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Elliptic curves and Fibonacci numbers over Lindenmayer system multicell model through Symbolic Computation

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By symbolic computation, we have derived algebraic equations including elliptic curves to obtain cell-diversity condition. The development of a multicellular organism is a marvelous phenomenon. Starting from one or a few cells, the organism becomes a set of cells with diverse cell types to serve various functions. To obtain conditions for cell-type diversity, we construct a model using a Lindenmayer system. In virtue of symbolic computation, *quantifier elimination*, we have successfully derived explicit relations of cell-type diversity under some constraints. The derived relations are, interestingly and remarkably, yield points deeply related to *Fibonacci numbers* and include *elliptic curves* which have been provided by symbolic computation tools for the first time. Survey of the rational points and quadratic irrational numbers on these curves has revealed periodic and quasi-periodic structures, respectively, bringing us possible relevance of these points to forms of organs.

In a multicellular organism, a single cell—an egg—or a group of cells develops into a multicell with various different cell types. The multicells are thought to form according to specific rules, wherein the first cell type transforms into one different cell type or into two or more different cell types. Such cell-type transformation rules can be described as a tree diagram, termed a *cell-lineage* diagram, while the developmental process of multicells is usually termed *cell differentiation* (*1*). In 1968, such a cell-differentiation process was modeled by Lindenmayer (*2,3*).

For simplicity, the model proposed in this paper is described by one-dimensional cell chains, and is based on a *Lindenmayer system* (L-system), where the cell-lineage diagram is described by relations between *rewriting rules*. L-system models for developmental systems with cell lineages and for biological tissues such as blood vessels of the eye have been studied over past decades (*4–6*). In these L-systems, *interactions* between cells have been ignored, forming a 0L-system, because of the complexity or difficulty in analysis. By contrast, in this paper, symbolic computation based on the quantifier elimination (QE) method is presented for the derivation of algebraic equations over an L-system with interactions ("IL-system" hereafter). Furthermore, stochastic aspects can be introduced into an L-system, termed a stochastic L-system (*7,8*). The stochastic IL-system with interactions (sIL-system) can account for the influences of cell-type-dependent proliferation and transition rates (*9*).

In this paper, we focus on the algebraic equations between the cell-type diversity and some constraints on the cell chain. Further, we consider three basic cell-lineage diagrams: (i) the branching cell-lineage $A \Rightarrow \{B, C\}$ without the cell-type order conservation rule, (ii) the linear diagram $A \Rightarrow B \Rightarrow C$ with the conservation rule and (iii) the linear diagram without the rule. Through various symbolic approaches, we have successfully obtained five equations including *elliptic curves,* for the first time. Analysis of these equations provides classification of the multicell features under some conditions.

Throughout the previous and the present paper, we derived algebraic numbers or equations instead of approximate values. The reason is illustrated by this simple example: consider the algebraic number $(1+\sqrt{5})/2$ and its approximate value 1.618. The former is one root of $x^2-x-1=0$ in x, which can be expressed as a continued fraction:

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}.$$

Appearance of only 1 in the above formula reveals the relationship between $(1+\sqrt{5})/2$ and *Fibonacci* numbers or self-similarity (quasi-periodicity) of the corresponding sequence (*11*,*12*). By contrast, from the latter, 1.618, we cannot submit such a strong proposal. This is why we put a lot of effort into obtaining algebraic numbers by QE.

Lindenmayer system model. In this paper, we extract two basic and essential subgraphs of general cell lineage diagrams: *linear* and *branching* cell differentiations with three cell types. This model is described by an sIL-system for three cell types, A, B and C. Although we study the case of three cell types because more than three types bring us highly complicated and intractable calculation, the present approach of discrete model and symbolic computation will shed some light on the mechanism of cell-type diversity within multicellular organisms.

We assume the following two basic cell-differentiation graphs as subgraphs of the cell-lineage diagram, *Branching* $A \Rightarrow \{B, C\}$ and *Linear* $A \Rightarrow B \Rightarrow C$. The proliferations and the transitions between types are denoted by $A \rightarrow AA, B \rightarrow BB$ or $C \rightarrow CC$ and $A \rightarrow B$, $A \rightarrow C$ (in the branching) or $A \rightarrow B$ (in the linear), respectively. Further, the proliferation and transition rates of branching diagram are defined as follows:

$$A \rightarrow \begin{cases} AA & p_{1,1}, \\ B & p_{1,2}, \\ C & p_{1,3}, \\ A & 1 - p_{1,1} - p_{1,2} - p_{1,3}, \end{cases}, B \rightarrow \begin{cases} BB & p_{2,2}, \\ B & 1 - p_{2,2}, \end{cases}, C \rightarrow \begin{cases} CC & p_{3,3}, \\ C & 1 - p_{3,3} \end{cases}$$

with $0 < p_{i,j} < 1$ $(1 \leq i \leq j \leq 3)$ and $p_{1,1} + p_{1,2} + p_{1,3} < 1$. The rules $A \rightarrow A, B \rightarrow B$ or $C \rightarrow C$ represents that the type does not change.

Likewise, the rates of the linear diagram are defined as:

$$A \rightarrow \begin{cases} AA & p_{1,1}, \\ B & p_{1,2}, \\ A & 1 - p_{1,1} - p_{1,2}, \end{cases}, B \rightarrow \begin{cases} BB & p_{2,2}, \\ C & p_{2,3}, \\ B & 1 - p_{2,2} - p_{2,3}, \end{cases}, C \rightarrow \begin{cases} CC & p_{3,3}, \\ C & 1 - p_{3,3} \end{cases}$$

with $0 < p_{i,j} < 1$ $(1 \leq i \leq j \leq 3)$ and $p_{i,i} + p_{i,i+1} < 1$ $(1 \leq i \leq 2)$. In addition to the rewriting rules above, we adopt another rewriting rule, termed a *cell-type order conservation rule*:

$$AC \rightarrow ABC,\ CA \rightarrow CBA,$$

which guarantees the contiguity of cell types. When we adopt this conservation rule is applied after each application of the Branching or Linear rewriting rules. The cell-type order conservation rule originates from the “intercalary regeneration” phenomenon in cockroach legs, as illustrated in Fig. 1. When portions of the legs with non-contiguous positional values are grafted together, new tissue is intercalated to fill the gap so that the non-contiguous positional values disappear (*13, 14*). Such a regeneration phenomenon may be common in wound repair in the early stages of the cell-lineage diagram (*15*). This is why we analyze the effect of the conservation rule. In this paper, we analyze the three cases: (i) a branching cell-lineage diagram without the cell-type order conservation rule, (ii) a linear cell-lineage diagram without the rule and (iii) a linear diagram with the rule. We have not analyzed the case of a branching diagram with the rule because we have not found a corresponding phenomenon in the actual cell lineage.

The one-dimensional cell chain becomes longer as these rules are applied. Using these two models, we estimate the cell-type diversity by calculating the following composition in the cell chain:

$$AA, AB, BA, (AC, CA), BB, BC, CB, CC,$$

where (AC, CA) does not exist when the cell-type order conservation rule is adopted and applied in the elongation of a cell chain. We assume the following constraint:

$$gN(AA) = N(BB) = N(CC) \wedge gN(AB) = N(BC), \tag{1}$$

where $\wedge$ denotes “And,” g designates the parameter in this constraint, and $N(XY)$ denotes the number of the two contiguous cell types XY in the cell chain. $N(AB) = N(BA)$ and $N(BC) = N(CB)$ hold because of the mirror symmetry of the rewriting rules. If A is regarded as the initial cell type, A corresponds to a stem cell, which is thought to be a minority in real biological mature tissues but a majority in the early developmental stage (*1*). Under these constraints, in this paper, we have calculated the maximum value of $N(AB)/N(AA)$, which represents the ratio of junctions to non-junctions between different cell types. We have adopted this maximum as the measure of *cell-type diversity* in the sense that the appearance frequencies of cell types become uniform as the ratio $N(AB)/N(AA)$ approaches 1 under the constraint (1).

Estimation of cell-type composition. We have calculated the *growth matrix* of the two contiguous cell types, which enables us to estimate its composition after n applications of the Branching or Linear rewriting rules and the cell-type order conservation rule. In this paper, we calculate three growth matrices for the branching cell lineage without the cell-type order conservation rule and the linear diagram without and with the conservation rule (Table 1).

Let M be the growth matrix of the two contiguous cell types under some condition. Here, we start with AA, that is, the axiom is AA, therefore, the composition at step n can be calculated by $(1, 0, 0, \ldots)M^n$ (*16*). In this paper, we study the composition as n approaches infinity—in other words, for sufficiently long cell chains. First, we derive the conditions for coexistence of the two contiguous cell types from the eigenvalues and the eigenvectors of the growth matrices. Second, under the constraint (1) and the coexistence condition, we estimate the composition using the growth matrix. Last, we derive the equation between $N(AB)/N(AA)$ and the constraint parameter g in (1) by QE.

Derivation of algebraic equations by QE. In this paper, we need to obtain the maximum value of $N(AB)/N(AA)$. For this purpose, we have used the quantifier elimination (QE) method. QE is one of the main subjects in computer algebra (*17, 18*). In general, QE deals with first-order formulas consisting of polynomial equations, inequalities, quantifiers $(\exists, \forall)$ and boolean operators such as $\wedge$(And), $\vee$(Or), and $\neg$(Negation). QE computes an equivalent quantifier-free formula for a given first-order formula over the real field. For instance, for the input ${}^{\forall}x(x^2 + bx + c > 0)$, QE outputs the equivalent quantifier-free formula $b^2 - 4c < 0$. QE obtains relations between unquantified variables that make the input formula true. We obtain the maximum value of an objective polynomial under some constraints by adding one extra variable, which is assigned to the objective polynomial (*10*). For instance, in order to calculate the maximum value of $x/(p + y)$ under the constraints $x^2 + py^2 \leq 1 \wedge y \leq x^2$, we provide QE with the following input, adding an extra variable h:

$$
{}^{\exists}y{}^{\exists}x \ \left(x^2 + py^2 \leq 1 \wedge y \leq x^2 \wedge h(p + y) == x \wedge p + y \neq 0\right).
$$

For this formula, QE outputs $p < 0 \vee h^2p^3 - p - h^2 < 0$, which indicates that the maximum value of $x/(p + y)$ is $\sqrt{p/(p^3 - 1)}$ when $p > 1$ and does not exist ($+\infty$) when $p \leq 1$. Thus, even when parameters exist, we can obtain the maximum value, in other words, the equation between the maximum and parameters by QE. We can utilize `QEPCAD-B` or `Mathematica` Ver. 6/7 as QE-implemented software.

Elliptic curves derived by QE. We first derive the coexistence condition of two contiguous cell types. Analysis of the eigenvalues and their corresponding eigenvectors of the growth matrix reveals the coexistence condition: $2p_{1,1} - p_{1,2} - p_{1,3} > 0 \wedge 2p_{1,1} - p_{1,2} - p_{1,3} > 2p_{2,2} \wedge 2p_{1,1} - p_{1,2} - p_{1,3} > 2p_{3,3}$ (See also (*10, Section 5.2*) for details). Under this condition and the constraint (1), the following relations hold:

$$\begin{array}{ccc}
N(AB)/N(AA) & = & (p_{1,2}(g(1-p_{1,2}-p_{1,3})-p_{1,3}))/(2p_{1,3}+g(p_{1,2}+p_{1,3})), \\
p_{1,1} & = & (-(1-p_{1,2}-p_{1,3})(2p_{1,3}+g(p_{1,2}+p_{1,3})))/(2(p_{1,3}-g(1-p_{1,2}-p_{1,3}))), \\
p_{2,2} & = & (g(-2+p_{1,2}+p_{1,3})(p_{1,3}(p_{1,2}^2+2p_{1,3})+g(p_{1,2}^3+p_{1,2}^2(-1+p_{1,3})+p_{1,2} \\
 & & p_{1,3}+p_{1,3}^2)))/(2(p_{1,3}+g(-1+p_{1,2}+p_{1,3}))(g^2(p_{1,2}^2+p_{1,2}(-2+ \\
 & & p_{1,3})-p_{1,3})+p_{1,2}p_{1,3}+g(-1+p_{1,2})(p_{1,2}+2p_{1,3}))), \\
p_{3,3} & = & (gp_{1,3}(-2+p_{1,2}+p_{1,3})(p_{1,3}(2+p_{1,3})+g(p_{1,2}+p_{1,2}p_{1,3}+p_{1,3}^2)))/ \\
 & & (2(p_{1,3}+g(-1+p_{1,2}+p_{1,3}))(g^2(p_{1,2}^2+p_{1,2}(-2+p_{1,3})-p_{1,3})+p_{1,3}^2 \\
 & & +gp_{1,3}(-3+2p_{1,2}+p_{1,3}))).
\end{array}$$

To obtain the maximum value of $N(AB)/N(AA)$ by QE, the following input is used:

$$\begin{gathered}
{}^{\exists}p_{1,3}{}^{\exists}p_{1,2}\Big(\psi(p_{1,2},p_{1,3},g) \wedge h == N(AB)/N(AA) \wedge 0 < p_{1,2} \wedge \\
0 < p_{1,3} \wedge 0 < p_{1,1} \wedge 0 < p_{2,2} \wedge 0 < p_{3,3} \wedge p_{1,1}+p_{1,2}+p_{1,3} < 1\Big),
\end{gathered}$$

where $\psi(p_{1,2}, p_{1,3}, g)$ is a formula derived by combining conjunctively all equations and inequalities appearing in the coexistence condition and the constraint (1). Unfortunately, one cannot obtain the maximum if one provides the QE command of `QEPCAD-B` or `Mathematica` 6.0.1 with the above formula as it is. This is because the computation requires too much memory and time. Therefore, we have transformed the above formula into the equivalent formulas of reduced form as shown in Supporting Online Material. After all, `Mathematica` 6.0.1 outputs: $0 < h < (\sqrt{8g+5g^2}-g)/(2(2+g))$. This reveals that the equation between g and the maximum value f of $N(AB)/N(AA)(= h)$ is $(g+2)f^2 + gf - g = 0$.

Likewise, for the *linear* cell-lineage diagram without and with the cell-type order conservation rule, we have also exhaustively used the QE method to derive the algebraic equations between f (the maximum of $N(AB)/N(AA)$) and g, as shown in Table 2.

Classification of cell chains. So far, we have derived the algebraic equations between the cell-type diversity measure f and the parameter g under the constraint (1). We summarize in Table 2 and illustrate in Fig. 2 the equations between g and f in the three cases: the branching and linear cell-lineage diagrams with and without the cell-type order conservation rule.

Let us look closely at the values of the natural number g that make f a rational number. The rational numbers of the maximum value of $N(AB)/N(AA)$ shows that the sequence becomes periodic with respect to the two contiguous cell types as the cell chain becomes sufficiently long. We have searched for $g \in \mathbf{N}$ with corresponding rational f on the three curves (Table 2 and Fig. 2). Interestingly, only on the curve of the branching diagram without the rule $(g+2)f^2 + gf - g = 0$, there exist a pair satisfying $\{(g, f) \mid g \in \mathbf{N}, f \in \mathbf{Q}\}$. The proof is given in Supporting Online Material. On this curve, f is rational when g is in the following set: $\{2, 18, 128, 882, 6050, 41472, \ldots\} = \{g_n | g_{n+2} = 7g_{n+1} - g_n + 4,\ g_0 = 0, g_1 = 2, n \geq 1\}$. Somewhat remarkably, the pair (g_n, f_n) can also be described as $(2F_{2n}^2, F_{2n}/F_{2n+1})$, where F_n are *Fibonacci* numbers defined as $\{F_n \mid F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1,\ n \geq 1\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots\}$. The features of $g \in \mathbf{N}$ and their corresponding values of f are summarized as follows.

(a) When $g = 1$, for the branching diagram without the rule: $f = (\sqrt{13} - 1)/6 \sim 0.434259$; linear without the rule: $f = -1/2 + \{(9 + \imath\sqrt{687})^{1/3} + (9 - \imath\sqrt{687})^{1/3}\}/12^{2/3} \sim 0.557454$; and linear with the rule: $f = (1 + \sqrt{17})/8 \sim 0.640388$.

(b) When $g \geq 2$ and $g \in \mathbf{N}$, for the branching diagram without the rule: f is a rational number if g is in $\{2F_{2n}^2 \mid n \geq 1\}$, otherwise f is a quadratic irrational number. For the linear diagram without and with the rule: f is always a quadratic irrational number (See also Supporting Online Material). A quadratic irrational number is a solution to a quadratic equation $ax^2 + bx + c = 0$, where a, b, c are integers, and $b^2 - 4ac$ is positive and not a perfect square. The cell chain associated with a quadratic irrational number is quasi-periodic (*11, 12*). Furthermore, based on (*20*), we can construct the rewriting rules that produce the pattern associated with a given quadratic irrational or rational number f. Indeed, we have constructed the sequences in the case of $g = 1, f = (1 + \sqrt{17})/8$ in the linear diagram with the rule and $g = 2, f = 1/2$ in the branching diagram, respectively, as shown in Fig. 3. In this figure, we laid the 2-dimensional space with 40000 colored cells spirally, where red, green and blue boxes denotes $A-$, $B-$ and $C-$ cell types. Figures 3 (B) and (C) show quasi-periodic and periodic structure, respectively.

From the viewpoint of the developmental process of multicells, A-type cells in the diagram can be regarded as stem cells, and B- and C-types can be regarded as differentiated cells. In this schema, g can be one measure of the ratio of stem cells to differentiated cells. Therefore, the tissues according to the constraint (1) with large g (≥ 1) are thought to be mature, and with small g (< 1) to be in the early developmental stage. It has been shown that only when g is one of some specific natural numbers, f is rational in the cell-lineage diagrams we have analyzed. In other words, the cell chain is periodic only when $g_n = 2F_{2n}^2$ in the branching cell-lineage diagram.

Relationship between classification and various forms when g is large: Mature multicells. It seems reasonable to suppose that multicell structures are classified as periodic and quasi-periodic structures when g is a natural number. Figure 4 shows the schematic structures of various organs inside a human body (A–D). The structure of muscular cells seems periodic (Fig. 4(A)), while that of lung cells seems quasi-periodic (B). Interestingly, the cochlea of the inner ear forms a golden spiral derived from *Fibonacci* rectangles (*21*) (C). The finger is also characterized by *Fibonacci* numbers so that the motion path of the digits follows a golden spiral (*22*) (D).

Relationship when g is small: Early stage. Now, let us consider the region of small $g < 1$ in the cell-lineage diagrams with and without the cell-type order conservation rule. The curve ‡ in Table 2 is transformed into the minimal model: $y^2 + xy = x^3 - 7x + 9$, by bi-rational transformation:

$$\begin{cases} f & = (x-2)/(x+2), \\ g & = (2-x)(3+y)/(2x(2+x)), \end{cases}, \begin{cases} x & = 2(1+f)/(1-f), \\ y & = (-3f+3f^2-4g-4fg)/((1-f)f). \end{cases}$$

Likewise, the curve † is transformed into: $y^2 + xy + y = x^3 - x^2 - 9x + 9$, by:

$$\begin{cases} f & = (3x+y-1)/(x+y-3), \\ g & = (3x+y-1)^2(x^2-4x+3-2y)/(2(x+y-3)(x-y+5)(2x+y-2)), \end{cases},$$

$$\begin{cases} x & = (f^2(3f+1)+2(f+1)(f-2)g)/(f^2(f-1)), \\ y & = 2(f+1)(2f^2-(f-2)(f-3)g)/(f^2(f-1)^2). \end{cases}$$

These curves are, interestingly, *elliptic curves* (*23, Ch. 3*). These two curves have Mordell–Weil rank $= 1$ and no torsion points over $\mathbf{Q}$. Using the features of an elliptic curve (We can calculate all the rational points by using the fact that the point $P = (2,1)$ generates all the points under the group operation on the curve), we easily find the rational points of (g,f) $(0 < g \leq 1, 0 < f < 1)$ on the elliptic curve ‡:

$$\begin{aligned} (g,f) \quad = \quad & (1/4, 1/3), (725/3588, 145/483), (8832/9019493, 552/25201), \\ & (3466764335501/11363197974660, 127769297/347639747), \\ & \left(\frac{84858714933103932390656051 2}{2071989543607159017101756689 05}, \frac{525449923546336144}{11833844877112783169}\right), \\ & \left(\frac{302625468129280607689457377536931382 05817}{822504717497984192215154857880027459705 64}, \frac{302713566908857083166867201}{751660468695296029317353763}\right), \ldots . \end{aligned}$$

We can prove that the rational points are dense. Further, focusing on points such that $1/g \in \mathbf{N}$ and $f \in \mathbf{Q}(g > 0, 0 < f < 1)$, we have proved that there does not exist such a point in the branching diagram and that there exist only $(g,f) = (1/7, 1/4)$ and $(1/4, 1/3)$ on the curve † (the linear diagram without the rule) and the curve ‡ (the linear diagram with the rule), respectively. The finiteness of these points follows from Siegel's theorem on the finiteness of integral points (*23, Ch. 9*). In the above cases, we transformed the curves into quartic elliptic equations on which we can compute all integral points based on (*24, 25*).

After all, when $g < 1$ and $1/g \in \mathbf{N}$, the corresponding f is usually not rational, indicating that the resultant cell chain is usually aperiodic. This indicates that there are few periodic structures of multicells in the early developmental stage over this sIL-system.

Conclusion. In this paper, by symbolic computation, we have derived Fibonacci-number and Elliptic-curve relations between a measure of cell-type diversity and the cell-type ratio constraint. We have focused on patterns of cell chains with the "maximum" value. It is not certain that living things aim at the "maximum" structure. However, the form of living things can often be explained well by the optimality principle (*26*). For this reason, we have tried to obtain the equations between the maximum value (f) and the constraint on the structure (g). A survey of rational numbers and quadratic irrational numbers on these equations revealed remarkable features: rare *Fibonacci*-related periodicity and frequent quasi-periodicity. A more elaborate study on the relationship between these features and actual algae or organs may provide us with a more profound understanding of multicells.

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Table 1: The three growth matrices; we have calculated the matrices to estimate the composition of two contiguous cell types at step n. The top and middle matrices are of 9×9 form $(AA, AB, BA, AC, CA, BB, BC, CB, CC)$ because of the non-existence of the cell-type conservation rule, while the bottom is of 7×7 form $(AA, AB, BA, BB, BC, CB, CC)$ with the rule.

Branching diagram without the cell-type order conservation rule

$$\begin{pmatrix}
m_{1,1} & m_{1,2} & m_{1,3} & m_{1,2} & p_{1,2}^2 & p_{1,2}p_{1,3} & m_{1,3} & p_{1,2}p_{1,3} & p_{1,3}^2 \\
p_{1,1} & m_{2,2} & 0 & 0 & p_{1,2}+p_{2,2} & 0 & 0 & p_{1,3} & 0 \\
p_{1,1} & 0 & m_{2,2} & 0 & 0 & p_{1,2} & 0 & 0 & p_{1,3}+p_{3,3} \\
p_{1,1} & 0 & 0 & m_{2,2} & p_{1,2}+p_{2,2} & p_{1,3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1+2p_{2,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{2,2} & 1 & 0 & 0 & p_{3,3} \\
p_{1,1} & 0 & 0 & 0 & 0 & 0 & m_{2,2} & p_{1,2} & p_{1,3}+p_{3,3} \\
0 & 0 & 0 & 0 & p_{2,2} & 0 & 0 & 1 & p_{3,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+2p_{3,3}
\end{pmatrix}$$

with $m_{1,1} = 2p_{1,1} + (1 - p_{1,2} - p_{1,3})^2, m_{1,2} = p_{1,2}(1 - p_{1,2} - p_{1,3}),$
$m_{1,3} = p_{1,3}(1 - p_{1,2} - p_{1,3}), m_{2,2} = 1 - p_{1,2} - p_{1,3}.$

Linear diagram without the rule

$$\begin{pmatrix}
m'_{1,1} & m'_{1,2} & 0 & m'_{1,2} & p_{1,2}^2 & 0 & 0 & 0 & 0 \\
p_{1,1} & m'_{2,2} & p_{2,3}-p_{1,2}p_{2,3} & 0 & m'_{2,5} & p_{1,2}p_{2,3} & 0 & 0 & 0 \\
p_{1,1} & 0 & 1-p_{1,2} & 0 & 0 & p_{1,2} & 0 & 0 & p_{3,3} \\
p_{1,1} & 0 & 0 & m'_{2,2} & m'_{2,5} & 0 & p_{2,3}-p_{1,2}p_{2,3} & p_{1,2}p_{2,3} & 0 \\
0 & 0 & 0 & 0 & m'_{5,5} & m'_{5,6} & 0 & m'_{5,6} & p_{2,3}^2 \\
0 & 0 & 0 & 0 & p_{2,2} & 1-p_{2,3} & 0 & 0 & p_{2,3}+p_{3,3} \\
p_{1,1} & 0 & 0 & 0 & 0 & 0 & 1-p_{1,2} & p_{1,2} & p_{3,3} \\
0 & 0 & 0 & 0 & p_{2,2} & 0 & 0 & 1-p_{2,3} & p_{2,3}+p_{3,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+2p_{3,3}
\end{pmatrix}$$

with $m'_{1,1} = p_{1,1} + (1 - p_{1,2})^2, m'_{1,2} = p_{1,2}(1 - p_{1,2}), m'_{2,2} = (1 - p_{1,2})(1 - p_{2,3}),$
$m'_{2,5} = p_{1,2} + p_{2,2} - p_{1,2}p_{2,3}, m'_{5,5} = 2p_{2,2} + (1 - p_{2,3})^2, m'_{5,6} = p_{2,3}(1 - p_{2,3}).$

Linear diagram with the rule

$$\begin{pmatrix}
m''_{1,1} & m''_{1,2} & m''_{1,2} & p_{1,2}^2 & 0 & 0 & 0 \\
p_{1,1} & 1-p_{1,2} & 0 & m''_{2,4} & p_{2,3} & 0 & 0 \\
p_{1,1} & 0 & 1-p_{1,2} & m''_{2,4} & 0 & p_{2,3} & 0 \\
0 & 0 & 0 & m''_{3,4} & m''_{3,5} & m''_{3,5} & p_{2,3}^2 \\
0 & 0 & 0 & p_{2,2} & 1-p_{2,3} & 0 & p_{2,3}+p_{3,3} \\
0 & 0 & 0 & p_{2,2} & 0 & 1-p_{2,3} & p_{2,3}+p_{3,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 1+2p_{3,3}
\end{pmatrix}$$

with $m''_{1,1} = 2p_{1,1} + (1 - p_{1,2})^2, m''_{1,2} = (1 - p_{1,2})p_{1,2}, m''_{2,4} = p_{1,2} + p_{2,2} - p_{1,2}p_{2,3},$
$m''_{3,4} = 2p_{2,2} + (1 - p_{2,3})^2, m''_{3,5} = (1 - p_{2,3})p_{2,3}.$

Table 2: The algebraic equations between g and f; g denotes the parameter in the constraint (1) and f the measure of cell-type diversity. *root*$_{-1}$ denotes the largest real root of the equation $2(g+1)f^4 + g(g+1)f^3 - g(2g+5)f^2 - g^2 f + 2g^2 = 0$ in g. g_1 is nearly 1.29661, which is exactly the intersection of the two curves within $0 < g, 0 < f < 1$, the largest real root of $1 - 67g + 1224g^2 - 4008g^3 + 4599g^4 - 1701g^5 = 0$. Likewise, g_0 is nearly 1.14254 and exactly the real root of $1 - 3g + 11g^2 - 8g^3 = 0$.

Branching without the cell-type order conservation rule
$(g+2)f^2 + gf - g = 0$
Linear without the rule
$\begin{cases} g = \textit{root}_{-1} 2(g+1)f^4 + g(g+1)f^3 - g(2g+5)f^2 - g^2 f + 2g^{2\dagger}, & (0 \le g < g_1) \\ 2(3g-1)f^2 + (3g-1)f - 3g = 0 & (g \ge g_1) \end{cases}$
Linear with the rule
$\begin{cases} 2(g-1)f^3 - g(g+3)f^2 + gf + g^2 = 0^{\ddagger}, & (0 \le g < g_0) \\ 2gf^2 + (g-1)f - g = 0 & (g \ge g_0) \end{cases}$

Figure legends

Figure 1: Intercalary regeneration in cockroach legs (*27*). When mismatched portions of the growing legs are grafted together, new tissue is intercalated to fill the gap so that the non-contiguous positional values disappear. In this figure, we exemplify the rule: $I_8I_4 \rightarrow I_8I_7I_6I_5I_4$.

Figure 2: The equations between g and f. The gray line denotes the equation in the branching cell-lineage diagram without the cell-type order conservation rule. The black and broken lines denote the equations in the linear diagram with and without the rule, respectively. The derivative discontinuities are nearly $1.14254\,(g_0)$ and $1.29661\,(g_1)$. The intersection of the broken and gray lines is nearly 3.28773 (g_2), and that of the black and gray lines is nearly 3.83118 (g_3). The f-value of the broken and black lines approaches $1/2$ as $g \rightarrow \infty$, while that of the gray line approaches the reciprocal of the golden ratio $1/\varphi \sim 0.61803$ $\left(\varphi = (1+\sqrt{5})/2\right)$ as $g \rightarrow \infty$.

Figure 3: Spiral tiling of cells. (A) Schematic illustration of spiral tiling of one-dimensional cell chain. (B) $g = 1, f = (1+\sqrt{17})/8$ in the linear diagram with the rule. (C) $g = 2, f = 1/2$ in the branching diagram.

Figure 4: The structures of the organs. (A) Muscular cells seem periodic. (B) Lung cells seem quasi-periodic. (C) The cochlea of the inner ear forms a golden spiral related to *Fibonacci* numbers. (D) The motion paths of the digits also form a golden spiral (*22*).

Supporting Online Material

We show the reduced form of complicated formulas in the case of Branching and Linear diagrams. Via factorization, we have derived the formulas with respect to the branching cell-lineage as follows:

$$
\begin{gathered}
{}^{\exists}p_{1,3}{}^{\exists}p_{1,2}\left(0 < p_{1,2} \wedge 0 < p_{1,3} \wedge\ - h(2p_{1,3} + g(p_{1,2} + p_{1,3})) == p_{1,2}(p_{1,3} + g(-1 + p_{1,2} + p_{1,3}) \wedge \right. \\
\left. e_1 < 0 \wedge e_2 < 0 \wedge e_3 < 0 \wedge e_4 < 0 \wedge e_5 < 0 \wedge e_6 < 0 \right)
\end{gathered}
$$

with:

$$
\begin{array}{rcc}
e_1 & := & p_{1,3} + g(-1 + p_{1,2} + p_{1,3}), \\
e_2 & := & (g^2(p_{1,2}^2 + p_{1,2}(-2 + p_{1,3}) - p_{1,3}) + p_{1,2}p_{1,3} + g(-1 + p_{1,2})(p_{1,2} + 2p_{1,3})) \\
 & & (p_{1,3}(p_{1,2}^2 + 2p_{1,3}) + g(p_{1,2}^3 + p_{1,2}^2(-1 + p_{1,3}) + p_{1,2}p_{1,3} + p_{1,3}^2)), \\
e_3 & := & (g^2(p_{1,2}^2 + p_{1,2}(-2 + p_{1,3}) - p_{1,3}) + p_{1,3}^2 + gp_{1,3}(-3 + 2p_{1,2} + p_{1,3})), \\
e_4 & := & (4p_{1,3} + g(-2 + 3p_{1,2} + 3p_{1,3})), \\
e_5 & := & g^2(p_{1,2}^2 + p_{1,2}(-2 + p_{1,3}) - p_{1,3}) + p_{1,2}p_{1,3} + g(-1 + p_{1,2})(p_{1,2} + 2p_{1,3}), \\
e_6 & := & g^2(p_{1,2}^2 + p_{1,2}(-2 + p_{1,3}) - p_{1,3}) + p_{1,3}^2 + gp_{1,3}(-3 + 2p_{1,2} + p_{1,3}),
\end{array}
$$

under the physiological condition: $0 < h < 1, g > 0$. After all, `Mathematica` 6.0.1 outputs: $0 < h < (\sqrt{8g + 5g^2} - g)/(2(2+g))$. This reveals that the equation between g and the maximum value f of $N(AB)/N(AA)(= h)$ is $(g + 2)f^2 + gf - g = 0$.

Likewise, for the *linear* cell-lineage diagram without and with the cell-type order conservation rule, we have also successfully used the QE method to derive the algebraic equations between f (the maximum of $N(AB)/N(AA)$) and g, as shown in Table 2. The procedure for deriving these equations follows. The eigenvalues of the growth matrix in the middle row of Table 1 are:

$$
\begin{gathered}
1 - p_{1,2}, 1 + 2p_{1,1} - p_{1,2}, (1 - p_{1,2})^2, 1 - p_{2,3}, 1 + 2p_{2,2} - p_{2,3}, \\
(1 - p_{1,2})(1 - p_{2,3}), (1 - p_{2,3})^2, 1 + 2p_{3,3}.
\end{gathered}
$$

The coexistence condition is calculated as:

$$
2p_{1,1} > p_{1,2} \wedge 2p_{1,1} - p_{1,2} > 2p_{2,2} - p_{2,3} \wedge 2p_{1,1} - p_{1,2} > 2p_{3,3}.
$$

Under the constraint and the coexistence condition, we obtain the following relations:

$$\begin{array}{rcc}
N(AB)/N(AA) & = & ((1-p_{1,2})p_{1,2})/(2p_{1,1}+(1-p_{1,2})p_{2,3}), \\
p_{2,2} & = & (p_{1,1}(2p_{1,1}-p_{1,2}+p_{2,3})(p_{1,2}^2(-2p_{1,1}-(-1+p_{1,2})(-2+p_{2,3}))g \\
 & & (4p_{1,1}^2+-2p_{1,1}((-3+p_{2,3})p_{2,3}+p_{1,2}(1+p_{2,3}))+p_{2,3}(p_{1,2}^2-(- \\
 & & 2+p_{2,3})p_{2,3}+p_{1,2}(-1-2p_{2,3}+p_{2,3}^2))))))/((-1+p_{1,2})p_{1,2}(-4 \\
 & & p_{1,1}^2+(-1+p_{1,2})p_{1,2}p_{2,3}-2p_{1,1}(p_{1,2}(-1+p_{2,3})+p_{2,3}))2gp_{1,1}(\\
 & & 4p_{1,1}^2-2p_{1,1}((-3+p_{2,3})p_{2,3}+p_{1,2}(1+p_{2,3}))+p_{2,3}(p_{1,2}^2-(-2+ \\
 & & p_{2,3})p_{2,3}+p_{1,2}(-1-2p_{2,3}+p_{2,3}^2)))), \\
p_{3,3} & = & ((1+2p_{1,1}-p_{1,2})(-1+p_{1,2})p_{1,2}^2p_{2,3}^2+gp_{1,1}(8p_{1,1}^3-4p_{1,1}^2((-2+ \\
 & & p_{2,3})p_{2,3}+p_{1,2}(2+p_{2,3}))+2p_{1,1}(-p_{2,3}^2+p_{1,2}^2(1+2p_{2,3})+p_{1,2}p_{2,3} \\
 & & (-3+p_{2,3}^2))+p_{2,3}(-p_{1,2}^3+(-2+p_{2,3})p_{2,3}^2+p_{1,2}^2(1+p_{2,3}-p_{2,3}^2)- \\
 & & p_{1,2}p_{2,3}(1-3p_{2,3}+p_{2,3}^2)))))/((2p_{1,1}+p_{2,3}-p_{1,2}p_{2,3})(-(-1+p_{1,2} \\
 & &)p_{1,2}p_{2,3}+2gp_{1,1}(2p_{1,1}-p_{1,2}+2p_{2,3}-p_{2,3}^2))).
\end{array}$$

To obtain the maximum value of $N(AB)/N(AA)$, we provide QE with the following input:

$$\begin{gathered}
{}^{\exists}p_{1,1}{}^{\exists}p_{1,2}{}^{\exists}p_{2,3}\Big(\psi'(p_{1,1},p_{1,2},p_{2,3},g)\wedge h == N(AB)/N(AA)\wedge 0<p_{1,2}<1\wedge \\
0<p_{2,3}<1\wedge 0<p_{1,1}\wedge 0<p_{2,2}\wedge 0<p_{3,3}<1\wedge p_{1,1}+p_{1,2}<1\wedge p_{2,2}+p_{2,3}<1\Big),
\end{gathered}$$

where $\psi'(p_{1,1},p_{1,2},p_{2,3},g)$ is a formula derived by combining conjunctively all equations and inequalities appearing in the coexistence condition and the constraint. The above formula was, unfortunately, too complicated to analyze by the QE program (`QEPCAD-B` or `Mathematica`).

Therefore, we transform the formula in the following manner.

- From $h = N(AB)/N(AA)$,

  $$p_{1,1} = \frac{(1-p_{1,2})(p_{1,2}-hp_{2,3})}{2h} \tag{2}$$

  holds. Using 2, we eliminate one variable, $p_{1,1}$.

- Next, via factorization, the inequalities with $p_{1,1}$ eliminated, can be transformed into the following formulas:

  $$e'_1<0\wedge e'_2<0\wedge e'_3>0\wedge e'_4<0\wedge e'_5<0\wedge e'_6<0\wedge e'_7<0 \tag{3}$$

with:

$$
\begin{array}{rcc}
e'_1 & := & -p_{1,2} + hp_{2,3}, \\
e'_2 & := & (1 - h - p_{1,2} + hp_{2,3})(-2h^2 p_{1,2} + gp_{1,2} - hgp_{1,2} - hp_{1,2}^2 - gp_{1,2}^2 + hgp_{2,3} + 2h^2 \\
 & & p_{1,2}p_{2,3} + hgp_{1,2}p_{2,3} - hgp_{2,3}^2)(-hp_{1,2}^2 + h^2 p_{1,2}^2 - gp_{1,2}^2 + hgp_{1,2}^2 + hp_{1,2}^3 + gp_{1,2}^3 \\
 & & +h^2 p_{1,2}p_{2,3} - 2h^3 p_{1,2}p_{2,3} - h^2 gp_{1,2}p_{2,3} - 3h^2 p_{1,2}^2 p_{2,3} - 2hgp_{1,2}^2 p_{2,3} + h^2 gp_{2,3}^2 \\
 & & +2h^3 p_{1,2}p_{2,3}^2 + hgp_{1,2}p_{2,3}^2 + h^2 gp_{1,2}p_{2,3}^2 - h^2 gp_{2,3}^3), \\
e'_3 & := & gp_{1,2}^3 - 2hgp_{1,2}^3 + h^2 gp_{1,2}^3 - 2gp_{1,2}^4 + 2hgp_{1,2}^4 + gp_{1,2}^5 - 2hgp_{1,2}^2 p_{2,3} + 3h^2 gp_{1,2}^2 \\
 & & p_{2,3} - h^3 gp_{1,2}^2 p_{2,3} + 5hgp_{1,2}^3 p_{2,3} - 4h^2 gp_{1,2}^3 p_{2,3} - 3hgp_{1,2}^4 p_{2,3} - 2h^4 p_{1,2}p_{2,3}^2 \\
 & & -h^2 gp_{1,2}p_{2,3}^2 - h^3 gp_{1,2}p_{2,3}^2 - 2h^3 p_{1,2}^2 p_{2,3}^2 - hgp_{1,2}^2 p_{2,3}^2 - 3h^2 gp_{1,2}^2 p_{2,3}^2 + 2h^3 \\
 & & gp_{1,2}^2 p_{2,3}^2 + hgp_{1,2}^3 p_{2,3}^2 + 3h^2 gp_{1,2}^3 p_{2,3}^2 + 2h^3 gp_{2,3}^3 + 2h^4 p_{1,2}p_{2,3}^3 + 3h^2 gp_{1,2}p_{2,3}^3 \\
 & & -2h^2 gp_{1,2}^2 p_{2,3}^3 - h^3 gp_{1,2}^2 p_{2,3}^3 - 2h^3 gp_{2,3}^4 + h^3 gp_{1,2}p_{2,3}^4, \\
e'_4 & := & -2h^2 p_{1,2}^2 + 2h^3 p_{1,2}^2 - 2hgp_{1,2}^2 + 2h^2 gp_{1,2}^2 + 2h^3 p_{1,2}^3 + gp_{1,2}^3 + h^2 gp_{1,2}^3 - hp_{1,2}^4 \\
 & & +3h^2 p_{1,2}^4 - 2gp_{1,2}^4 + 2hgp_{1,2}^4 + hp_{1,2}^5 + gp_{1,2}^5 + 2h^3 p_{1,2}p_{2,3} - 4h^4 p_{1,2}p_{2,3} - 2h^3 \\
 & & gp_{1,2}p_{2,3} + 2h^2 p_{1,2}^2 p_{2,3} - 6h^3 p_{1,2}^2 p_{2,3} - 2h^4 p_{1,2}^2 p_{2,3} + 2hgp_{1,2}^2 p_{2,3} - 5h^2 gp_{1,2}^2 \\
 & & p_{2,3} - h^3 gp_{1,2}^2 p_{2,3} + h^2 p_{1,2}^3 p_{2,3} - 7h^3 p_{1,2}^3 p_{2,3} + hgp_{1,2}^3 p_{2,3} - 4h^2 gp_{1,2}^3 p_{2,3} - 4 \\
 & & h^2 p_{1,2}^4 p_{2,3} - 3hgp_{1,2}^4 p_{2,3} + 2h^3 gp_{2,3}^2 - 2h^3 p_{1,2}p_{2,3}^2 + 8h^4 p_{1,2}p_{2,3}^2 + h^2 gp_{1,2}p_{2,3}^2 \\
 & & +5h^3 gp_{1,2}p_{2,3}^2 + 4h^3 p_{1,2}^2 p_{2,3}^2 + 4h^4 p_{1,2}^2 p_{2,3}^2 - hgp_{1,2}^2 p_{2,3}^2 + 5h^2 gp_{1,2}^2 p_{2,3}^2 + 2h^3 \\
 & & gp_{1,2}^2 p_{2,3}^2 + 5h^3 p_{1,2}^3 p_{2,3}^2 + hgp_{1,2}^3 p_{2,3}^2 + 3h^2 gp_{1,2}^3 p_{2,3}^2 - 4h^3 gp_{2,3}^3 - 4h^4 p_{1,2}p_{2,3}^3 \\
 & & -h^2 gp_{1,2}p_{2,3}^3 - 4h^3 gp_{1,2}p_{2,3}^3 - 2h^4 p_{1,2}^2 p_{2,3}^3 - 2h^2 gp_{1,2}^2 p_{2,3}^3 - h^3 gp_{1,2}^2 p_{2,3}^3 + 2 \\
 & & h^3 gp_{2,3}^4 + h^3 gp_{1,2}p_{2,3}^4, \\
e'_5 & := & -1 + h + p_{1,2} - hp_{2,3}, \\
e'_6 & := & -hp_{1,2}^2 + h^2 p_{1,2}^2 - gp_{1,2}^2 + hgp_{1,2}^2 + hp_{1,2}^3 + gp_{1,2}^3 + h^2 p_{1,2}p_{2,3} - 2h^3 p_{1,2}p_{2,3} - h^2 \\
 & & gp_{1,2}p_{2,3} - 3h^2 p_{1,2}^2 p_{2,3} - 2hgp_{1,2}^2 p_{2,3} + h^2 gp_{2,3}^2 + 2h^3 p_{1,2}p_{2,3}^2 + hgp_{1,2}p_{2,3}^2 + \\
 & & h^2 gp_{1,2}p_{2,3}^2 - h^2 gp_{2,3}^3, \\
e'_7 & := & -gp_{1,2}^2 + hgp_{1,2}^2 + gp_{1,2}^3 - h^2 p_{1,2}p_{2,3} - h^2 gp_{1,2}p_{2,3} - 2hgp_{1,2}^2 p_{2,3} + h^2 gp_{2,3}^2 + \\
 & & hgp_{1,2}p_{2,3}^2 + h^2 gp_{1,2}p_{2,3}^2 - h^2 gp_{2,3}^3.
\end{array}
$$

Notice that in the above formulas, we show only those inequalities that maximize $N(AB)/N(AA)$; it is relatively easy to find these via polynomial factorization and confirmation by QE input for fixed values of g.

- Furthermore, the equality $gN(AB) = N(BC)$ in the constraint condition yields:

$$
\begin{array}{rcc}
0 & = & -4gp_{1,1}^2 p_{1,2} + 2gp_{1,1}p_{1,2}^2 + 4gp_{1,1}^2 p_{1,2}^2 - 2gp_{1,1}p_{1,2}^3 + 4gp_{1,1}^2 p_{2,3} - 2gp_{1,1}p_{1,2} \\
 & & p_{2,3} + p_{1,2}^2 p_{2,3} + 2p_{1,1}p_{1,2}^2 p_{2,3} + 2gp_{1,1}p_{1,2}^2 p_{2,3} - 2p_{1,2}^3 p_{2,3} - 2p_{1,1}p_{1,2}^3 p_{2,3} + p_{1,2}^4 \\
 & & p_{2,3} + 2gp_{1,1}p_{2,3}^2 - 4gp_{1,1}^2 p_{2,3}^2 - 2gp_{1,1}p_{1,2}p_{2,3}^2 - 2gp_{1,1}p_{2,3}^3 + 2gp_{1,1}p_{1,2}p_{2,3}^3.
\end{array}
$$

Substituting 2 into the above equality yields:

$$
g = \frac{hp_{1,2}p_{2,3}(-h - p_{1,2} + hp_{2,3})}{(-p_{1,2} + hp_{2,3})(p_{1,2} - hp_{1,2} - p_{1,2}^2 - p_{2,3} + hp_{1,2}p_{2,3} + p_{2,3}^2)}. \tag{4}
$$

We therefore eliminate one more variable, g.

- By eliminating g from the equalities and inequalities and factorizing, we obtain the following reduced formulas suitable for QEPCAD-B:

  $$^{\exists}p_{2,3}{}^{\exists}p_{1,2}(0 < h < 1 \wedge g > 0 \wedge 0 < p_{1,2} < 1 \wedge 0 < p_{2,3} < 1 \wedge e_{p0} == 0 \wedge e_{p1} < 0 \wedge e_{p2} < 0)$$

  with:

  $$\begin{array}{rcc} e_{p0} & := & hp_{1,2}p_{2,3}(-h - p_{1,2} + hp_{2,3}) - g(-p_{1,2} + hp_{2,3})(p_{1,2} - hp_{1,2} - p_{1,2}^2 \\ & & -p_{2,3} + hp_{1,2}p_{2,3} + p_{2,3}^2) \\ e_{p1} & := & 2hp_{1,2}^2 - 2h^2p_{1,2}^2 + p_{1,2}^3 - 3hp_{1,2}^3 - p_{1,2}^4 - 3hp_{1,2}p_{2,3} - h^2p_{1,2}p_{2,3} + \\ & & 2h^3p_{1,2}p_{2,3} - 2p_{1,2}^2p_{2,3} - hp_{1,2}^2p_{2,3} + 7h^2p_{1,2}^2p_{2,3} + p_{1,2}^3p_{2,3} + 4hp_{1,2}^3 \\ & & p_{2,3} + h^2p_{2,3}^2 + 5hp_{1,2}p_{2,3}^2 - 4h^3p_{1,2}p_{2,3}^2 + p_{1,2}^2p_{2,3}^2 - 2hp_{1,2}^2p_{2,3}^2 - 5h^2 \\ & & p_{1,2}^2p_{2,3}^2 - 2h^2p_{2,3}^3 - 2hp_{1,2}p_{2,3}^3 + h^2p_{1,2}p_{2,3}^3 + 2h^3p_{1,2}p_{2,3}^3 + h^2p_{2,3}^4, \\ e_{p2} & := & -p_{1,2} + hp_{1,2} + p_{1,2}^2 + hp_{2,3} + 2h^2p_{2,3} - hp_{1,2}p_{2,3} + hp_{2,3}^2. \end{array}$$

- Last, because of the RAM limitation (up to 4 GB) of 32-bit software, we provide the above formulas for the QE program (We used `QEPCAD-B` Ver. 1.48 with `measure-zero-error` and $+N670000000$ option. It took from 4.5 to 12 hours and 3.7 GB RAM with an `Intel Xeon` CPU 2.33 GHz processor), with g-space divided into $(0, 1/7]$, $(1/7, 1/4]$, $(1/4, 1)$, $[1, 2]$,$(2, \infty)$.

Summing up, with respect to the *linear* cell-lineage diagram without the rule, we obtain the following equivalent quantifier-free formulas as QE outputs:

$$\begin{cases} g \geq \textit{root}_{-1}\, h^3g^2 - 2h^2g^2 - hg^2 + 2g^2 + 2h^4g + h^3g - 5h^2g + 2h^4 & \begin{array}{c}\text{For } (0, 1/7], \\ (1/7, 1/4], (1/4, 1),\end{array} \\ \left(\begin{array}{c} h^3g^2 - 2h^2g^2 - hg^2 + 2g^2 + 2h^4g + h^3g - 5h^2g + 2h^4 \geq 0 \\ \wedge 6h^2g + 3hg - 3g - 2h^2 - h \leq 0 \end{array} \right. & \text{For } [1, 2], \\ 6h^2g + 3hg - 3g - 2h^2 - h \leq 0 & \text{For } (2, \infty), \end{cases}$$

where *root*$_{-1}$ denotes the largest real root of the equation $h^3g^2 - 2h^2g^2 - hg^2 + 2g^2 + 2h^4g + h^3g - 5h^2g + 2h^4 = 0$ in g. The middle row of Table 2 has been constructed from the above QE output. The equation for the linear diagram with the rule is the same as in (*19*).

Proof of the existence or non-existence of rational numbers f.

- For the branching cell-lineage diagram without the cell-type order conservation rule, we obtain the equation between f and g:

  $$(g+2)f^2 + gf - g = 0. \tag{5}$$

  The discriminant of the above equation is $D = 5g^2 + 8g$. We survey (g_n, f_n) such that g_n and f_n are natural and positive rational numbers, respectively. If and only if $D = m^2, m \in \mathbf{N}$, f is rational. $D = m^2$ yields $(5g+4)^2 - 5m^2 = 4^2$. Let X be $5g+4$. Then the equation $X^2 - 5m^2 = 4^2$ is a Pell's equation. It follows that $X + \sqrt{5}m = \pm 4\left((3+\sqrt{5})/2\right)^n$, $n = 0, 1, 2, \cdots$ yields a pair $\{(X, m) \,|\, X \in \mathbf{Z}, m \in \mathbf{Z}\}$. Therefore, from $X = 5g+4$, $g + \sqrt{5}m' = 4\left\{\left((3+\sqrt{5})/2\right)^{2n} - 1\right\}/5,\ n = 1, 2, 3, \ldots$ yields a natural number sequence $\{g_n \mid g_n \in \mathbf{N}\}$. Moreover, g_n can more explicitly be described as $2\left\{\left((3+\sqrt{5})/2\right)^{2n} + \left((3-\sqrt{5})/2\right)^{2n} - 2\right\}/5$. It follows from the golden ratio $\varphi = (1+\sqrt{5})/2$ that g_n is $2\left(\varphi^{4n} + \varphi^{-4n} - 2\right)/5$. Because *Fibonacci* number F_n can be described as $\left(\varphi^n - (-\varphi)^{-n}\right)/\sqrt{5}$, $g_n = 2F_{2n}^2$ *q.e.d.*

  The recurrence formula of g_n can be derived as follows:

  $$\begin{aligned} g_{n+2} &= 2F_{2(n+2)}^2 = 2(F_{2n+3} + F_{2(n+1)})^2 \\ &= 2(2F_{2(n+1)} + F_{2n+1})^2 \\ &= 2(3F_{2(n+1)} - F_{2n})^2 \\ &= 2(9F_{2(n+1)}^2 + F_{2n}^2 - 6F_{2(n+1)}F_{2n}) \\ &\quad \text{From the formula } 3F_{2(n+1)}F_{2n} = F_{2(n+1)}^2 + F_{2n}^2 - 1, \\ &= 2(7F_{2(n+1)}^2 - F_{2n}^2 + 2) \\ &= 7g_{n+1} - g_n + 4.\ q.e.d. \end{aligned}$$

  The corresponding f_n with respect to g_n on the curve (5) is calculated as:

  $$f_n = \frac{F_{2n}\left(\sqrt{4 + 5F_{2n}^2} - F_{2n}\right)}{2(1 + F_{2n}^2)}.$$

  The formulae on *Fibonacci* and *Lucas* numbers F_n and L_n:

  $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, 5F_n^2 - L_n^2 = 4(-1)^{n+1}, F_n + L_n = 2F_{n+1}$$

reveal that $f_n = F_{2n}/F_{2n+1}$ $q.e.d.$

- With respect to the cell-lineage diagram without the rule, we obtain:

$$2(3g-1)f^2 + (3g-1)f - 3g = 0. \tag{6}$$

The discriminant of the above equation is $D = 81g^2 - 30g + 1$. We survey (g_n, f_n) such that g_n and f_n are natural and positive rational numbers, respectively. If and only if $D = m^2, m \in \mathbf{N}$, f is rational. $D = m^2$ yields $(27g-5)^2 - (3m)^2 = 4^2$, which can be factorized as $(27g+3m-5)(27g-3m-5) = 4^2$. From $g \geq 1$ and $m \geq 1$, $27g + 3m - 5 \geq 5^2$. Therefore, there exists no rational f_n for natural number g_n $q.e.d.$

- For the linear diagram with the rule, we obtain:

$$2gf^2 + (g-1)f - g = 0. \tag{7}$$

We survey (g_n, f_n) such that g_n and f_n are natural numbers ($g > 0$) and positive rational numbers ($0 < f < 1$), respectively.

We prove the non-existence of such pairs in $0 < g, 0 < f < 1$. Assume that $f = n/m$, gcd$(m,n) = 1,\ m \in \mathbf{N}, n \in \mathbf{N}$ because it is a positive rational number. Then, the equation (7) can be transformed into:

$$g(2n-m)(m+n) = mn. \tag{8}$$

From $0 < g$ and $0 < f < 1$, it follows that $m/2 < n < m$. Furthermore, from (8), $2n^2g \equiv 0 \text{mod } m, and\ m^2g \equiv 0 \text{mod } n$. It follows from gcd$(m,n) = 1$ that $2g \equiv 0 \pmod{m}$ and $g \equiv 0 \pmod{n}$, revealing $2g \equiv 0 \pmod{mn}$. This fact, together with $g = mn/((2n-m)(m+n)), m/2 < n < m$, implies that the only possible value of g is $mn/2$. $g = mn/2$ yields an equation $(2n-m)(m+n) = 2$, which has a single solution $m = n = 1$, contradicting the assumption $f = n/m < 1$ $q.e.d.$

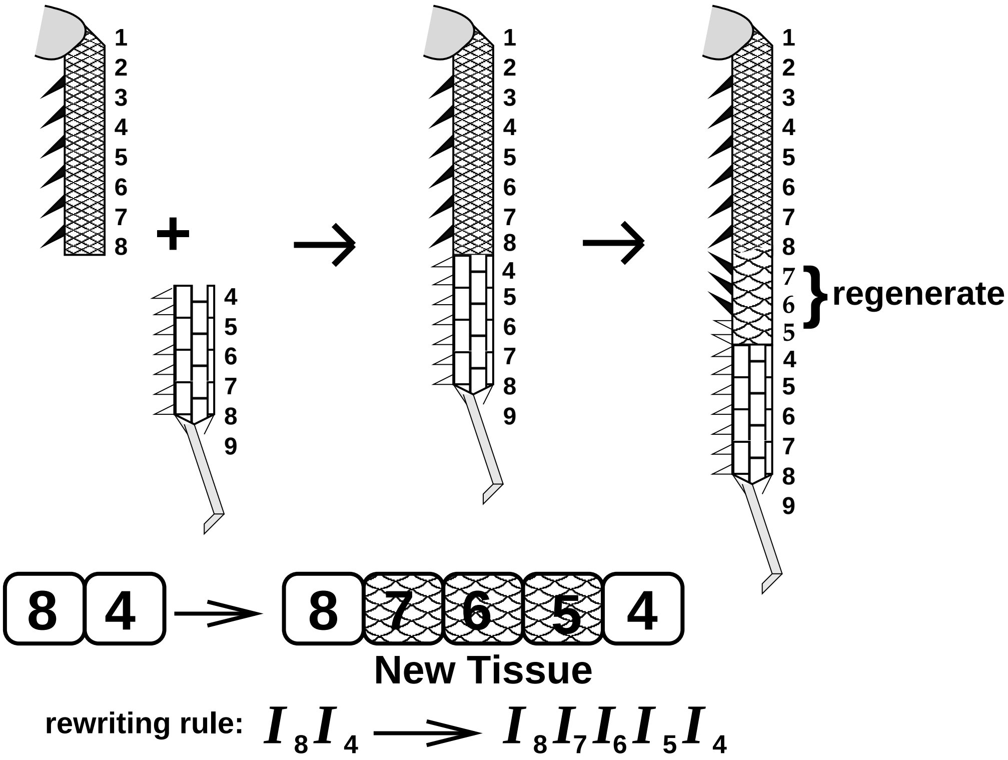


Figure 1: Intercalary regeneration in cockroach legs (*27*). When mismatched portions of the growing legs are grafted together, new tissue is intercalated to fill the gap so that the non-contiguous positional values disappear. In this figure, we exemplify the rule: $I_8 I_4 \to I_8 I_7 I_6 I_5 I_4$.

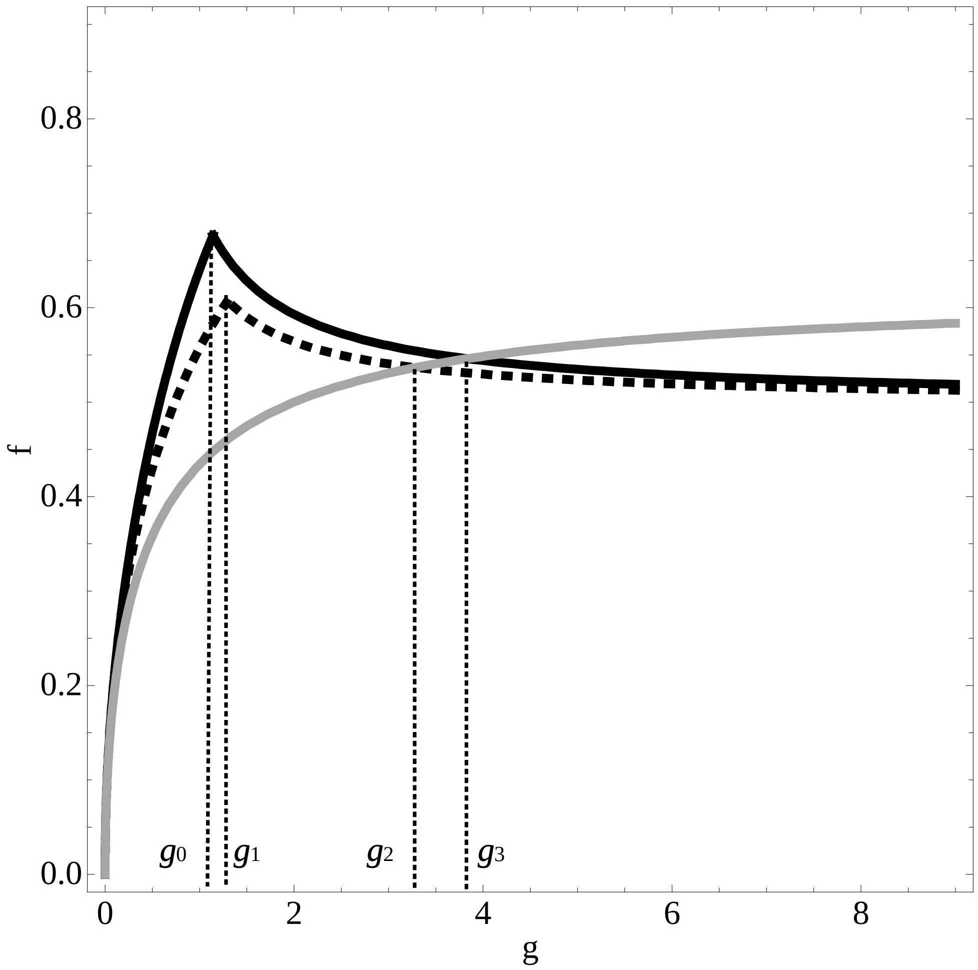
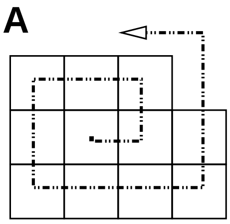
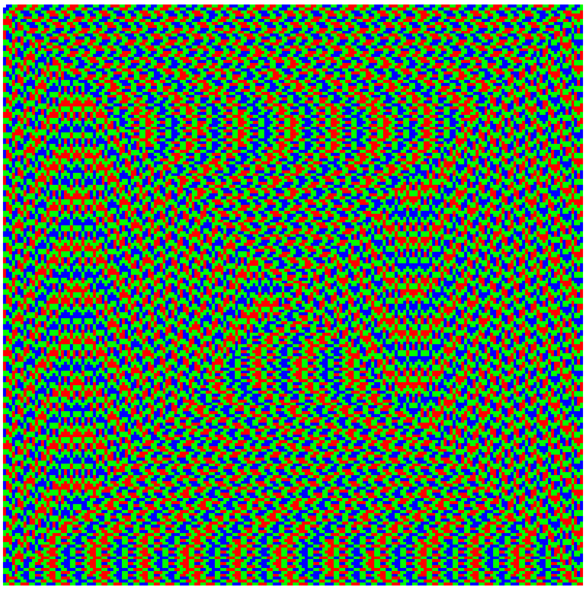


Figure 2: The equations between g and f. The gray line denotes the equation in the branching cell-lineage diagram without the cell-type order conservation rule. The black and broken lines denote the equations in the linear diagram with and without the rule, respectively. The derivative discontinuities are nearly $1.14254 (g_0)$ and $1.29661 (g_1)$. The intersection of the broken and gray lines is nearly $3.28773\ (g_2)$, and that of the black and gray lines is nearly $3.83118\ (g_3)$. The f-value of the broken and black lines approaches $1/2$ as $g \to \infty$, while that of the gray line approaches the reciprocal of the golden ratio $1/\varphi \sim 0.61803\ \left(\varphi = (1+\sqrt{5})/2\right)$ as $g \to \infty$.



B g=1 (Linear with the rule)



C g=2 (Branching)

Figure 3: Spiral tiling of cells. (A) Schematic illustration of spiral tiling of one-dimensional cell chain. (B) $g = 1, f = (1 + \sqrt{17})/8$ in the linear diagram with the rule. (C) $g = 2, f = 1/2$ in the branching diagram.

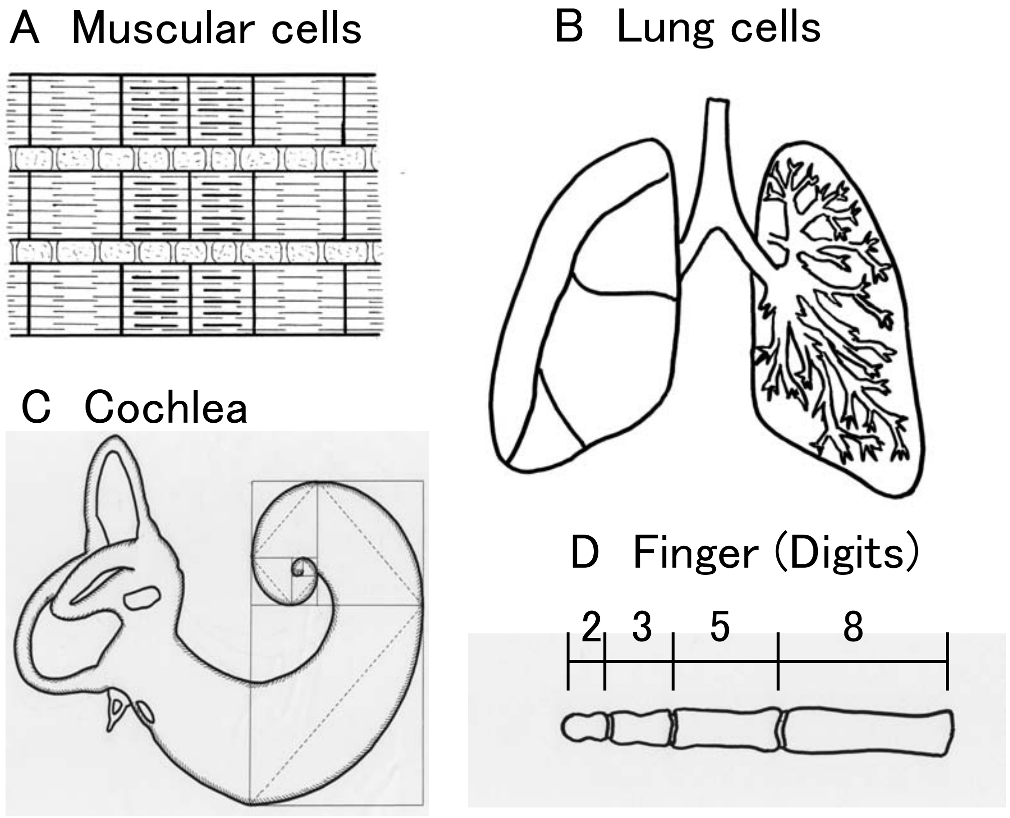


Figure 4: The structures of the organs. (A) Muscular cells seem periodic. (B) Lung cells seem quasi-periodic. (C) The cochlea of the inner ear forms a golden spiral related to *Fibonacci* numbers. (D) The motion paths of the digits also form a golden spiral (*22*).