11}{6}\partial(E_4^2\psi_k) + \Delta\partial^2(\psi_{k-6})\\
&= \frac{2k-13}{24}E_4[E_4, \psi_k] - \frac{2k+11}{6}\Big(\frac{E_4}{4}[E_4, \psi_k] - \frac{2k+15}{6}E_4E_6\psi_k\Big)\\
&\qquad\qquad + \Delta\partial^2(\psi_{k-6})\\
&= -E_4[E_4, \psi_k] + \kappa_{k+6}E_4E_6\psi_k + \kappa_{k-6}\Delta E_4\psi_{k-6}.
\end{aligned}$$

Hence, we obtain

$$\partial^2(\psi_{k+6}) - \kappa_{k+6}E_4\psi_{k+6} = -E_4\Big\{[E_4, \psi_k] + \frac{4(2k+1)}{3}\Delta\psi_{k-6}\Big\}. \qquad (14)$$

The lemma follows from (14) together with (13) (for $k+6$). $\square$

Proposition 5.11. *Let ϕ_k be a solution of $(\flat_k)$. Define the functions ϕ_{k+6i} $(i = -1, 1, 2, 3, \cdots)$ as*

$$\phi_{k-6} := -\frac{2k-11}{576(2k-9)(2k-1)}\frac{[E_4, \phi_k]}{\Delta},$$
$$\phi_{k+6i+6} := E_6\phi_{k+6i} + \mu_i^{(k)}\Delta\phi_{k+6i-6} \quad (i = 0, 1, 2, \cdots),$$

where

$$\mu_i^{(k)} := 432\frac{(2k+12i-9)(2k+12i-1)}{(2k+12i-11)(2k+12i+1)} \ (i = 0, 1, 2, \cdots).$$

Then the function ϕ_{k+6i} is a solution of $(\flat_{k+6i})$ for every $i \geq -1$.

Proof. From the assumption that ϕ_k is a solution of $(\flat_k)$ and Lemma 5.9 (iii), it is sufficient to show that ϕ_{k+6i} is a solution of $(\flat_{k+6i})$ for every $i \geq 1$. We prove this claim by induction on $i \geq 1$. When $i = 1$, by a direct calculation, we have

$$[E_4, \phi_k] = -\frac{4(2k+1)}{3}\Delta(\mu_0^{(k)}\phi_{k-6}).$$

Hence, from the relation and Lemma 5.10, the function $\phi_{k+6} = E_6\phi_k + \mu_0^{(k)}\Delta\phi_{k-6}$ is a solution of $(\flat_{k+6})$. Next, assume that $i \geq 1$ and ϕ_{k+6h} is a solution of $(\flat_{k+6h})$ for every $1 \leq h \leq i$. We show that $\phi_{k+6i+6} = E_6\phi_{k+6i} + \mu_i^{(k)}\Delta\phi_{k+6i-6}$ is a solution of $(\flat_{k+6(i+1)})$. From the assumption that $\phi_{k+6i} = E_6\phi_{k+6i-6} + \mu_{i-1}^{(k)}\Delta\phi_{k+6i-12}$ is a solution of $(\flat_{k+6i})$ and Lemma 5.10, we obtain the relation

$$\mu_{i-1}^{(k)}\Delta\phi_{k+6i-12} = -\frac{3}{4\{2(k+6i)-11\}}[E_4, \phi_{k+6i-6}].$$

Then, using this relation, we have

$$\begin{aligned}
[E_4, \phi_{k+6i}] &= [E_4, E_6\phi_{k+6i-6}] + [E_4, \mu_{i-1}^{(k)}\Delta\phi_{k+6i-12}] \\
&= [E_4, E_6\phi_{k+6i-6}] - \frac{3}{4\big\{2(k+6i)-11\big\}}[E_4, [E_4, \phi_{k+6i-6}]] \\
&= -8E_4^3\phi_{k+6i-6} + 4E_4E_6\partial(\phi_{k+6i-6}) + \frac{2\big\{2(k+6i)-1\big\}}{3}E_6^2\phi_{k+6i-6} \\
&\quad - 4E_4E_6\partial(\phi_{k+6i-6}) - \frac{\big\{2(k+6i)-21\big\}\big\{2(k+6i)-13\big\}}{3\big\{2(k+6i)-11\big\}}E_4^3\phi_{k+6i-6} \\
&\qquad\qquad - \frac{\big\{2(k+6i)-13\big\}\big\{2(k+6i)-1\big\}}{3\big\{2(k+6i)-11\big\}}E_6^2\phi_{k+6i-6} \\
&= -\frac{\big\{2(k+6i)-9\big\}\big\{2(k+6i)-1\big\}}{3\big\{2(k+6i)-11\big\}}(E_4^3 - E_6^2)\phi_{k+6i-6} \\
&= -576\frac{\big\{2(k+6i)-9\big\}\big\{2(k+6i)-1\big\}}{\big\{2(k+6i)-11\big\}}\Delta\phi_{k+6i-6} \\
&= -\frac{4\big\{2(k+6i)+1\big\}}{3}\Delta(\mu_0^{(k+6i)}\phi_{k+6i-6}).
\end{aligned}$$

Therefore, using Lemma 5.10 again, we see that $E_6\phi_{k+6i} + \mu_0^{(k+6i)}\Delta\phi_{k+6i-6}$ is a solution of $(\flat_{k+6(i+1)})$. Because $\mu_0^{(k+6i)} = \mu_i^{(k)}$, we arrive at the claim. □

5.1.4 Proof of Theorem 5.8

Let us prove Theorem 5.8. First, note that the Jacobi forms ϕ_k satisfy the recursion

$$\phi_{4+6(i+1)} = E_6\phi_{4+6i} + \lambda_i\Delta\phi_{4+6(i-1)} \ (i = 1, 2, \cdots). \tag{15}$$

From Proposition 5.11, to prove Theorem 5.8, we only need to show that
(i) $\phi_4 = E_{4,1}$ is a solution of $(\flat_4)$,
(ii) ϕ_{16} is a solution of $(\flat_{16})$ and

$$\phi_{10} = -\frac{2\cdot 16 - 11}{576(2\cdot 16 - 9)(2\cdot 16 - 1)}\frac{[E_4, \phi_{16}]}{\Delta},$$

and
(iii) $\phi_{22+6i} = E_6\phi_{16+6i} + \mu_i^{(16)}\Delta\phi_{10+6i} \quad (i = 0, 1, 2, \cdots)$.

Since $\dim_{\mathbb{C}} J_4 = 1$ and thus $J_4 = \mathbb{C}\cdot E_{4,1}$, the assertion (i) is obvious. Using $\partial(E_{4,1}) = -7E_{6,1}/6$ and $\partial(E_{6,1}) = -11E_4E_{4,1}/6$, we obtain the assertion (ii). Because the identity

$$\mu_i^{(16)} = 48\frac{(12i+23)(12i+31)}{(4i+7)(4i+11)} = \lambda_{i+2} \quad (i \geq 0)$$

and the recursion (15) hold, we obtain the assertion (iii). This completes the proof of Theorem 5.8.

5.2 The case of general index m

In this subsection, we consider the equation $(\flat_{k,m})$ which is a generalization of $(\flat_k)$. First, let us derive the equation $(\flat_{k,m})$. In order to derive $(\flat_{k,m})$, we define the modified heat operator $\partial_{k,m}$ of weight k $(\in \mathbb{Z})$ and index m [R09]:

$$L_m := \frac{1}{(2\pi\sqrt{-1})^2}\Big(8\pi\sqrt{-1}m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\Big),$$
$$\partial_{k,m} := L_m - \frac{(2k-1)m}{6}E_2 : J_{k,m} \longrightarrow J_{k+2,m}.$$

Note that the Leibniz rule

$$\partial_{k+l,m}(f\phi) = 4m\vartheta_k(f)\phi + f\partial_{l,m}(\phi) \quad \text{for } f \in M_k \text{ and } \phi \in J_{l,m} \tag{16}$$

holds (note the factor $4m$ on the right). Using the dimension formulas (3) for $J_{k,m}$, for $k \geq m \geq 1$, we see that the endomorphism

$$\varpi_{k,m} := \frac{1}{E_4}\partial_{k+2,m} \circ \partial_{k,m} : J_{k,m} \longrightarrow J_{k,m}$$

is well-defined only if $k \equiv 1 \pmod 6$ and $m = 1$, $k \equiv 3 \pmod 6$ and $m \in \{1,2,3\}$, $k \equiv 4 \pmod 6$ and $m = 1$, or $k \equiv 5 \pmod 6$ and $m \in \{1,2\}$. But, because the codimension of subspace of Jacobi cusp forms of odd weight and index $m \in \{1,2,3\}$ is 0 (cf. [EZ85, a paragraph before Theorem 2.3]), we can not assert that the value $\kappa_{k,m} := (2k-1)(2k+3)m^2/36$ is an eigenvalue of $\varpi_{k,m}$ by the argument used in Subsection 5.1.1 unless $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 4 \pmod 6$ and $m = 1$. However, extending the domain $J_{k,m}$ of $\varpi_{k,m}$ to the space of holomorphic functions $\phi(\tau, z)$ on $\mathfrak{H} \times \mathbb{C}$, we can consider the equation

$$\varpi_{k,m}(\phi)(\tau, z) = \kappa_{k,m}\phi(\tau, z).$$

Then, using $E_2' = (E_2^2 - E_4)/12$, we find that the above equation is equivalent to the equation $(\flat_{k,m})$.

We mention two propositions on the equation $(\flat_{k,m})$. The first proposition is an extension of Proposition 5.3 to the case of general index m:

Proposition 5.12. *If $\phi(\tau, z)$ is a solution of $(\flat_{k,m})$, then the two functions*

$$(c\tau + d)^{-k}\mathbf{e}\Big(\frac{-cmz^2}{c\tau + d}\Big)\phi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) \text{ and } \mathbf{e}(\lambda^2 m\tau + 2\lambda mz)\phi(\tau, z + \lambda\tau + \mu)$$

are also solutions of $(\flat_{k,m})$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$.

Proposition 5.5 is also extended to the case of general index m, which describes a characterization of the equation $(\flat_{k,m})$. Let us consider the equation (5) appearing in Subsection 5.1.1. For fixed integers k and m, we impose the following condition on the equation (5):

Condition 5.13. (i) *If $\phi(\tau, z)$ is a solution of* (5)*, then the two functions*

$$(c\tau+d)^{-k}\mathbf{e}\Big(\frac{-cmz^2}{c\tau + d}\Big)\phi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) \text{ and } \mathbf{e}(\lambda^2 m\tau+2\lambda mz)\phi(\tau, z+\lambda\tau+\mu)$$

are also solutions of (5) *for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$.*
(ii) *The functions $A_i(\tau)$ $(1 \le i \le 5)$ are bounded when $y = \mathrm{Im}(\tau) \to \infty$.*

Proposition 5.14. *If a function ϕ is a solution of the equation* (5) *satisfying Condition* 5.13*, then the function $\Delta^\beta\phi$ for a suitable constant β is a solution of the equation $(\flat_{12\beta+k,m})$.*

Proposition 5.12 (resp. Proposition 5.14) is proved in a similar way to Proposition 5.3 (resp. Proposition 5.5), so we omit the proof.

5.3 A connection with $(\flat_{k,m})$, $(\sharp_k)$ and the heat equation

In this subsection, we consider a connection with $(\flat_{k,m})$, $(\sharp_k)$ and the heat equation

$$L_m(\psi) = \partial_{\frac{1}{2},m}(\psi) = 0 \tag{17}$$

on the space of holomorphic functions $\psi : \mathfrak{H}\times\mathbb{C} \to \mathbb{C}$. Using Proposition 5.16 given below, solutions of (17) (i.e. the theta series $\theta_{m,\mu}(\tau, z)$) and solutions of $(\sharp_k)$ in Subsection 3.2, we can construct a solution of $(\flat_{k,m})$. Let us start with an easy lemma.

Lemma 5.15. *Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function and $\psi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ be a solution of the heat equation* (17). Then we have

$$\partial_{k+\frac{1}{2},m}(f\psi) = 4m\vartheta_k(f)\psi,$$

and thus

$$\varpi_{k+\frac{1}{2},m}(f\psi) = 16m^2\varphi_k(f)\psi.$$

Proof. The lemma follows from the Leibniz rule (16) immediately. $\square$

Proposition 5.16. *Let $\psi_i : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ $(1 \le i \le N)$ be linearly independent over the ring of holomorphic functions on $\mathfrak{H}$. For holomorphic functions $f_i : \mathfrak{H} \to \mathbb{C}$ $(1 \le i \le N)$, define the function*

$$\phi := \sum_{i=1}^{N} f_i\psi_i.$$

If the functions ψ_i $(1 \le i \le N)$ are solutions of the heat equation (17), then the function ϕ is a solution of the equation $(\flat_{k+1/2,m})$ if and only if the function f_i is a solution of the K-Z equation $(\sharp_k)$ for all i.

Proof. Using the linearity of $\varpi_{k+1/2,m}$, Lemma 5.15, $\kappa_{k+1/2,m} = k(k+2)m^2/9$ and the linear independence of ψ_i, we have

$$\begin{aligned}
\phi \text{ is a solution of } (\flat_{k+\frac{1}{2},m}) &\iff \varpi_{k+\frac{1}{2},m}(\phi) = \kappa_{k+\frac{1}{2},m}\phi \\
&\iff \sum_{i=1}^{N} \varpi_{k+\frac{1}{2},m}(f_i\psi_i) = \kappa_{k+\frac{1}{2},m}\sum_{i=1}^{N} f_i\psi_i \\
&\iff \sum_{i=1}^{N}\Big\{16m^2\varphi_k(f_i) - \frac{k(k+2)m^2}{9}f_i\Big\}\psi_i = 0 \\
&\iff \varphi_k(f_i) - \frac{k(k+2)}{144}f_i = 0 \quad (1 \le i \le N) \\
&\iff f_i \text{ is a solution of } (\sharp_k) \quad (1 \le i \le N).
\end{aligned}$$

This completes the proof of Proposition 5.16. $\square$

We end the paper by proving two theorems on solutions of $(\flat_{k,m})$. First, we prove the existence of Fourier series solutions of the equation $(\flat_{k,m})$ for suitable $k \in \mathbb{Q}$ and all $m \in \mathbb{N}$:

Theorem 5.17. (i) *For all $k \in \mathbb{Z}_{\geq 0}$ with $k \equiv 1 \pmod 3$ and $m \in \mathbb{N}$, there exists a solution $\phi(\tau, z)$ of $(\flat_{k,m})$ having a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn \geq r^2}} c(n,r) q^n \zeta^r.$$

(ii) *For all $k \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$, there exists a solution $\phi(\tau, z)$ of $(\flat_{k+1/2,m})$ having a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn \geq r^2}} c(n,r) q^n \zeta^r.$$

Proof. (i) By Theorem 3.9, we can take the solution $f_{k-1/2}$ of $(\sharp_{k-1/2})$ such that $f_{k-1/2} = 1 + O(q)$. Using Proposition 5.16 for $N = 1$, we see that the function $f_{k-1/2}\theta_{m,0}$ is a solution of $(\flat_{k,m})$, where

$$\theta_{m,0}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{mn^2} \zeta^{2mn}$$

is the theta series introduced in Section 4.
(ii) Using Theorem 3.8 and the same argument as in the proof of (i), we arrive at the claim. This completes the proof of Theorem 5.17. $\square$

Next, let us consider the following formal expression in terms of the theta series:

$$\phi(\tau, z) := \sum_{\mu \pmod{2m}} h_\mu(\tau) \theta_{m,\mu}(\tau, z),$$

where $h_\mu : \mathfrak{H} \to \mathbb{C}$ are holomorphic functions. Because $\theta_{m,\mu}$ are solutions of (17) and linearly independent, we obtain the following theorem from Proposition 5.16:

Theorem 5.18. *The function ϕ is a solution of the equation $(\flat_{k+1/2,m})$ if and only if the function h_μ is a solution of the K-Z equation $(\sharp_k)$ for all $\mu \pmod{2m}$.*

References

[C75] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters,* Math. Ann. **217** (1975), 271–285.

[DMZ] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms,* to appear in Cambridge Monographs in Mathematical Physics.

[EZ85] M. Eichler and D. Zagier, *The Theory of Jacobi Forms,* Birkhäuser (1985), Boston-Basel-Stuttgart.

[Ga1812] C. F. Gauss, *Disquisitiones generales circa serien infinitam* $1+\frac{\alpha\beta}{1\cdot\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+$ *etc.,* Pars prior (1812), Werke III, 125–162.

[Gu15] P. Guerzhoy, *A mixed mock modular solution of the Kaneko-Zagier equation,* Ramanujan J. **36** (2015), 149–164.

[I12] T. Ibukiyama, *Vector valued Siegel modular forms of symmetric tensor weight of small degrees,* Comment. Math. Univ. St. Pauli. **61** (2012), 51–75.

[KK03] M. Kaneko and M. Koike, *On modular forms arising from a differential equation of hypergeometric type,* Ramanujan J. **7** (2003), 145–164.

[KNS13] M. Kaneko, K. Nagatomo and Y. Sakai, *Modular forms and second order ordinary differential equations: applications to vertex operator algebras,* Lett. Math. Phys. **103** (2013), no. 4, 439–453.

[KZ98] M. Kaneko and D. Zagier, *Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials,* AMS/IP Stud. Adv. Math. **7** (1998), 97–126.

[Ki14] T. Kiyuna, *Kaneko-Zagier type equation for Jacobi forms of index 1,* Ramanujan J. (2014), doi:10.1007/s11139-014-9641-0.

[Ko80] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249–266.

[R09] O. Richter, *The action of the heat operator on Jacobi forms,* Proc. Amer. Math. Soc. **137** (2009), 869–875.

[Z77] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields,* in Modular Functions of One Variable, VI, Lecture Notes in Math. **627**, Springer-Verlag, Berlin-Heidelberg-New York (1977), pp. 105–169.