A Study on a Game of Pursuit and Evasion on a Cycle Graph

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A Study on a game of Pursuit and Evasion on a Cycle Graph

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Abstract

We analyze the Hunter vs Rabbit game on graph, which is a kind of model of communication in an adhoc mobile network. Let G be a cycle graph with N nodes. The hunter can move from a vertex to another vertex on the graph along an edge. The rabbit can move to any vertex on graph at once. We formalized the game using the random walk framework. The strategy of rabbit is formalized using a one dimensional random walk over $\mathbb{Z}$. We classify strategies using the order $O(k^{-\beta-1})$ of their Fourier transformation. We investigate lower bounds and upper bounds of a probability the rabbit is caught. We found a constant lower bound if $\beta \in (0, 1)$. This constant does not depend on the size N of the given graph. We show the order is equivalent to $O(1/\log N)$ if $\beta = 1$ and a lower bound is $1/N^{(\beta-1)/\beta}$ if $\beta \in (1, 2]$. Those results assist to choose the parameter β of a rabbit strategy according to the size N of the given graph. We introduce a formalization of strategies using a random walk, theoretical estimation of bounds of a probability the rabbit caught, and we also show computing simulation results.

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Contents

Chapter 1

Introduction

We consider a game played by two players the hunter and the rabbit. This game is described using a graph $G(V, E)$ where V is a set of vertices and E is a set of edges. Both players may use a randomized strategy. If the hunter moves to a vertex that the rabbit is staying, we say that the hunter catches the rabbit. This game finishes when the hunter catches the rabbit. The hunter can move from a vertex to another vertex along an edge. The rabbit can move to any vertex at once. The hunter's purpose is that the hunter catch the rabbit as few steps as possible. On the other hand, the rabbit considers a strategy that maximizes the time until the hunter catch the rabbit.

The Hunter vs Rabbit game model is used for analyzing transmission procedures in mobile adhoc network[5, 6]. This model helps to send an electronic message efficiently between people using mobile phones. The expected value of time until the hunter catches the rabbit is equal to the expected time until the recipient get mails. One of the our goals is to improve these procedure.

We introduce some game resembling Hunter vs Rabbit game. First one is Princess-Monster game. This game is that the Monster tries to catch the Princess in area D. Difference of Hunter vs Rabbit game is that the

Monster catches the Princess if distance of two players smaller than the some value. The Monster moves constant speed, but the Princess can move at any speed. This game played on a cycle graph is introduced by Isaacs[11]. After that, this game has been investigated by Alpern, Zelekin, and so on. Gal analyzed this Princess-Monster game on convex multidimensional domain. Next one is Deterministic pursuit-evasion game. In this game, we consider a runaway hide dark spot, for example a tunnel. Parsons innovated the search number of a graph[16, 17]. The search number of a graph is least number of persons that is able to catch a runaway moving at any speed. LaPaugh proved that we can make a runaway can not pass in the edge after we know that a runaway does not stay in this edge, if we assure many searcher[13]. Meggido showed that computation time of the search number of a graph is NP-hard[15]. If an edge can be cleared without moving along it, but it suffices to 'look into' an edge from a vertex, then the minimum number of guards needed to catch the fugitive is called the node search number of graph [12]. The pursuit evasion problem in the plane were introduced by Suzuki and Yamashita [19]. They gave necessary and sufficient conditions for a simple polygon to be searchable by a single pursuer. Later Guibas et al. [9] presented a complete algorithm and showed that the problem of determining the minimal number of pursuers needed to clear a polygonal region with holes is NP-hard. Park et al. gave 3 necessary and sufficient conditions for a polygon to be searchable and showed that there is $O(n^2)$ time algorithm for constructing a search path for an n-sided polygon. Efrat et al. [7] gave a polynomial time algorithm for the problem of clearing a simple polygon with a chain of k pursuers when the first and last pursuer have to move on the boundary of the polygon.

A first study of the Hunter vs. Rabbit game can be found in .The presented hunter strategy is based on random walk on a graph and it is shown that the hunter catches an unrestricted rabbit within $O(nm^2)$ rounds, where n and m denote the number of nodes and edges, respectively.

Adler et al. showed that if the hunter chooses a good strategy, the upper bound of the expected time of the hunter catches the rabbit is $O(n \log(diam(G)))$ where $diam(G)$ is a diameter of a graph G, and if the rabbit chooses a good strategy, the lower bound of the expected time of the hunter catches the rabbit is $\Omega(n \log(diam(G)))$ [1].

Babichenko et al. showed Adler's strategies yield a Kakeya set consisting of $4n$ triangles with minimal area [4].

In this paper,we propose three assumptions for a strategy of the rabbit. We have the general lower bound formula of a probability that the hunter catches the rabbit. The strategy of the rabbit is formalized using a one dimensional random walk over $\mathbb{Z}$. We classify strategies using the order $O(k^{-\beta-1})$ of their Fourier transform. If $\beta = 1$, the lower bound of a probability that the hunter catches the rabbit is $((c_*\pi)^{-1} \log N + c_2)^{-1}$ where c_2 and c_* are constants defined by the given strategy. If $\beta \in (1, 2]$, the lower bound of a probability that the hunter catches the rabbit is $c_4 N^{-(\beta-1)/\beta}$ where $c_4 > 0$ is are constant defined by the given strategy.

We show experimental results for three examples of the rabbit strategy.

1. $$P\{X_t = k\} = \begin{cases} \dfrac{1}{2a(|k|+1)(|k|+2)} & (k \in \mathbb{Z} \setminus \{0\}) \\ 1 - \dfrac{1}{2a} & (k = 0) \end{cases}$$

2. $$P\{X_t = k\} = \begin{cases} \dfrac{1}{2a|k|^{\beta+1}} & (k \in \mathbb{Z} \setminus \{0\}) \\ 1 - \dfrac{1}{a} \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{k^{\beta+1}} & (k = 0) \end{cases}$$

3. $$P\{X_t = k\} = \begin{cases} \dfrac{1}{3} & (k \in \{-1, 0, 1\}) \\ 0 & (k \notin \{-1, 0, 1\}). \end{cases}$$

We can confirm our bounds formula, and the asymptotic behavior of those bounds by the results of simulations.

We develop a simulation program to simulate the Hunter vs Rabbit game using C++. In this program, we compute random walks using the digamma function. Andrews et al. showed many property of the digamma function [3]. We use those propertied and show the transition probability for given strategies. To simulate the Hunter vs Rabbit game, we have to use the digamma function. Usually, a transition probability that the rabbit move from a vertex i to a vertex $i + d$ is consists of the infinite sum in a cycle graph. But we show that this probability is denoted by a finite sum by digamma functions. We introduce those expansion formulas in Chapter 3. We also summarize the property of the digamma function and the relation between transition matrices of random walks and the digamma function. And we show experimental results with this application. We can confirm our bounds formulas and asymptotic behavior of those bound by the results of simulations.

Chapter 2

The Random Walk on a Cycle Graph

In this chapter, we formalize the Hunter vs Rabbit game using the random walk framework. First, we show a lower bound and an upper bound of a probability that the hunter catches the rabbit for any strategy of the hunter and the rabbit. Then, strategies of the rabbit is formalize a one-dimensional random walk. Next, we introduce some results concerning one-dimensional random walk using Fourier transform. Finally, we calculate a lower bound of this probability for given strategies.

2.1 Statements of Results

We consider the Hunter vs Rabbit game on a cycle graph. To explain the Hunter vs Rabbit game, we introduce some notation. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in the integer lattice $\mathbb{Z}$. A one-dimensional random walk $\{S_n\}_{n=1}^{\infty}$ is defined by

$$S_n = \sum_{j=1}^{n} X_j.$$

Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables defined on a probability space $(\Omega_{\mathcal{H}}, \mathcal{F}_{\mathcal{H}}, P_{\mathcal{H}})$ taking values in the integer lattice $\mathbb{Z}$ with

$$P_{\mathcal{H}}\{|Y_1| \leq 1\} = 1.$$

Let $N \in \mathbb{N}$ be fixed. We denote by $X_0^{(N)}$ a random variable defined on a probability space $(\Omega_N, \mathcal{F}_N, \mu_N)$ taking values in $V_N := \{0, 1, 2, \ldots, N-1\}$ with

$$\mu_N\{X_0^{(N)} = l\} = \frac{1}{N} \quad (l \in V_N).$$

For $b \in \mathbb{Z}$, we denote by $(b \mod N)$ the remainder when b is divided by N. A rabbit's strategy $\{\mathcal{R}_n^{(N)}\}_{n=0}^{\infty}$ is defined by

$$\mathcal{R}_0^{(N)} = X_0^{(N)} \text{ and } \mathcal{R}_n^{(N)} = (X_0^{(N)} + S_n \mod N).$$

$\mathcal{R}_n^{(N)}$ indicates the position of the rabbit at time n on V_N. Hunter's strategy $\{\mathcal{H}_n^{(N)}\}_{n=0}^{\infty}$ is defined by

$$\mathcal{H}_0^{(N)} = 0 \text{ and } \mathcal{H}_n^{(N)} = \left(\sum_{j=1}^{n} Y_j \mod N\right).$$

$\mathcal{H}_n^{(N)}$ indicates the position of the hunter at time n on V_N. Put

$$\mathbb{P}_{\mathcal{R}}^{(N)} = \mu_N \times P \text{ and } \tilde{\mathbb{P}}^{(N)} = P_{\mathcal{H}} \times \mathbb{P}_{\mathcal{R}}^{(N)}.$$

The hunter catches the rabbit when in some round the hunter and the rabbit are both located on the same place. We will discuss the probability that the hunter catches the rabbit by time N on V_N, that is,

$$\tilde{\mathbb{P}}^{(N)}\left(\bigcup_{n=1}^{N}\{\mathcal{H}_n^{(N)} = \mathcal{R}_n^{(N)}\}\right).$$

We investigate the asymptotic estimate of this probability as $N \to \infty$.

Definition 2.1.1. We define conditions (A1), (A2) and (A3) as follows.

$(A1)$ random walk $\{S_n\}_{n=1}^{\infty}$ is strongly aperiodic, i.e. for each $y \in \mathbb{Z}$, the smallest subgroup containing the set

$$\{y + k \in \mathbb{Z} \mid P\{X_1 = k\} > 0\}$$

is $\mathbb{Z}$,

$(A2)$ $P\{X_1 = k\} = P\{X_1 = -k\} \quad (k \in \mathbb{Z})$,

$(A3)$ There exist $\beta \in (0, 2]$, $c_* > 0$ and $\varepsilon > 0$ such that

$$\phi(\theta) := \sum_{k \in \mathbb{Z}} e^{i\theta k} P\{X_1 = k\} = 1 - c_* |\theta|^{\beta} + O(|\theta|^{\beta+\varepsilon}).$$

We denote the β in $(A3)$ as $\beta_{\mathcal{R}}$.

Theorem 2.1.1. Assume that X_1 satisfies $(A1) - (A3)$.

(I) If $\beta_{\mathcal{R}} \in (0, 1)$, then there exists a constant $c_1 > 0$ such that for $N \in \mathbb{N} \setminus \{1\}$ and $y_1, y_2, \ldots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \leq 1$ $(n = 1, 2, \ldots, N-1)$,

$$c_1 \leq \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = (y_n \mod N) \right\} \right). \tag{2.1}$$

(II) If $\beta_{\mathcal{R}} = 1$, then there exist constants $c_2 > 0$ and $c_3 > 0$ such that for $N \in \mathbb{N} \setminus \{1\}$ and $y_1, y_2, \ldots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \leq 1$ $(n = 1, 2, \ldots, N-1)$,

$$\begin{aligned} \frac{1}{\frac{1}{c_* \pi} \log N + c_2} &\leq \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = (y_n \mod N) \right\} \right) \\ &\leq \frac{c_3}{\log N}. \end{aligned} \tag{2.2}$$

(III) If $\beta_{\mathcal{R}} \in (1, 2]$, then there exists a constant $c_4 > 0$ such that for $N \in \mathbb{N} \setminus \{1\}$ and $y_1, y_2, \ldots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \leq 1$ $(n = 1, 2, \ldots, N-1)$,

$$\begin{aligned} \frac{c_4}{N^{(\beta-1)/\beta}} &\leq \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = (y_n \mod N) \right\} \right) \\ &\leq 1. \end{aligned} \tag{2.3}$$

The following bounds are obtained as a corollary of Theorem 1.

Corollary 2.1.1. Assume $(A1) - (A3)$.

If $\beta_{\mathcal{R}} \in (0,1)$, then there exists a constant $c_1 > 0$ such that for $N \in \mathbb{N} \setminus \{1\}$,

$$c_1 \leq \tilde{\mathbb{P}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{H}_n^{(N)} = \mathcal{R}_n^{(N)}\} \right).$$

If $\beta_{\mathcal{R}} = 1$, then there exist constants $c_2 > 0$ and $c_3 > 0$ such that for $N \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned} \frac{1}{\frac{1}{c_* \pi} \log N + c_2} &\leq \tilde{\mathbb{P}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{H}_n^{(N)} = \mathcal{R}_n^{(N)} \right\} \right) \\ &\leq \frac{c_3}{\log N}. \end{aligned} \tag{2.4}$$

If $\beta_{\mathcal{R}} \in (1,2]$, then there exists a constant $c_4 > 0$ such that for $N \in \mathbb{N} \setminus \{1\}$,

$$\frac{c_4}{N^{(\beta-1)/\beta}} \leq \tilde{\mathbb{P}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{H}_n^{(N)} = \mathcal{R}_n^{(N)}\} \right).$$

Remark 1. Adler, Räcke, Sivadasan, Sohler and Vöcking considered $\tilde{\mathbb{P}}^{(N)}(\cup_{n=1}^{N} \{\mathcal{H}_n^{(N)} = \mathcal{R}_n^{(N)}\})$ in the case of

$$P\{X_1 = k\} = \begin{cases} \dfrac{1}{2(|k|+1)(|k|+2)} & (k \in \mathbb{Z} \setminus \{0\}) \\ \dfrac{1}{2} & (k = 0). \end{cases}$$

In this case, X_1 satisfies $(A1)$, $(A2)$ and

$$\phi(\theta) = 1 - \frac{\pi}{2}|\theta| + O(|\theta|^{3/2})$$

($(A3)$ with $\beta = 1$), and we have (2.4) in Corollary 1 which coincides with the result of Lemma 3 in [1].

Remark 2. For $\beta \in (0,2)$, put

$$P\{X_1 = k\} = \begin{cases} \dfrac{1}{2a|k|^{\beta+1}} & (k \in \mathbb{Z} \setminus \{0\}) \\ 1 - \dfrac{1}{a} \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\beta+1}} & (k = 0) \end{cases}$$

with a constant a satisfying $a > \sum_{k=1}^{\infty}(1/k^{\beta+1})$. Then $(A3)$ becomes

$$\phi(\theta) = 1 - \frac{\pi}{2a}\frac{|\theta|^{\beta}}{\Gamma(\beta+1)\sin(\beta\pi/2)} + O(|\theta|^{\beta+(2-\beta)/2}), \tag{2.5}$$

where Γ is the gamma function. X_1 satisfies $(A1)$, $(A2)$ and (2.5).

To obtain (2.5), we use the formula

$$\int_0^{+\infty} \frac{\sin bx}{x^{\alpha}}\, dx = \frac{\pi b^{\alpha-1}}{2\Gamma(\alpha)\sin(\alpha\pi/2)} \tag{2.6}$$

for $\alpha \in (0,2)$ and $b > 0$. From the definition of X_1,

$$1 - \phi(\theta) = \frac{1}{a}\sum_{k=1}^{\infty}(1-\cos|\theta|k)\frac{1}{k^{\beta+1}}.$$

A simple calculation shows that the absolute value of the difference between the right-hand side of the above and

$$\frac{1}{a}\int_0^{+\infty}\frac{1-\cos|\theta|x}{x^{\beta+1}}\, dx$$

is bounded by a constant multiple of $|\theta|^{\beta+(2-\beta)/2}$. It remains to show that

$$\frac{1}{a}\int_0^{+\infty}\frac{1-\cos|\theta|x}{x^{\beta+1}}\, dx = \frac{\pi}{2a}\frac{|\theta|^{\beta}}{\Gamma(\beta+1)\sin(\beta\pi/2)}. \tag{2.7}$$

We perform integration by part for the left-hand side of (2.7) and use (2.6). Then we have (2.7) and (2.5).

If X_1 takes three values $-1, 0, 1$ with equal probability, then X_1 satisfies $(A1)$, $(A2)$ and

$$\phi(\theta) = 1 - \frac{1}{3}|\theta|^2 + O(|\theta|^4)$$

($(A3)$ with $\beta = 2$).

Remark 3. (2.3) seems to be sharp, because the powers of upper and lower bound appearing in (2.3) cannot be improved. Indeed, we have the following estimates.

Assume $(A1) - (A3)$. If $\beta_{\mathcal{R}} \in (1,2]$, then there exist constants $c_5, c_6 > 0$ such that for $N \in \mathbb{N}$,

$$\frac{c_5}{N^{(\beta-1)/\beta}} \leq \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\{\mathcal{R}_n^{(N)} = 0\}\right) \leq \frac{c_6}{N^{(\beta-1)/\beta}}. \tag{2.8}$$

Proof of (2.8). The first inequality in (2.8) comes from (2.3) in Theorem 2.1.1. To prove the last inequality in (2.8), we will use Corollary 2.2.1 and 2.3.2 instead of Proposition 2.2.1 and Corollary 2.3.1. The same argument as showing the last inequality in (2.3) gives the last inequality in (2.8).

If X_1 takes three values $-1, 0, 1$ with equal probability, then there exists a constant $c_7 > 0$ such that for $N \in \mathbb{N}$,

$$c_7 \leq \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{R}_n^{(N)} = n\} \right). \tag{2.9}$$

Proof of (2.9). We consider the case when X_1 takes three values $-1, 0, 1$ with equal probability. In this case, X_1 satisfies $(A1) - (A3)$ and

$$\phi(\theta) = 1 - \frac{1}{3}|\theta|^2 + O(|\theta|^4).$$

From (2.32), there exist $\tilde{C}_1 > 0$ and $\tilde{N}_1 \in \mathbb{N}$ such that for $i \geq \tilde{N}_1$ and $l \in \mathbb{Z}$,

$$P\{S_i = l\} \leq \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{1}{i^{1/2}} \exp\left(-\frac{3l^2}{4i}\right) + \tilde{C}_1 i^{-1}. \tag{2.10}$$

By noticing $P\{|X_1| \leq 1\} = 1$, we obtain that for $N \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned} &1 + \sum_{i=1}^{N-1} P\{S_i \in [i]_N\} \\ &= 1 + \sum_{i=1}^{N-1} P\{S_i = i\} + \sum_{N/2 \leq i \leq N-1} P\{S_i = i - N\} \end{aligned}$$

and

$$\sum_{i=1}^{N-1} P\{S_i = i\} = \sum_{i=1}^{N-1} \left(\frac{1}{3}\right)^i \leq \frac{1}{2}.$$

With the help of $e^{-x} \le 1/x$ $(x > 0)$, (2.10) implies that for $N \ge 2\tilde{N}_1$,

$$\begin{aligned}
&\sum_{N/2 \le i \le N-1} P\{S_i = i - N\} \\
&\le \sum_{N/2 \le i \le N-1} \left\{ \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{1}{i^{1/2}} \exp\left(-\frac{3(i-N)^2}{4i} \right) + \tilde{C}_1 i^{-1} \right\} \\
&\le \sqrt{\frac{3}{2\pi}} \frac{1}{N^{1/2}} \sum_{1 \le i \le N/2} \exp\left(-\frac{3i^2}{4N} \right) + \tilde{C}_1 \sum_{1 \le i \le N/2} \frac{2}{N} \\
&\le \sqrt{\frac{3}{2\pi}} \frac{1}{N^{1/2}} \left(\sum_{1 \le i \le N^{1/2}} 1 + \sum_{N^{1/2} < i} \frac{4N}{3i^2} \right) + 2\tilde{C}_1 \\
&\le \sqrt{\frac{3}{2\pi}} + \frac{2\sqrt{2}}{\sqrt{3\pi}} N^{1/2} \left(\frac{1}{N} + \int_{N^{1/2}}^{+\infty} \frac{1}{x^2}\, dx \right) + 2\tilde{C}_1 \\
&\le c_{13},
\end{aligned}$$

where $c_{13} = \sqrt{3/(2\pi)} + 4\sqrt{2}/\sqrt{3\pi} + 2\tilde{C}_1$. Thus for $N \in \mathbb{N} \setminus \{1\}$,

$$1 + \sum_{i=1}^{N-1} P\{S_i \in [i]_N\} \le \max\{2\tilde{N}_1, (3/2) + c_{13}\}.$$

Combining the above inequality with Corollary 2.2.2, we have (2.9).

Remark 4. Assume $(A1)$ and $(A2)$. If there exist $c_* > 0$ and $\varepsilon > 0$ such that

$$\phi(\theta) = 1 - c_*|\theta| + O(|\theta|^{1+\varepsilon})$$

($(A3)$ with $\beta = 1$). Then

$$\lim_{N\to\infty} \left(\frac{1}{c_*\pi} \log N \right) \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{R}_n^{(N)} = 0\} \right) = 1. \tag{2.11}$$

Proof of (2.11). Let $\epsilon > 0$ be fixed. From Corollary 4, there exist $C_2 > 0$ and $N_2 \in \mathbb{N}$ such that for $i \ge N_2$,

$$P\{S_i = 0\} \ge \frac{1}{c_*\pi} \frac{1}{i} - C_2 i^{-1-\delta}. \tag{2.12}$$

(2.12) implies that for $N \ge (4/\epsilon)(N_2 + 1)$,

$$1 + \sum_{1 \le i \le (\epsilon/4)N} P\{S_i \in [0]_N\} \ge \sum_{N_2 \le i \le (\epsilon/4)N} P\{S_i = 0\}$$

$$\geq \sum_{N_2 \leq i \leq (\epsilon/4)N} \left(\frac{1}{c_* \pi} \frac{1}{i} - C_2 i^{-1-\delta} \right)$$

$$\geq \frac{1}{c_* \pi} \int_{N_2}^{(\epsilon/4)N} \frac{1}{x} \, dx - C_2 \left(\frac{1}{N_2^{1+\delta}} + \int_{N_2}^{+\infty} x^{-1-\delta} \, dx \right)$$

$$= \frac{1}{c_* \pi} \log N + \frac{1}{c_* \pi} \log \epsilon - c_{14}, \tag{2.13}$$

where $c_{14} = (1/(c_* \pi)) \log 4 + (1/(c_* \pi)) \log N_2 + C_2 \{1/N_2^{1+\delta} + 1/(\delta N_2^{\delta})\}$.

We can choose $N_4 \in \mathbb{N}$ which satisfies

$$\min \left\{ \frac{1}{2}, \frac{\epsilon}{8} \right\} \frac{1}{c_* \pi} \log N_4 \geq \left| - \frac{1}{c_* \pi} \log \epsilon + c_{14} \right| \tag{2.14}$$

and

$$\frac{\epsilon}{4} \frac{1}{c_* \pi} \log N_4 \geq c_2, \tag{2.15}$$

where c_2 is the same constant in (2.2).

Combining Remark 5 with (2.13) and using the left-hand side of (2.2), we obtain that for $N \geq \max\{N_4, (4/\epsilon)(N_2 + 1)\}$,

$$\frac{1}{\frac{1}{c_* \pi} \log N + c_2} \leq \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{ \mathcal{R}_n^{(N)} = 0 \} \right)$$

$$\leq \frac{1 + (\epsilon/4)}{\frac{1}{c_* \pi} \log N + \frac{1}{c_* \pi} \log \epsilon - c_{14}}.$$

Hence for $N \geq \max\{N_4, (4/\epsilon)(N_2 + 1)\}$,

$$\left| \left(\frac{1}{c_* \pi} \log N \right) \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{ \mathcal{R}_n^{(N)} = 0 \} \right) - 1 \right| \leq E_N^{(1)} + E_N^{(2)},$$

where

$$E_N^{(1)} = \left| \frac{\frac{1}{c_* \pi} \log N}{\frac{1}{c_* \pi} \log N + c_2} - 1 \right|$$

and

$$E_N^{(2)} = \left| \frac{(1 + (\epsilon/4)) \frac{1}{c_* \pi} \log N}{\frac{1}{c_* \pi} \log N + \frac{1}{c_* \pi} \log \epsilon - c_{14}} - 1 \right|.$$

The proof is complete if we show that for $N \geq \max\{N_4, (4/\epsilon)(N_2 + 1)\}$,

$$E_N^{(1)} + E_N^{(2)} \leq \epsilon. \tag{2.16}$$

By (2.15) ,

$$E_N^{(1)} \leq \frac{c_2}{\frac{1}{c_*\pi}\log N} \leq \frac{\epsilon}{4}$$

for $N \geq max\{N_4, (4/\epsilon)(N_2+1)\}$. By (2.14),

$$\begin{aligned} E_N^{(2)} &\leq \frac{(\epsilon/4)\frac{1}{c_*\pi}\log N + \left|-\frac{1}{c_*\pi}\log\epsilon + c_{14}\right|}{\frac{1}{c_*\pi}\log N - \left|-\frac{1}{c_*\pi}\log\epsilon + c_{14}\right|} \\ &\leq \frac{\epsilon}{2} + \frac{\left|-\frac{1}{c_*\pi}\log\epsilon + c_{14}\right|}{(1/2)\frac{1}{c_*\pi}\log N} \leq \frac{3\epsilon}{4} \end{aligned}$$

for $N \geq max\{N_4, (4/\epsilon)(N_2+1)\}$. The above two inequalities yield (2.16).

2.2 Upper bounds and Lower bounds

In this section, we give a relation between

$$\mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)} = (y_n \mod N)\right\}\right)$$

and one-dimensional random walk $\{S_n\}_{n=1}^{\infty}$.

Proposition 2.2.1. For $N \in \mathbb{N}\setminus\{1\}$ and $y_1, y_2, \ldots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \leq 1$ $(n = 1, 2, \ldots, N-1)$,

$$\begin{aligned} \frac{1}{\sum_{i=0}^{N-1} p_i^{(N)}} &\leq \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)} = (y_n \mod N)\right\}\right) \\ &\leq \frac{2}{\sum_{i=0}^{N-1} q_i^{(N)}}, \end{aligned} \tag{2.17}$$

where

$$[y]_N = \{y + kN \mid k \in \mathbb{Z}\},$$

$$p_i^{(N)} = \begin{cases} 1 & (i = 0) \\ \max_{|y|\leq i,\ y\in\mathbb{Z}} P\{S_i \in [y]_N\} & (i \in \mathbb{N}) \end{cases}$$

and

$$q_i^{(N)} = \begin{cases} 1 & (i = 0) \\ \min_{|y|\leq i,\ y\in\mathbb{Z}} P\{S_i \in [y]_N\} & (i \in \mathbb{N}). \end{cases}$$

Proof. From the definition of $\left\{\mathcal{R}_n^{(N)}\right\}_{n=0}^{\infty}$,

$$\begin{aligned}
&\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=(y_n \mod N)\right\}\\
&=\bigcup_{l=0}^{N-1}\bigcup_{n=1}^{N}\left\{X_0^{(N)}=l,\ l+S_n\in[y_n]_N\right\}\\
&=\bigcup_{l=0}^{N-1}\bigcup_{n=1}^{N}\left\{\begin{array}{ll} X_0^{(N)}=l, & l+S_n\in[y_n]_N,\\ l+S_i\notin[y_i]_N, & 1\le i\le n-1\end{array}\right\}.
\end{aligned}$$

From $\mathbb{P}_{\mathcal{R}}^{(N)}=\mu_N\times P$ and the above relation,

$$\begin{aligned}
&\mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=(y_n \mod N)\right\}\right)\\
&=\sum_{l=0}^{N-1}\sum_{n=1}^{N}\frac{1}{N}P\left\{\begin{array}{ll} l+S_i\notin[y_i]_N, & 1\le i\le n-1,\\ l+S_n\in[y_n]_N & \end{array}\right\}. \qquad (2.18)
\end{aligned}$$

For $l\in\{0,1,\dots,N-1\}$ and $n\in\{2,3,\dots,N\}$, we decompose the event $\{l+S_n\in[y_n]_N\}$ according to the value of the first hitting time for $[y_1]_N,[y_2]_N,\dots,[y_n]_N$ and the hitting place to obtain

$$\begin{aligned}
&P\{l+S_n\in[y_n]_N\}\\
&=\sum_{j=1}^{n}\sum_{m\in\mathbb{Z}}P\left\{\begin{array}{ll} l+S_i\notin[y_i]_N, & 1\le i\le j-1,\\ l+S_j=y_j+mN, & \\ y_j+mN+X_{j+1}+ & \cdots+X_n\in[y_n]_N\end{array}\right\}.
\end{aligned}$$

By the Markov property, the probability in the double summation on the right-hand side above is equal to

$$\begin{aligned}
&P\left\{\begin{array}{ll} l+S_i\notin[y_i]_N, & 1\le i\le j-1,\\ l+S_j=y_j+mN, & \end{array}\right\}\\
&\times P\left\{y_j+mN+S_{n-j}\in[y_n]_N\right\}.
\end{aligned}$$

Since $|y_n-y_j|\le n-j$, it is easy to verify that for any $m\in\mathbb{Z}$,

$$P\left\{y_j+mN+S_{n-j}\in[y_n]_N\right\}=P\left\{S_{n-j}\in[y_n-y_j]_N\right\}\le p_{n-j}^{(N)}.$$

Therefore

$$P\{l+S_n \in [y_n]_N\} \le \sum_{j=1}^{n} P\left\{\begin{array}{ll} l+S_i \notin [y_i]_N, & 1\le i\le j-1, \\ l+S_j = [y_j]_N & \end{array}\right\} p_{n-j}^{(N)}, \tag{2.19}$$

for $l \in \{0,1,\ldots,N-1\}$ and $n \in \{1,2,\ldots,N\}$. By multiplying (2.19) by $1/N$ and summing (l,n) over $\{0,1,\ldots,N-1\}\times\{1,2,\ldots,N\}$,

$$\begin{aligned} &\sum_{l=0}^{N-1}\sum_{n=1}^{N}\frac{1}{N}P\{l+S_n\in[y_n]_N\} \\ &\le \sum_{l=0}^{N-1}\sum_{j=1}^{N}\frac{1}{N}P\left\{\begin{array}{ll} l+S_i \notin [y_i]_N, & 1\le i\le j-1, \\ l+S_j = [y_j]_N & \end{array}\right\}\cdot\left(\sum_{i=0}^{N-j}p_i^{(N)}\right) \\ &\le \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=(y_n \mod N)\right\}\right)\left(\sum_{i=0}^{N-1}p_i^{(N)}\right). \end{aligned} \tag{2.20}$$

Here we used (2.18).

From $\sum_{l=0}^{N-1} P\{l+S_n\in[y]_N\} = P\{S_n\in\mathbb{Z}\} = 1$ $(n\in\mathbb{N},\ y\in\mathbb{Z})$,

$$\sum_{l=0}^{N-1}\sum_{n=1}^{N}\frac{1}{N}P\{l+S_n\in[y_n]_N\} = 1. \tag{2.21}$$

(2.20) and (2.21) imply

$$1 \le \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=(y_n \mod N)\right\}\right)\left(\sum_{i=0}^{N-1}p_i^{(N)}\right) \tag{2.22}$$

that is the first inequality in (2.17).

For the last inequality in (2.17), let $y_{N+j} = y_N$ $(j=1,2,\ldots,N)$. The same argument as showing (2.22) (we use $q_i^{(N)}$ instead of $p_i^{(N)}$) gives

$$\begin{aligned} 2 &= \sum_{l=0}^{N-1}\sum_{n=1}^{2N}\frac{1}{N}P\{l+S_n\in[y_n]_N\} \\ &\ge \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=(y_n \mod N)\right\}\right)\left(\sum_{i=0}^{N-1}q_i^{(N)}\right). \end{aligned}$$

□

Corollary 2.2.1. For $N \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned} \frac{1}{1+\sum_{i=1}^{N-1} P\{S_i \in [0]_N\}} &\leq \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=0\right\}\right) \\ &\leq \frac{2}{1+\sum_{i=1}^{N-1} P\{S_i \in [0]_N\}}. \end{aligned} \tag{2.23}$$

Proof. Put $y_1 = y_2 = \cdots = y_{2N} = 0$ in the proof of Proposition 2.2.1. Then the same argument as showing (2.17) gives (2.23). □

Corollary 2.2.2. For $N \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned} \frac{1}{1+\sum_{i=1}^{N-1} P\{S_i \in [i]_N\}} &\leq \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=(n \mod N)\right\}\right) \\ &\leq \frac{2}{1+\sum_{i=1}^{N-1} P\{S_i \in [i]_N\}}. \end{aligned} \tag{2.24}$$

Proof. Put $y_j = j$ $(j = 1, 2, \ldots, 2N)$ in the proof of Proposition 2.2.1. Then the same argument as showing (2.17) gives (2.24). □

Remark 5. By the same argument as showing (2.23), we obtain that for $\tilde{\epsilon} > 0$ and $N \geq 1/\tilde{\epsilon}$,

$$\mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)}=0\right\}\right) \leq \frac{1+\tilde{\epsilon}}{1+\sum_{i=1}^{\tilde{\epsilon}N} P\{S_i \in [0]_N\}}.$$

2.3 Fourier transform

In this section, we introduce some results concerning one-dimensional random walk.

Proposition 2.3.1. If a one-dimensional random walk satisfies $(A1)$ and $(A3)$, then there exist $C_1 > 0$ and $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$\sup_{l\in\mathbb{Z}}\left|n^{1/\beta}P\{S_n = l\} - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-c_*|x|^\beta}\exp\left(-i\frac{xl}{n^{1/\beta}}\right)\,dx\right| \leq C_1 n^{-\delta},$$

where $\delta = \min\{\varepsilon/(2\beta), 1/2\}$.

Proof. Proposition 2 can be proved by the same procedure as in Theorem 1.2.1 of [14].

The Fourier inversion formula for $\phi^n(\theta)$ is

$$n^{1/\beta} P\{S_n = l\} = \frac{n^{1/\beta}}{2\pi} \int_{-\pi}^{\pi} \phi^n(\theta) e^{-i\theta l} \, d\theta. \tag{2.25}$$

From $(A3)$, there exist $C_* > 0$ and $r \in (0, \pi)$ such that for $|\theta| < r$,

$$|\phi(\theta) - (1 - c_*|\theta|^\beta)| \le C_* |\theta|^{\beta + \varepsilon} \tag{2.26}$$

and

$$|\phi(\theta)| \le 1 - \frac{c_*}{2} |\theta|^\beta. \tag{2.27}$$

With r, we decompose the right-hand side of (2.25) to obtain

$$n^{1/\beta} P\{S_n = l\} = I(n, l) + J(n, l),$$

where

$$\begin{aligned} I(n, l) &= \frac{n^{1/\beta}}{2\pi} \int_{|\theta| < r} \phi^n(\theta) e^{-i\theta l} \, d\theta, \\ J(n, l) &= \frac{n^{1/\beta}}{2\pi} \int_{r \le |\theta| \le \pi} \phi^n(\theta) e^{-i\theta l} \, d\theta. \end{aligned}$$

A strongly aperiodic random walk $(A1)$ has the property that $|\phi(\theta)| = 1$ only when θ is a multiple of 2π (see §7 Proposition 8 of [18]). From the definition of $\phi(\theta)$, $|\phi(\theta)|$ is a continuous function on the bounded closed set $[-\pi, -r] \cup [r, \pi]$, and $|\phi(\theta)| \le 1$ $(\theta \in [-\pi, \pi])$. Hence, there exists a $\rho < 1$, depending on $r \in (0, \pi]$, such that

$$\max_{r \le |\theta| \le \pi} |\phi(\theta)| \le \rho. \tag{2.28}$$

By using the above inequality,

$$|J(n, l)| \le \frac{n^{1/\beta}}{2\pi} \int_{r \le |\theta| \le \pi} |\phi(\theta)|^n \, d\theta \le n^{1/\beta} \rho^n.$$

We perform the change of variables $\theta = x/n^{1/\beta}$, so that

$$I(n, l) = \frac{1}{2\pi} \int_{|x| < r n^{1/\beta}} \phi^n \left(\frac{x}{n^{1/\beta}} \right) \exp \left(-i \frac{xl}{n^{1/\beta}} \right) \, dx.$$

Put

$$\gamma = \min\left\{\frac{\varepsilon}{2\beta(\beta+\varepsilon+1)},\ \frac{1}{2(2\beta+1)}\right\}.$$

We decompose $I(n,l)$ as follows:

$$\begin{aligned}
&I(n,l) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-c_*|x|^\beta} \exp\left(-i\frac{xl}{n^{1/\beta}}\right)\ dx + I_1(n,l) + I_2(n,l) + I_3(n,l),
\end{aligned}$$

where

$$\begin{aligned}
I_1(n,l) &= \frac{1}{2\pi}\int_{|x|\le n^\gamma} \left\{\phi^n\left(\frac{x}{n^{1/\beta}}\right) - e^{-c_*|x|^\beta}\right\} \exp\left(-i\frac{xl}{n^{1/\beta}}\right)\ dx, \\
I_2(n,l) &= -\frac{1}{2\pi}\int_{n^\gamma<|x|} e^{-c_*|x|^\beta} \exp\left(-i\frac{xl}{n^{1/\beta}}\right)\ dx
\end{aligned}$$

and

$$I_3(n,l) = \frac{1}{2\pi}\int_{n^\gamma<|x|<rn^{1/\beta}} \phi^n\left(\frac{x}{n^{1/\beta}}\right) \exp\left(-i\frac{xl}{n^{1/\beta}}\right)\ dx.$$

Therefore,

$$\begin{aligned}
&\left| n^{1/\beta}P\{S_n = l\} - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-c_*|x|^\beta} \exp\left(-i\frac{xl}{n^{1/\beta}}\right)\ dx\right| \\
&\le |J(n,l)| + \sum_{k=1}^{3} |I_k(n,l)|.
\end{aligned}$$

The proof of Proposition 2.3.1 will be complete if we show that each term in the right-hand side of the above inequality is bounded by a constant (independent of l) multiple of $n^{-\delta}$.

If n is large enough, then the bound $|J(n,l)| \le n^{1/\beta}\rho^n$, which has already been shown above, yields

$$|J(n,l)| \le n^{-\delta}.$$

With the help of

$$|a^n - b^n| = |a-b|\left|\sum_{j=0}^{n-1} a^{n-1-j}b^j\right| \le n|a-b| \quad (a,b\in[-1,1]) \qquad (2.29)$$

and $|\phi(\theta)| \le 1$ $(\theta \in [-\pi, \pi])$, (2.26) implies that for $|x|(< n^{\gamma}) < rn^{1/\beta}$,

$$\begin{aligned}
&\left|\phi^n\left(\frac{x}{n^{1/\beta}}\right) - e^{-c_*|x|^\beta}\right| \\
&\le n\left|\phi\left(\frac{x}{n^{1/\beta}}\right) - e^{-c_*|x|^\beta/n}\right| \\
&\le n\left|\phi\left(\frac{x}{n^{1/\beta}}\right) - \left(1 - c_*\frac{|x|^\beta}{n}\right)\right| + n\left|\left(1 - c_*\frac{|x|^\beta}{n}\right) - e^{-c_*|x|^\beta/n}\right| \\
&\le C_*|x|^{\beta+\varepsilon}n^{-\varepsilon/\beta} + \frac{c_*^2}{2}|x|^{2\beta}n^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
|I_1(n,l)| &\le \frac{1}{2\pi}\int_{|x|\le n^\gamma}\left|\phi^n\left(\frac{x}{n^{1/\beta}}\right) - e^{-c_*|x|^\beta}\right| \, d\theta \\
&\le \frac{1}{\pi}\left(\frac{C_*}{\beta+\varepsilon+1} + \frac{c_*^2}{2(2\beta+1)}\right)n^{-\delta}.
\end{aligned}$$

From (2.27), it is easy to verify that for $|x| < rn^{1/\beta}$,

$$\left|\phi^n\left(\frac{x}{n^{1/\beta}}\right)\right| \le \left(1 - \frac{c_*}{2}\frac{|x|^\beta}{n}\right)^n \le e^{-c_*|x|^\beta/2},$$

and we obtain that

$$\begin{aligned}
|I_3(n,l)| &\le \frac{1}{2\pi}\int_{n^\gamma<|x|<rn^{1/\beta}}\left|\phi^n\left(\frac{x}{n^{1/\beta}}\right)\right| \, dx \\
&\le \frac{1}{2\pi}\int_{n^\gamma<|x|} e^{-c_*|x|^\beta/2} \, dx.
\end{aligned} \tag{2.30}$$

Moreover, if n is large enough, then

$$e^{-c_*|x|^\beta/2} \le \frac{2^s}{c_*^s}|x|^{-s\beta} \quad (|x| > n^\gamma),$$

where $s = (1/\beta)(1 + 1/(2\gamma))$. By replacing the integrand in the right-hand side of the last inequality of (2.30) with the right-hand side of the above inequality, we obtain

$$|I_3(n,l)| \le \frac{2^{s+1}\gamma}{\pi c_*^s}n^{-1/2} \le \frac{2^{s+1}\gamma}{\pi c_*^s}n^{-\delta}. \tag{2.31}$$

The same argument as showing (2.31) gives

$$|I_2(n,l)| \le \frac{1}{2\pi}\int_{n^\gamma\le|\theta|} e^{-c_*|x|^\beta} \, dx \le \frac{2^{s+1}\gamma}{\pi c_*^s}n^{-\delta}.$$

□

Put

$$I_0(n,l:\beta,c_*) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-c_*|x|^\beta} \exp\left(-i\frac{xl}{n^{1/\beta}}\right)\, dx$$

appearing in Proposition 2.3.1.

Remark 6. When a one-dimensional random walk is the strongly aperiodic $(A1)$ with $E[X_1]=0$ and $E[|X_1|^{2+\varepsilon}]<\infty$ for some $\varepsilon\in(0,1)$, it is verified that

$$\phi(\theta) = 1-\frac{E[X_1^2]}{2}|\theta|^2 + O(|\theta|^{2+\varepsilon}).$$

In this case, $I_0(n,l:2,E[X_1^2]/2)$ can be computed and Proposition 2.3.1 gives the following.

(Local Central Limit Theorem) There exist $\tilde{C}_1>0$ and $\tilde{N}_1\in\mathbb{N}$ such that for $n\geq\tilde{N}_1$,

$$\sup_{l\in\mathbb{Z}}\left|n^{1/2}P\{S_n=l\} - \frac{1}{\sqrt{2E[X_1^2]\pi}}\exp\left(-\frac{l^2}{2E[X_1^2]n}\right)\right| \leq \tilde{C}_1 n^{-\delta}, \quad (2.32)$$

where $\delta=\min\{\varepsilon/4,1/2\}$. (See Remark after Proposition 7.9 in [18].)

It is easy to see

$$I_0(n,l:1,c_*) = \frac{1}{\pi}\frac{c_*}{c_*^2+(l/n)^2} \quad (n\in\mathbb{N}, l\in\mathbb{Z}, c_*>0)$$

and we have the following corollary of Proposition 2.3.1.

Corollary 2.3.1. If a one-dimensional random walk satisfies $(A1)$ and $(A3)$ with $\beta=1$, then there exist $C_2>0$ and $N_2\in\mathbb{N}$ such that for $n\geq N_2$,

$$\sup_{l\in\mathbb{Z}}\left|nP\{S_n=l\}-\frac{1}{\pi}\frac{c_*}{c_*^2+(l/n)^2}\right| \leq C_2 n^{-\delta},$$

where $\delta=\min\{\varepsilon/2,1/2\}$.

We perform the change of variables $t=c_*x^\beta$, so that

$$I_0(n,0:\beta,c_*) = \frac{1}{\pi}\int_0^{+\infty} e^{-c_*x^\beta}\, dx = \frac{1}{\beta c_*^{1/\beta}\pi}\Gamma\left(\frac{1}{\beta}\right).$$

With the help of the above calculation, Proposition 2.3.1 gives the following corollary.

Corollary 2.3.2. If a one-dimensional random walk satisfies $(A1)$ and $(A3)$, then there exist $C_3 > 0$ and $N_3 \in \mathbb{N}$ such that for $n \geq N_3$,

$$\left| n^{1/\beta} P\{S_n = 0\} - \frac{1}{\beta c_*^{1/\beta} \pi} \Gamma\left(\frac{1}{\beta}\right) \right| \leq C_3 n^{-\delta},$$

where $\delta = \min\{\varepsilon/2\beta, 1/2\}$.

Proposition 2.3.2. If a one-dimensional random walk satisfies $(A2)$, then for $l \in \mathbb{Z}$ and $n \in \{0\} \cup \mathbb{N}$,

$$P\{S_n \in [l]_N\} = \frac{1}{N} + \frac{2}{N} \sum_{1 \leq j \leq (N-1)/2} \phi^n\left(\frac{2j\pi}{N}\right) \cos\left(\frac{2j\pi}{N} l\right) + J_N(n,l), \tag{2.33}$$

where

$$J_N(n,l) = \begin{cases} (1/N)\phi^n(\pi)\cos(\pi l) & (\text{ if } N \text{ is even }) \\ 0 & (\text{ if } N \text{ is odd }). \end{cases}$$

Proof. From the definition of $\phi(\theta)$,

$$\phi^n(\theta) = \sum_{k \in \mathbb{Z}} e^{i\theta k} P\{S_n = k\}.$$

Thus

$$\begin{aligned} \phi^n\left(\frac{2j\pi}{N}\right) &= \sum_{k \in \mathbb{Z}} e^{2ij\pi k/N} P\{S_n = k\} \\ &= \sum_{\tilde{l}=0}^{N-1} \sum_{m \in \mathbb{Z}} e^{2ij\pi(\tilde{l}+mN)/N} P\left\{S_n = \tilde{l} + mN\right\} \\ &= \sum_{\tilde{l}=0}^{N-1} e^{2ij\pi\tilde{l}/N} P\left\{S_n \in [\tilde{l}]_N\right\}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{j=0}^{N-1} e^{-2ij\pi l/N} \phi^n\left(\frac{2j\pi}{N}\right) &= \sum_{\tilde{l}=0}^{N-1} \sum_{j=0}^{N-1} e^{2ij\pi(\tilde{l}-l)/N} P\left\{S_n \in [\tilde{l}]_N\right\} \\ &= NP\{S_n \in [l]_N\} \end{aligned}$$

since

$$\sum_{j=0}^{N-1} e^{2ij\pi(\tilde{l}-l)/N} = \begin{cases} N & \tilde{l} = l \\ 0 & \tilde{l} \neq l. \end{cases}$$

Therefore,

$$\begin{aligned} P\left\{S_n \in [l]_N\right\} &= \frac{1}{N}\sum_{j=0}^{N-1} \phi^n\left(\frac{2j\pi}{N}\right) e^{-2j\pi i l/N} \\ &= \frac{1}{N}\sum_{j=0}^{N-1} \phi^n\left(\frac{2j\pi}{N}\right) \cos\left(\frac{2j\pi l}{N}\right). \end{aligned}$$

From $(A2)$, $\phi^n(\theta) \in \mathbb{R}$ and

$$\frac{1}{N}\sum_{j=0}^{N-1} \phi^n\left(\frac{2j\pi}{N}\right) \cos\left(\frac{2j\pi l}{N}\right) \in \mathbb{R}.$$

So we have

$$\phi^n\left(\frac{2m\pi}{N}\right)\cos\left(\frac{2m\pi l}{N}\right) = \phi^n\left(\frac{2(N-m)\pi}{N}\right)\cos\left(\frac{2(N-m)\pi l}{N}\right). \quad (2.34)$$

Let N be an even number. Then, from (2.34),

$$\begin{aligned} P\left\{S_n \in [l]_N\right\} &= \frac{1}{N}\phi^n(0)\cos(0) + \frac{2}{N}\sum_{1\le j\le (N-1)/2} \phi^n\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi l}{N}\right) \\ &\quad + \frac{1}{N}\phi^n(\pi)\cos(\pi l) \\ &= \frac{1}{N} + \frac{2}{N}\sum_{1\le j\le (N-1)/2} \phi^n\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi l}{N}\right) \\ &\quad + \frac{1}{N}\phi^n(\pi)\cos(\pi l). \end{aligned}$$

Therefore, we have (2.33) for every even number N. The proof of (2.33) for odd number is similar and is omitted. □

2.4 Proof of Theorem 2.1.1

In this section we prove Theorem 2.1.1. To prove it, we introduce the following Proposition.

Proposition 2.4.1. Assume $(A1)-(A3)$.

If $\beta \in (0,1)$, then there exists a constant $c_8 > 0$ such that

$$\sum_{i=0}^{N-1} p_i^{(N)} \leq c_8. \tag{2.35}$$

If $\beta = 1$, then there exists a constant $c_9 > 0$ such that

$$\sum_{i=0}^{N-1} p_i^{(N)} \leq \frac{1}{c_* \pi} \log N + c_9. \tag{2.36}$$

If $\beta \in (1,2]$, then there exists a constant $c_{10} > 0$ such that

$$\sum_{i=0}^{N-1} p_i^{(N)} \leq c_{10} N^{(\beta-1)/\beta}. \tag{2.37}$$

Proof. From $(A3)$, there exist C_* and $r \in (0,\pi)$ such that for $|\theta| < r$,

$$|\phi(\theta) - (1 - c_*|\theta|^\beta)| \leq C_* |\theta|^{\beta+\varepsilon} \tag{2.38}$$

We can choose $r_* \in (0,r]$ small enough so that

$$C_* r_*^\varepsilon \leq \frac{1}{2} c_* \quad \text{and} \quad c_* r_*^\beta \leq \frac{1}{3}. \tag{2.39}$$

Then for $|\theta| < r_*$,

$$\frac{1}{2} c_* |\theta|^\beta \leq |1 - \phi(\theta)| \tag{2.40}$$

and

$$|1 - \phi(\theta)| \leq \frac{3}{2} c_* |\theta|^\beta \leq \frac{1}{2}. \tag{2.41}$$

By the same reason as (2.28), there exists a $\rho_* \in [0,1)$, depending on r_*, such that

$$\max_{r_* \leq |\theta| \leq \pi} |\phi(\theta)| \leq \rho_*. \tag{2.42}$$

(Here we used the condition $(A1)$.)

Using Proposition 2.3.2 and (2.42), we obtain that for $n \in \{0\} \cup \mathbb{N}$,

$$\begin{aligned}
p_i^{(N)} &= \max_{|l| \leq i} P\{S_i \in [l]_N\} \\
&\leq \frac{1}{N} + \sum_{1 \leq j \leq (N-1)/2} \frac{2}{N} \left| \phi\left(\frac{2j\pi}{N}\right) \right|^i + |J_N(i,0)| \\
&\leq \frac{1}{N} + \sum_{1 \leq j < (r_*/(2\pi))N} \frac{2}{N} \left| \phi\left(\frac{2j\pi}{N}\right) \right|^i + \rho_*^i.
\end{aligned}$$

Therefore

$$\sum_{i=0}^{N-1} p_i^{(N)} \le 1 + \Phi_N + \frac{1}{1-\rho_*}, \tag{2.43}$$

where

$$\Phi_N = \sum_{1 \le j < (r_*/(2\pi))N} \frac{2}{N} \frac{1 - \left|\phi\left(\frac{2j\pi}{N}\right)\right|^N}{1 - \left|\phi\left(\frac{2j\pi}{N}\right)\right|}.$$

Because of $(A2)$, $\phi(\theta)$ takes a real number. Then (2.40), (2.41) and $(A1)$ mean that

$$\frac{1}{2} < \phi(\theta) = |\phi(\theta)| < 1 \quad (\theta \in (-r_*, 0) \cup (0, r_*)) \tag{2.44}$$

and

$$\Phi_N \le \sum_{1 \le j < (r_*/(2\pi))N} \frac{2}{N} \frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)}. \tag{2.45}$$

We will calculate Φ_N in the case $\beta \in (0, 1]$. From (2.45), we decompose the right-hand side of the above to obtain

$$\sum_{1 \le j < (r_*/(2\pi))N} \frac{2}{N} \frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)} = \tilde{\Phi}_N + E_N, \tag{2.46}$$

where

$$\tilde{\Phi}_N = \frac{2^{1-\beta}}{\pi^\beta c_*} N^{\beta-1} \sum_{1 \le j < (r_*/(2\pi))N} j^{-\beta},$$

$$E_N = \sum_{1 \le j < (r_*/(2\pi))N} \frac{2}{N} \left(\frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)} - \frac{1}{c_*\left(\frac{2j\pi}{N}\right)^\beta} \right).$$

To estimate E_N, we use (2.38) and (2.40) which imply that for $j \in [1, (r_*/(2\pi))N) \cap \mathbb{Z}$,

$$\begin{aligned} \frac{2}{N} \left| \frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)} - \frac{1}{c_*\left(\frac{2j\pi}{N}\right)^\beta} \right| &= \frac{2}{N} \frac{\left| 1 - \phi\left(\frac{2j\pi}{N}\right) - c_*\left(\frac{2j\pi}{N}\right)^\beta \right|}{\left| 1 - \phi\left(\frac{2j\pi}{N}\right) \right| \cdot \left| c_*\left(\frac{2j\pi}{N}\right)^\beta \right|} \\ &\le c_{11} N^{\beta-\varepsilon-1} j^{\varepsilon-\beta}, \end{aligned}$$

where $c_{11} = 2^{2+\varepsilon-\beta}\pi^{\varepsilon-\beta}C_*/c_*^2$. By noticing that $1+\varepsilon-\beta > 0$,

$$\sum_{1\le j<(r_*/(2\pi))N} j^{\varepsilon-\beta} \le \int_0^N x^{\varepsilon-\beta}\, dx = \frac{N^{1+\varepsilon-\beta}}{1+\varepsilon-\beta}.$$

Thus

$$|E_N| \le c_{11}/(1+\varepsilon-\beta). \tag{2.47}$$

It is easy to see that

$$\begin{aligned} \tilde{\Phi}_N &\le \frac{2^{1-\beta}}{\pi^\beta c_*} N^{\beta-1}\left(1+\int_1^N x^{-\beta}\, dx\right) \\ &\le \begin{cases} \dfrac{2^{1-\beta}}{\pi^\beta c_*(1-\beta)} & (\beta\in(0,1)) \\ \dfrac{1}{\pi c_*}\log N + \dfrac{1}{\pi c_*} & (\beta = 1). \end{cases} \end{aligned} \tag{2.48}$$

Put the pieces ((2.43), (2.45)-(2.48)) together, we have (2.35) and (2.36).

In the case $\beta \in (1,2]$, we use (2.44) to obtain

$$\Phi_N \le \Phi_N^{(1)} + \Phi_N^{(2)}, \tag{2.49}$$

where $N(\beta) = \min\{N^{(\beta-1)/\beta}, (r_*/(2\pi))N\}$ and

$$\begin{aligned} \Phi_N^{(1)} &= \sum_{1\le j<N(\beta)} \frac{2}{N} \frac{\left|1-\phi\left(\frac{2j\pi}{N}\right)^N\right|}{\left|1-\phi\left(\frac{2j\pi}{N}\right)\right|}, \\ \Phi_N^{(2)} &= \sum_{N(\beta)\le j<(r_*/(2\pi))N} \frac{2}{N} \frac{1}{\left|1-\phi\left(\frac{2j\pi}{N}\right)\right|}. \end{aligned}$$

From (2.29)(set $n = N$ and $a = 1, b = \phi\left(\frac{2j\pi}{N}\right)$),

$$\Phi_N^{(1)} \le 2N(\beta) \le 2N^{(\beta-1)/\beta}. \tag{2.50}$$

By noticing that $\beta - 1 > 0$, (2.40) gives

$$\begin{aligned} \Phi_N^{(2)} &\le \frac{2^{2-\beta}}{c_*\pi^\beta} N^{\beta-1}\left(\sum_{N(\beta)\le j<(r_*/(2\pi))N} j^{-\beta}\right) \\ &\le \frac{2^{2-\beta}}{c_*\pi^\beta} N^{\beta-1}\left(N^{-\beta+1} + \int_{N^{(\beta-1)/\beta}}^{+\infty} x^{-\beta}\, dx\right) \\ &\le \frac{2^{2-\beta}}{c_*\pi^\beta}\left(1+\frac{1}{\beta-1}\right) N^{(\beta-1)/\beta}. \end{aligned} \tag{2.51}$$

Put the pieces ((2.43), (2.49)-(2.51)) together, we have (2.37). □

It remains to show the last inequality in (2.2). To achieve this, we will use Proposition 2.2.1 and Corollary 2.3.1.

From Corollary 2.3.1, there exist $C_2 > 0$ and $N_2 \in \mathbb{N}$ such that for $i \geq N_2$ and $l \in \mathbb{Z}$,

$$P\{S_i = l\} \geq \frac{1}{\pi} \frac{c_*}{c_*^2 + (l/i)^2} \frac{1}{i} - C_2 i^{-1-\delta}.$$

Let

$$c_{12} := \frac{1}{\pi} \frac{c_*}{c_*^2 + 1} \log N_2 + C_2 \sum_{i=N_2}^{\infty} i^{-1-\delta}.$$

We can choose $N_* \in \mathbb{N}$ large enough so that

$$\frac{1}{2} \frac{1}{\pi} \frac{c_*}{c_*^2 + 1} \log N_* \geq c_{12}.$$

Then for $N \geq N_* + 1$,

$$\begin{aligned} \sum_{i=0}^{N-1} q_i^{(N)} &\geq \sum_{i=N_2}^{N-1} \min_{|l| \leq i} P\{S_i = l\} \\ &\geq \frac{1}{\pi} \frac{c_*}{c_*^2 + 1} \sum_{i=N_2}^{N-1} \frac{1}{i} - C_2 \sum_{i=N_2}^{\infty} i^{-1-\delta} \\ &\geq \frac{1}{\pi} \frac{c_*}{c_*^2 + 1} \log N - c_{12} \\ &\geq \frac{1}{2} \frac{1}{\pi} \frac{c_*}{c_*^2 + 1} \log N. \end{aligned} \tag{2.52}$$

It follows from Proposition 2.2.1 and (2.52) that for $N \in [N_* + 1, +\infty) \cap \mathbb{N}$ and $y_1, y_2, \ldots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \leq 1$ $(n = 1, 2, \ldots, N-1)$,

$$\mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{R}_n^{(N)} = (y_n \mod N)\} \right) \leq \frac{\frac{4\pi(c_*^2+1)}{c_*}}{\log N}.$$

It is clear that $\mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{R}_n^{(N)} = (y_n \mod N)\} \right)$ is bounded by 1. Put $c_3 = \max\{4\pi(c_*^2 + 1)/c_*, \log N_*\}$. The last inequality in (2.2) holds.

The proof of Theorem 2.1.1 is complete.

Chapter 3

Simulation of Hunter vs Rabbit Game

In this chapter, we introduce the digamma function. The digamma function is defined by the gamma function. We describe a transition probability matrix of a random walk using those functions. To simulate a random walk, we compute the matrix using C++. We also use a 'random' sampling function in discrete_distribution C++ Library.

3.1 Digamma Function

In this section, we introduce the digamma function and property of the digamma function.

Definition 3.1.1. The digamma function $\psi(x)$ is defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x)$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

is the gamma function.

Lemma 3.1.1. For $x \neq 0, -1, -2, \ldots$,

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k} \right), \tag{3.1}$$

and

$$\frac{d}{dx}\psi(x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2}, \tag{3.2}$$

where $m \geq 1$ and γ is the Euler's constant.

Proof. From Theorem 1.2.5 of [3], we have (3.1) and (3.2). □

Lemma 3.1.2. (See proof of Theorem 1.2.7 of [3])

$$\psi(1-x) - \psi(x) = \pi \cot(\pi x) = \pi \frac{\cos(\pi x)}{\sin(\pi x)}.$$

Lemma 3.1.3. (Theorem 1.2.7 of [3].)

$$\psi(1+z) = \psi(z) + \frac{1}{z}.$$

Lemma 3.1.4. (Theorem 1.2.7 of [3]) For $0 < p < q$,

$$\begin{aligned} \psi\left(\frac{p}{q}\right) &= -\gamma - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) \\ &\quad - \log q + \frac{1}{2}\sum_{n=1}^{q-1} \cos\left(\frac{2\pi n p}{q}\right) \log\left(2 - 2\cos\left(\frac{2\pi n}{q}\right)\right). \end{aligned}$$

Lemma 3.1.5. (Theorem 1.2.5 of [3].) $\psi(1) = -\gamma$.

3.2 Transition Matrix for the Random Walk on a Cycle Graph

We show the theory of using the simulation of the Hunter vs Rabbit Game.

Definition 3.2.1. Let $N = |V|$ and P be a probability over $\mathbb{Z}$. A transition matrix $\mathbb{P}(N, P)$ is defined by

$$\mathbb{P}(N, P) = \{\mathbb{P}_{i,j}\}_{0 \leq i,j \leq N-1}$$

where

$$\begin{aligned}\mathbb{P}_{i,i+d} &= \mathbb{P}_{(i \mod N),(i+d \mod N)} \\ &= \sum_{k=0}^{\infty}(P(x_t = kN+d) + P(x_t = (k+1)N-d)).\end{aligned}$$

Lemma 3.2.1. $\mathbb{P}(N, P)$ is a symmetric matrix.

Proof. From definition of $\mathbb{P}_{i,i+d}$, we have $\mathbb{P}_{i,i+d} = \mathbb{P}_{i+d,i}$. □

Lemma 3.2.2. For any $0 \le i \le N-1$, $d \in \mathbb{Z}$, $\mathbb{P}_{i,i+d} = \mathbb{P}_{i+1,i+d+1}$.

Proof. From the definition of $\mathbb{P}_{i,j}$, we have Lemma 3.2.2. □

Lemma 3.2.3. Let $n \in \mathbb{N}$ and $\mathbb{P}^n(N, P) = \left\{\mathbb{P}^{(n)}_{i,j}\right\}$. Then, $\mathbb{P}^{(2)}_{i,j} = \mathbb{P}^{(2)}_{i+1,j+1}$.

Proof. For any $0 \le i, j \le N-1$,

$$\mathbb{P}^{(2)}_{i,j} = \sum_{k=0}^{N-1} \mathbb{P}_{i,k}\mathbb{P}_{k,j}.$$

By Lemma 3.2.2,

$$\sum_{k=0}^{N-1} \mathbb{P}_{i,k}\mathbb{P}_{k,j} = \sum_{k=0}^{N-1} \mathbb{P}_{i+1,k+1}\mathbb{P}_{k+1,j+1} = \mathbb{P}^{(2)}_{i+1,j+1}.$$

□

Lemma 3.2.4. Let $m \in \mathbb{N}$. For $m \ge 2$, $\mathbb{P}^{(m)}_{i,j} = \mathbb{P}^{(m)}_{i+1,j+1}$.

Proof. By using mathematical induction, we have Lemma 3.2.4. □

Example 3.2.1. Let

$$P\{x_t = k\} = P\{x_t = -k\} = \begin{cases} \dfrac{1}{ak^2}, & k \ne 0 \ and, \\ 1 - \dfrac{\pi^2}{3a}, & k = 0, \end{cases}$$

where $a \ge \frac{\pi^2}{3}$.

In this case, we have Lemma as following.

Lemma 3.2.5. For $d \neq 0$,

$$\mathbb{P}_{i,i+d} = \frac{1}{aN^2}\left(\psi'\left(\frac{d}{N}\right) + \psi'\left(1 - \frac{d}{N}\right)\right).$$

Proof. From Definition 3.2.1,

$$\mathbb{P}_{i,i+d} = \sum_{k=0}^{\infty}(P\left\{x_t = kN + d\right\} + P\left\{x_t = (k+1)N - d\right\}).$$

And from definition of $P\left\{x_t = k\right\}$,

$$\begin{aligned}
\mathbb{P}_{i,i+d} &= \sum_{k=0}^{\infty}\left(\frac{1}{a(kN+d)^2} + \frac{1}{a((k+1)N-d)^2}\right) \\
&= \frac{1}{aN^2}\sum_{k=0}^{\infty}\left(\frac{1}{(k+\frac{d}{N})^2} + \frac{1}{(k+\frac{N-d}{N})^2}\right) \\
&= \frac{1}{aN^2}\left(\sum_{k=0}^{\infty}\frac{1}{(k+\frac{d}{N})^2} + \sum_{k=0}^{\infty}\frac{1}{(k+\frac{N-d}{N})^2}\right).
\end{aligned}$$

From Lemma 3.1.1,

$$\begin{aligned}
\mathbb{P}_{i,i+d} &= \frac{1}{aN^2}\left(\psi'\left(\frac{d}{N}\right) + \psi'\left(\frac{N-d}{N}\right)\right) \\
&= \frac{1}{aN^2}\left(\psi'\left(\frac{d}{N}\right) + \psi'\left(1 - \frac{d}{N}\right)\right).
\end{aligned}$$

□

Proposition 3.2.1. Let

$$P\left\{x_t = k\right\} = P\left\{x_t = -k\right\} = \begin{cases} \dfrac{1}{ak^2}, & k \neq 0, \text{and} \\ 1 - \dfrac{\pi^2}{3a}, & k = 0, \end{cases}$$

where $a \geq \frac{\pi^2}{3}$. For $d \neq 0$,

$$\mathbb{P}_{i,i+d} = \frac{\pi^2}{an^2\sin^2\left(\frac{\pi d}{n}\right)}, \text{and} \tag{3.3}$$

$$\mathbb{P}_{i,i} = 1 - \frac{\pi^2}{3a} + \frac{\pi^2}{3an^2}. \tag{3.4}$$

Proof. (3.3). From Lemma 3.2.5,

$$\mathbb{P}_{i,i+d} = \frac{1}{aN^2}\left(\psi'\left(\frac{d}{N}\right) + \psi'\left(1 - \frac{d}{N}\right)\right).$$

From Lemma 3.1.2,

$$\mathbb{P}_{i,i+d} = \frac{\pi^2}{aN^2 \sin^2 \frac{\pi d}{N}}.$$

(3.4). For $d = 0$,

$$\begin{aligned}\mathbb{P}_{i,i} &= \sum_{k=0}^{\infty}(P\{x_t = kN\} + P\{x_t = (k+1)N\}) \\ &= P\{x_t = 0\} + 2\sum_{k=1}^{\infty} P\{x_t = kN\}.\end{aligned}$$

From definition of $P\{X_t = k\}$,

$$\mathbb{P}_{i,i} = 1 - \frac{\pi^2}{3a} + 2\sum_{k=1}^{\infty}\frac{1}{a(kN)^2} = 1 - \frac{\pi^2}{3a} + \frac{\pi^2}{3aN^2}$$

□

Example 3.2.2. Let

$$P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$.

Lemma 3.2.6.

$$\begin{aligned}\mathbb{P}_{i,i} &= \frac{1}{aN}\left(\psi\left(\frac{N+2}{N}\right) - \psi\left(\frac{N+1}{N}\right)\right) \\ &\quad -\frac{1}{a(N+1)(N+2)} + 1 - \frac{1}{2a}, \qquad (3.5)\\ \mathbb{P}_{i,i+d} &= \frac{1}{aN}\left(\psi\left(\frac{d+2}{N}\right) - \psi\left(\frac{d+1}{N}\right)\right. \\ &\quad \left. +\psi\left(\frac{N-d+2}{N}\right) - \psi\left(\frac{N-d+1}{N}\right)\right). \qquad (3.6)\end{aligned}$$

Proof. From Definition 3.2.1,

$$\mathbb{P}_{i,i+d} = \sum_{k=0}^{\infty}(P\{x_t = kN + d\} + P\{x_t = (k+1)N - d\}).$$

(3.5). When $d = 0$,

$$\begin{aligned}\mathbb{P}_{i,i} &= \sum_{k=0}^{\infty}(P(x_t = kN) + P(x_t = (k+1)N)) \\ &= 2\sum_{k=0}^{\infty} P(x_t = (k+1)N) + P(x_t = 0).\end{aligned}$$

From Definition of $P\{X_t = k\}$,

$$\begin{aligned}\mathbb{P}_{i,i} &= 2\sum_{k=0}^{\infty}\frac{1}{2a((k+1)N+1)((k+1)N+2)} + 1 - \frac{1}{2a} \\ &= \frac{1}{a}\sum_{k=0}^{\infty}\left(\frac{1}{(k+1)N+1} - \frac{1}{(k+1)N+2}\right) + 1 - \frac{1}{2a} \\ &= \frac{1}{aN}\sum_{k=0}^{\infty}\left(\frac{1}{k+\frac{N+1}{N}} - \frac{1}{k+\frac{N+2}{N}}\right) + 1 - \frac{1}{2a} \\ &= \frac{1}{aN}\sum_{k=0}^{\infty}\left(\frac{1}{k+\frac{N+1}{N}} - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+\frac{N+2}{N}}\right) + 1 - \frac{1}{2a} \\ &= \frac{1}{aN}\left(\sum_{k=0}^{\infty}\left(\frac{1}{k} - \frac{1}{k+\frac{N+2}{N}}\right) - \sum_{k=0}^{\infty}\left(\frac{1}{k} - \frac{1}{k+\frac{N+1}{N}}\right)\right) + 1 - \frac{1}{2a}.\end{aligned}$$

From Lemma 3.1.1,

$$\sum_{k=0}^{\infty}\left(\frac{1}{k} - \frac{1}{k+\frac{N+2}{N}}\right) = \gamma + \frac{N}{N+2} - 1 + \psi\left(\frac{N+2}{N}\right)$$

and

$$\sum_{k=0}^{\infty}\left(\frac{1}{k} - \frac{1}{k+\frac{N+1}{N}}\right) = \gamma + \frac{N}{N+1} - 1 + \psi\left(\frac{N+1}{N}\right).$$

Thus

$$\mathbb{P}_{i,i} = \frac{1}{aN}\left(\psi\left(\frac{N+2}{N}\right) - \psi\left(\frac{N+1}{N}\right)\right) - \frac{1}{a(N+1)(N+2)} + 1 - \frac{1}{2a}.$$

The proof of (3.6) is similar and is omitted. □

Proposition 3.2.2. Let

$$P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$. Then,

$$\begin{aligned}
&\mathbb{P}_{i,i} \\
&= \frac{1}{aN}\left(\frac{\pi}{2}\left(\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{2\pi}{N}\right)\right) - \frac{N}{2}\right. \\
&\quad \left. + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{4\pi m}{N}\right) - \cos\left(\frac{2\pi m}{N}\right)\right)\log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right) \\
&\quad - \frac{1}{a(N+1)(N+2)} + 1 - \frac{1}{2a}.
\end{aligned}$$

Proof. From Lemma 3.2.6,

$$\mathbb{P}_{i,i} = \frac{1}{aN}\left(\psi\left(\frac{N+2}{N}\right) - \psi\left(\frac{N+1}{N}\right)\right) - \frac{1}{a(N+1)(N+2)} + 1 - \frac{1}{2a}.$$

From Lemma 3.1.3 and Lemma 3.1.4,

$$\begin{aligned}
\psi\left(\frac{N+2}{N}\right) &= \psi\left(\frac{2}{N}\right) + \frac{N}{2} \\
&= -\gamma - \frac{\pi}{2}\cot\left(\frac{2\pi}{N}\right) - \log N \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-1}\cos\left(\frac{4\pi m}{N}\right)\log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right) + \frac{N}{2}, \\
\psi\left(\frac{N+1}{N}\right) &= \psi\left(\frac{1}{N}\right) + N \\
&= -\gamma - \frac{\pi}{2}\cot\left(\frac{\pi}{N}\right) - \log N \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-1}\cos\left(\frac{2\pi m}{N}\right)\log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right) + N.
\end{aligned}$$

Thus

$$\begin{aligned}
&\psi\left(\frac{N+2}{N}\right) - \psi\left(\frac{N+1}{N}\right) \\
&= \frac{\pi}{2}\left(\cot\left(\frac{\pi}{n}\right) - \cot\left(\frac{2\pi}{N}\right)\right) \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{4\pi m}{N}\right) - \cos\left(\frac{2\pi m}{N}\right)\right)\log\left(2 - 2\cos\left(\frac{2\pi m}{n}\right)\right) - \frac{N}{2}.
\end{aligned}$$

So we have

$$
\begin{aligned}
&P_{i,i}(N) \\
&= \frac{1}{aN}\left(\frac{\pi}{2}\left(\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{2\pi}{N}\right)\right)\right. \\
&\left. + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{4\pi m}{N}\right) - \cos\left(\frac{2\pi m}{N}\right)\right)\log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right) - \frac{N}{2}\right) \\
&- \frac{1}{a(N+1)(N+2)} + 1 - \frac{1}{2a}.
\end{aligned}
$$

□

Proposition 3.2.3. Let

$$
P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}
$$

where $a \geq \frac{1}{2}$. For $2 < d < N-2$,

$$
\begin{aligned}
\mathbb{P}_{i,i+d} &= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{d+2}{N}\pi\right) + \cot\left(\frac{d+1}{N}\pi\right)\right.\right. \\
&\quad \left. + \cot\left(\frac{d-2}{N}\pi\right) - \left(\cot\frac{d-1}{N}\pi\right)\right) \\
&\quad \left. + \frac{1}{2}\sum_{m=1}^{N-1}\left(C_d(m)\log\left(2 - 2\cos\frac{2\pi m}{N}\right)\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
C_d(m) &= \left(\cos\left(\frac{2m(d+2)}{N}\pi\right) - \cos\left(\frac{2m(d+1)}{N}\pi\right)\right. \\
&\quad \left. + \cos\left(\frac{2m(d-2)}{N}\pi\right) - \cos\left(\frac{2m(d-1)}{N}\pi\right)\right).
\end{aligned}
$$

Proof. From Lemma 3.2.6,

$$
\begin{aligned}
\mathbb{P}_{i,i+d} &= \frac{1}{aN}\left(\psi\left(\frac{d+2}{N}\right) - \psi\left(\frac{d+1}{N}\right)\right. \\
&\quad \left. + \psi\left(\frac{N-d+2}{N}\right) - \psi\left(\frac{N-d+1}{N}\right)\right).
\end{aligned}
$$

And from Lemma 3.1.4,

$$\begin{aligned}
\psi\left(\frac{d+2}{N}\right) &= -\gamma-\frac{\pi}{2}\cot\left(\frac{d+2}{N}\pi\right)-\log N \\
&\quad +\frac{1}{2}\sum_{m=1}^{N-1}\cos\left(\frac{2m(d+2)}{N}\pi\right)\log\left(2-2\cos\frac{2\pi m}{N}\right), \\
\psi\left(\frac{d+1}{N}\right) &= -\gamma-\frac{\pi}{2}\cot\left(\frac{d+1}{N}\pi\right)-\log N \\
&\quad +\frac{1}{2}\sum_{m=1}^{N-1}\cos\left(\frac{2m(d+1)}{N}\pi\right)\log\left(2-2\cos\frac{2\pi m}{N}\right), \\
\psi\left(\frac{N-d+2}{N}\right) &= -\gamma+\frac{\pi}{2}\cot\left(\frac{d-2}{N}\pi\right)-\log N \\
&\quad +\frac{1}{2}\sum_{m=1}^{N-1}\cos\left(\frac{2m(d-2)}{N}\pi\right)\log\left(2-2\cos\frac{2\pi m}{N}\right), \text{and} \\
\psi\left(\frac{N-d+1}{N}\right) &= -\gamma+\frac{\pi}{2}\cot\left(\frac{d-1}{N}\pi\right)-\log N \\
&\quad +\frac{1}{2}\sum_{m=1}^{N-1}\cos\left(\frac{2m(d-1)}{N}\pi\right)\log\left(2-2\cos\frac{2\pi m}{N}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{P}_{i,i+d} &= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{d+2}{N}\pi\right)+\cot\left(\frac{d+1}{N}\pi\right)\right.\right. \\
&\quad \left.+\cot\left(\frac{d-2}{N}\pi\right)-\left(\cot\frac{d-1}{N}\pi\right)\right) \\
&\quad \left.+\frac{1}{2}\sum_{m=1}^{N-1}\left(C_d(m)\log\left(2-2\cos\frac{2\pi m}{N}\right)\right)\right)
\end{aligned}$$

where

$$\begin{aligned}
C_d(m) &= \left(\cos\left(\frac{2m(d+2)}{n}\pi\right)-\cos\left(\frac{2m(d+1)}{n}\pi\right)\right. \\
&\quad \left.+\cos\left(\frac{2m(d-2)}{n}\pi\right)-\cos\left(\frac{2m(d-1)}{n}\pi\right)\right).
\end{aligned}$$

□

Proposition 3.2.4. Let

$$P\{x_t=k\}=\begin{cases}\frac{1}{2a(|k|+1)(|k|+2)} & k\neq 0, \text{and}\\ 1-\frac{1}{2a} & k=0,\end{cases}$$

where $a \geq \frac{1}{2}$. For $N > 3$,

$$\begin{aligned}
&\mathbb{P}_{i,i+N-1} \\
&= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{3\pi}{N}\right) + \cot\left(\frac{2\pi}{N}\right)\right) - \log N + N\right. \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{6\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right)\right) \\
&\quad\quad \left.\times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right).
\end{aligned}$$

Proof. From Lemma 3.2.6,

$$P_{i,i+n-1}(N) = \frac{1}{aN}\left(\psi\left(\frac{N+1}{N}\right) - \psi(1) + \psi\left(\frac{3}{N}\right) - \psi\left(\frac{2}{N}\right)\right).$$

And Lemma 3.1.4 and Lemma 3.1.5,

$$\begin{aligned}
&\mathbb{P}_{i,i+n-1} \\
&= \frac{1}{aN}\left(\psi\left(\frac{N+1}{N}\right) - \psi(1) + \psi\left(\frac{3}{N}\right) - \psi\left(\frac{2}{N}\right)\right) \\
&= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{3\pi}{N}\right) + \cot\left(\frac{2\pi}{N}\right)\right) - \log N + N\right. \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{6\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right)\right) \\
&\quad\quad \left.\times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right).
\end{aligned}$$

□

Proposition 3.2.5. Let

$$P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$. For $N > 4$,

$$\begin{aligned}
&\mathbb{P}_{i,i+N-2} \\
&= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{4\pi}{N}\right) + \cot\left(\frac{3\pi}{N}\right)\right) + \log N\right. \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-1}\left(-\cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{8\pi m}{N}\right) - \cos\left(\frac{6\pi m}{N}\right)\right) \\
&\quad\quad \left.\times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right).
\end{aligned}$$

Proof. This Proposition is proved by the same way as the proof of Proposition 3.2.4. □

Proposition 3.2.6. Let

$$P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$. For $N > 4$,

$$\begin{aligned} &\mathbb{P}_{i,i+2} \\ &= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{4\pi}{N}\right) + \cot\left(\frac{3\pi}{N}\right) - \cot\left(\frac{\pi}{N}\right)\right) + \log N \right. \\ &\quad + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{8\pi m}{N}\right) - \cos\left(\frac{6\pi m}{N}\right) - \cos\left(\frac{2\pi m}{N}\right)\right) \\ &\quad \left.\times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right). \end{aligned}$$

Proof. This Proposition is proved by the same way as the proof of Proposition 3.2.4. □

Proposition 3.2.7. Let

$$P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$. For $N > 3$,

$$\begin{aligned} &\mathbb{P}_{i,i+1} \\ &= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{3\pi}{N}\right) + \cot\left(\frac{2\pi}{N}\right) - \cot\left(\frac{\pi}{N}\right)\right) - \log N + N \right. \\ &\quad + \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{6\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right) + \cos\left(\frac{2\pi m}{N}\right)\right) \\ &\quad \left.\times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right). \end{aligned}$$

Proof. This Proposition is proved by the same way as the proof of Proposition 3.2.4. □

Proposition 3.2.8. Let $N > 4$. Let

$$P\{x_t = k\} = \begin{cases} \frac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \text{and} \\ 1 - \frac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$. For $2 < d < N-2$,

(I) $\mathbb{P}_{i,i+N-1}$

$$= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{3\pi}{N}\right) + \cot\left(\frac{2\pi}{N}\right)\right) - \log N + N \right.$$
$$+ \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{6\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right)\right)$$
$$\left. \times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right),$$

(II) $\mathbb{P}_{i,i+N-2}$

$$= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{\pi}{N}\right) - \cot\left(\frac{4\pi}{N}\right) + \cot\left(\frac{3\pi}{N}\right)\right) + \log N \right.$$
$$+ \frac{1}{2}\sum_{m=1}^{N-1}\left(-\cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{8\pi m}{N}\right) - \cos\left(\frac{6\pi m}{N}\right)\right)$$
$$\left. \times \log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right),$$

(III) $\mathbb{P}_{i,i+d}$

$$= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{d+2}{N}\pi\right) + \cot\left(\frac{d+1}{N}\pi\right) \right.\right.$$
$$\left. + \cot\left(\frac{d-2}{N}\pi\right) - \cot\left(\frac{d-1}{N}\pi\right)\right)$$
$$\left. + \frac{1}{2}\sum_{m=1}^{N-1}\left(C_d(m)\log\left(2 - 2\cos\left(\frac{2\pi m}{N}\right)\right)\right)\right),$$

(IV) $\mathbb{P}_{i,i+2}$

$$= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{4\pi}{N}\right) + \cot\left(\frac{3\pi}{N}\right) - \cot\left(\frac{\pi}{N}\right)\right) + \log N \right.$$
$$+ \frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{8\pi m}{n}\right) - \cos\left(\frac{6\pi m}{n}\right) - \cos\left(\frac{2\pi m}{n}\right)\right)$$
$$\left. \times \log\left(2 - 2\cos\left(\frac{2\pi m}{n}\right)\right)\right),$$

(V) $\mathbb{P}_{i,i+1}$

$$= \frac{1}{aN}\left(\frac{\pi}{2}\left(-\cot\left(\frac{3\pi}{N}\right) + \cot\left(\frac{2\pi}{N}\right) + \cot\left(\frac{\pi}{N}\right)\right) - \log N + N\right.$$

$$+\frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{6\pi m}{N}\right)-\cos\left(\frac{4\pi m}{N}\right)-\cos\left(\frac{2\pi m}{N}\right)\right)$$
$$\times\log\left(2-2\cos\left(\frac{2\pi m}{N}\right)\right)\Bigg), \text{ and}$$

(VI) $\mathbb{P}_{i,i}$

$$=\frac{1}{aN}\left(\frac{\pi}{2}\left(\cot\left(\frac{\pi}{N}\right)-\cot\left(\frac{2\pi}{N}\right)\right)-\frac{N}{2}\right.$$
$$+\frac{1}{2}\sum_{m=1}^{N-1}\left(\cos\left(\frac{4\pi m}{N}\right)-\cos\left(\frac{2\pi m}{N}\right)\right)\log\left(2-2\cos\left(\frac{2\pi m}{N}\right)\right)\Bigg)$$
$$-\frac{1}{a(n+1)(n+2)}+1-\frac{1}{2a}$$

where

$$\begin{aligned} C_d(m) &= \left(\cos\left(\frac{2m(d+2)}{n}\pi\right)-\cos\left(\frac{2m(d+1)}{n}\pi\right)\right. \\ &\quad \left.+\cos\left(\frac{2m(d-2)}{n}\pi\right)-\cos\left(\frac{2m(d-1)}{n}\pi\right)\right). \end{aligned}$$

Proof. From Proposition 3.2.2 - Proposition 3.2.7, we have Proposition 3.2.8.

□

3.3 Computer simulation

In this section, we show some experimental results about the Hunter vs Rabbit game on a cycle graph. We compute $P\{S_n \mod N = k\}$ by using the gamma function and the class `discrete_distribution` in `C++`. We can show the probability the rabbit is caught and the expected value of the time until the rabbit is caught using this application.

Example 3.3.1. We consider the generalization of the case of [1]. We put

$$P\{X_t = k\} = \begin{cases} \dfrac{1}{2a(|k|+1)(|k|+2)} & (k \in \mathbb{Z}\setminus\{0\}) \\ 1-\dfrac{1}{2a} & (k=0) \end{cases}$$

where $a \geq \frac{1}{2}$. From Remark 1, $\beta = 1$, $c_* = \frac{\pi}{2a}$ and $\varepsilon = \frac{1}{2}$. If $a = 1$, then this is the case in [1].

$$P\{X_t = k\} = \begin{cases} \dfrac{1}{2(|k|+1)(|k|+2)} & (k \in \mathbb{Z}\setminus\{0\}) \\ \dfrac{1}{2} & (k=0). \end{cases}$$

In this case, $\beta = 1$, $c_* = \frac{\pi}{2a}$ and $\varepsilon = \frac{1}{2}$. From (2.47),

$$|E_N| = 2c_{11}.$$

We notice

$$c_{11} = \frac{2^{2+\varepsilon-\beta}\pi^{\varepsilon-\beta}C_*}{c_*^2} = \frac{2^{3/2}\pi^{-1/2}C_*}{c_*^2}.$$

From (2.39), $C_* \le \frac{r_*^{-1/2}}{2}c_*$. So we have

$$c_{11} \le \frac{2^{1/2}}{c_*\pi^{1/2}r_*^{1/2}} = \frac{2^{3/2}a}{\pi^{3/2}r_*^{1/2}}.$$

Therefore,

$$|E_N| \le \frac{2^{5/2}a}{\pi^{3/2}r_*^{1/2}}.$$

From (2.48),

$$\tilde{\Phi}_N \le \frac{2a}{\pi^2}\log N + \frac{2a}{\pi^2}.$$

By (2.43), (2.45) and (2.46),

$$\begin{aligned}\sum_{i=0}^{N-1} p_i^{(N)} &\le 1 + \tilde{\Phi}_N + |E_N| + \frac{1}{1-\rho_*} \\ &\le 1 + \frac{2a}{\pi^2}\log N + \frac{2a}{\pi^2} + \frac{2^{5/2}a}{\pi^{3/2}r_*^{1/2}} + \frac{1}{1-\rho_*}.\end{aligned}$$

By Proposition 2.2.1,

$$\frac{1}{\sum_{i=0}^{N-1} p_i^{(N)}} \ge \frac{1}{1 + \frac{2a}{\pi^2}\log N + \frac{2a}{\pi^2} + \frac{2^{5/2}a}{\pi^{3/2}r_*^{1/2}} + \frac{1}{1-\rho_*}}.$$

Let

$$L(N,a,r,\rho_*) = \frac{1}{1 + \frac{2a}{\pi^2}\log N + \frac{2a}{\pi^2} + \frac{2^{5/2}a}{\pi^{3/2}r_*^{1/2}} + \frac{1}{1-\rho_*}}.$$

Then, r_* and ρ_* is unknown parameter. So, we maximize $L(N, a, r_*, \rho_*)$ about r_* and ρ_*, and we confirm that the average of the probability that the hunter catches the rabbit is bounded below by $\max_{r_*,\rho_*} L(N, a, r_*, \rho_*)$. Therefore,

$$\max_{r_*,\rho_*}\left(\frac{1}{1+\frac{2a}{\pi^2}\log N+\frac{2a}{\pi^2}+\frac{2^{5/2}a}{\pi^{3/2}r_*^{1/2}}+\frac{1}{1-\rho_*}}\right)$$
$$<\frac{1}{\frac{2a}{\pi^2}\log N+\frac{2a}{\pi^2}+\frac{2^{5/2}a}{\pi^2}+2}.$$

Let

$$L(N,a)=\frac{1}{\frac{2a}{\pi^2}\log N+\frac{2a}{\pi^2}+\frac{2^{5/2}a}{\pi^2}+2}.$$

Figure 3.1 shows an experimental result of the probabilities for all initial positions of the rabbit with $N = 100$ and $a = 1$. The horizontal axis is the initial position of the rabbit, and the vertical axis shows the probability the rabbit is caught. The red line in the Figure is a probability that the hunter catches the rabbit for an initial position of the rabbit. The blue line is the average of probabilities that the hunter catches the rabbit. The green line is $L(N, a)$. In this case, the hunter does not move from the initial position 0. As you can see, the average of the probability that the hunter catches the rabbit is bounded below by $L(N, a)$.

In this case, the average of the probability that the hunter catches the rabbit each initial position of the rabbit nearly equals 0.42745, so we have

$$\frac{1}{L(100,1)}\fallingdotseq 3.709,$$

and

$$\frac{1}{L(100,1)}\mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\{\mathcal{R}_n^{(N)}=0\}\right)\fallingdotseq 1.58541.$$

Table 3.3.1 is the experimental results of Example 3.3.1 with $a = 1$ and $N = 100, 500, 1000$. This table shows the asymptotic behavior of (2.11).

Table 3.1: This table is experimental results of Example 3.3.1 with $a = 1$ and $N = 100, 500$ and 1000. A is the average of the probability that the hunter catches the rabbit.

	$1/L(N, a)$	A	$A/L(N.a)$
$N = 100$	3.709	0.42745	1.5854
$N = 500$	4.03514	0.3919	1.5814
$N = 1000$	4.17561	0.3775	1.5763

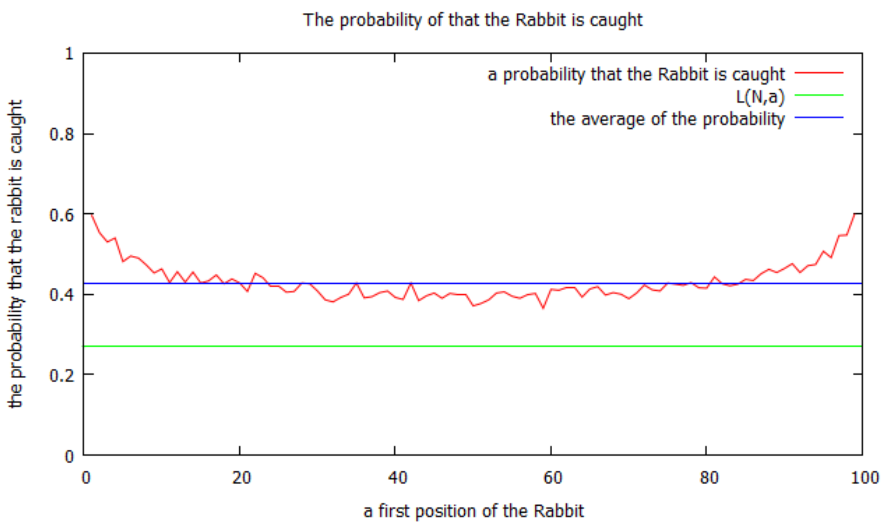

Figure 3.1: This is an experimental result of Example 3.3.1. In this case, $a = 1$. The hunter does not move from an initial position 0. For graph $G = (V, E)$, $V = 100$ and G is cycle graph.

Example 3.3.2. We consider the case of $\beta \in (0,2)$. We put

$$P\{X_t = k\} = \begin{cases} \dfrac{1}{2a|k|^{\beta+1}} & (k \in \mathbb{Z} \setminus \{0\}) \\ 1 - \dfrac{1}{a}\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{k^{\beta+1}} & (k = 0) \end{cases}$$

where $a > \sum_{k=1}^{\infty} \frac{1}{k^{\beta+1}}$. From Remark 2, $c_* = \frac{\pi}{2a\Gamma(\beta+1)\sin(\beta\pi/2)}$ and $\varepsilon = \frac{2-\beta}{2}$. Then, the lower bound of the probability that the hunter catches the rabbit $L_2(N,a)$ is

$$\begin{aligned} &L_2(N,a) \\ &= \begin{cases} \frac{1}{2+2^{2-\beta}\pi^{-\beta-1}a\left(2^{1-\beta/2}+(1-\beta)^{-1}\right)\gamma(\beta+1)\sin(\beta\pi/2)} \\ \qquad\qquad (\beta \in (0,1)) \\ L(N,a) \\ \qquad\qquad (\beta = 1) \\ \frac{1}{2+\left(2^{3-\beta}a\pi^{-\beta-1}(1-(\beta-1)^{-1})\gamma(\beta+1)\sin(\beta\pi/2)+2\right)N^{(\beta-1)/\beta}} \\ \qquad\qquad (\beta \in (1,2)) \end{cases} \end{aligned}$$

Figure 3.2 is an experimental result with $\beta = 1$, $N = 100$ and $a = 1$. In this case, the average of the probability that the hunter catches the rabbit nearly equals 0.43511, so we have

$$\frac{1}{L(100,2)} \fallingdotseq 5.41801,$$

and

$$\frac{1}{L(100,2)}\mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\{\mathcal{R}_n^{(N)} = 0\}\right) \fallingdotseq 2.35743.$$

Table 3.3.2 is the experimental results of Example 3.3.2 with $\beta = 1$, $a = 2$ and $N = 100, 500$ and 1000. This table shows that the value of $A/L(N,a)(> 1)$ is decreasing.

Example 3.3.3. We put

$$P\{X_t = k\} = \begin{cases} \dfrac{1}{3} & (k \in \{-1,0,1\}) \\ 0 & (k \notin \{-1,0,1\}). \end{cases}$$

Table 3.2: This table is experimental results of Example 3.3.2 with $\beta = 1$, $a = 2$ and $N = 100, 500$ and 1000. A is the average of the probability that the hunter catches the rabbit.

	$1/L(N,a)$	A	$A/L(N.a)$
$N = 100$	5.41801	0.43511	2.35743
$N = 500$	6.07029	0.3619	2.19683
$N = 1000$	6.35121	0.3364	2.13655

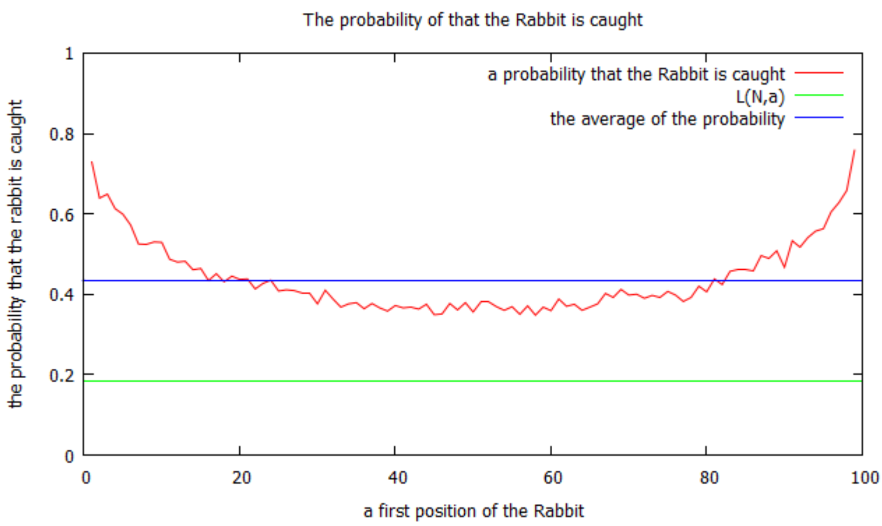

Figure 3.2: This is an experimental result of Example 3.3.2. In this case, $a = 2$. The hunter does not move from an initial position 0. For graph $G = (V, E)$, $V = 100$ and G is cycle graph.

From Remark 2, $\beta = 2$, $c_* = \frac{1}{3}$ and $\varepsilon = 2$. We notice

$$\frac{c_4}{N^{(\beta-1)/\beta}} \leq L'(N)$$

where

$$L'(N) = \frac{1}{(2 + \frac{6}{\pi^2})N^{1/2} + 2}.$$

(We can prove this using in the same way Example 3.3.1.) Figure 3.3 is an experimental result of Example 3.3.3. The green line in Figure 3.3 is $L'(N)$.

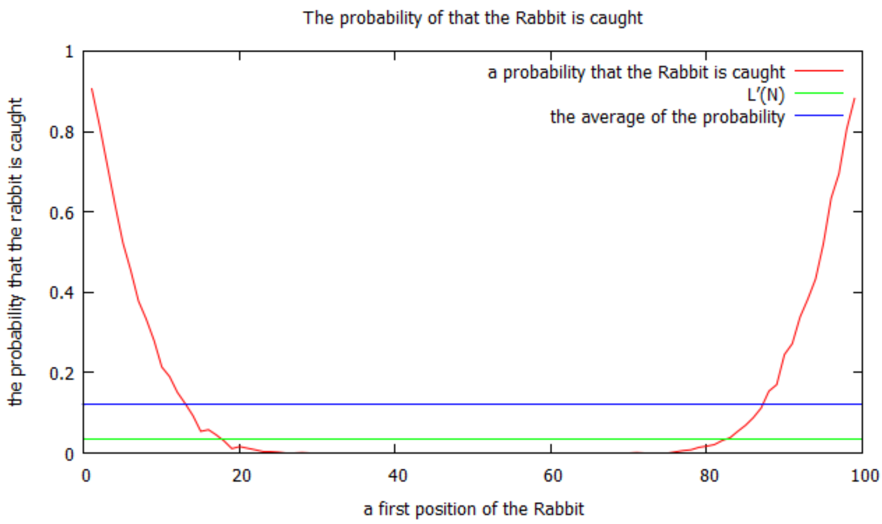

Figure 3.3: This is an experimental result of Example 3.3.3. The hunter does not move from an initial position 0. For graph $G = (V, E)$, $V = 100$ and G is cycle graph.

By those examples, we could have a concrete lower bound of the average of a probability that the hunter catches the rabbit.

Chapter 4

Conclusion

We formalized the Hunter vs Rabbit game using the random walk framework. We generalize a probability distribution of the rabbit's strategy using four assumptions. We have the general lower bound formula of a probability that the rabbit is caught. Let $P\{X_1 = k\} = O(k^{-\beta-1})$. If $\beta \in (0, 1)$, the lower bound of a probability that the hunter catches the rabbit is c_1 where $c_1 > 0$ is a constant. If $\beta = 1$, the lower bound of a probability that the rabbit is caught is $1/(\frac{1}{c_*\pi} \log N + c_2)$ where C_1 and c_* are constants defined by the given strategy. If $\beta \in (1, 2]$, the lower bound of a probability that the rabbit is caught is $c_4 N^{-(\beta-1)/\beta}$ where $c_4 > 0$ is a constant defined by the given strategy.

We show experimental results for three examples of the rabbit strategies. We can confirm our bounds formula, and asymptotic behavior of those bounds

$$\lim_{N\to\infty} \left(\frac{1}{c_*\pi} \log N\right) \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \{\mathcal{R}_n^{(N)} = 0\}\right) = 1.$$

We develop a simulation program to simulate the Hunter vs Rabbit game using C++. In this program, we compute random walks using the digamma function. To simulate the Hunter vs Rabbit game, we have to use the digamma function in a cycle graph. In a cycle graph, calculating of $\mathbb{P}_{i,j}$

is difficult because $\mathbb{P}_{i,j}$ consists of the infinite sum. We show that $\mathbb{P}_{i,j}$ can be denoted by a finite sum using the digamma function. And we can confirm our bounds formulas and asymptotic behavior of those bounds by the results of simulations.

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