Recursive Computation for Subspace Identification Methods

Takei, Yoshinori
Department of Electrical and Electronic Systems Engineering, Kyushu University : Graduate Student

Imai, Jun
Department of Electrical and Electronic Systems Engineering, Kyushu University

Wada, Kiyoshi
Department of Electrical and Electronic Systems Engineering, Kyushu University

https://doi.org/10.15017/1500432

出版情報：九州大学大学院システム情報科学紀要. 5 (1), pp.19-24, 2000-03-24. 九州大学大学院システム情報科学研究院
バージョン：
権利関係：

九州大学大学院
システム情報科学研究科報告
第5巻 第1号 平成12年3月

Research Reports on Information Science and
Electrical Engineering of Kyushu University
Vol.5, No.1, March 2000

Recursive Computation for Subspace Identification Methods

Yoshinori TAKEI* , Jun IMAI** and Kiyoshi WADA**

(Received December 10, 1999)

Abstract: In this paper, we present a recursive computation for Subspace-based State Space System IDentification (4SID) methods. We develope another interpretation of the method by using Schur complement, and propose a recursive formula for the error covariance matrix in the 4SID procedure. In the ordinary MIMO Output-Error State space model identification (MOESP) algorithm, we show the estimate of the extended observability matrix is obtained by Schur complement of input-output data matrix. For the noisy case interpretation of the 4SID procedure is also presented. Finally, we illustrate a numerical example for the proposed algorithm.

Keywords: Recursive computation, Subspace identification, Schur complement

1. Introduction

The 4SID methods have attracted much attention because of being essentially suitable for multivariable system identification. The methods have been demonstrated to perform well in a number of applications, but the properties of these have not been fully analyzed nor understood yet. For applying the methods, no assumptions on structure of realization are needed and any coordinate transformation is allowed for the estimates. This is one reason why many kinds of properties expected for identification procedures have not been clarified yet. The methods are essentially characterized by determination of the extended observability matrix from input-output data by using QR factorization and singular value decomposition. Recursive computations for 4SID methods seem to suffer from the fact that the SVD is not easily updated in a recursive manner. Then we consider the QR factorization on the procedure of the 4SID methods[1),3)], and show a relationship between the least squares residual and the matrix obtained by using Schur complement[3)] in the 4SID procedure. We will propose a recursive formula for the error covariance matrix in the 4SID method.

2. Problem statement

We consider a discrete time linear time-invariant system represented by

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$
$$y_k = Cx_k + Du_k \tag{2}$$

* Dept. of Electrical and Electronic Systems Engineering, Graduate Student

** Dept. of Electrical and Electronic Systems Engineering

where x_k is an n dimensional state vector, u_k m dimensional input and y_k l dimensional output, respectively. The system matrices A, B, C, and D have appropriate dimensions. Furthermore, it is assumed that the model is minimal that is, the system is completely reachable and observable.

Let $i > n$, $N \gg i$, and Hankel matrix $U_{k,i,N}$ of $\{u_k\}$ is defined by

$$U_{k,i,N} := \begin{bmatrix} u_k & u_{k+1} & \cdots & u_{k+N-1} \\ u_{k+1} & u_{k+2} & \cdots & u_{k+N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k+i-1} & u_{k+i} & \cdots & u_{k+N+i-2} \end{bmatrix} \tag{3}$$

and $Y_{k,i,N}$ is defined in a similar way. Then, we have

$$Y_{k,i,N} = \Gamma_i X_{k,N} + H_i U_{k,i,N} \tag{4}$$

from equation (1) and (2) where

$$X_{k,N} := [x_k \ x_{k+1} \ \cdots \ x_{k+N-1}]$$
$$\Gamma_i := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}$$
$$H_i := \begin{bmatrix} D & & O \\ CB & D & \\ \vdots & \ddots & \ddots \\ CA^{i-2}B & \cdots \ CB & D \end{bmatrix}$$

where Γ_i is called the extended observability matrix. We will omit the subscripts for U and Y unless otherwise mentioned. An estimated realization of the system matrices are defined as

$$[A_T, B_T, C_T, D_T] = [TAT^{-1}, TB, CT^{-1}, D]$$

where T is a similarity transformation. Let the input u_k be such that the following condition is satisfied:

$$\text{rank}\begin{bmatrix} U \\ X \end{bmatrix} = mi + n \tag{5}$$

An ordinary MOESP algorithm can be described as follows.

Algorithm A[1)]

step1 Compute QR factorization of a matrix $[U^T\ Y^T]^T$, as in

$$\begin{bmatrix} U \\ Y \end{bmatrix} = \begin{bmatrix} R_{11} & O \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{6}$$

where R_{11} and R_{22} are lower triangular, and $Q_1 Q_1^T = I_{mi}$, $Q_2 Q_2^T = I_{li}$, $Q_1 Q_2^T = O$.

step2 Compute Singular value decomposition of R_{22} as given in (6) of step1, i.e.,

$$R_{22} = [E_n\ E_n^{\perp}] \begin{bmatrix} S_n & O \\ O & S_2 \end{bmatrix} \begin{bmatrix} F_n^T \\ (F_n^{\perp})^T \end{bmatrix} \tag{7}$$

Here the dimension of S_n is equal to the one of the system.

step3 Compute the C_T and A_T from E_n as given in (7) of step2

$$C_T = E_n\,(1{:}l,\ {:}) \tag{8}$$
$$E_n^{(1)} A_T = E_n^{(2)} \tag{9}$$

where $E_n\,(1{:}l,\ {:})$ denotes the first l rows of E_n, $E_n^{(1)}$ is the submatrix composed of the first $(i-1)l$ rows of the matrix E_n and $E_n^{(2)}$ is constructed by the last rows in a similar way.

step4 Solve the following equation for the B_T and D:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_i \end{bmatrix} = \Psi \begin{bmatrix} D \\ B_T \end{bmatrix} \tag{10}$$

where ξ_j, ψ_j, and Ψ are defined by the following relations:

$$[\,\xi_1\ \xi_2\ \cdots\ \xi_i\,] := (E_n^{\perp})^T R_{21} R_{11}^{-1}$$
$$[\,\psi_1\ \psi_2\ \cdots\ \psi_i\,] := (E_n^{\perp})^T$$

$$\Psi := \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_i \\ \psi_2 & & \psi_i & \\ \vdots & \psi_i & & \\ \psi_i & & & O \end{bmatrix} \begin{bmatrix} I_l & O \\ O & E_n^{(1)} \end{bmatrix}$$

Here, the size of ξ_j $(1 \le j \le i)$ and ψ_j $(1 \le j \le i)$ is $(li-n) \times m$ and $(li-n) \times l$, respectively.

3. Another interpretation of 4SID method

In this section, we show another interpretation of the 4SID method via the Schur complement.

Definition of the Schur complement[3)]

Suppose we partition A represented by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Asssume that A_{11} is nonsingular. Then the matrix $S = A_{22} - A_{21} A_{11}^{-1} A_{12}^T$ is called the Schur complement of A_{11} in A.

3.1 Noise-free case

First, we consider the following matrix constructed from input-output data

$$\begin{bmatrix} U \\ Y \end{bmatrix} [\,U^T\ Y^T\,] = \begin{bmatrix} UU^T & UY^T \\ YU^T & YY^T \end{bmatrix} \tag{11}$$

The Schur complement S_1 of UU^T in (11) is represented by

$$S_1 = Y(I - U^T(UU^T)^{-1}U)Y^T$$
$$= Y\Pi_U^{\perp} Y^T \tag{12}$$

where $\Pi_U = U^T(UU^T)^{-1}U$ and $\Pi_U^{\perp} = I - \Pi_U$. Here S_1 in (12) can be rewritten by

$$S_1 = R_{22} Q_2^T Q_2 R_{22}^T$$
$$= R_{22} R_{22}^T \tag{13}$$

using (4) and (6). From results of (12) and (13), we have

$$Y\Pi_U^{\perp} Y^T = R_{22} R_{22}^T \tag{14}$$

Therefore we see that the matrix E_n in equation (7) is yielded by computing singular value decomposition of $Y\Pi_U^{\perp} Y^T$.

3.2 Noisy case

It is assumed that the output of the system is perturbed by the noise v_k, where the input u_k and the noise v_k are independent. Then the output equation reads

$$z_k = y_k + v_k \tag{15}$$

Hankel matrices of $\{z_k\}$ and $\{v_k\}$ is represented by Z and V ,then we have

$$Z = Y + V \tag{16}$$

Using a matrix $\widehat{R}_{22}$ yielded by computing the QR factorization of a matrix $\left[U^T \ Z^T\right]^T$ in a similar way as in the noise-free case, the following relation is obtained from equation (14).

$$\frac{1}{N}\widehat{R}_{22}\widehat{R}_{22}^T \Rightarrow \frac{1}{N}Y\Pi_U^{\perp}Y^T + R_{vv} \tag{17}$$

where R_{vv} is a covariance matrix of v_k, and the notation A $\Rightarrow$ B means that A asymptotically converges to B if the number of data N tends to infinity. From equation (17) the estimate of the extended observability matrix is not obtained by computing singular value decomposition of $Z\Pi_U^{\perp}Z^T$ asymptotically. Therefore we introduce an instrumental variable Φ, satisfying the following conditions:

$$\lim_{N\to\infty}\frac{1}{N}\Phi V^T = O, \tag{18}$$

$$\mathrm{rank}\begin{bmatrix} U \\ \Phi \end{bmatrix} = mi + p \tag{19}$$

where the size of Φ is $p \times N$ and $p > n$.

A 4SID procedure using the instrumental variable Φ is described as follows.

Algorithm B[2)]

step1$'$ Compute QR factorization of a matrix consisted of input-output data U, Z and an instrumental variable Φ, i.e.

$$\begin{bmatrix} U \\ Y \\ \Phi \end{bmatrix} = \begin{bmatrix} \widehat{R}_{11} & & O \\ \widehat{R}_{21} & \widehat{R}_{22} & \\ \widehat{R}_{31} & \widehat{R}_{32} & \widehat{R}_{33} \end{bmatrix}\begin{bmatrix} \widehat{Q}_1 \\ \widehat{Q}_2 \\ \widehat{Q}_3 \end{bmatrix} \tag{20}$$

step2$'$ Compute Singular value decomposition of $\widehat{R}_{22}\widehat{R}_{32}^T$ as given in (20) of step1$'$, as in

$$\widehat{R}_{22}\widehat{R}_{32}^T = \left[E_n \ E_n^{\perp}\right]\begin{bmatrix} S_n & O \\ O & S_2 \end{bmatrix}\begin{bmatrix} F_n^T \\ (F_n^{\perp})^T \end{bmatrix} \tag{21}$$

step3$'$ Using E_n, $E_n^{\perp}$, $\widehat{R}_{11}$, and $\widehat{R}_{21}$ yielded in step1$'$ and step2$'$, compute the quadruple of the system matrices $[A,\ B,\ C,\ D]$ in a similar way in *Algorithm A*.

We consider a matrix $\widehat{Z}$ which is a linear combination of U and Φ. This is represented by

$$\begin{aligned}\widehat{Z} &:= L_1U + L_2\Phi \\ &= \left[L_1 \ L_2\right]\begin{bmatrix} U \\ \Phi \end{bmatrix} \\ &= L\Omega \end{aligned} \tag{22}$$

where $L := \left[L_1 \ L_2\right]$, $\Omega := \left[U^T \ \Phi^T\right]^T$.

Since the condition (19) is satisfied, $\widehat{L}$ which minimize $\| Z - \widehat{Z} \|_F^2$ for Ω exists uniquely. Here $\| \cdot \|_F$ denotes the Frobenius norm. Then $\widehat{L}$ is represented by

$$\widehat{L} = Z\Omega^T(\Omega\Omega^T)^{-1}. \tag{23}$$

From the result of (23), we substitute $\widehat{L}$ for L in (22) to obtain

$$\begin{aligned}\widehat{Z} &= Z\Omega^T(\Omega\Omega^T)^{-1}\Omega \\ &= Z\Pi_{\Omega}\end{aligned} \tag{24}$$

where $\Pi_{\Omega} = \Omega^T(\Omega\Omega^T)^{-1}\Omega$. We consider a matrix $\left[U^T \ \widehat{Z}^T\right]^T$. From the assumption, $\widehat{Z}$ and V are uncorrelated. Therefore we have

$$\lim_{N\to\infty}\frac{1}{N}V\left[U^T \ \widehat{Z}^T\right] = O \tag{25}$$

Suppose the following matrix as in (11)

$$\begin{aligned}\begin{bmatrix} U \\ \widehat{Z} \end{bmatrix}\left[U^T \ \widehat{Z}^T\right] &= \begin{bmatrix} U \\ Z \end{bmatrix}\Pi_{\Omega}\left[U^T \ Z^T\right] \\ &= \begin{bmatrix} UU^T & UZ^T \\ ZU^T & Z\Pi_{\Omega}Z^T \end{bmatrix}\end{aligned} \tag{26}$$

where $U\Pi_{\Omega} = U$. Thus the Schur complement S_3 of UU^T in (26) is represented by

$$\begin{aligned}S_3 &= Z(\Pi_{\Omega} - U^T(UU^T)^{-1}U)Z^T \\ &= \widehat{Z}\Pi_U^{\perp}\widehat{Z}^T.\end{aligned} \tag{27}$$

Here

$$\begin{aligned}\Pi_{\Omega} &= \Omega^T(\Omega\Omega^T)^{-1}\Omega \\ &= \Pi_U + \Pi_U^{\perp}\Phi^T(\Phi\Pi_U^{\perp}\Phi^T)^{-1}\Phi\Pi_U^{\perp}.\end{aligned} \tag{28}$$

Then we have

$$S_3 = Z\Pi_U^{\perp}\Phi^T(\Phi\Pi_U^{\perp}\Phi^T)^{-1}\Phi\Pi_U^{\perp}Z^T. \tag{29}$$

From the result of (20), the equation (27) can be rewritten as

$$S_3 = \widehat{R}_{22}\widehat{R}_{32}^T\Lambda^{-1}\widehat{R}_{32}\widehat{R}_{22}^T \tag{30}$$

where $\Lambda = \widehat{R}_{32}\widehat{R}_{32}^T + \widehat{R}_{33}\widehat{R}_{33}^T$. S_3 converges for $N \to \infty$, that is,

$$\frac{1}{N^2}S_3 \Rightarrow$$
$$\Gamma_i \frac{1}{N} X\Pi_U^{\perp}\Phi^T(\Phi\Pi_U^{\perp}\Phi^T)^{-1}\frac{1}{N}\Phi\Pi_U^{\perp}X^T\Gamma_i^T. \quad (31)$$

4. Recursive computation

In this section, we will show a relationship between the least squares residual and the matrix yielded by the Schur complement, and derive a recursive formula for an error covariance matrix.

4.1 Noise-free case

We have obtained the following relation

$$Y\Pi_U^{\perp}Y^T = R_{22}R_{22}^T$$

The left hand side of (14) can be rewritten as

$$Y\Pi_U^{\perp}Y^T = Y\Pi_U^{\perp}(Y\Pi_U^{\perp})^T \quad (32)$$

Then we consider the following matrix

$$\begin{aligned} Y\Pi_U^{\perp} &= Y - YU^T(UU^T)^{-1}U \\ &= Y - \widehat{G}_N U \end{aligned} \quad (33)$$

where $\widehat{G}$ is defined as

$$\widehat{G}_N = YU^T(UU^T)^{-1}, \quad (34)$$

and we denote an error $Y - \widehat{G}_N U$ by $\widehat{E}$.

We can regard a matrix $\widehat{E}$ as a least squares residual, then $Y\Pi_U^{\perp}Y^T$ a squared sum of residuals denoted by an error covariance matrix $\widehat{E}\widehat{E}^T$. The nomal equation can be represented by

$$(Y - \widehat{G}_N U)U^T = 0 \quad (35)$$

Equation (32) can be rewritten as

$$\begin{aligned} \widehat{E}\widehat{E}^T &= (Y - \widehat{G}_N U)(Y - \widehat{G}_N U)^T \\ &= YY^T - \widehat{G}UY^T \end{aligned} \quad (36)$$

From the results of (35) and (36), the extended normal equation can be represented by

$$\begin{bmatrix} -\widehat{G}_N & I \end{bmatrix} \begin{bmatrix} UU^T & UY^T \\ YU^T & YY^T \end{bmatrix} = \begin{bmatrix} O & \widehat{E}_N\widehat{E}_N^T \end{bmatrix} \quad (37)$$

The submatrix of the Hankel matrix $U_{k,i,N+1}$ is defined as

$$u_i(k+N) := \begin{bmatrix} u_{k+N}^T & u_{k+N+1}^T & \cdots & u_{k+N+i-1}^T \end{bmatrix}^T \quad (38)$$

and y_i is defined in a similar way. Then the Hankel matrices $U_{k,i,N+1}$ and $Y_{k,i,N+1}$ are partitioned such as

$$U_{k,i,N+1} = \begin{bmatrix} U_{k,i,N} \mid u_i(k+N) \end{bmatrix} \quad (39)$$
$$Y_{k,i,N+1} = \begin{bmatrix} Y_{k,i,N} \mid y_i(k+N) \end{bmatrix} \quad (40)$$

For brevity we denote $U_{k,i,N}$, $Y_{k,i,N}$ by U_N, Y_N, and $u_i(k+N)$, $y_i(k+N)$ by u_i, y_i, respectively. We consider a matrix consisted of the Hankel matrix U_{N+1} and Y_{N+1}. The following relation is obtained from (40).

$$\begin{aligned} &\begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} \\ &= \begin{bmatrix} U_N \\ Y_N \end{bmatrix} \begin{bmatrix} U_N^T & Y_N^T \end{bmatrix} + \begin{bmatrix} u_i \\ y_i \end{bmatrix} \begin{bmatrix} u_i^T & y_i^T \end{bmatrix} \end{aligned} \quad (41)$$

Equation (41) is mutiplied by a matrix $\begin{bmatrix} -\widehat{G}_N & I \end{bmatrix}$, and then the following is obtained

$$\begin{aligned} &\begin{bmatrix} -\widehat{G}_N & I \end{bmatrix} \begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} \\ &= \begin{bmatrix} O & \widehat{E}_N\widehat{E}_N^T \end{bmatrix} + e_i \begin{bmatrix} u_i^T & y_i^T \end{bmatrix} \end{aligned} \quad (42)$$

where

$$e_i = y_i - \widehat{G}_N u_i \quad (43)$$

We introduce the vectors $\boldsymbol{k}_{N+1}$ and $\widehat{y}_i(k+N)$ which satisfy the following equation;

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{k}_{N+1}^T & O \end{bmatrix} \begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} \\ &= \begin{bmatrix} u_i^T & \widehat{y}_i(k+N)^T \end{bmatrix} \end{aligned} \quad (44)$$

Equations (42) and (44) multiplied by e_i give the following:

$$\begin{aligned} &\begin{bmatrix} -\widehat{G}_N - e_i\boldsymbol{k}_{N+1}^T & I \end{bmatrix} \begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} \\ &= \begin{bmatrix} O & \widehat{E}_N\widehat{E}_N^T \end{bmatrix} + \begin{bmatrix} O & e_i\{y_i - \widehat{y}_i(k+N)\}^T \end{bmatrix} \end{aligned} \quad (45)$$

Equation (37) leads to equation (46), if the number of data is $N+1$.

$$\begin{aligned} &\begin{bmatrix} -\widehat{G}_{N+1} & I \end{bmatrix} \begin{bmatrix} U_{N+1}U_{N+1}^T & U_{N+1}Y_{N+1}^T \\ Y_{N+1}U_{N+1}^T & Y_{N+1}Y_{N+1}^T \end{bmatrix} \\ &= \begin{bmatrix} O & \widehat{E}_{N+1}\widehat{E}_{N+1}^T \end{bmatrix} \end{aligned} \quad (46)$$

Comparing equation (45) with (46), an error covariance $\widehat{E}_{N+1}\widehat{E}_{N+1}^T$ and estimates $\widehat{G}_{N+1}$ are obtained as

$$\widehat{G}_{N+1} = \widehat{G}_N + e_i\boldsymbol{k}_{N+1}^T \quad (47)$$
$$\widehat{E}_{N+1}\widehat{E}_{N+1}^T = \widehat{E}_N\widehat{E}_N^T + e_i\{y_i - \widehat{y}_i(k+N)\}^T \quad (48)$$

From equation (44), k_{N+1} and $\widehat{y}_i(k+N)$ are obtained as follows

$$\boldsymbol{k}_{N+1}^T = u_i^T P_{N+1} \quad (49)$$
$$\widehat{y}_i(k+N) = \widehat{G}_{N+1}u_i \quad (50)$$

where the matrix P_{N+1} is defined as

$$P_{N+1} = (U_{N+1}U_{N+1}^T)^{-1}. \tag{51}$$

Using the matrix inversion lemma, P_{N+1} can be represented by

$$P_{N+1} = P_N - P_N u_i u_i^T P_N/\alpha \tag{52}$$

where α is defined by

$$\alpha = 1 + u_i^T P_N u_i. \tag{53}$$

Using equations (49) and (52), the relation between $\widehat{G}_N$ and $\widehat{G}_{N+1}$ can be represented by

$$\begin{aligned}\widehat{G}_{N+1} &= \widehat{G}_N + e_i u_i^T P_{N+1}\\ &= \widehat{G}_N + e_i u_i^T P_N/\alpha.\end{aligned} \tag{54}$$

Therefore, a recursive formula is summarized as follows;

$$\begin{aligned}\alpha &= 1 + u_i^T P_N u_i\\ e_i &= y_i - \widehat{G}_N u_i\\ \widehat{G}_{N+1} &= \widehat{G}_N + e_i u_i^T P_N/\alpha\\ \widehat{E}_{N+1}\widehat{E}_{N+1}^T &= \widehat{E}_N\widehat{E}_N^T + e_i e_i^T/\alpha\\ P_{N+1} &= P_N - P_N u_i u_i^T P_N/\alpha\\ \mathcal{R}_{N+1} &= \mathcal{R}_N + \frac{e_i e_i^T}{1 + u_i^T P_N u_i}\end{aligned} \tag{55}$$

where the error covariance $\mathcal{R}_N = \widehat{E}_N\widehat{E}_N^T$.

4.2 Noisy case

We have obtained the following result

$$\begin{aligned}\widehat{Z}\Pi_U^\perp\widehat{Z}^T &= \widehat{Z}\Pi_U^\perp(\widehat{Z}\Pi_U^\perp)^T\\ &= \widehat{R}_{22}\widehat{R}_{32}^T\Lambda^{-1}\widehat{R}_{32}\widehat{R}_{22}^T\end{aligned} \tag{56}$$

We consider a recursive formula of the matix $\widehat{Z}\Pi_U^\perp\widehat{Z}^T$. The matrix $\widehat{Z}\Pi_U^\perp$ can be rewritten as

$$\begin{aligned}\widehat{Z}\Pi_U^\perp &= Z\Pi_\Omega\Pi_U^\perp\\ &= Z\Pi_\Omega - Z\Pi_U\\ &= \widehat{Z} - \widehat{G}_N U\\ &= \widehat{G}_N^*\Omega - \widehat{G}_N U\\ &= Y - \widehat{G}_N U - (Y - \widehat{G}_N^*\Omega)\\ &= \widehat{E}_N - \widehat{E}_N^*\end{aligned} \tag{57}$$

where

$$\widehat{G}_N^* = Y\Omega^T(\Omega\Omega^T)^{-1} \tag{58}$$
$$\widehat{E}_N^* = Y - \widehat{G}_N^*\Omega \tag{59}$$

We denote $\widehat{Z}\Pi_U^\perp$ by $\widetilde{E}_N$, then $\widehat{Z}\Pi_U^\perp\widehat{Z}^T$ is represented by $\widetilde{E}_N\widetilde{E}_N^T$. From equation (57), the following equation is obtained.

$$\begin{aligned}\widetilde{E}_N\widetilde{E}_N^T &= (\widehat{E}_N - \widehat{E}_N^*)(\widehat{E}_N - \widehat{E}_N^*)^T\\ &= \widehat{E}_N\widehat{E}_N^T - \widehat{E}_N^*\widehat{E}_N^{*T}\end{aligned} \tag{60}$$

where

$$\begin{aligned}\widehat{E}_N\widehat{E}_N^{*T} &= E_N^*\widehat{E}_N^T\\ &= \widehat{E}_N^*\widehat{E}_N^{*T}\end{aligned} \tag{61}$$

Therefore we see that $\widetilde{E}_N\widetilde{E}_N^T$ is yeilded by computing $\widehat{E}_N\widehat{E}_N^T$ and $\widehat{E}_N^*\widehat{E}_N^{*T}$.
Here the matrix Ω is defined as the following equation

$$\Omega = \left[\,\omega_i(k)\;\omega_i(k+1)\;\cdots\;\omega_i(k+N-1)\,\right] \tag{62}$$

where

$$\omega_i(k) = \left[\,u_i(k)^T\;\phi_k^T\;\phi_{k+1}^T\;\cdots\;\phi_{k+i-1}^T\,\right]^T \tag{63}$$

and ϕ is an element of the instrumental variable Φ. $\widehat{E}_N^*$, $\widehat{G}_N^*$ and Ω are corresponded $\widehat{E}_N$, $\widehat{G}_N$ and U, then a recursive formula of the error covariance matrix $\widehat{E}_N^*\widehat{E}_N^{*T}$ is summarized as follows;

$$\begin{aligned}\beta &= 1 + \omega_i^T Q_N \omega_i\\ e_i^* &= y_i - \widehat{G}_N^*\omega_i\\ \widehat{G}_{N+1}^* &= \widehat{G}_N^* + e_i^*\omega_i^T Q_N/\beta\\ \widehat{E}_{N+1}^*\widehat{E}_{N+1}^{*T} &= \widehat{E}_N^*\widehat{E}_N^{*T} + e_i^* e_i^{*T}/\beta\\ Q_{N+1} &= Q_N - Q_N\omega_i\omega_i^T Q_N/\beta\\ \mathcal{R}_{N+1}^* &= \mathcal{R}_N^* + \frac{e_i^* e_i^{*T}}{1 + \omega_i^T Q_N\omega_i}\end{aligned} \tag{64}$$

where $\mathcal{R}_N^* = \widehat{E}_N^*\widehat{E}_N^{*T}$, $Q_N = (\Omega_N\Omega_N^T)^{-1}$. Therefore a recursive formula of $\widetilde{E}\widetilde{E}^T$ is obtained

$$\widetilde{\mathcal{R}}_{N+1} = \mathcal{R}_{N+1} - \mathcal{R}_{N+1}^* \tag{65}$$

where $\widetilde{\mathcal{R}}_N = \widetilde{E}_N\widetilde{E}_N^T$

4.3 Numerical example

In this section, we applied the recursive method presented in this paper to identify the following discrete time linear system;

$$\begin{aligned}x_{k+1} &= \begin{bmatrix}0.8 & -0.4 & 0.2\\ 0 & 0.3 & -0.5\\ 0 & 0 & 0.5\end{bmatrix}x_k + \begin{bmatrix}0 & 0\\ 0 & -0.6\\ 0.5 & 0\end{bmatrix}u_k\\ y_k &= \begin{bmatrix}0.5 & 0.5 & 0\\ 0 & 0 & 1\end{bmatrix}x_k + \begin{bmatrix}0.05\\ 0.02\end{bmatrix}v_k\end{aligned} \tag{66}$$

where u_k is constructed by 2 inputs, y_k by 2 outputs and x_k is 3 state vector, respectively. In order to use our recursive procedure we generated 1500 samples of input-output data and took the input u_k equal to a zero-mean white noise of unit variance, the noise v_k is as in a similar way. The experiment was conducted with MATLAB package. Using the recursive formula for noisy case as in equation (64), we estimated a sequence of state space models. The instrumental variable is selected as

$$\Phi = U_{k-i,i,N} \tag{67}$$

and an auxiliary order $i = 7$. To start up the recursive operation, we used the first 50 samples to produce an estimate as initial value in off-line. The initially obtained factors $\mathcal{R}_N$, $\mathcal{R}_N^*$, etc are updated for 1500 steps. Computing the SVD for the error covariance matrix $\widetilde{\mathcal{R}}_{N+1}$, the system of matrices A_T, B_T, C_T and D_T is obtained. In **Fig.1** and **Fig.2**, we denote the ith largest eigenvalues of A_T by λ_i, and the singular value of $\widetilde{\mathcal{R}}$ by σ_i in a similar way.

We have shown the behaviour of singular values of $\widetilde{\mathcal{R}}$ in equation (65) in **Fig.1**. We can see that the order of the system $n = 3$ from the result in **Fig.1**. **Fig.2** has shown that eigenvalues of A_T asymptotically tend to the true values for increasing step number. In **Fig.3** and **Fig.4**, we have shown that step responses for two elements of output y_k of the system and estimated model. The system of matrices estimated from input-output measurements for 1500 steps is used in this simulation. Then the quadruple of the system matrices is obtained as in equation (68).

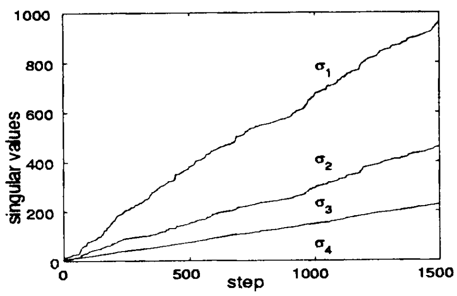

Fig.1 Singular values of $\widetilde{\mathcal{R}}$

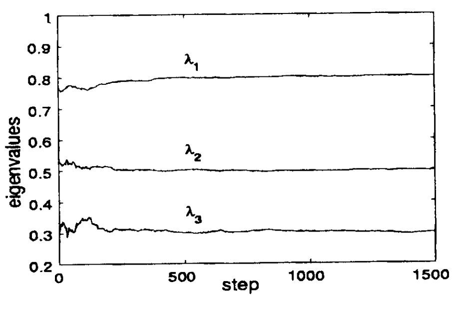

Fig.2 Eigenvalues of A_T

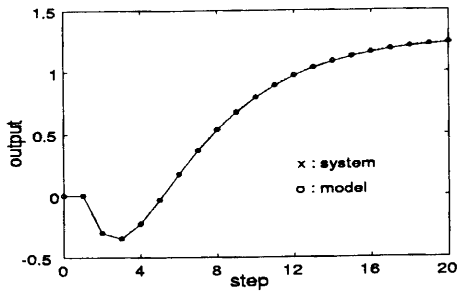

Fig.3 First element of output y_k

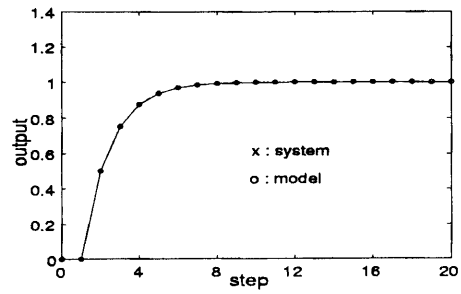

Fig.4 Second element of output y_k

$$A_T = \begin{bmatrix} 0.7289 & -0.0753 & -0.1350 \\ -0.3039 & 0.7323 & 0.4244 \\ -0.0564 & -0.1980 & 0.1367 \end{bmatrix} \tag{68-a}$$

$$B_T = \begin{bmatrix} 0.5460 & 0.1199 \\ 0.2677 & 0.0341 \\ 0.1184 & -0.3412 \end{bmatrix} \tag{68-b}$$

$$C_T = \begin{bmatrix} 0.0532 & -0.4810 & 0.8465 \\ 0.6441 & 0.4361 & 0.2692 \end{bmatrix} \tag{68-c}$$

$$D_T = \begin{bmatrix} 0.0011 & 0.0003 \\ 0.0007 & 0.0000 \end{bmatrix} \tag{68-d}$$

5. Conclusion

In this paper, we focused on QR factorization in the 4SID procedure, and showed another interpretation of the 4SID method by using the Schur complement for the input-output data matrix. We have also proposed a recursive formula for the error covariance matrix in the 4SID method. The results are useful for analysing properties of parameters obtained by 4SID methods.

References

1) M. Verhaegen and Dewilde, "Subspace Model Identification. Part1. the Output-Error State-Space Model Identification Class of Algorithms," *Int. J. Control*, vol56-5, pp1187-1210, 1992

2) M. Verhaegen, "Subspace Model Identification. Part3. analysis of the ordinary output-error state space model identification algorithm, " *Int. J. Control*, vol58-3, pp555-586, 1993

3) G. H. Golub and C. F. Van Loan, "Matrix Computations, " *THE JOHNS HOPKINS UNIVERSITY PRESS*, 1991