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Abstract: This paper is concerned with the stability of a parallel flow of the compressible Navier-Stokes equation in a cylindrical domain. The spectrum of the linearized operator is analyzed for the purpose of the study of the nonlinear stability. It is shown that if the Reynolds and Mach numbers are sufficiently small, then the linearized semigroup is decomposed into two parts; one behaves like a solution of a one dimensional heat equation as time goes to infinity and the other one decays exponentially. Some estimates related to the spectral projections are established, which will also be useful for the study of the nonlinear problem.

Keywords: Compressible Navier-Stokes equation, parallel flow, cylindrical domain, linearized semigroup, asymptotic behavior

1   Introduction

We consider the initial boundary value problem for the equations for a barotropic motion of viscous compressible fluid

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div} v + \nabla p(\rho) = \rho g, \tag{1.2}$$

$$v \mid_{\partial D} = 0, \tag{1.3}$$

$$(\rho, v) \mid_{t=0} = (\rho_0, v_0) \tag{1.4}$$

in a cylindrical domain $\Omega = D \times \mathbf{R}$:

$$\Omega = \{x = (x', x_3); \, x' = (x_1, x_2) \in D, x_3 \in \mathbf{R}\}.$$

Here D is a bounded and connected domain in $\mathbf{R}^2$ with a smooth boundary ∂D; $\rho = \rho(x,t)$ and $v = {}^T(v^1(x,t), v^2(x,t), v^3(x,t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively; $P(\rho)$ is the pressure that is a smooth function of ρ and satisfies

$$p'(\rho_*) > 0$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that satisfy

$$\mu > 0, \quad \tfrac{2}{3}\mu + \mu' \geq 0;$$

and g is an external force of the form $g = {}^T(g^1(x'), g^2(x'), g^3(x'))$ with g^1 and g^2 satisfying

$$\big(g^1(x'), g^2(x')\big) = \big(\partial_{x_1}\Phi(x'), \partial_{x_2}\Phi(x')\big),$$

where Φ and g^3 are given smooth functions of x'. Here and in what follows ${}^T\!\cdot$ stands for the transposition.

It is known that problem (1.1)-(1.3) has the stationary solution $\overline{u}_s = {}^T(\overline{\rho}_s(x'), \overline{v}_s(x'))$; $\overline{\rho}_s$ is determinated by

$$\begin{cases} \text{Const.} - \Phi(x') = \int_{\rho_*}^{\overline{\rho}_s} \frac{p'(\eta)}{\eta} d\eta, \\ \int_D \overline{\rho}_s - \rho_* dx' = 0; \end{cases}$$

and $\overline{v}_s$ takes the form

$$\overline{v}_s = {}^T\big(0, 0, \overline{v}_s^3(x')\big),$$

where $\overline{v}_s^3(x')$ is the solution of

$$\begin{cases} -\mu\Delta'\overline{v}_s^3 = \overline{\rho}_s g^3, \\ \overline{v}_s^3 \mid_{\partial D} = 0. \end{cases}$$

Here $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$. The stationary solution $\overline{u}_s$ represents a parallel flow in Ω.

We are interested in the large time behavior of solutions to problem (1.1)-(1.4) when the initial value $(\rho, v) \mid_{t=0} = (\rho_0, v_0)$ is sufficiently close to the stationary solution $\overline{u}_s = {}^T(\overline{\rho}_s, \overline{v}_s)$. In [1] the decay estimates of the linearized semigroup for (1.1) - (1.4) were established. In this paper we study the spectral properties of the linearized semigroup in more detail, which play an important role in the analysis of the nonlinear problem.

As for the asymptotic behavior of multidimensional compressible Navier-Stokes equations on unbounded domains, a lot of results have been obtained. See, e.g., [5, 6, 11, 13, 14, 15, 16, 17, 19] and references therein.

For the stability of parallel flow, detailed descriptions of large time behavior of disturbances have been obtained in the case of an n dimensional infinite layer $\mathbf{R}^{n-1} \times (0,1) = \{x = (x_h, x_n); \, x_h = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}, 0 < x_n < 1\}$. It

was proved in [10] that asymptotic behavior of solutions of the linearized problem is described by an $n-1$ dimensional linear heat equation, if Reynolds and Mach numbers are sufficiently small. The nonlinear problem was then studied in [9]; it was shown that if Reynolds and Mach numbers are sufficiently small, then the parallel flow is stable under sufficiently small initial disturbances in some Sobolev space. In the case of $n \geq 3$, the disturbance $u(t)$ behaves like a solution of an $n-1$ dimensional linear heat equation as $t \to \infty$, while, in the case of $n = 2$, $u(t)$ behaves like a solution of a one dimensional viscous Burgers equation. See also [2, 3, 4] for the stability of time periodic parallel flow.

As for the case of the cylindrical domain Ω, Iooss and Padula [7] studied the linearized stability of a stationary parallel flow under the periodic boundary condition in x_3 with vanishing average condition on the basic period cell for the density-disturbance and proved that if the Reynolds number is suitably small, then the linearized semigroup decays exponentially as $t \to \infty$.

On the other hand, stability under the non-periodic but local disturbances (i.e., belonging to some L^2 Sobolev space on Ω) was studied in [12] in the case of the motionless state $\widetilde{u}_s = {}^T(\rho_*, 0)$. It was shown that the disturbance decays in $L^2(\Omega)$ in the order $t^{-\frac{1}{4}}$ and its asymptotic leading part is given by a solution of a one dimensional linear heat equation. (See also [8] for the analysis in $L^p(\Omega)$.)

The linearized stability of parallel flow under non-periodic local disturbances on Ω was then studied in [1]. It was shown that the linearized semigroup e^{-tL} satisfies the decay estimate

$$\|\partial_{x'}^k \partial_{x_3}^l e^{-tL} u_0\|_{L^2(\Omega)} \leq C\big\{(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}: L^2(D))} + e^{-dt}\|u_0\|_{H^1(\Omega)}\big\} \qquad (1.5)$$

for $t \geq 0$ and $0 \leq k + l \leq 1$, provided that the Reynolds and Mach numbers are sufficiently small and that $\overline{\rho}_s$ is sufficiently close to ρ_*. In view of the argument in [9], estimate (1.5) is not enough to show the global in time solvability of the nonlinear problem. The purpose of this paper is to derive more detailed spectral information of the linearized operator which will be available for the nonlinear analysis.

The main results of this paper are summarized as follows. We consider the linearized problem whose non-dimensional form is written as

$$\partial_t u + Lu = 0, \quad u \mid_{t=0} = u_0. \qquad (1.6)$$

Here $u = {}^T(\phi, w)$ is the unknown; and $u_0 = {}^T(\phi_0, w_0)$ is a given initial value; and L denotes the linearized operator on $L^2(\widetilde{\Omega})$ defined by

$$L = \begin{pmatrix} v_s \cdot \nabla & \gamma^2 \mathrm{div}(\rho_s \cdot) \\ \nabla\left(\frac{\widetilde{P}'(\rho_s)}{\gamma^2 \rho_s}\cdot\right) + \frac{\nu \Delta v_s}{\gamma^2 \rho_s^2} & -\frac{\nu}{\rho_s}\Delta I_3 - \frac{\nu+\nu'}{\rho_s}\nabla\mathrm{div} + v_s \cdot \nabla + {}^T(\nabla v_s) \end{pmatrix},$$

with domain

$$D(L) = \{u = {}^T(\phi, w) \in L^2(\widetilde{\Omega}); \ w \in H_0^1(\widetilde{\Omega}), \ Lu \in L^2(\widetilde{\Omega})\},$$

where $\widetilde{\Omega}$, $\widetilde{D}$, ρ_s, v_s and $\widetilde{P}(\rho_s)$ are the non-dimensional forms of Ω, D, $\overline{\rho}_s$, $\overline{v}_s$ and $p(\overline{\rho}_s)$ respectively; I_3 denotes the 3×3 identity matrix; ν, ν' and γ are some positive constants.

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We will show that there exists a bounded projection P_0 satisfying $P_0 e^{-tL} = e^{-tL} P_0$ such that if Reynolds and Mach numbers are sufficiently small, then

$$\|e^{-tL} P_0 u_0 - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{3}{4}} \|u_0\|_{L^1(\Omega)}. \tag{1.7}$$

Here $u^{(0)}$ is some function of x', and σ is a function of x_3 and t; and σ is a solution of one dimensional linear heat equation

$$\begin{cases} \partial_t \sigma - \kappa_1 \partial_{x_3}^2 \sigma + \kappa_0 \partial_{x_3} \sigma = 0, \\ \sigma \mid_{t=0} = \int_0^1 \phi_0(x', x_3) dx_3 \end{cases} \tag{1.8}$$

with some constants $\kappa_0 \in \mathbf{R}$ and $\kappa_1 > 0$. Some estimates for operators related to P_0 are established; and, as in [2, 4], we will give a factorization of $e^{-tL} P_0$ which is useful in the study of nonlinear problem. As for the $I - P_0$ part of e^{-tL}, we will establish the exponential decay estimate

$$\|e^{-tL}(I - P_0) u_0\|_{H^1(\Omega)} \leq C e^{-dt} \{\|u_0\|_{H^1(\Omega) \times \widetilde{H}^1(\Omega)} + t^{-\frac{1}{2}} \|w_0\|_{L^2(\Omega)}\} \tag{1.9}$$

for a positive constant d. Here $\widetilde{H}^1(\Omega)$ is the set of all locally H^1 functions in $L^2(\Omega)$ whose tangential derivatives near $\partial\Omega$ belong to $L^2(\Omega)$.

The linear problem (1.6) will be investigated through the Fourier transform $\mathcal{F}$ in $x_3 \in \mathbf{R}$ that leads to the problem on D, written in the form,

$$\partial_t u + L_\xi u = 0, \quad u \mid_{t=0} = u_0,$$

where $\xi \in \mathbf{R}$ denotes the dual variable. The operator L_ξ has different properties $|\xi| \leq r_0$ and $|\xi| > r_0$, where r_0 is a positive number sufficiently small.

The spectrum of $-L_\xi$ for $|\xi| \leq r_0$ can be regarded as a perturbation from the one with $\xi = 0$, and we will show that the spectrum near the origin is given by a simple eigenvalue $\lambda_0(\xi) = -i\kappa_0 \xi - \kappa_1 \xi^2 + \mathcal{O}(|\xi|^3)$ as $|\xi| \to 0$. Furthermore, we will establish the boundedness of the eigenprojection $\Pi(\xi)$ for the eigenvalue $\lambda_0(\xi)$ in some Sobolev space by investigating the regularity of the corresponding eigenfunctions. Setting $P_0 = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \Pi(\xi) \mathcal{F}$ with a frequency cut-off function $\mathbf{1}_{\{|\xi| \leq r_0\}}$ such that $\mathbf{1}_{\{|\xi| \leq r_0\}} = 1$ for $|\xi| \leq r_0$ and $\mathbf{1}_{\{|\xi| \leq r_0\}} = 0$ for $|\xi| > r_0$, we find the desired asymptotic behavior of $e^{-tL} P_0$ as described in (1.7) and (1.8). As for the complimentary part $e^{-tL}(I - P_0)$, we have already shown in [1] that

$$\|e^{-tL}(I - P_0) u_0\|_{H^1(\Omega)} \leq e^{-dt} \|u_0\|_{H^1(\Omega)}.$$

In this paper we will improve the class of initial value u_0 and will show that the exponential decay estimate holds for $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$ as in (1.9). This improvement will be also useful in the study of the nonlinear problem.

This paper is organized as follows. In Section 2 we first rewrite the problem into the system of equations in a non-dimensional form and then present the existence of a stationary solution of parallel flow. We state the main result of this paper in Section 3. In Section 4 we will investigate the spectrum of $-L_\xi$ for $|\xi| \leq r_0$, and in Section 5 we will establish a factorization of $e^{-tL} P_0$ and prove (1.7). Section 6 is devoted to the proof of (1.9).

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2 Preliminaries

In this section we introduce notations and then rewrite the problem into a non-dimensional form. In the end of this section we state the existence of stationary solution which represents parallel flow.

2.1 Notation

We first introduce some notations which will be used throughout the paper. For $1 \leq p \leq \infty$ we denote by $L^p(X)$ the usual Lebesgue space on a domain X and its norm is denoted by $\| \cdot \|_{L^p(X)}$. Let m be a nonnegative integer. $H^m(X)$ denotes the m th order L^2 Sobolev space on X with norm $\| \cdot \|_{H^m(X)}$. In particular, we write $L^2(X)$ for $H^0(X)$.

We denote by $C_0^m(X)$ the set of all C^m functions with compact support in X. $H_0^m(X)$ stands for the completion of $C_0^m(X)$ in $H^m(X)$. We denote by $H^{-1}(X)$ the dual space of $H_0^1(X)$ with norm $\| \cdot \|_{H^{-1}(X)}$.

We simply denote by $L^p(X)$ (resp., $H^m(X)$) the set of all vector fields $w = {}^T(w^1, w^2, w^3)$ on X and its norm is denoted by $\| \cdot \|_{L^p(X)}$ (resp., $\| \cdot \|_{H^m(X)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(X)$ and $w = {}^T(w^1, w^2, w^3) \in H^m(X)$, we define $\|u\|_{H^k(X) \times H^m(X)}$ by $\|u\|_{H^k(X) \times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^m(X)}$.

When $X = \Omega$ we abbreviate $L^p(\Omega)$ as L^p, and likewise, $H^m(\Omega)$ as H^m. The norm $\| \cdot \|_{L^p(\Omega)}$ is written as $\| \cdot \|_{L^p}$, and likewise, $\| \cdot \|_{H^m(\Omega)}$ as $\| \cdot \|_{H^m}$.

In the case $X = D$ we denote the norm of $L^p(D)$ by $| \cdot |_p$. The norm of $H^m(D)$ is denoted by $| \cdot |_{H^m}$, respectively. The inner product of $L^2(D)$ is denoted by

$$(f, g) = \int_D f(x') \overline{g(x')} dx', \quad f, g \in L^2(D).$$

Here $\bar{g}$ denotes the complex conjugate of g. For $u_j = {}^T(\phi_j, w_j)$ $(j = 1, 2)$, we also define a weighted inner product $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \frac{1}{\gamma^2} \int_D \phi_1 \overline{\phi}_2 \frac{P'(\rho_s)}{\gamma^2 \rho_s} dx' + \int_D w_1 \cdot \overline{w}_2 \rho_s dx',$$

where $\rho_s = \rho_s(x')$ is the density of the parallel flow u_s. As will be seen in Proposition 2.1 below, $\rho_s(x')$ and $\frac{P'(\rho_s(x'))}{\rho_s(x')}$ are strictly positive in D.

For $f \in L^1(D)$ we denote the mean value of f over D by $\langle f \rangle$:

$$\langle f \rangle = (f, 1) = \frac{1}{|D|} \int_D f dx',$$

where $|D| = \int_D dx'$. For $u = {}^T(\phi, w) \in L^1(D)$ with $w = {}^T(w^1, w^2, w^3)$ we define $\langle u \rangle$ by

$$\langle u \rangle = \langle \phi \rangle + \langle w_1 \rangle + \langle w_2 \rangle + \langle w_3 \rangle.$$

Partial derivatives of a function u in x, x', x_3 and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_3} u$ and $\partial_t u$. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

We denote the $n \times n$ identity matrix by I_n. We define 4×4 diagonal matrices Q_0 and $\widetilde{Q}$ by

$$Q_0 = \mathrm{diag}(1,0,0,0), \qquad\qquad \widetilde{Q} = \mathrm{diag}(0,1,1,1).$$

It then follows that for $u = {}^T(\phi, w)$ with $w = {}^T(w^1, w^2, w^3)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We denote the Fourier transform of $f = f(x_3)$ $(x_3 \in \mathbf{R})$ by f or $\mathcal{F}[f]$:

$$f(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbf{R}} f(x_3) e^{-i\xi x_3} dx_3, \qquad \xi \in \mathbf{R}.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x_3) = (2\pi)^{-1} \int_{\mathbf{R}} f(\xi) e^{i\xi x_3} d\xi, \qquad x_3 \in \mathbf{R}.$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum by $\sigma(A)$.

We finally introduce a function space which consists of locally H^1 functions in $L^2(\Omega)$ whose tangential derivatives near ∂D belong to $L^2(\Omega)$. To do so, we first introduce a local curvilinear coordinate system. For any $\overline{x}_0' \in \partial D$, there exist a neighborhood $\widetilde{\mathcal{O}}_{\overline{x}_0}$ of $\overline{x}_0'$ and a smooth diffeomorphism map $\Psi = (\Psi_1, \Psi_2) : \widetilde{\mathcal{O}}_{\overline{x}_0'} \to B_1(0) = \{z' = (z_1, z_2) : |z'| < 1\}$ such that

$$\begin{cases} \Psi\big(\widetilde{\mathcal{O}}_{\overline{x}_0'} \cap D\big) = \{z' \in B_1(0) : z_1 > 0\}, \\ \Psi\big(\widetilde{\mathcal{O}}_{\overline{x}_0'} \cap \partial D\big) = \{z' \in B_1(0) : z_1 = 0\}, \\ \det \nabla_{x'} \Psi \neq 0 \quad \text{on} \quad \overline{\widetilde{\mathcal{O}}_{\overline{x}_0'} \cap D}. \end{cases}$$

By the tubular neighborhood theorem, there exist a neighborhood $\mathcal{O}_{\overline{x}_0'}$ of $\overline{x}_0'$ and a local curvilinear coordinate system $y' = (y_1, y_2)$ on $\mathcal{O}_{\overline{x}_0'}$ defined by

$$x' = y_1 a_1(y_2) + \Psi^{-1}(0, y_2) : \ \mathcal{R} \to \mathcal{O}_{\overline{x}_0'}, \tag{2.1}$$

where $\mathcal{R} = \{y' = (y_1, y_2) : |y_1| \leq \widetilde{\delta}_1, \ |y_2| \leq \widetilde{\delta}_2\}$ for some $\widetilde{\delta}_1, \widetilde{\delta}_2 > 0$; $a_1(y_2)$ is the unit inward normal to ∂D that is given by

$$a_1(y_2) = \frac{\nabla_{x'} \Psi_1}{|\nabla_{x'} \Psi_1|}.$$

Setting $y_3 = x_3$ we obtain

$$\nabla_x = e_1(y_2)\partial_{y_1} + J(y')e_2(y_2)\partial_{y_2} + e_3 \partial_{y_3},$$

$$\nabla_y = \begin{pmatrix} {}^T e_1(y_2) \\ \frac{1}{J(y')} {}^T e_2(y_2) \\ {}^T e_3 \end{pmatrix} \nabla_x,$$

where

$$e_1(y_2) = \begin{pmatrix} a_1(y_2) \\ 0 \end{pmatrix}, \quad e_2(y_2) = \begin{pmatrix} a_2(y_2) \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \tag{2.2}$$

$$J(y') = |\det \nabla_{x'} \Psi|, \quad a_2(y_2) = \frac{-\nabla_{x'}^\perp \Psi_1}{|\nabla_{x'}^\perp \Psi_1|}$$

with $\nabla_{x'}^\perp \Psi_1 = {}^T(-\partial_{x_2} \Psi_1, \partial_{x_1} \Psi_1)$. Note that ∂_{y_1} and ∂_{y_2} are the inward normal derivative and tangential derivative at $x' = \Psi^{-1}(0, y_2) \in \partial D \cap \mathcal{O}_{\overline{x}_0'}$, respectively. Let us denote the normal and tangential derivatives by ∂_n and ∂, i.e.,

$$\partial_n = \partial_{y_1}, \quad \partial = \partial_{y_2}.$$

Since ∂D is compact, there are bounded open sets $\mathcal{O}_m$ $(m = 1, \dots, N)$ such that $\partial D \subset \cup_{m=1}^N \mathcal{O}_m$ and for each $m = 1, \dots, N$, there exists a local curvilinear coordinate system $y' = (y_1, y_2)$ as defined in (2.1) with $\mathcal{O}_{\overline{x}_0'}$, Ψ and $\mathcal{R}$ replaced by $\mathcal{O}_m$, Ψ^m and $\mathcal{R}_m = \left\{ y' = (y_1, y_2) : |y_1| < \widetilde{\delta}_1^m, |y_2| < \widetilde{\delta}_2^m \right\}$ for some $\widetilde{\delta}_1^m, \widetilde{\delta}_2^m > 0$. At last, we take an open set $\mathcal{O}_0 \subset D$ such that

$$\cup_{m=0}^N \mathcal{O}_m \supset D, \quad \overline{\mathcal{O}}_0 \cap \partial D = \emptyset.$$

We set a local coordinate $y' = (y_1, y_2)$ such that $y_1 = x_1$, $y_2 = x_2$ on $\mathcal{O}_0$. We note that if $h \in H^2(D)$, then $h|_{\partial D} = 0$ implies that $\partial^k h|_{\partial D \cap \mathcal{O}^m} = 0$ $(k = 0, 1)$.

Let us introduce a partition of unity $\{\chi_m\}_{m=0}^N$ subordinate to $\{\mathcal{O}_m\}_{m=0}^N$, satisfying

$$\sum_{m=0}^N \chi_m = 1 \text{ on } D, \quad \chi_m \in C_0^\infty(\mathcal{O}_m) \ (m = 0, 1, 2 \cdots, N).$$

We denote by $\widetilde{H}^1(\Omega)$ the set of all locally H^1 functions in $L^2(\Omega)$ whose tangential derivatives near $\partial\Omega$ belong to $L^2(\Omega)$, and its norm is denoted by $\|w\|_{\widetilde{H}^1(\Omega)}$:

$$\|w\|_{\widetilde{H}^1(\Omega)} = \|w\|_2 + \|\partial_{x_3} w\|_2 + \|\chi_0 \partial_{x'} w\|_2 + \sum_{m=1}^N \|\chi_m \partial w\|_2.$$

Note that $H_0^1(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$.

2.2 Stationary solution

In this subsection we rewrite the problem into the one in a non-dimensional form and state the existence of stationary solution which represents parallel flow. Let k_0 be an integer satisfying $k_0 \geq 3$. We introduce the following non-dimensional variables:

$$x = \ell \widetilde{x}, \quad v = V \widetilde{v}, \quad \rho = \rho_* \widetilde{\rho}, \quad t = \frac{\ell}{V} \widetilde{t},$$

$$p = \rho_* V^2 \widetilde{P}, \quad \Phi = \frac{V^2}{\ell}\widetilde{\Phi}, \quad g^3 = \frac{V^2}{\ell}\widetilde{g}^3,$$

$$V = |\overline{v}_s^3|_{C_*^{k_0}(D)} = \sum_{k=0}^{k_0} \sup_{x' \in D} \ell^k |\partial_{x'}^k \overline{v}_s^3(x')|, \quad \ell = \left(\int_D dx' \right)^{\frac{1}{2}}.$$

The problem (1.1)-(1.4) is then transformed into the following non-dimensional problem on $\widetilde{\Omega} = \widetilde{D} \times \mathbf{R}$:

$$\partial_{\widetilde{t}}\widetilde{\rho} + \mathrm{div}_{\widetilde{x}}(\widetilde{\rho}\widetilde{v}) = 0, \tag{2.3}$$

$$\widetilde{\rho}(\partial_{\widetilde{t}}\widetilde{v} + \widetilde{v} \cdot \nabla_{\widetilde{x}}\widetilde{v}) - \nu\Delta_{\widetilde{x}}\widetilde{v} - (\nu + \nu')\nabla_{\widetilde{x}}\mathrm{div}_{\widetilde{x}}\widetilde{v} + \widetilde{P}'(\widetilde{\rho})\nabla_{\widetilde{x}}\widetilde{\rho} = \widetilde{\rho}\widetilde{g}, \tag{2.4}$$

$$\widetilde{v}\,|_{\partial\widetilde{D}} = 0, \tag{2.5}$$

$$(\widetilde{\rho}, \widetilde{v})\,|_{\widetilde{t}=0} = (\widetilde{\rho}_0, \widetilde{v}_0). \tag{2.6}$$

Here $\widetilde{D}$ is a bounded and connected domain in $\mathbf{R}^2$; $\widetilde{g} = {}^T(\partial_{\widetilde{x}_1}\widetilde{\Phi}, \partial_{\widetilde{x}_2}\widetilde{\Phi}, \widetilde{g}^3)$; and ν and ν' are non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}.$$

We also introduce a parameter γ:

$$\gamma = \sqrt{\widetilde{P}'(1)} = \frac{\sqrt{p'(\rho_*)}}{V}.$$

Note that the Reynolds and Mach numbers are given by $1/\nu$ and $1/\gamma$, respectively.

In what follows, for simplicity, we omit tildes of $\widetilde{x}$, $\widetilde{t}$, $\widetilde{v}$, $\widetilde{\rho}$, $\widetilde{g}$, $\widetilde{P}$, $\widetilde{\Phi}$, $\widetilde{D}$ and $\widetilde{\Omega}$ and write them as x, t, v, ρ, g, P, Φ, D and Ω. Observe that, due to the non-dimensionalization, we have

$$|D| = \int_D dx' = 1,$$

and thus,

$$\langle f \rangle = \int_D f(x')\, dx'.$$

Let us state the existence of a stationary solution which represents parallel flow.

Proposition 2.1. *If* $\Phi \in C^{k_0}(\overline{D})$, $g^3 \in H^{k_0}(D)$ *and* $|\Phi|_{C^{k_0}}$ *is sufficiently small, then* (2.3)-(2.5) *has a stationary solution* $u_s = {}^T(\rho_s, v_s) \in C^{k_0}(\overline{D})$. *Here* ρ_s *satisfies*

$$\begin{cases} \mathrm{Const.} - \Phi(x') = \int_1^{\rho_s(x')} \frac{P'(\eta)}{\eta}\,d\eta, \\ \int_D \rho_s dx' = 1, \ \rho_1 < \rho_s(x') < \rho_2 \quad (\rho_1 < 1 < \rho_2) \end{cases}$$

for some constants ρ_1, $\rho_2 > 0$; *and* v_s *is a function of the form* $v_s = {}^T(0, 0, v_s^3)$ *with* $v_s^3 = v_s^3(x')$ *being the solution of*

$$\begin{cases} -\nu\Delta' v_s^3 = \rho_s g^3, \\ v_s^3\,|_{\partial D} = 0. \end{cases}$$

Furthermore, $u_s = {}^T(\rho_s, v_s)$ satisfies the estimates:

$$|\rho_s(x') - 1|_{C^k} \le C|\Phi|_{C^k}(1 + |\Phi|_{C^k})^k,$$

$$|v_s^3|_{C^k} \le C|v_s^3|_{H^{k+2}} \le C|\Phi|_{C^k}(1 + |\Phi|_{C^k})^k|g^3|_{H^k}$$

for $k = 3, 4, \cdots, k_0$.

Proposition 2.1 can be proved in a similar manner to the proof of [18, Lemma 2.1].

3 Main results

We set $\rho = \rho_s + \gamma^{-2}\phi$ and $v = v_s + w$ in (2.3)-(2.6) (without tildes) and omit the nonlinear terms of ϕ and w. We then arrive at the linearized problem

$$\partial_t u + Lu = 0, \ u = {}^T(\phi, w), \ w\,|_{\partial D} = 0, \ u\,|_{t=0} = u_0, \tag{3.1}$$

where

$$L = \begin{pmatrix} v_s \cdot \nabla & \gamma^2 \mathrm{div}(\rho_s\cdot) \\ \nabla\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\cdot\right) + \frac{\nu\Delta' v_s}{\gamma^2 \rho_s^2} & -\frac{\nu}{\rho_s}\Delta I_3 - \frac{\nu+\nu'}{\rho_s}\nabla\mathrm{div} + v_s \cdot \nabla + e_3 \otimes (\nabla v_s^3) \end{pmatrix}.$$

Here, $e_3 = {}^T(0,0,1)$; and, for $\mathbf{a} = {}^T(a_1, a_2, a_3)$ and $\mathbf{b} = {}^T(b_1, b_2, b_3)$, $\mathbf{a} \otimes \mathbf{b}$ is the 3×3 matrix $(a_i b_j)$.

We consider L as an operator on $L^2(\Omega)$ with domain

$$D(L) = \left\{u = {}^T(\phi, w) \in L^2(\Omega); \ w \in H_0^1(\Omega), \ Lu \in L^2(\Omega)\right\}.$$

As was shown in [1] (see also [7]), $-L$ generates a C_0-semigroup e^{-tL} on $L^2(\Omega)$. Furthermore, if $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$, then

$$\begin{aligned} e^{-tL}u_0 &\in C\big([0,T]; H^1(\Omega) \times \widetilde{H}^1(\Omega)\big) \cap C\big((0,T]; H^1(\Omega) \times H_0^1(\Omega)\big), \\ \nabla\widetilde{Q}e^{-tL}u_0 &\in L^2\big(0,T; \widetilde{H}^1(\Omega)\big) \end{aligned} \tag{3.2}$$

for all $T > 0$. The regularity property (3.2) of e^{-tL} can be proved as follows. It is not difficult to see that if $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$, then $e^{-tL}u_0$ satisfies

$$\begin{aligned} e^{-tL}u_0 &\in C\big([0,T]; H^1(\Omega) \times H_0^1(\Omega)\big), \\ \nabla\widetilde{Q}e^{-tL}u_0 &\in L^2\big(0,T; H^1(\Omega)\big). \end{aligned} \tag{3.3}$$

Since $H_0^1(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, one can see from (3.3), Lemma 6.5 and Lemma 6.6 below that $e^{-tL}u_0$ satisfies (3.2) if $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$.

In what follows we set

$$\omega = \|\rho_s - 1\|_{C^{k_0}}.$$

We have the following estimates for $e^{-tL}u_0$.

Theorem 3.1. *There exist positive constants ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{2\nu+\nu'} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then $e^{-tL}u_0$ is decomposed as*

$$e^{-tL}u_0 = e^{-tL}P_0 u_0 + e^{-tL}P_\infty u_0.$$

Here P_0 and P_∞ are projections satisfying

$$P_0 + P_\infty = I, \quad P^2 = P,$$

$$PL \subset LP, \quad Pe^{-tL} = e^{-tL}P$$

for $P \in \{P_0, P_\infty\}$; and $e^{-tL}P_0$ and $e^{-tL}P_\infty$ have the following properties.
(i) If $u_0 \in L^1(\Omega) \cap L^2(\Omega)$, then $e^{-tL}P_0 u_0$ satisfies the following estimates

$$\|\partial_{x'}^k \partial_{x_3}^l e^{-tL}P_0 u_0\|_2 \leq C_{k,l}(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u_0\|_1 \tag{3.4}$$

uniformly for $t \geq 0$ and for $k = 0, 1, \cdots, k_0$ and $l = 0, 1, \cdots$;

$$\|e^{-tL}P_0 u_0 - [\mathcal{H}(t)\langle\phi_0\rangle]u^{(0)}\|_2 \leq Ct^{-\frac{3}{4}}\|u_0\|_1 \tag{3.5}$$

uniformly for $t > 0$. Here

$$\mathcal{H}(t)\langle\phi_0\rangle = \mathcal{F}^{-1}[e^{-(i\kappa_0\xi+\kappa_1\xi^2)t}\langle\phi_0\rangle],$$

where $u^{(0)} = u^{(0)}(x')$ is the function given in Lemma 4.1 below; and $\kappa_0 \in \mathbf{R}$ and $\kappa_1 > 0$ are some constants satisfying

$$\kappa_0 = \mathcal{O}(1),$$

$$\kappa_1 = C\frac{\gamma^2}{\nu}\left\{1 + \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{2\nu+\nu'}{\gamma^2}\right)\right\},$$

where C is a positive constant.
(ii) If $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$, then there exists a constant $d > 0$ such that $e^{-tL}P_\infty u_0$ satisfies

$$\|e^{-tL}P_\infty u_0\|_{H^1} \leq Ce^{-dt}\left(\|u_0\|_{H^1\times\widetilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_2\right) \tag{3.6}$$

uniformly for $t > 0$.

Remark 3.2. *It is well-known that if $u_0 = {}^T(\phi_0, w_0) \in L^1(\Omega)$, then $\|\mathcal{H}(t)\langle\phi_0\rangle\|_2 = \mathcal{O}\left(t^{-\frac{1}{4}}\right)$, and $\sigma = \sigma(x_3, t) = \mathcal{H}(t)\langle\phi_0\rangle$ satisfies*

$$\begin{cases} \partial_t\sigma - \kappa_1\partial_{x_3}^2\sigma + \kappa_0\partial_{x_3}\sigma = 0, \\ \sigma\mid_{t=0} = \int_D \phi_0(x', x_3)dx'. \end{cases}$$

More detailed properties of P_0 and $e^{-tL}P_0$ will be given in Section 5 below, where we will establish a factorization of $e^{-tL}P_0$ which will be useful in the nonlinear analysis.

To prove Theorem 3.1, we consider the Fourier transform of (3.1) in x_3 variable which is written as

$$\partial_t \phi + i\xi v_s^3 \phi + \gamma^2 \nabla' \cdot (\rho_s w') + \gamma^2 i\xi \rho_s w^3 = 0, \tag{3.7}$$

$$\partial_t w' - \tfrac{\nu}{\rho_s}(\Delta' - \xi^2)w' - \tfrac{\nu+\nu'}{\rho_s}\nabla'(\nabla' \cdot w' + i\xi w^3) + \nabla'\left(\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}\phi\right) + i\xi v_s^3 w' = 0, \tag{3.8}$$

$$\partial_t w^3 - \tfrac{\nu}{\rho_s}(\Delta' - \xi^2)w^3 - \tfrac{\nu+\nu'}{\rho_s}i\xi(\nabla' \cdot w' + i\xi w^3) + i\xi\left(\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}\phi\right) + i\xi v_s^3 w^3$$
$$+ \tfrac{\nu}{\gamma^2 \rho_s^2}\Delta' v_s^3 \phi + w' \cdot \nabla' v_s^3 = 0, \tag{3.9}$$

$$w \mid_{\partial D} = 0 \tag{3.10}$$

for $t > 0$, and

$$^T(\phi, w) \mid_{t=0} = {}^T(\phi_0, w_0) = u_0. \tag{3.11}$$

Therefore, we arrive at the following problem

$$\frac{du}{dt} + L_\xi u = 0, \quad u \mid_{t=0} = u_0, \tag{3.12}$$

where $\xi \in \mathbf{R}$ is a parameter. Here $u = {}^T\big(\phi(x', t), w(x', t)\big) \in D\big(L_\xi\big)$ $(x' \in D,\ t > 0)$, u_0 is a given initial value, and L_ξ is the operator on $L^2(D)$ of the form

$$L_\xi = A_\xi + B_\xi + C_0,$$

where

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tfrac{\nu}{\rho_s}(\Delta' - |\xi|^2)I_2 - \tfrac{\nu+\nu'}{\rho_s}\nabla'\nabla'\cdot & -i\tfrac{\nu+\nu'}{\rho_s}\xi\nabla' \\ 0 & -i\tfrac{\nu+\nu'}{\rho_s}\xi\nabla'\cdot & -\tfrac{\nu}{\rho_s}(\Delta' - |\xi|^2) + \tfrac{\nu+\nu'}{\rho_s}|\xi|^2 \end{pmatrix},$$

$$B_\xi = \begin{pmatrix} i\xi v_s^3 & \gamma^2 \nabla'(\rho_s \cdot) & i\gamma^2 \rho_s \xi \\ \nabla'\left(\tfrac{P(\rho_s)}{\gamma^2 \rho_s}\cdot\right) & i\xi v_s^3 I_2 & 0 \\ i\xi \tfrac{P(\rho_s)}{\gamma^2 \rho_s} & 0 & i\xi v_s^3 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tfrac{\nu}{\gamma^2 \rho_s^2}\Delta' v_s^3 & {}^T(\nabla' v_s^3) & 0 \end{pmatrix}$$

with domain

$$D\big(L_\xi\big) = \big\{u = {}^T(\phi, w) \in L^2(D);\ w \in H_0^1(D),\ L_\xi u \in L^2(D)\big\}.$$

Note that $D\big(L_\xi\big) = D\big(L_0\big)$ for all $\xi \in \mathbf{R}$. Here and in what follows, we denote

$$\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2, \quad \nabla' = {}^T(\partial_{x_1}, \partial_{x_2}).$$

As in the case of L, we can see that $-L_\xi$ generates a C_0-semigroup on $L^2(D)$. Furthermore, if $u_0 \in H^1(D) \times \widetilde{H}^1(D)$, then

$$e^{-tL_\xi}u_0 \in C\big([0, T]; H^1(D) \times \widetilde{H}^1(D)\big) \cap C\big((0, T]; H^1(D) \times H_0^1(D)\big),$$
$$\partial_{x'}\widetilde{Q}e^{-tL_\xi}u_0 \in L^2\big(0, T; \widetilde{H}^1(D)\big) \tag{3.13}$$

for any $T > 0$.

In Section 4 we will investigate the spectrum of $-L_\xi$ for $|\xi| \ll 1$. In Section 5 we will give the proof of Theorem 3.1 (i). In Section 6 we will prove Theorem 3.1 (ii).

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4 Spectrum of $-L_\xi$ for $|\xi| \ll 1$

In this section, we consider the spectrum of $-L_\xi$ for $|\xi| \ll 1$. For simplicity, in what follows, we denote $\nu + \nu'$ by $\widetilde{\nu}$, i.e.,

$$\widetilde{\nu} = \nu + \nu'.$$

Let us consider the resolvent problem

$$(\lambda + L_\xi)u = f$$

with $|\xi| \ll 1$, where $u = {}^T(\phi, w) \in D(L_\xi) = D(L_0)$ and $f = {}^T(f^0, g) \in L^2(D)$.

We introduce the adjoint operator L_ξ^* of L_ξ with respect to the weighted inner product $\langle \cdot, \cdot \rangle$. The operator L_ξ^* is given by

$$L_\xi^* = A_\xi^* + B_\xi^* + C_0^*$$

with domain of definition

$$D\big(L_\xi^*\big) = \big\{u = {}^T(\phi, w) \in L^2(D); \, w \in H_0^1(D), \, L_\xi^* u \in L^2(D)\big\}.$$

Here

$$A_\xi^* = A_\xi, \quad B_\xi^* = -B_\xi$$

and

$$C_0^* = \begin{pmatrix} 0 & 0 & \frac{\gamma^2 \nu \Delta' v_s^3}{P'(\rho_s)} \\ 0 & 0 & \nabla' v_s^3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $D(L_\xi) = D(L_\xi^*)$ for any $\xi \in \mathbf{R}$.

We begin with a lemma on the zero eigenvalue of L_0 and L_0^* which was proved in [1, Lemma 4.1]. Here L_0 and L_0^* stand for L_ξ and L_ξ^* with $\xi = 0$, respectively.

Lemma 4.1. ([1, Lemma 4.1]) (i) *There exists a constant $\omega_0 > 0$ such that if $\omega \le \omega_0$, then $\lambda = 0$ is a simple eigenvalue of L_0 and L_0^*.*
(ii) *The eigenspaces for $\lambda = 0$ of L_0 and L_0^* are spanned by $u^{(0)}$ and $u^{(0)*}$, respectively, where*

$$u^{(0)} = {}^T(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = {}^T(0, 0, w^{(0),3})$$

and

$$u^{(0)*} = {}^T(\phi^{(0)*}, 0).$$

Here

$$\phi^{(0)}(x') = \alpha_0 \frac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))}, \quad \alpha_0 = \left(\int_D \frac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))} dx'\right)^{-1};$$

and $w^{(0),3}$ is the solution of the following problem

$$\begin{cases} -\Delta' w^{(0),3} = -\frac{1}{\gamma^2 \rho_s} \Delta' v_s^3 \phi^{(0)}, \\ w^{(0),3} \big|_{\partial D} = 0; \end{cases}$$

and

$$\phi^{(0)*}(x') = \tfrac{\gamma^2}{\alpha_0}\phi^{(0)}(x').$$

Furthermore, $\phi^{(0)} = O(1)$, $\alpha_0 = O(1)$ and $w^{(0),3} = O(\gamma^{-2})$ as $\gamma \to \infty$.

(iii) *The eigenprojections $\Pi^{(0)}$ and $\Pi^{(0)*}$ for $\lambda = 0$ of L_0 and L_0^* are given by*

$$\Pi^{(0)}u = \langle u, u^{(0)*}\rangle u^{(0)} = \langle Q_0 u\rangle u^{(0)},$$

$$\Pi^{(0)*}u = \langle u, u^{(0)}\rangle u^{(0)*}$$

for $u = {}^T(\phi, w)$, respectively.

(iv) *Let $u^{(0)}$ be written as $u^{(0)} = u_0^{(0)} + u_1^{(0)}$, where*

$$u_0^{(0)} = {}^T(\phi^{(0)}, 0), \quad u_1^{(0)} = {}^T(0, w^{(0)}).$$

Then

$$u^{(0)*} = \tfrac{\gamma^2}{\alpha_0}u_0^{(0)}$$

and

$$\langle u, u^{(0)}\rangle = \tfrac{\alpha_0}{\gamma^2}\langle\phi\rangle + (w^3, w^{(0),3}\rho_s)$$

for $u = {}^T(\phi, w) = {}^T(\phi, w', w^3)$.

We next establish the resolvent estimate for $|\xi| \ll 1$. To do so, let us consider the resolvent problem for $\xi = 0$

$$(\lambda + L_0)u = f, \tag{4.1}$$

where $u = {}^T(\phi, w) \in D(L_0)$ and $f = {}^T(f^0, g) \in L^2(D)$. Decomposing u in (4.1) as

$$u = \langle\phi\rangle u^{(0)} + u_1$$

with

$$u_1 = \big(I - \Pi^{(0)}\big)u,$$

we obtain

$$\lambda\big(\langle\phi\rangle u^{(0)} + u_1\big) + L_0 u_1 = f.$$

Applying $\Pi^{(0)}$ and $I - \Pi^{(0)}$ to this equation, we have

$$\begin{cases} \lambda\langle\phi\rangle = \langle f^0\rangle, \\ \lambda u_1 + L_0 u_1 = f_1, \end{cases} \tag{4.2}$$

where $f_1 = (I - \Pi^{(0)})f$. We see from the first equation of (4.2) that if $\lambda \neq 0$, then

$$\langle\phi\rangle = \tfrac{1}{\lambda}\langle f^0\rangle.$$

This implies that

$$|\langle\phi\rangle| \leq \tfrac{1}{|\lambda|}|f^0|_2. \tag{4.3}$$

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On the other hand, the u_1-part has the following properties. The second equation of (4.2) is written as

$$\begin{cases} \lambda\phi_1 + \gamma^2 \nabla' \cdot (\rho_s w_1') = f_1^0, \\ \lambda w_1' - \frac{\nu}{\rho_s}\Delta' w_1' - \frac{\widetilde{\nu}}{\rho_s}\nabla'\nabla' \cdot w_1' + \nabla'\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\phi_1\right) = g_1', \\ \lambda w_1^3 - \frac{\nu}{\rho_s}\Delta' w_1^3 + \frac{\nu}{\gamma^2\rho_s^2}\Delta' v_s^3\phi_1 + w_1' \cdot \nabla' v_s^3 = g_1^3, \end{cases} \tag{4.4}$$

where $u_1 = {}^T(\phi_1, w_1) = {}^T(\phi_1, w_1', w_1^3)$ and $f_1 = {}^T(f_1^0, g_1) = {}^T(f_1^0, g_1', g_1^3)$.

To state the estimates for the u_1-part, we introduce a quantity $\widetilde{D}_0[w_1]$ defined by

$$\widetilde{D}_0[w_1] = |\nabla' w_1|_2^2 + |\nabla' \cdot w_1'|_2^2$$

for $w_1 = {}^T(w_1', w_1^3)$.

Proposition 4.2. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$ and $\omega_0 > 0$ and an energy functional $E_0[u_1]$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then there holds the estimate*

$$(\text{Re}\lambda)E_0[u_1] + c(|\phi_1|_2^2 + \widetilde{D}_0[w_1]) \leq C|f_1|_2|u_1|_2,$$

where c and C are positive constants independent of u_1 and λ; and $E_0[u_1]$ is equivalent to $|u_1|_2^2$.

Proposition 4.2 can be proved in a similar manner to the proof of [1, Proposition 4.7] by replacing $\frac{d}{dt}$ with $\text{Re}\lambda$ and taking $\xi = 0$ there.

The Poincaré inequality yields $\widetilde{D}_0[w_1] \geq C|w_1|_2^2$ with a positive constant C. Therefore, the resolvent estimates for $-L_0$ now follow from (4.3) and Proposition 4.2.

Proposition 4.3. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$ and $\omega_0 > 0$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then there is a positive constant $c_0 > 0$ such that*

$$\Sigma_0 \equiv \{\lambda \neq 0 : \text{Re}\lambda > -c_0\} \subset \rho(-L_0).$$

Furthermore, the following estimates

$$|(\lambda + L_0)^{-1}f|_2 \leq C\left\{\frac{1}{|\lambda|}|f^0|_2 + \frac{1}{(\text{Re}\lambda + c_0)}|f_1|_2\right\},$$

$$\left|\partial_{x'}\{\widetilde{Q}(\lambda + L_0)^{-1}f\}\right|_2 \leq C\left\{\frac{1}{|\lambda|}|f^0|_2 + \frac{1}{(\text{Re}\lambda + c_0)^{1/2}}|f_1|_2\right\}$$

hold uniformly for $\lambda \in \Sigma_0$. The same assertions also hold for $-L_0^*$.

Based on Proposition 4.3, we have the resolvent estimates for $-L_\xi$ with $|\xi| \ll 1$.

Theorem 4.4. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$ and $\omega_0 > 0$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then the following assertions hold. For any η satisfying $0 < \eta \leq \frac{c_0}{2}$ there is a number $r_0 = r_0(\eta)$ such that*

$$\Sigma_1 \equiv \{\lambda \neq 0 : |\lambda| \geq \eta, \text{Re}\lambda \geq -\frac{c_0}{2}\} \subset \rho(-L_\xi)$$

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for $|\xi| \leq r_0$. Furthermore, the following estimates

$$|(\lambda + L_\xi)^{-1}f|_2 \leq C|f|_2,$$

$$\left|\partial_{x'}\{\widetilde{Q}(\lambda + L_\xi)^{-1}f\}\right|_2 \leq C|f|_2$$

hold uniformly for $\lambda \in \Sigma_1$ and ξ with $|\xi| \leq r_0$. The same assertions also hold for $-L_\xi^*$.

Proof. Let us decompose L_ξ as

$$L_\xi = L_0 + \xi L^{(1)} + \xi^2 L^{(2)},$$

where

$$L^{(1)} = i \begin{pmatrix} v_s^3 & 0 & \gamma^2 \rho_s \\ 0 & v_s^3 I_2 & -\frac{\widetilde{\nu}}{\rho_s}\nabla' \\ \frac{P'(\rho_s)}{\gamma^2 \rho_s} & -\frac{\widetilde{\nu}}{\rho_s}\nabla'. & v_s^3 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}I_2 & 0 \\ 0 & 0 & \frac{\nu + \widetilde{\nu}}{\rho_s} \end{pmatrix}.$$

For $u = {}^T(\phi, w) \in L^2(D) \times H_0^1(D)$ we have

$$|L^{(1)}u|_2 \leq C|u|_{L^2 \times H^1}, \quad |L^{(2)}u|_2 \leq C|u|_2. \tag{4.5}$$

Therefore, we see from Proposition 4.3 that for any $0 < \eta \leq \frac{c_0}{2}$ there exists $r_0 > 0$ such that if $|\xi| \leq r_0$, then

$$\left|\left(\xi L^{(1)} + \xi^2 L^{(2)}\right)\left(\lambda + L_0\right)^{-1}f\right|_2 \leq \tfrac{1}{2}|f|_2. \tag{4.6}$$

It then follows that

$$\Sigma_1 \equiv \{\lambda : |\lambda| > \eta, \text{Re}\lambda \geq -\tfrac{c_0}{2}\} \subset \rho(-L_\xi),$$

and that, if $\lambda \in \Sigma_1$, then $\left(\lambda + L_\xi\right)^{-1}$ is given by the Neumann series expansion

$$\left(\lambda + L_\xi\right)^{-1} = (\lambda + L_0)^{-1} + \sum_{N=0}^{\infty} (-1)^N \left[\left(\xi L^{(1)} + \xi^2 L^{(2)}\right)\left(\lambda + L_0\right)^{-1}\right]$$

for $|\xi| \leq r_0$, and it holds that

$$|\left(\lambda + L_\xi\right)^{-1}f|_2 \leq C|f|_2 \tag{4.7}$$

for $\lambda \in \Sigma_1$ and $|\xi| \leq r_0$. We thus obtain the desired estimates. This completes the proof. $\square$

As for the spectrum of $-L_\xi$ near $\lambda = 0$, we have the following result.

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Theorem 4.5. *There exist positive constants ν_0, γ_0, ω_0 and r_0 such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then it holds that*

$$\sigma(-L_\xi) \cap \{\lambda : |\lambda| \leq \tfrac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for ξ with $|\xi| \leq r_0$, where $\lambda_0(\xi)$ is a simple eigenvalue of $-L_\xi$ that has the form

$$\lambda_0(\xi) = -i\kappa_0\xi - \kappa_1\xi^2 + \mathcal{O}(|\xi|^3)$$

as $\xi \to 0$. Here $\kappa_0 \in \mathbf{R}$ and $\kappa_1 > 0$ are the numbers given by

$$\kappa_0 = \langle v_s^3\phi^{(0)} + \gamma^2\rho_s w^{(0),3}\rangle = \mathcal{O}(1),$$

$$\kappa_1 = \tfrac{\gamma^2}{\nu}\left\{\alpha_0\big|(-\Delta')^{-\frac{1}{2}}\rho_s\big|_2^2 + \mathcal{O}\big(\tfrac{1}{\gamma^2}\big) + \big(\tfrac{\nu}{\gamma^2} + \tfrac{1}{\nu^2}\big) \times \mathcal{O}\big(\tfrac{\nu+\tilde{\nu}}{\gamma^2}\big)\right\},$$

where $-\Delta'$ denotes the Laplace operator on $L^2(D)$ under the zero Dirichlet boundary condition with domain

$$D(-\Delta') = H^2(D) \cap H_0^1(D).$$

Proof. For $u \in L^2(D) \times H_0^1(D)$ we see from Theorem 4.4 and (4.5) that

$$|L^{(1)}u|_2 \leq C\big(|L_0 u|_2 + |u|_2\big), \quad |L^{(2)}u|_2 \leq C|u|_2.$$

Therefore, since 0 is a simple eigenvalue of $-L_0$, we see from the analytic perturbation theory that there exists a positive constant r_0 such that

$$\sigma(-L_\xi) \cap \{\lambda : |\lambda| \leq \tfrac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for all ξ with $|\xi| \leq r_0$. Here $\lambda_0(\xi)$ is a simple eigenvalue of $-L_\xi$. Furthermore, $\lambda_0(\xi)$ and the eigenprojection $\Pi(\xi)$ for $\lambda_0(\xi)$ are expanded as

$$\lambda_0(\xi) = \lambda^{(0)} + \xi\lambda^{(1)} + \xi^2\lambda^{(2)} + \mathcal{O}(|\xi|^3),$$

$$\Pi(\xi) = \Pi^{(0)} + \xi\Pi^{(1)} + \mathcal{O}(|\xi|^2) \tag{4.8}$$

with

$$\lambda^{(0)} = 0,$$
$$\lambda^{(1)} = -\langle L^{(1)}u^{(0)}, u^{(0)*}\rangle,$$
$$\lambda^{(2)} = -\langle L^{(2)}u^{(0)}, u^{(0)*}\rangle + \langle L^{(1)}SL^{(1)}u^{(0)}, u^{(0)*}\rangle,$$
$$\Pi^{(1)} = -\Pi^{(0)}L^{(1)}S - SL^{(1)}\Pi^{(0)},$$

where

$$S = \left\{\big(I - \Pi^{(0)}\big)L_0\big(I - \Pi^{(0)}\big)\right\}^{-1}.$$

We first consider $\lambda^{(1)}$. Since

$$L^{(1)}u^{(0)} = i\begin{pmatrix} v_s^3\phi^{(0)} + \gamma^2\rho_s w^{(0),3} \\ -\frac{\tilde{\nu}}{\rho_s}\nabla' w^{(0),3} \\ \alpha_0 + v_s^3 w^{(0),3} \end{pmatrix},$$

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we obtain

$$\lambda^{(1)} = -\langle L^{(1)}u^{(0)}, u^{(0)*}\rangle = -\langle Q_0 L^{(1)}u^{(0)}\rangle = -i\langle v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3}\rangle = i\,\mathcal{O}(1)$$

as $\gamma^2 \to \infty$.

We next consider $\lambda^{(2)}$. Since $Q_0 L^{(2)}u^{(0)} = 0$, we have

$$\langle L^{(2)}u^{(0)}, u^{(0)*}\rangle = \langle Q_0 L^{(2)}u^{(0)}\rangle = 0.$$

It then follows that

$$\lambda^{(2)} = \langle L^{(1)}SL^{(1)}u^{(0)}, u^{(0)*}\rangle = \langle Q_0 L^{(1)}SL^{(1)}u^{(0)}\rangle.$$

We define $\widetilde{u}$ by

$$\widetilde{u} = SL^{(1)}u^{(0)},$$

which satisfies

$$\begin{cases} L_0\widetilde{u} = \left(I - \Pi^{(0)}\right)L^{(1)}u^{(0)} = L^{(1)}u^{(0)} + \lambda^{(1)}u^{(0)}, \\ \widetilde{w}\mid_{\partial D} = 0, \\ \langle\widetilde{\phi}\rangle = 0. \end{cases} \tag{4.9}$$

Note that $\widetilde{u} = {}^T(\widetilde{\phi}, \widetilde{w}) \in i\mathbf{R}^4$ and $\lambda^{(1)} \in i\mathbf{R}$. We rewrite $\lambda^{(2)}$ as

$$\lambda^{(2)} = \langle Q_0 L^{(1)}\widetilde{u}\rangle = \langle iv_s^3\widetilde{\phi} + i\gamma^2\rho_s\widetilde{w}^3\rangle,$$

where $\widetilde{u} = {}^T(\widetilde{\phi}, \widetilde{w}) = {}^T(\widetilde{\phi}, \widetilde{w}', \widetilde{w}^3)$. To show the strict negativity of $\lambda^{(2)}$, we estimate $\widetilde{u}$. The problem (4.9) is written as

$$\begin{cases} \gamma^2\nabla'\cdot(\rho_s\widetilde{w}') = i\xi v_s^3\phi^{(0)} + i\gamma^2\rho_s w^{(0),3} + \lambda^{(1)}\phi^{(0)}, \\ -\frac{\nu}{\rho_s}\Delta'\widetilde{w}' - \frac{\widetilde{\nu}}{\rho_s}\nabla'\nabla'\cdot\widetilde{w}' + \nabla'\left(\frac{P'(\rho_s)}{\gamma^2\rho_s}\widetilde{\phi}\right) = -i\frac{\widetilde{\nu}}{\rho_s}\nabla' w^{(0),3}, \\ -\frac{\nu}{\rho_s}\Delta'\widetilde{w}^3 + \frac{\nu\Delta' v_s^3}{\gamma^2\rho_s^2}\widetilde{\phi} + \widetilde{w}'\cdot\nabla' v_s^3 = i\frac{P'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} + iv_s^3 w^{(0),3} + \lambda^{(1)}w^{(0),3}, \\ \widetilde{w}\mid_{\partial D} = 0, \\ \langle\widetilde{\phi}\rangle = 0, \end{cases}$$

i.e., $\widetilde{u} = {}^T(\widetilde{\phi}, \widetilde{w}) = {}^T(\widetilde{\phi}, \widetilde{w}', \widetilde{w}^3)$ is a solution of

$$\begin{cases} \nabla'\cdot\widetilde{w}' = F^0[\widetilde{w}'], \\ -\nu\Delta'\widetilde{w}' + \nabla'\widetilde{\phi} = G'[\widetilde{\phi}, \widetilde{w}'], \\ \widetilde{w}'\mid_{\partial D} = 0, \\ \langle\widetilde{\phi}\rangle = 0 \end{cases} \tag{4.10}$$

and

$$\begin{cases} -\nu\Delta'\widetilde{w}^3 = G^3[\widetilde{\phi}, \widetilde{w}'], \\ \widetilde{w}^3\mid_{\partial D} = 0, \end{cases} \tag{4.11}$$

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where $F^0[\widetilde{w}']$, $G'[\widetilde{\phi}, \widetilde{w}']$ and $G^3[\widetilde{\phi}, \widetilde{w}']$ are defined as

$$F^0[\widetilde{w}'] = \tfrac{1}{\gamma^2}\big\{iv_s^3\phi^{(0)} + i\gamma^2\rho_s w^{(0),3} + \lambda^{(1)}\phi^{(0)}\big\} - \nabla' \cdot \big((1-\rho_s)\widetilde{w}'\big),$$

$$G'[\widetilde{\phi}, \widetilde{w}'] = -i\widetilde{\nu}\nabla' w^{(0),3} + \widetilde{\nu}\nabla' F^0[\widetilde{w}'] + \nabla'\big((1-\rho_s)\widetilde{\phi}\big)$$
$$+ (\nabla'\rho_s)\widetilde{\phi} + \rho_s\nabla'\Big\{\Big(1 - \tfrac{P'(\rho_s)}{\gamma^2\rho_s}\Big)\Big\}\widetilde{\phi},$$

$$G^3[\widetilde{\phi}, \widetilde{w}'] = \rho_s\Big\{i\tfrac{P'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} + iv_s^3 w^{(0),3} + \lambda^{(1)}w^{(0),3}\Big\} - \rho_s\Big\{\tfrac{\nu\Delta'v_s^3}{\gamma^2\rho_s^2}\widetilde{\phi} + \widetilde{w}' \cdot \nabla'v_s^3\Big\}.$$

As for the problem (4.10), since $\lambda^{(1)} = -i\langle v_s^3\phi^{(0)} + \gamma^2\rho_s w^{(0),3}\rangle$, it holds that $\langle F^0[\widetilde{w}']\rangle = 0$. Furthermore, we have

$$|F^0[\widetilde{w}']|_2 \le C\big\{\tfrac{1}{\gamma^2}\big(|\lambda^{(1)}||\phi^{(0)}|_2 + |\phi^{(0)}|_2 + \gamma^2|w^{(0),3}|_2\big) + \omega|\nabla'\widetilde{w}'|_2\big\}$$
$$\le C\omega|\nabla'\widetilde{w}'|_2 + \mathcal{O}\big(\tfrac{1}{\gamma^2}\big),$$

$$|G'[\widetilde{\phi}, \widetilde{w}']|_{H^{-1}} \le C\big\{\widetilde{\nu}|\nabla'w^{(0),3}|_{H^{-1}} + \widetilde{\nu}|\nabla'F^0[\widetilde{w}']|_{H^{-1}} + \big|\nabla'\big((1-\rho_s)\widetilde{\phi}\big)\big|_{H^{-1}}$$
$$+ |\nabla'\rho_s\widetilde{\phi}|_{H^{-1}} + \big|\rho_s\big(\big(1 - \tfrac{P'(\rho_s)}{\gamma^2\rho_s}\big)\widetilde{\phi}\big)\big|_{H^{-1}}\big\}$$
$$\le C\omega\{|\widetilde{\phi}|_2 + \widetilde{\nu}|\nabla'\widetilde{w}'|_2\} + \mathcal{O}\big(\tfrac{\widetilde{\nu}}{\gamma^2}\big).$$

Since $(\widetilde{\phi}, \widetilde{w}') \in \dot{X} \equiv \{(p, v') \in L^2(D) \times H_0^1(D) : \langle p\rangle = 0\}$ and it is a solution of the Stokes system (4.10), we see from estimate for the Stokes system (see, e.g., [20]) that there holds the estimate

$$|\widetilde{\phi}|_2^2 + \nu^2|\nabla'\widetilde{w}'|_2^2 \le \nu^2\big\{C\omega^2|\widetilde{w}'|_2^2 + \mathcal{O}\big(\tfrac{1}{\gamma^4}\big)\big\} + \big\{C\omega^2\big(|\widetilde{\phi}|_2^2 + \widetilde{\nu}^2|\nabla'\widetilde{w}'|_2^2\big) + \mathcal{O}\big(\tfrac{\widetilde{\nu}^2}{\gamma^4}\big)\big\}$$
$$\le C_1\omega^2\big\{|\widetilde{\phi}|_2^2 + (\nu+\widetilde{\nu})^2|\nabla'\widetilde{w}'|_2^2\big\} + \mathcal{O}\big(\tfrac{(\nu+\widetilde{\nu})^2}{\gamma^4}\big).$$

Therefore, if ω is so small that $\omega^2 < \tfrac{1}{2C_1}\min\big\{1, \big(\tfrac{\nu}{\nu+\widetilde{\nu}}\big)^2\big\}$, then

$$|\widetilde{\phi}|_2^2 + \nu^2|\nabla'\widetilde{w}'|_2^2 = \mathcal{O}\big(\tfrac{(\nu+\widetilde{\nu})^2}{\gamma^4}\big). \tag{4.12}$$

As for the problem (4.11), since

$$|G^3[\widetilde{\phi}, \widetilde{w}']|_2 \le C\big\{|\lambda^{(1)}||w^{(0),3}|_2 + \tfrac{1}{\gamma^2}|\phi^{(0)}|_2 + |w^{(0),3}|_2 + \tfrac{\nu}{\gamma^2}|\widetilde{\phi}|_2 + |\widetilde{w}'|_2\big\}$$
$$\le C\big\{\tfrac{\nu}{\gamma^2}|\widetilde{\phi}|_2 + |\widetilde{w}'|_2\big\} + \mathcal{O}\big(\tfrac{1}{\gamma^2}\big),$$

we have $G^3[\widetilde{\phi}, \widetilde{w}'] \in L^2(D)$. It then follows that

$$\widetilde{w}^3 = \tfrac{1}{\nu}(-\Delta')^{-1}G^3[\widetilde{\phi}, \widetilde{w}'].$$

Since $\phi^{(0)} = \alpha_0\tfrac{\gamma^2\rho_s}{P'(\rho_s)}$ (see Lemma 4.1 (ii)), we find that

$$\langle\rho_s\widetilde{w}^3\rangle = \tfrac{1}{\nu}\big\langle\rho_s(-\Delta')^{-1}G^3[\widetilde{\phi}, \widetilde{w}']\big\rangle$$
$$= \tfrac{1}{\nu}\big\langle\rho_s(-\Delta')^{-1}(i\alpha_0\rho_s)\big\rangle$$
$$+ \tfrac{1}{\nu}\big\langle\rho_s(-\Delta')^{-1}\big\{i\rho_s v_s^3 w^{(0),3} + \rho_s\lambda^{(1)}w^{(0),3} - \tfrac{\nu\Delta'v_s^3}{\gamma^2\rho_s}\widetilde{\phi} - \rho_s\widetilde{w}' \cdot \nabla'v_s^3\big\}\big\rangle$$
$$= i\tfrac{\alpha_0}{\nu}\big|(-\Delta')^{-\frac{1}{2}}\rho_s\big|_2^2$$
$$+ \tfrac{1}{\nu}\big\langle\rho_s(-\Delta')^{-1}\big\{i\rho_s v_s^3 w^{(0),3} + \rho_s\lambda^{(1)}w^{(0),3} - \tfrac{\nu\Delta'v_s^3}{\gamma^2\rho_s}\widetilde{\phi} - \rho_s\widetilde{w}' \cdot \nabla'v_s^3\big\}\big\rangle.$$

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Furthermore, since $\widetilde{u} = {}^T(\widetilde{\phi}, \widetilde{w}') \in i\mathbf{R}^4$ and $\lambda^{(1)} \in i\mathbf{R}$, we see from (4.12) that

$$\begin{aligned}
&\left\langle \rho_s(-\Delta')^{-1}\{i\rho_s v_s^3 w^{(0),3} + \rho_s \lambda^{(1)} w^{(0),3} - \tfrac{\nu\Delta' v_s^3}{\gamma^2 \rho_s}\widetilde{\phi} - \rho_s \widetilde{w}' \cdot \nabla' v_s^3\}\right\rangle \\
&= i\mathcal{O}\big(\tfrac{1}{\gamma^2}\big) + i\big(\tfrac{\nu}{\gamma^2} + \tfrac{1}{\nu^2}\big) \times \mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big).
\end{aligned}$$

It then follows that

$$\langle \rho_s \widetilde{w}^3 \rangle = i\tfrac{\alpha_0}{\nu}\big|(-\Delta')^{-\frac{1}{2}}\rho_s\big|_2^2 + i\tfrac{1}{\nu}\Big\{\mathcal{O}\big(\tfrac{1}{\gamma^2}\big) + \big(\tfrac{\nu}{\gamma^2} + \tfrac{1}{\nu^2}\big) \times \mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big)\Big\}.$$

By (4.12) we also have

$$\langle v_s^3 \widetilde{\phi} \rangle = i\mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big).$$

We conclude that

$$\begin{aligned}
\lambda^{(2)} &= \langle iv_s^3 \widetilde{\phi} + i\gamma^2 \rho_s \widetilde{w}^3 \rangle \\
&= i\gamma^2\Big[i\tfrac{\alpha_0}{\nu}\big|(-\Delta')^{-\frac{1}{2}}\rho_s\big|_2^2 + i\tfrac{1}{\nu}\big\{\mathcal{O}\big(\tfrac{1}{\gamma^2}\big) + \big(\tfrac{\nu}{\gamma^2} + \tfrac{1}{\nu^2}\big) \times \mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big)\big\}\Big] + i \cdot i\mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big) \\
&= -\tfrac{\gamma^2}{\nu}\Big[\alpha_0\big|(-\Delta')^{-\frac{1}{2}}\rho_s\big|_2^2 + \big\{\mathcal{O}\big(\tfrac{1}{\gamma^2}\big) + \big(\tfrac{1}{\nu^2} + \tfrac{\nu}{\gamma^2}\big) \times \mathcal{O}\big(\tfrac{\nu+\widetilde{\nu}}{\gamma^2}\big)\big\}\Big] \\
&< 0
\end{aligned}$$

for sufficiently small $\tfrac{1}{\nu}$ and $\tfrac{\nu+\widetilde{\nu}}{\gamma^2}$. We thus obtain the desired estimates. This completes the proof. $\qquad\square$

We next establish some estimates related to $\Pi(\xi)$ in $H^k(D)$. We first consider estimates for higher order derivatives of $(\lambda + L_0)^{-1}f$.

Proposition 4.6. *For any $f = {}^T(f^0, g) \in H^k(D) \times H^{k-1}(D)$. There exist positive constants ν_0, γ_0, ω_0 and c_1 such that if $\nu \geq \nu_0$, $\tfrac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_0^2$, $\omega \leq \omega_0$ and $\lambda \in \Sigma_2 \equiv \{\lambda \neq 0 : |\lambda| \leq c_1\}$, then $(\lambda + L_0)^{-1}f \in H^k(D) \times \big(H^{k+1}(D) \cap H_0^1(D)\big)$ for $k = 0, 1, \cdots, k_0$. Furthermore, the following estimate holds:*

$$|(\lambda + L_0)^{-1}f|_{H^k \times H^{k+1}} \leq C\big(1 + \tfrac{1}{|\lambda|}\big)|f|_{H^k \times H^{k-1}},$$

where C is a positive constant independent of $\lambda \in \Sigma_2$. The same assertions also hold for $-L_0^*$.

Proof. For a given $f = {}^T(f^0, g) \in H^k(D) \times H^{k-1}(D)$, we consider the problem

$$\begin{cases} (\lambda + \mathcal{L}_0)U = f, \\ W\,|_{\partial D} = 0 \end{cases} \tag{4.13}$$

for $U = {}^T(\Phi, W)$. Here $\mathcal{L}_0$ is differential operator given by

$$\mathcal{L}_0 U = \begin{pmatrix} \gamma^2 \nabla' \cdot (\rho_s W') \\ -\tfrac{\nu}{\rho_s}\Delta' W' - \tfrac{\widetilde{\nu}}{\rho_s}\nabla'\nabla' \cdot W' + \nabla'\big(\tfrac{P'(\rho_s)}{\gamma^2 \rho_s}\Phi\big) \\ -\tfrac{\nu}{\rho_s}\Delta' W^3 + \tfrac{\nu\Delta' v_s^3}{\gamma^2 \rho_s^2}\Phi + W' \cdot \nabla' v_s^3 \end{pmatrix}$$

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for $U = {}^T(\Phi, W)$. To solve the problem (4.13), we decompose Φ and f^0 as

$$\Phi = \Phi_1 + \sigma, \qquad f^0 = f_1^0 + \langle f^0 \rangle,$$

where $\sigma = \langle \Phi \rangle$, $\Phi_1 = \Phi - \sigma$ and $f_1^0 = f^0 - \langle f^0 \rangle$. Note that

$$\langle \Phi_1 \rangle = 0, \qquad \langle f_1^0 \rangle = 0.$$

Then (4.13) is equivalent to the problem

$$\lambda \sigma = \langle f^0 \rangle, \tag{4.14}$$

$$\lambda \Phi_1 + \gamma^2 \nabla' \cdot (\rho_s W') = f_1^0, \tag{4.15}$$

$$\lambda W' - \frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla'\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}(\sigma + \Phi_1)\right) = g', \tag{4.16}$$

$$\lambda W^3 - \frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2}(\sigma + \Phi_1) - W' \cdot \nabla' v_s^3 = g^3 \tag{4.17}$$

with $W \mid_{\partial D} = 0$. If $\lambda \neq 0$, then we find from (4.14) that

$$\sigma = \frac{1}{\lambda} \langle f^0 \rangle. \tag{4.18}$$

Substituting $\sigma = \frac{1}{\lambda} \langle f^0 \rangle$ into (4.16) and (4.17), we obtain

$$\begin{cases} \lambda \Phi_1 + \gamma^2 \nabla' \cdot (\rho_s W') = f_1^0, \\ \lambda W' - \frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla'\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\Phi_1\right) = g' - \frac{1}{\lambda}\langle f^0 \rangle \nabla'\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\right), \\ \lambda W^3 - \frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2}\Phi_1 - W' \cdot \nabla' v_s^3 = g^3 - \frac{1}{\lambda}\langle f^0 \rangle \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \end{cases} \tag{4.19}$$

with $W \mid_{\partial D} = 0$. Let us write the problem (4.19) as

$$\begin{cases} \nabla' \cdot W' = F^0[\Phi_1, W' : f_1^0], \\ -\nu \Delta' W' + \nabla' \Phi_1 = G'[\Phi_1, W' : f^0, g'], \\ W' \mid_{\partial D} = 0 \end{cases} \tag{4.20}$$

and

$$\begin{cases} -\nu \Delta' W^3 = G^3[\Phi_1, W', W^3 : f^0, g^3], \\ W^3 \mid_{\partial D} = 0. \end{cases} \tag{4.21}$$

Here

$$F^0[\Phi_1, W' : f_1^0] = -\frac{1}{\gamma^2}\lambda \Phi_1 + \nabla' \cdot \left((1 - \rho_s)W'\right) + \frac{1}{\gamma^2} f_1^0,$$

$$G'[\Phi_1, W' : f^0, g'] = -\lambda \rho_s W' + \tilde{\nu}\nabla' F^0[\Phi_1, W' : f_1^0] + \nabla'\left((1 - \rho_s)\Phi_1\right) + \nabla' \rho_s \Phi_1$$
$$- \frac{1}{\lambda}\langle f^0 \rangle \rho_s \nabla'\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\right) + \rho_s \nabla'\left(\left(1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s}\right)\Phi_1\right) + \rho_s g',$$

$$G^3[\Phi_1, W', W^3 : f^0, g^3] = -\lambda \rho_s W^3 - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2}\frac{1}{\lambda}\langle f^0 \rangle - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2}\Phi_1 - \rho_s W' \cdot \nabla' v_s^3 + \rho_s g^3.$$

We now define a set $\dot{X}_k$ by

$$\dot{X}_k = \left\{(p, v') \in H^k(D) \times \left(H^{k+1}(D) \cap H_0^1(D)\right) : \langle p \rangle = 0\right\}$$

with norm
$$|(p, v')|_{\dot{X}_k} = |p|_{H^k} + \nu |v'|_{H^{k+1}}.$$

For a given $(\widetilde{\Phi}_1, \widetilde{W}') \in \dot{X}_k$, we consider the problem

$$\begin{cases} \nabla' \cdot W' = F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0], \\ -\nu \Delta' W' + \nabla' \Phi_1 = G'[\widetilde{\Phi}_1, \widetilde{W}' : f^0, g'], \\ W' \mid_{\partial D} = 0. \end{cases} \tag{4.22}$$

It holds that

$$\langle F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0] \rangle = 0, \quad F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0] \in H^k(D),$$

$$G'[\widetilde{\Phi}_1, \widetilde{W}' : f^0, g'] \in H^{k-1}(D).$$

In fact, we see that

$$\langle F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0] \rangle = -\frac{1}{\gamma^2} \lambda \langle \widetilde{\Phi}_1 \rangle + \langle \nabla' \cdot ((1 - \rho_s) \widetilde{W}') \rangle + \frac{1}{\gamma^2} \langle f_1^0 \rangle = 0,$$

$$\left| F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0] \right|_{H^k} \le C \{ \frac{1}{\gamma^2} |\lambda| |\widetilde{\Phi}_1|_{H^k} + \omega |\widetilde{W}'|_{H^{k+1}} + \frac{1}{\gamma^2} |f_1^0|_{H^k} \}$$

and

$$\left| G'[\widetilde{\Phi}_1, \widetilde{W}' : f^0, g'] \right|_{H^{k-1}}$$
$$\le C \{ |\lambda| |\widetilde{W}'|_{H^{k-1}} + \widetilde{\nu} |F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0]|_{H^k} + \omega |\widetilde{\Phi}_1|_{H^k} + \frac{1}{|\lambda|} |\langle f^0 \rangle| + |g'|_{H^{k-1}} \}$$
$$\le C \{ (\frac{\widetilde{\nu}}{\gamma^2} |\lambda| + \omega) |\widetilde{\Phi}_1|_{H^k} + \nu (\frac{1}{\nu} |\lambda| + \frac{\widetilde{\nu}}{\nu} \omega) |\widetilde{W}'|_{H^{k+1}} + (\frac{\widetilde{\nu}}{\gamma^2} + \frac{1}{|\lambda|}) |f^0|_{H^k} + |g'|_{H^{k-1}} \}$$

for a positive constant C independent of λ. From [20], we see that there is a unique solution $(\Phi_1, W') \in \dot{X}_k$ of (4.22) and there holds the estimate

$$\begin{aligned} &|\Phi|_{H^k} + \nu |W'|_{H^{k+1}} \\ &\le C \{ \nu |F^0[\widetilde{\Phi}_1, \widetilde{W}' : f_1^0]|_{H^k} + |G'[\widetilde{\Phi}_1, \widetilde{W}' : f^0, g']|_{H^{k-1}} \} \\ &\le C \{ (\frac{\nu + \widetilde{\nu}}{\gamma^2} |\lambda| + \omega) |\widetilde{\Phi}_1|_{H^k} + \nu (\frac{1}{\nu} |\lambda| + \frac{\nu + \widetilde{\nu}}{\nu} \omega) |\widetilde{W}'|_{H^{k+1}} \\ &\quad + (\frac{\nu + \widetilde{\nu}}{\gamma^2} + \frac{1}{|\lambda|}) |f^0|_{H^k} + |g'|_{H^{k-1}} \} \end{aligned} \tag{4.23}$$

for a positive constant C independent of λ. Let us define a map $\Gamma_1 : \dot{X}_k \to \dot{X}_k$ such that
$$\Gamma_1(\widetilde{\Phi}_1, \widetilde{W}') = (\Phi_1, W'),$$

where $(\Phi_1, W') \in \dot{X}_k$ is a solution of (4.22). From (4.23), for $(\widetilde{\Phi}_{1,1}, \widetilde{W}_1'), (\widetilde{\Phi}_{1,2}, \widetilde{W}_2') \in \dot{X}_k$, the estimate

$$\begin{aligned} &\left| \Gamma_1(\widetilde{\Phi}_{1,1}, \widetilde{W}_1') - \Gamma_1(\widetilde{\Phi}_{1,2}, \widetilde{W}_2') \right|_{H^k \times H^{k+1}} \\ &\le C_1 \{ (\frac{\nu + \widetilde{\nu}}{\gamma^2} + \frac{1}{\nu}) |\lambda| + (\frac{\nu + \widetilde{\nu}}{\nu} + 1) \omega \} \left| (\widetilde{\Phi}_{1,1} - \widetilde{\Phi}_{1,2}, \widetilde{W}_1' - \widetilde{W}_2') \right|_{\dot{X}_k} \end{aligned}$$

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holds for a positive constant C_1 independent of λ. If ω and $|\lambda|$ are so small that $\omega < \frac{1}{2C_1}\frac{\nu}{\nu+\widetilde{\nu}}$ and $|\lambda| < \frac{1}{2C_1}$, then $\Gamma_1 : \dot{X}_k \to \dot{X}_k$ is a contraction map. This implies that there is a unique $(\Phi_1, W') \in \dot{X}_k$ such that $\Gamma_1(\Phi_1, W') = (\Phi_1, W')$, i.e., there is a unique solution $(\Phi_1, W') \in \dot{X}_k$ of (4.20). Futhermore, from (4.23), (Φ_1, W') satisfies the estimate

$$|\Phi_1|_{H^k} + |W'|_{H^{k+1}} \leq C\big\{\big(1 + \tfrac{1}{|\lambda|}\big)|f^0|_{H^k} + |g'|_{H^{k-1}}\big\}, \tag{4.24}$$

where C is a positive constant independent of λ.

As for (4.21), for a given $\widetilde{W^3} \in H^{k+1}(D) \cap H_0^1(D)$, we consider the problem

$$\begin{cases} -\nu\Delta' W^3 = G^3[\Phi_1, W', \widetilde{W^3} : f^0, g^3], \\ W^3\mid_{\partial D} = 0, \end{cases} \tag{4.25}$$

where $(\Phi_1, W') \in \dot{X}_k$ is a solution of (4.20). It holds that

$$G^3[\Phi_1, W', \widetilde{W^3} : f^0, g^3] \in H^{k-1}(D).$$

In fact, we have

$$\begin{aligned}
&\big|G^3[\Phi_1, W', \widetilde{W^3} : f^0, g^3]\big|_{H^{k-1}} \\
&\leq C\big\{|\lambda|\big|\widetilde{W^3}\big|_{H^{k-1}} + \big|\Phi_1\big|_{H^{k-1}} + \big|W'\big|_{H^{k-1}} + |g^3|_{H^{k-1}} + \tfrac{1}{|\lambda|}|\langle f^0\rangle|\big\} \\
&\leq C_2\big\{|\lambda|\big|\widetilde{W^3}\big|_{H^{k-1}} + \big(1 + \tfrac{1}{|\lambda|}\big)|f^0|_{H^k} + |g|_{H^{k-1}}\big\}
\end{aligned} \tag{4.26}$$

for a positive constant C_2 independent of λ. If $|\lambda|$ is sufficiently small satisfying $|\lambda| < \min\big\{\frac{1}{2C_1}, \frac{1}{C_2}\big\}$, then there is a unique solution $W^3 \in H^{k+1}(D) \cap H_0^1(D)$ of (4.21). Furthermore, from (4.26), W^3 satisfies the estimate

$$|W^3|_{H^{k+1}} \leq C\big\{\big(1 + \tfrac{1}{|\lambda|}\big)|f^0|_{H^k} + |g|_{H^{k-1}}\big\}, \tag{4.27}$$

where C is a positive constant independent of λ.

Now we set

$$\Sigma_2 \equiv \big\{\lambda \neq 0 : |\lambda| < \min\big\{\tfrac{1}{2C_1}, \tfrac{1}{C_2}\big\}\big\}.$$

Since $\Phi = \sigma + \Phi_1$, we see that if $\omega < \frac{1}{2C_1}\frac{\nu}{\nu+\widetilde{\nu}}$ and $\lambda \in \Sigma_2$, then there is a unique solution $(\Phi, W) \in H^k(D) \times \big(H^{k+1}(D) \cap H_0^1(D)\big)$ of (4.13). Moreover, from (4.18), (4.24) and (4.27), Φ and W satisfies the estimate

$$\begin{aligned}
|\Phi|_{H^k} + |W|_{H^{k+1}} &\leq |\sigma| + |\Phi_1|_{H^k} + |W'|_{H^{k+1}} + |W^3|_{H^{k+1}} \\
&\leq C\big\{\big(1 + \tfrac{1}{|\lambda|}\big)|f^0|_{H^k} + |g|_{H^{k-1}}\big\}
\end{aligned}$$

for a positive constant C independent of $\lambda \in \Sigma_2$.

Since $D(L_0) \supset H^k(D) \times \big(H^{k+1}(D) \cap H_0^1(D)\big)$, we can replace $\mathcal{L}_0$ with L_0; and we find that if $\omega < \frac{1}{2C_1}\frac{\nu}{\nu+\widetilde{\nu}}$ and $\lambda \in \Sigma_2$, then $(\lambda+L_0)^{-1}f \in H^k(D) \times \big(H^{k+1}(D) \cap H_0^1(D)\big)$. Furthermore, $(\lambda + L_0)^{-1}f$ satisfies the estimate

$$|(\lambda + L_0)^{-1}f|_{H^k \times H^{k+1}} \leq C\big\{\big(1 + \tfrac{1}{|\lambda|}\big)|f^0|_{H^k} + |g|_{H^{k-1}}\big\},$$

where C is a positive constant independent of $\lambda \in \Sigma_2$. We thus obtain the desired estimates. The assertions for L_0^* can be proved in a similar manner. This completes the proof. $\qquad\square$

We finally obtain the following estimates for the eigenfunctions u_ξ and u_ξ^* associated with $\lambda_0(\xi)$ and $\overline{\lambda}_0(\xi)$, respectively, which yields the boundedness of $\Pi(\xi)$ on $H^k(D)$.

Theorem 4.7. *There exist positive constants ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu + \widetilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then there exists a positive constant r_0 such that for any $\xi \in \mathbf{R}$ with $|\xi| \leq r_0$ the following assertions hold. There exist u_ξ and u_ξ^* eigenfunctions associated with $\lambda_0(\xi)$ and $\overline{\lambda}_0(\xi)$, respectively, that satisfy*

$$\langle u_\xi, u_\xi^* \rangle = 1,$$

and the eigenprojection $\Pi(\xi)$ for $\lambda_0(\xi)$ is given by

$$\Pi(\xi)u = \langle u, u_\xi^* \rangle u_\xi.$$

Furthermore, u_ξ and u_ξ^* are written in the form

$$u_\xi(x') = u^{(0)}(x') + i\xi u^{(1)}(x') + |\xi|^2 u^{(2)}(x', \xi),$$
$$u_\xi^*(x') = u^{*(0)}(x') + i\xi u^{*(1)}(x') + |\xi|^2 u^{*(2)}(x', \xi),$$

and the following estimates hold

$$|u|_{H^{k+2}} \leq C_{k,r_0}$$

for $u \in \{u_\xi, u_\xi^*, u^{(1)}, u^{*(1)}, u^{(2)}, u^{*(2)}\}$ and $k = 0, 1, \cdots, k_0$: and a positive constant C_{k,r_0} depending on k and r_0.

We can prove Theorem 4.7 by using Proposition 4.6, similarly to the proof of [9, Lemma 4.3]. We thus omit the proof.

5 Spectral properties of $e^{-tL}P_0$

In this section we give a a factorization of $e^{-tL}P_0$ and prove Theorem 3.1 (i).

We denote the characteristic function of a set $\{\xi \in \mathbf{R} : |\xi| \leq r_0\}$ by $\mathbf{1}_{\{|\xi| \leq r_0\}}$, i.e.,

$$\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) = \begin{cases} 1, & |\xi| \leq r_0, \\ 0, & |\xi| > r_0. \end{cases}$$

We define the projection P_0 by

$$P_0 = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \Pi(\xi) \mathcal{F}.$$

P_0 is a bounded projection on $L^2(\Omega)$ satisfying

$$P_0 L \subset L P_0, \quad P_0 e^{-tL} = e^{-tL} P_0.$$

As in [2, 4], to investigate $e^{-tL} P_0$, we introduce operators related to u_ξ and u_ξ^*. We define the operators $\mathcal{T} : L^2(\mathbf{R}) \to L^2(\Omega)$, $\mathcal{P} : L^2(\Omega) \to L^2(\mathbf{R})$ and $\Lambda : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by

$$\mathcal{T}\sigma = \mathcal{F}^{-1}[\mathcal{T}_\xi \sigma], \qquad \mathcal{T}_\xi \sigma = \mathbf{1}_{\{|\xi| \le r_0\}} u_\xi \sigma;$$

$$\mathcal{P}u = \mathcal{F}^{-1}[\mathcal{P}_\xi u], \qquad \mathcal{P}_\xi u = \mathbf{1}_{\{|\xi| \le r_0\}} \langle u, u_\xi^* \rangle;$$

$$\Lambda\sigma = \mathcal{F}^{-1}[\mathbf{1}_{\{|\xi| \le r_0\}} \lambda_0(\xi) \sigma]$$

for $u \in L^2(\Omega)$ and $\sigma \in L^2(\mathbf{R})$. It then follows that

$$P_0 = \mathcal{T}\mathcal{P}, \quad e^{-tL} P_0 = \mathcal{T} e^{t\Lambda} \mathcal{P}.$$

We investigate boundedness properties of $\mathcal{T}$, $\mathcal{P}$ and $e^{t\Lambda}$.

As for $\mathcal{T}$, we have the following

Proposition 5.1. *The operator $\mathcal{T}$ has the following properties:*
(i) $\partial_{x_3}^l \mathcal{T} = \mathcal{T} \partial_{x_3}^l$ *for* $l = 0, 1, \cdots$.
(ii) $\|\partial_{x'}^k \partial_{x_3}^l \mathcal{T}\sigma\|_2 \le C\|\sigma\|_{L^2(\mathbf{R})}$ *for* $k = 0, 1, \cdots k_0$, $l = 0, 1, \cdots$ *and* $\sigma \in L^2(\mathbf{R})$.
(iii) $\mathcal{T}$ *is decomposed as*

$$\mathcal{T} = \mathcal{T}^{(0)} + \partial_{x_3} \mathcal{T}^{(1)} + \partial_{x_3}^2 \mathcal{T}^{(2)}.$$

Here $\mathcal{T}^{(j)}\sigma = \mathcal{F}^{-1}[\mathcal{T}^{(j)}\sigma]$ $(j = 0, 1, 2)$ *with*

$$\mathcal{T}^{(0)}\sigma = \mathbf{1}_{\{|\xi| \le r_0\}} \sigma u^{(0)},$$
$$\mathcal{T}^{(1)}\sigma = \mathbf{1}_{\{|\xi| \le r_0\}} \sigma u^{(1)}(\cdot),$$
$$\mathcal{T}^{(2)}\sigma = -\mathbf{1}_{\{|\xi| \le r_0\}} \sigma u^{(2)}(\cdot, \xi),$$

where $u^{(j)}$ $(j = 0, 1, 2)$ *are the functions given in Theorem 4.7. The assertions* (i) *and* (ii) *hold with $\mathcal{T}$ replaced by $\mathcal{T}^{(j)}$* $(j = 0, 1, 2)$.

Proof. It is clear that (i) is true. As for (ii), we can prove the estimates by using the properties of u_ξ in Theorem 4.7 and the Sobolev inequality. From the expansion of u_ξ given in Theorem 4.7, we can expand $\mathcal{T}$ as in (iii). $\qquad \square$

As for $\mathcal{P}$, we have the following properties.

Proposition 5.2. *The operator $\mathcal{P}$ has the following properties:*
(i) $\partial_{x_3}^l \mathcal{P} = \mathcal{P} \partial_{x_3}^l$ *for* $l = 0, 1, \cdots$.
(ii) $\|\partial_{x_3}^l \mathcal{P}u\|_{L^2(\mathbf{R})} \le C\|u\|_2$ *for* $k = 0, 1, \cdots k_0$, $l = 0, 1, \cdots$ *and* $u \in L^2(\Omega)$.
    Furthermore, $\|\mathcal{P}u\|_{L^2(\mathbf{R})} \le C\|u\|_1$ *for* $u \in L^1(\Omega)$.
(iii) $\mathcal{P}$ *is decomposed as*

$$\mathcal{P} = \mathcal{P}^{(0)} + \partial_{x_3} \mathcal{P}^{(1)} + \partial_{x_3}^2 \mathcal{P}^{(2)}.$$

Here $\mathcal{P}^{(j)}u = \mathcal{F}^{-1}[\mathcal{P}^{(j)}u]$ $(j = 0, 1, 2)$ with

$$\mathcal{P}^{(0)}u = \mathbf{1}_{\{|\xi| \leq r_0\}}\langle u, u^{*(0)}\rangle = \mathbf{1}_{\{|\xi| \leq r_0\}}\langle Q_0 u\rangle,$$

$$\mathcal{P}^{(1)}u = \mathbf{1}_{\{|\xi| \leq r_0\}}\langle u, u^{*(1)}\rangle,$$

$$\mathcal{P}^{(2)}u = -\mathbf{1}_{\{|\xi| \leq r_0\}}\langle u, u^{*(2)}(\xi)\rangle,$$

where $u^{(j)*}$ $(j = 0, 1, 2)$ are the functions given in Theorem 4.7. The assertions (i) *and* (ii) *hold with $\mathcal{P}$ replaced by $\mathcal{P}^{(j)}$ $(j = 0, 1, 2)$.*

Proof. It is clear that (i) holds true. As for (ii), we can prove the estimates by using the properties of u^*_ξ in Theorem 4.7 and the Sovolev inequality. From the expansion of u^*_ξ given in Theorem 4.7, we can expand $\mathcal{P}$ as in (iii). $\qquad\square$

As for Λ, we have the following decay estimates for $e^{t\Lambda}$.

Proposition 5.3. *The operator $e^{t\Lambda}$ satisfies the following decay estimates.*
(i) $\|\partial^l_{x_3} e^{t\Lambda}\mathcal{P}u\|_{L^2(\mathbf{R})} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u\|_1$,
(ii) $\|\partial^l_{x_3} e^{t\Lambda}\mathcal{P}^{(j)}u\|_{L^2(\mathbf{R})} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u\|_1$, $\quad j = 0, 1, 2$,
(iii) $\|\partial^l_{x_3}(\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P}u\|_2 \leq C(1+t)^{-\frac{3}{4}-\frac{l}{2}}\|u\|_1$,
for $u \in L^1(\Omega)$ and $l = 0, 1, 2\cdots$.

Proof. Since $\lambda_0(\xi) = -i\kappa_0\xi - \kappa_1\xi^2 + \mathcal{O}(|\xi|^3)$, we see from Theorem 4.7 that

$$\|\partial^l_{x_3} e^{t\Lambda}\mathcal{P}^{(j)}u\|_{L^2(\mathbf{R})} \leq C \int_{\mathbf{R}} \mathbf{1}_{\{|\xi| \leq r_0\}}|\xi|^{2l} e^{-t(i\kappa_0\xi + \kappa_1\xi^2)}|\langle u(\xi), u^{*(j)}\rangle|^2 d\xi$$

$$\leq C \int_{\mathbf{R}} \mathbf{1}_{\{|\xi| \leq r_0\}}|\xi|^{2l} e^{-t(i\kappa_0\xi + \kappa_1\xi^2)}|u(\xi)|^2_1 d\xi \qquad (5.1)$$

$$\leq C \begin{cases} \|u\|^2_1, \\ t^{-\frac{1}{2}-l}\|u\|^2_1. \end{cases}$$

This implies (i) and (ii). As for (iii), since $\mathcal{T} - \mathcal{T}^{(0)} = \partial_{x_3}\mathcal{T}^{(1)} + \partial^2_{x_3}\mathcal{T}^{(2)}$, we obtain the desired estimate from (i) and Proposition 5.1. $\qquad\square$

The estimate (3.4) in Theorem 3.1 follows from Propositions 5.1 and 5.3.

We next investigate the asymptotic behavior of e^{-tL}. Recall that $\mathcal{H}(t)$ is defined by

$$\mathcal{H}(t)\sigma = \mathcal{F}^{-1}[e^{-(i\kappa_0\xi + \kappa_1\xi^2)t}\sigma]$$

for $\sigma \in L^2(\mathbf{R})$, where $\kappa_0 \in \mathbf{R}$ and $\kappa_1 > 0$ are given in Theorem 4.5. We first introduce the well-known decay estimate for $\mathcal{H}(t)$.

Proposition 5.4. *There holds the estimate*

$$\|\partial^l_{x_3}(\mathcal{H}(t)\sigma)\|_{L^2(\mathbf{R})} \leq Ct^{-\frac{1}{4}-\frac{l}{2}}\|\sigma\|_{L^1(\mathbf{R})} \quad (l = 0, 1, \cdots)$$

for $\sigma \in L^1(\mathbf{R})$.

We next consider the asymptotic behavior of $e^{t\Lambda}$. The asymptotic leading part of $e^{t\Lambda}\mathcal{P}$ is given by $\mathcal{H}(t)$. In fact, we have the following

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Proposition 5.5. *For $u \in L^2(\Omega)$, we set $\sigma = \langle Q_0 u \rangle$. If $u \in L^1(\Omega)$, then there holds the estimate*

$$\|\partial_{x_3}^l \left(e^{t\Lambda} \mathcal{P} u - \mathcal{H}(t)\sigma \right)\|_{L^2(\mathbf{R})} \leq C t^{-\frac{3}{4}-\frac{l}{2}} \|u\|_1 \quad (l = 0, 1, \cdots).$$

Proof. By Proposition 5.2 we have

$$e^{t\Lambda} \mathcal{P} = e^{t\Lambda} \mathcal{P}^{(0)} + \partial_{x_3} e^{t\Lambda} \mathcal{P}^{(1)} + \partial_{x_3}^2 e^{t\Lambda} \mathcal{P}^{(2)}.$$

Set $\sigma = \langle Q_0 u \rangle$. Since $e^{t\Lambda} \mathcal{P}^{(0)} u = \mathcal{F}^{-1}[\mathbf{1}_{\{|\xi| \leq r_0\}} e^{\lambda_0(\xi)t} \sigma]$, we see that

$$\mathcal{F}[e^{t\Lambda} \mathcal{P}^{(0)} u - \mathcal{H}(t)\sigma] = (\mathbf{1}_{\{|\xi| \leq r_0\}} - 1) e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \sigma + \mathbf{1}_{\{|\xi| \leq r_0\}} \left(e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \right) \sigma.$$

By using the relation

$$\lambda_0(\xi) + (i\kappa_0 \xi + \kappa_1 \xi^2) = \mathcal{O}(|\xi|^3)$$

we obtain

$$e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} = e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \left(e^{(\lambda_0(\xi) + i\kappa_0 \xi + \kappa_1 \xi^2)t} - 1 \right)$$
$$= e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \mathcal{O}(|\xi|^3) t.$$

It then follows that

$$\int_{|\xi| \leq r_0} |\xi|^{2l} \left| \left(e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \right) \sigma \right|^2 d\xi \leq C \int_{|\xi| \leq r_0} |\xi|^{2(l+3)} t^2 e^{-2\kappa_1 \xi^2 t} d\xi \|\sigma\|_{L^1(\mathbf{R})}^2$$
$$\leq C \int_{|\xi| \leq r_0} (|\xi|^2 t)^2 e^{-\kappa_1 \xi^2 t} |\xi|^{2(l+1)} e^{-\kappa_1 \xi^2 t} d\xi \|\sigma\|_{L^1(\mathbf{R})}^2$$
$$\leq C \int_{|\xi| \leq r_0} |\xi|^{2(l+1)} e^{-\kappa_1 \xi^2 t} d\xi \|\sigma\|_{L^1(\mathbf{R})}^2$$
$$\leq C t^{-\frac{3}{2}-l} \|\sigma\|_{L^1(\mathbf{R})}^2.$$

On the other hand, we also have

$$\int_{|\xi| \leq r_0} |\xi|^{2l} \left| \left(e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \right) \sigma \right|^2 d\xi \leq C \|\sigma\|_{L^1(\mathbf{R})}^2.$$

We thus obtain

$$\int_{|\xi| \leq r_0} |\xi|^{2l} \left| \left(e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \right) \sigma \right|^2 d\xi \leq C(1+t)^{-\frac{3}{2}-l} \|\sigma\|_{L^1(\mathbf{R})}^2.$$

Similarly, we have

$$\|(\mathbf{1}_{\{|\xi| \leq r_0\}} - 1) e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \sigma\|_2^2 \leq C t^{-\frac{1}{2}-l} e^{-\kappa_1 r_0^2 t} \|\sigma\|_{L^1(\mathbf{R})}^2.$$

We thus see that

$$\|e^{t\Lambda} \mathcal{P}^{(0)} u - \mathcal{H}(t)\sigma\|_{L^2(\mathbf{R})} \leq C t^{-\frac{3}{4}-\frac{l}{2}} \|u_0\|_1.$$

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This estimate and Proposition 5.3 (ii) give the desired estimate. This completes the proof. □

We are now in a position to prove estimate (3.5) in Theorem 3.1 (i). In fact, we have

$$e^{-tL}P_0 u - [\mathcal{H}(t)\sigma]u^{(0)} = (\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P}u + \left[e^{t\Lambda}\mathcal{P}u - \mathcal{H}(t)\sigma\right]u^{(0)}.$$

This, together with Proposition 5.3 (iii) and Proposition 5.5, yields the desired estimate (3.5).

We finally state the estimates for the projection P_0.

Theorem 5.6. *The projection P_0 has the following properties:*
(i) $\partial_{x_3}^l P_0 = P_0 \partial_{x_3}^l$ *for $l = 0, 1, \cdots$.*
(ii) $\|\partial_{x'}^k \partial_{x_3}^l P_0 u\|_2 \leq C_k \|u\|_1$ *for $k = 0, 1, \cdots k_0$, $l = 0, 1, \cdots$ and $u \in L^1(\Omega)$.*
(iii) P_0 *is decomposed as*

$$P_0 = P_0^{(0)} + \partial_{x_3} P_0^{(1)} + \partial_{x_3}^2 P_0^{(2)},$$

where $P_0^{(j)} u = \mathcal{F}^{-1}[P_0^{(j)} u]$ $(j = 0, 1, 2)$ with

$$P_0^{(0)} = \mathcal{T}^{(0)}\mathcal{P}^{(0)} = \mathbf{1}_{\{|\xi| \leq r_0\}}\Pi^{(0)}, \tag{5.2}$$

$$P_0^{(1)} = \mathcal{T}^{(0)}\mathcal{P}^{(1)} + \mathcal{T}^{(1)}\mathcal{P}^{(0)} = -i\mathbf{1}_{\{|\xi| \leq r_0\}}\Pi^{(1)}, \tag{5.3}$$

$$P_0^{(2)} = \mathcal{T}^{(0)}\mathcal{P}^{(2)} + \mathcal{T}^{(1)}\{\mathcal{P}^{(1)} + \partial_{x_3}\mathcal{P}^{(2)}\} + \mathcal{T}^{(2)}\{\mathcal{P}^{(0)} + \partial_{x_3}\mathcal{P}^{(1)} + \partial_{x_3}^2\mathcal{P}^{(2)}\}. \tag{5.4}$$

Furthermore, $P_0^{(j)}$ $(j = 0, 1, 2)$ satisfy assertions (i) *and* (ii) *by replacing P_0 with $P_0^{(j)}$.*

Proof. It is clear that (i) is true. Estimates in (ii) are given by Propositions 5.1, 5.2. As for (iii), it is easy to see that $\partial_{x_3}^l P_0^{(j)} = P_0^{(j)} \partial_{x_3}^l$. The estimates

$$\|\partial_{x'}^k \partial_{x_3}^l P_0^{(j)} u\|_2 \leq C_k \|u\|_1$$

can also be obtained by Propositions 5.1, 5.2. The relations (5.3) and (5.4) can be verified by equating the coefficients of each power of ξ in the expansions of $\Pi(\xi)$ in (4.8) and $\langle u, u_\xi^* \rangle u_\xi$. This completes the proof. □

6 Decay estimate for $e^{-tL}(I - P_0)$

In this section we prove Theorem 3.1 (ii). We set

$$P_\infty = I - P_0.$$

To prove Theorem 3.1 (ii), we first introduce the decay estimate of $e^{-tL}P_\infty u_0$ for $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$.

Proposition 6.1. *There exist constants ν_0, γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then $e^{-tL}P_\infty u_0$ have the following properties. If $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$, then there exists a constant $d > 0$ such that $e^{-tL}P_\infty u_0$ satisfies*

$$\|e^{-tL}P_\infty u_0\|_{H^1} \leq C e^{-dt}\|u_0\|_{H^1} \tag{6.1}$$

for $t \geq 0$.

Proof. P_∞ is written as

$$P_\infty = P_{\infty,0} + \widetilde{P}_\infty,$$

where

$$P_{\infty,0}u = \mathcal{F}^{-1}[\widehat{P}_{\infty,0}u], \quad \widehat{P}_{\infty,0}u = \mathbf{1}_{\{|\xi|\leq r_0\}}(I - P_0)u,$$

$$\widetilde{P}_\infty u = \mathcal{F}^{-1}[\widehat{\widetilde{P}}_\infty u], \quad \widehat{\widetilde{P}}_\infty u = (1 - \mathbf{1}_{\{|\xi|\leq r_0\}})u.$$

The estimate $\|e^{-tL}\widetilde{P}_\infty u_0\|_{H^1} \leq C e^{-dt}\|u_0\|_{H^1}$ was proved in [1, Theorem 3.3]. As for $P_{\infty,0}$ part, since $\rho(-L_\xi \mid_{(I-\Pi_0(\xi))L^2}) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq -\frac{c_0}{2}\}$ by Theorem 4.4, we have

$$|e^{-tL_\xi}\widehat{P}_{\infty,0}u_0|_2 \leq C e^{-\frac{c_0}{4}t}|u_0|_2. \tag{6.2}$$

We now apply the argument of the proof of [1, Proposition 4.20] to $u(t) = e^{-tL}P_{\infty,0}u_0$. Due to (6.2), one can replace $e^{-\frac{d_2}{2}|\xi|^2 t}|u_0|_2^2$ in the inequality (4.72) of [1] by $e^{-\frac{c_0}{2}t}|u_0|_2^2$ to obtain $E_{4,1}^{(0)}[u](t) \leq C e^{-2\widetilde{d}_1 t}|u_0|_{H^1}^2$ for a positive constant $\widetilde{d}_1$. Integrating this over $|\xi| \leq r_0$ and using the Plancherel Theorem, we have

$$\|e^{-tL}P_{\infty,0}u_0\|_{H^1} \leq C e^{-\widetilde{d}t}\|u_0\|_{H^1}$$

for a positive constant $\widetilde{d}$. Combining the estimates for $e^{-tL}\widetilde{P}_\infty u_0$ and $e^{-tL}P_{\infty,0}u_0$ we obtain the desired estimate. This completes the proof. $\qquad\square$

We next consider the estimate for $e^{-tL}u$ for $0 < t \leq 1$.

Proposition 6.2. *Let $T > 0$. If $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$, then $e^{-tL}u_0$ satisfies $e^{-tL}u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ for $t > 0$ and*

$$\|e^{-tL}u_0\|_{H^1} \leq C_T\big\{\|u_0\|_{H^1\times\widetilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_2\big\} \tag{6.3}$$

for $0 < t \leq T$.

Let $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$. Applying Proposition 6.2 with $t = 1$, we have $u_1 = e^{-tL}u_0|_{t=1} \in H^1(\Omega) \times H_0^1(\Omega)$ and

$$\|u_1\|_{H^1} \leq C\|u_0\|_{H^1\times\widetilde{H}^1}.$$

This, together with Proposition 6.1 and Proposition 6.2, implies Theorem 3.1 (ii).

It remains to prove Proposition 6.2.

Lemma 6.3. *Let $T > 0$. There hold the following estimates for $0 \leq t \leq T$:*
(i) *for $\ell = 0, 1$,*

$$\|\partial_{x_3}^l u(t)\|_2^2 + c \int_0^t \|\nabla \partial_{x_3}^l w\|_2^2 + \|\mathrm{div}\, \partial_{x_3}^l w\|_2^2 d\tau \leq C_T \|\partial_{x_3}^l u_0\|_2^2,$$

(ii)

$$\|\chi_0 \partial_{x'} u(t)\|_2^2 + c \int_0^t \|\chi_0 \nabla \partial_{x'} w(\tau)\|_2^2 + \|\chi_0 \mathrm{div}\, \partial_{x'} w\|_2^2 d\tau$$

$$\leq C_T \Big\{ \|u_0\|_2^2 + \|\partial_{x_3} u_0\|_2^2 + \|\chi_0 \partial_{x'} u_0\|_2^2 + \int_0^t \|\partial_{x'}\phi(\tau)\|_2^2 d\tau \Big\},$$

(iii) *for $1 \leq m \leq N$,*

$$\|\chi_m \partial u(t)\|_2^2 + c \int_0^t \|\chi_m \nabla \partial w\|_2^2 + \|\chi_m \mathrm{div}\, \partial w\|_2^2 d\tau$$

$$\leq C_T \Big\{ \|u_0\|_2^2 + \|\partial_{x_3} u_0\|_2^2 + \|\chi_m \partial u_0\|_2^2 + \int_0^t \|\partial_{x'}\phi\|_2^2 d\tau \Big\}.$$

Lemma 6.3 can be proved by the energy method as those in the proof of [1, Propositions 4.7, 4.15, 4.17]. Note that here are no restrictions on ν, $\widetilde{\nu}$ and γ but C_T depends on T.

We next consider the L^2 estimate of the normal derivative for ϕ.

Lemma 6.4. *Let $T > 0$. For $1 \leq m \leq N$, there holds the estimate for $0 \leq t \leq T$:*

$$\|\chi_m \partial_n \phi(t)\|_2^2$$

$$\leq C_T \Big\{ \|u_0\|_2^2 + \|\partial_{x_3} u_0\|_2^2 + \|\chi_m \partial u_0\|_2^2 + \|\chi_m \partial_n \phi_0\|_2^2 + \int_0^t \|\partial_{x'}\phi\|_2^2 d\tau \Big\}.$$

Proof. Let us transform a scalar field $p(x')$ on $D \cap \mathcal{O}_m$ as

$$\widetilde{p}(y') = p(x') \quad \big(y' = \Psi^m(x'),\ x' \in D \cap \mathcal{O}_m\big),$$

where $\Psi^m(x')$ is a function given in Section 2. Similarly we transform a vector field $h(x') = {}^T\big(h^1(x'), h^2(x'), h^3(x')\big)$ into $\widetilde{h}(y') = {}^T\big(\widetilde{h}^1(y'), \widetilde{h}^2(y'), \widetilde{h}^3(y')\big)$ as

$$h(x') = E(y')\widetilde{h}(y')$$

where $E(y') = \big(e_1(y'), e_2(y'), e_3\big)$ with $e_1(y')$, $e_2(y')$ and e_3 given in Section 2. From the proof of [1, Proposition 4.16], we have

$$\partial_\tau \partial_{y_1}\widetilde{\phi} + \big(a + b\partial_{y_3}\big)\partial_{y_1}\widetilde{\phi} = \widetilde{\rho}_s I - \frac{\gamma^2 \widetilde{\rho}_s^2}{\nu + \widetilde{\nu}} \partial_\tau \widetilde{w}^1, \tag{6.4}$$

where

$$a(y') = \frac{\widetilde{\rho}_s \widetilde{P}'(\widetilde{\rho}_s)}{\nu + \widetilde{\nu}}, \quad b(y') = \widetilde{v}_s^3(y'),$$

$$I = -\frac{\gamma^2}{\nu+\widetilde{\nu}}\Big\{\nu\big(\mathrm{rot}_y\mathrm{rot}_y\widetilde{w}\big)^1 + \widetilde{\rho}_s\partial_{y_1}\Big(\frac{\widetilde{P}'(\widetilde{\rho}_s)}{\gamma^2\widetilde{\rho}_s}\Big)\widetilde{\phi} + \frac{\nu}{\gamma^2}\widetilde{\rho}_s\big(\Delta_{y'}\widetilde{v}_s\big)^1\widetilde{\phi} + \widetilde{\rho}_s\widetilde{v}_s^{\ 3}\partial_{y_3}\widetilde{w}^1\Big\}$$
$$-\Big\{\frac{1}{\widetilde{\rho}_s}\partial_{y_1}\widetilde{v}_s^{\ 3}\partial_{y_3}\widetilde{\phi} + \gamma^2\frac{1}{\widetilde{\rho}_s}\partial_{y_1}\big(\mathrm{div}_y(\widetilde{\rho}_s\widetilde{w})\big) - \gamma^2\partial_{y_1}\mathrm{div}_y\widetilde{w}\Big\}.$$

Here $\big(\mathrm{rot}_y\widetilde{w}\big)^1$ denotes the $e_1(y')$ component of $\mathrm{rot}_y\widetilde{w}$, and so on. We note that $\big(\mathrm{rot}_y\mathrm{rot}_y\widetilde{w}\big)^1$ does not contain $\partial_{y_1}^2$. See the proof of [1, Proposition 4.16]. We also note that there is a positive constant a_0 such that

$$a(y') \geq a_0 > 0$$

for any $y' \in \Psi^m(D)$.

We denote by $e^{-t(a+b\partial_{y_3})}$ the semigroup generated by $-(a+b\partial_{y_3})$, i.e,

$$e^{-t(a+b\partial_{y_3})}\widetilde{\phi}_0 = \mathcal{F}^{-1}\big[e^{-(a(y')+i\xi b(y'))t}\widehat{\widetilde{\phi}_0}\big].$$

Then it is easy to see that

$$\|\widetilde{\chi}_m e^{-t(a+b\partial_{y_3})}\widetilde{\phi}_0\|_2 \leq e^{-a_0 t}\|\widetilde{\chi}_m\widetilde{\phi}_0\|_2.$$

In terms of $e^{-t(a+b\partial_{y_3})}$, $\partial_{y_1}\widetilde{\phi}$ is written as

$$\begin{aligned}
\partial_{y_1}\widetilde{\phi}(t) &= e^{-t(a+b\partial_{y_3})}\partial_{y_1}\widetilde{\phi}_0 + \int_0^t e^{-(t-\tau)(a+b\partial_{y_3})}\widetilde{\rho}_s\widetilde{I}(\tau)d\tau \\
&\quad -\frac{\gamma^2\widetilde{\rho}_s^2}{\nu+\widetilde{\nu}}\int_0^t e^{-(t-\tau)(a+b\partial_{y_3})}\partial_\tau\widetilde{w}^1 d\tau \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}$$

As for J_1 and J_2, we have

$$\|\widetilde{\chi}_m J_1\|_2 \leq e^{-a_0 t}\|\widetilde{\chi}_m\partial_{y_1}\widetilde{\phi}_0\|_2,$$
$$\|\widetilde{\chi}_m J_2\|_2 \leq C\int_0^t e^{-a_0(t-\tau)}\|\widetilde{\chi}_m\widetilde{I}(\tau)\|_2 d\tau.$$

As for J_3, integrating by parts, we have

$$J_3 = \frac{\gamma^2\widetilde{\rho}_s^2}{\nu+\widetilde{\nu}}\Big[e^{-t(a+b\partial_{y_3})}\widetilde{w}_0^1 - \widetilde{w}^1(t) + (a+b\partial_{y_3})\int_0^t e^{-(t-\tau)(a+b\partial_{y_3})}\widetilde{w}^1(\tau)d\tau\Big].$$

We thus obtain

$$\|\widetilde{\chi}_m J_3\|_2 \leq C\Big\{e^{-a_0 t}\|\widetilde{\chi}_m\widetilde{w}_0^1\|_2 + \|\widetilde{\chi}_m\widetilde{w}^1(t)\|_2 + \int_0^t e^{-a_0(t-\tau)}\|\widetilde{\chi}_m\partial_{y_3}\widetilde{w}^1(\tau)\|_2 d\tau\Big\}.$$

Furthermore, we have

$$\begin{aligned}
\|\widetilde{\chi}_m I(\tau)\|_2 \leq C\big\{&\|\widetilde{\chi}_m\widetilde{\phi}(\tau)\|_2 + \|\widetilde{\chi}_m\partial_{y_3}\widetilde{\phi}(\tau)\|_2 + \|\widetilde{\chi}_m\widetilde{w}(\tau)\|_2 \\
&+ \|\widetilde{\chi}_m\nabla_y\widetilde{w}(\tau)\|_2 + \|\widetilde{\chi}_m\nabla_y\partial_{y_2}\widetilde{w}(\tau)\|_2 + \|\widetilde{\chi}_m\nabla_y\partial_{y_3}\widetilde{w}(\tau)\|_2\big\}.
\end{aligned}$$

It then follows that

$$\|\widetilde{\chi}_m \partial_{y_1} \widetilde{\phi}(t)\|_2 \le C \Big[e^{-a_0 t} \big(\|\widetilde{\chi}_m \partial_{y_1} \widetilde{\phi}_0\|_2 + \|\widetilde{\chi}_m \widetilde{w}_0^1\|_2 \big) + \|\widetilde{\chi}_m \widetilde{w}^1(t)\|_2$$
$$+ \int_0^t e^{-a_0(t-\tau)} \Big\{ \|\widetilde{\chi}_m \widetilde{\phi}(\tau)\|_2 + \|\widetilde{\chi}_m \partial_{y_3} \widetilde{\phi}(\tau)\|_2 + \|\widetilde{\chi}_m \widetilde{w}(\tau)\|_2$$
$$+ \|\widetilde{\chi}_m \nabla_y \widetilde{w}(\tau)\|_2 + \|\widetilde{\chi}_m \nabla_y \partial_{y_2} \widetilde{w}(\tau)\|_2 + \|\widetilde{\chi}_m \nabla_y \partial_{y_3} \widetilde{w}(\tau)\|_2 \Big\} d\tau \Big].$$

Inverting to the original coordinates x' and noting that $\partial_{y_1} = \partial_n$, $\partial_{y_2} = \partial$, we see that

$$\|\chi_m \partial_n \phi(t)\|_2 \le C \Big\{ e^{-a_0 t} \big(\|\chi_m \partial_n \phi_0\|_2 + \|\chi_m w_0^1\|_2 \big) + \|\chi_m w^1(t)\|_2$$
$$+ \int_0^t \|\chi_m \phi(\tau)\|_2 + \|\chi_m \partial_{x_3} \phi(\tau)\|_2 + \|\chi_m w(\tau)\|_2$$
$$+ \|\chi_m \nabla w(\tau)\|_2 + \|\chi_m \nabla \partial w(\tau)\|_2 + \|\chi_m \nabla \partial_{x_3} w(\tau)\|_2 d\tau \Big\}.$$

This, together with Lemma 6.3, yields the desired estimate. This completes the proof. $\qquad\square$

By Lemma 6.3 and Lemma 6.4, we have the following estimate.

Lemma 6.5. *Let $T > 0$. There exists a positive constant c such that the estimate*

$$\|u(t)\|_{H^1 \times \widetilde{H}^1}^2 + c \int_0^t \|\nabla w(\tau)\|_2^2 + \|\mathrm{div} w(\tau)\|_2^2 + \|\nabla \partial_{x_3} w(\tau)\|_2^2 + \|\mathrm{div} \partial_{x_3} w(\tau)\|_2^2$$
$$+ \|\chi_0 \nabla \partial_{x'} w(\tau)\|_2^2 + \|\chi_0 \mathrm{div} \partial_{x'} w(\tau)\|_2^2 + \sum_{m=1}^N \big\{ \|\chi_m \nabla \partial w(\tau)\|_2^2 + \|\chi_m \mathrm{div} \partial w(\tau)\|_2^2 \big\} d\tau$$
$$\le C_T \|u_0\|_{H^1 \times \widetilde{H}^1}^2$$

holds for $0 \le t \le T$.

We finally consider the L^2 estimate for $\partial_{x'} w$.

Lemma 6.6. *Let $T > 0$. There holds the estimate*

$$\|\partial_{x'} w(t)\|_2 \le C_T \{ \|u_0\|_{H^1 \times \widetilde{H}^1} + t^{-\frac{1}{2}} \|w_0\|_2 \}$$

for $0 < t \le T$.

Proof. We see that w satisfies the equation

$$\partial_t w + \overline{A} w + \overline{B} u = 0,$$

where $\overline{A}$ is the 3×3 operator defined by

$$\overline{A} = -\frac{\nu}{\rho_s} \Delta - \frac{\nu + \widetilde{\nu}}{\rho_s} \nabla \mathrm{div},$$

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$\overline{B}$ is the 3×4 operator defined by

$$\overline{B} = \begin{pmatrix} \nabla'\left(\frac{P(\rho_s)}{\gamma^2 \rho_s}\cdot\right) & v_s^3 \partial_{x_3} I_2 & 0 \\ \partial_{x_3}\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\cdot\right) + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} & {}^T(\nabla' v_s^3) & v_s^3 \partial_{x_3}\cdot \end{pmatrix}$$

We write $w(t)$ as

$$w(t) = e^{-t\overline{A}} w_0 + \int_0^t e^{-(t-\tau)\overline{A}} \overline{B} u(\tau) d\tau.$$

Then

$$\nabla' w(t) = \nabla' e^{-t\overline{A}} w_0 + \int_0^t \nabla' e^{-(t-\tau)\overline{A}} \overline{B} u(\tau) d\tau. \tag{6.5}$$

Since $\overline{A}$ is strongly elliptic, we have

$$\left\| \nabla' e^{-t\overline{A}} w_0 \right\|_2 \le C t^{-\frac{1}{2}} \| w_0 \|_2$$

for $0 < t \le T$. Furthermore, we see from Lemma 6.3 and Lemma 6.5 that

$$\begin{aligned}
\left\| \int_0^t \nabla' e^{-(t-\tau)\overline{A}} \overline{B} u(\tau) d\tau \right\|_2 &\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \| \overline{B} u(\tau) \|_2 d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \| u(\tau) \|_{H^1 \times \widetilde{H}^1} d\tau \\
&\le C \| u_0 \|_{H^1 \times \widetilde{H}^1} \int_0^t (t-\tau)^{-\frac{1}{2}} d\tau \\
&\le C T^{\frac{1}{2}} \| u_0 \|_{H^1 \times \widetilde{H}^1}
\end{aligned} \tag{6.6}$$

for $0 < t \le T$. It then follows from (6.5) and (6.6) that

$$\| \partial_{x'} w(t) \|_2 \le C_T \left\{ \| u_0 \|_{H^1 \times \widetilde{H}^1} + t^{-\frac{1}{2}} \| w_0 \|_2 \right\} \tag{6.7}$$

for $0 < t \le T$. This completes the proof. $\qquad\square$

Proof of Proposition 6.2. Let $u(t) = e^{-tL} u_0$. It is not difficult to see that if $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$, then $u(t)$ satisfies

$$u \in C\big([0, T]; H^1(\Omega) \times H_0^1(\Omega)\big), \quad \widetilde{Q} u \in L^2\big(0, T; H^2(\Omega)\big). \tag{6.8}$$

Using Lemma 6.5 and Lemma 6.6, we obtain the estimate

$$\| u(t) \|_{H^1}^2 + c \int_0^t \overline{D}_1[w](\tau) d\tau \le C_T \{ \| u_0 \|_{H^1 \times \widetilde{H}^1}^2 + t^{-1} \| w_0 \|_2^2 \}$$

for $0 < t \le T$. Here

$$\begin{aligned}
\overline{D}_1[w] = &\big(\| \nabla w \|_2^2 + \| \mathrm{div} w \|_2^2 \big) + \big(\| \nabla \partial_{x_3} w \|_2^2 + \| \mathrm{div} \partial_{x_3} w \|_2^2 \big) \\
&+ \big(\| \chi_0 \nabla \partial_{x'} w \|_2^2 + \| \chi_0 \mathrm{div} \partial_{x'} w \|_2^2 \big) + \sum_{m=1}^{N} \big(\| \chi_m \nabla \partial w \|_2^2 + \| \chi_m \mathrm{div} \partial w \|_2^2 \big).
\end{aligned}$$

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We thus obtain estimate (6.3) if $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, one can see from Lemma 6.5, (6.3) and (6.8) that if $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$, then $u(t)$ satisfies

$$u \in C\big([0,T]; H^1(\Omega) \times \widetilde{H}^1(\Omega)\big) \cap C\big((0,T]; H^1(\Omega) \times H_0^1(\Omega)\big)$$

and estimate (6.3). This completes the proof. $\qquad\square$

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