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Nozaki, Hiroshi
Graduate School of Mathematics, Kyushu University: Student (M2) : Algebraic Combinatorics

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A NOTE ON ODD UNIMODULAR EUCLIDEAN LATTICES

HIROSHI NOZAKI

Graduate School of Mathematics Kyushu University
Hakozaki 6-10-1 Higashi-ku, Fukuoka, 812-8581 Japan
ma204018@math.kyushu-u.ac.jp

Abstract. We prove that the theta series of any odd unimodular Euclidean Lattice is not congruent to 1 modulo any odd prime p.

1. Introduction

It is known that there are many even unimodular lattices in R^n whose theta series are congruent to 1 modulo prime p. See e.g., Bayer-Fluckiger [1], Dummigan and Tiep [3] and the papers referred there. The following problem was recently asked by Yuichiro Taguchi.

Problem 1.1. *Is there any unimodular lattice $\Lambda \subset R^n$ with n odd whose theta series Θ_Λ satisfy*

$$\Theta_\Lambda \equiv 1 \pmod p$$

for an odd prime p?

In the meantime, Taguchi noticed himself that it is an implicit consequence of the paper of Koblitz [4, pages 202-203] that there exists no such lattice. However, it seems that the proof is difficult since it heavily depends on the deep results of Katz and Kohnen. The purpose of this paper is to give an elementary and simple proof of this solution. Also, we prove it in a slightly general form as follows.

Theorem 1.1. *For any odd unimodular Euclidean lattice and for any odd prime p, its theta series cannot be congruent to 1 modulo p.*

2. Preliminaries

We have several definitions as follows.

The dual of a lattice $\Lambda \subset R^n$ is the lattice

$$\Lambda^\# := \{y \in R^n \mid \langle y \mid x \rangle \in \mathbb{Z}, \forall x \in \Lambda\}.$$

The lattice Λ is called integral if $\Lambda \subset \Lambda^\#$. An integral lattice Λ is called even if $\langle x \mid x \rangle \in 2\mathbb{Z}$ for all $x \in \Lambda$. It is called odd otherwise. An integral lattice is called selfdual (or unimodular) if $\Lambda^\# = \Lambda$. The shadow $S(\Lambda)$ of a selfdual lattice Λ is

$$S(\Lambda) = \begin{cases} \Lambda, & \text{if } \Lambda \text{ is even} \\ \Lambda_0^\# \setminus \Lambda, & \text{if } \Lambda \text{ is odd} \end{cases}$$

where $\Lambda_0 = \{x \in \Lambda \mid \langle x \mid x \rangle \equiv 0 \bmod 2\}$.

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We write $q := e^{\pi i z}$.

$$\theta_3(z) := \sum_{m\in\mathbb{Z}} q^{m^2} = 1 + 2q + 2q^4 \cdots$$

$$\theta_2(z) := \sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{m^2} = 2q^{\frac{1}{4}}(1 + q^2 + \cdots)$$

$$\theta_4(z) := \sum_{m\in\mathbb{Z}} (-q)^{m^2} = 1 - 2q + 2q^4 - \cdots$$

The theta series of Λ is

$$\Theta_\Lambda(z) := \sum_{x\in\Lambda} q^{\langle x|x\rangle}$$

We collect the results we need.

Theorem 2.1. (*A special case of Dummigan-Tiep* [3, Theorem 3.2]) *Let p be an odd prime and φ denote the Euler function. Define $M = l\varphi(p)$ as follows :*
(1) *If* $p \equiv 1 \pmod 4$, *then* $M = 2\varphi(p)$.
(2) *If* $p \equiv 3 \pmod 4$, *then* $M = 4\varphi(p)$.
Then there is an even unimodular lattice Λ of rank M such that $\Theta_\Lambda \equiv 1 \pmod p$.

Theorem 2.2. (*Originally due to Hecke, See Conway-Sloane* [2, pages 187-188]) *If Λ is a unimodular lattice with rank n then*

$$\Theta_\Lambda = \theta_3^n + c_1\theta_3^{n-8}\Delta_8 + \cdots + c_{[n/8]}\theta_3^{n-8[n/8]}\Delta_8^{[n/8]}$$

where the c_i are integers and $\Delta_8(z)$ is the cusp form

$$\Delta_8(z) = \frac{1}{16}\theta_2(z)^4\theta_4(z)^4 = q\prod_{m=1}^{\infty}\{(1-q^{2m-1})(1-q^{4m})\}^8.$$

Theorem 2.3. (*Pache* [5, Proposition 13]) *In the notation of Theorem 2.2,*

$$\Theta_\Lambda = \theta_3^n + c_1\theta_3^{n-8}\Delta_8 + \cdots + c_k\theta_3^{n-8k}\Delta_8^k$$

where $k = \frac{1}{8}(n - \sigma(\Lambda))$ ($\sigma(\Lambda) := 4\min\{\langle x\,|\,x\rangle \mid x \in S(\Lambda)\}$ where $S(\Lambda)$ means the shadow of the lattice Λ). Moreover, we have $c_k = (-1)^k\,2^{-n+12k}\,|S(\Lambda)_{(n-8k)/4}|$, where $S(\Lambda)_m := \{x \in S(\Lambda) \mid \langle x\,|\,x\rangle = m\}$.

Since $\varphi(p) = p - 1$,

$$M = \begin{cases} 8t, & \text{if } p = 4t+1,\ t \in \mathbb{N} \\ 8(2t-1), & \text{if } p = 4t-1,\ t \in \mathbb{N} \end{cases}$$

For every odd prime p, we can pick out an even unimodular lattice Λ of rank M such that $\Theta_\Lambda \equiv 1 \pmod p$(by Theorem 2.1). Since Λ is a unimodular lattice, there exist $c_i \in \mathbb{Z}$ such that

$$\Theta_\Lambda = \theta_3^M + c_1\theta_3^{M-8}\Delta_8 + \cdots + c_k\theta_3^{M-8k}\Delta_8^k$$

where $\sigma(\Lambda) = M - 8k$ and $c_k = (-1)^k\,2^{-M+12k}\,|S(\Lambda)_{(M-8k)/4}|$ (by Theorem 2.3).

Moreover Λ is even. Hence $S(\Lambda) = \Lambda$ and $\sigma(\Lambda) = 0$. Therefore
$c_k = (-1)^k\,2^{-M+12k}\,|\Lambda_0| = (-1)^k\,2^{-M+12k}$.

(1)If $p = 4t + 1$, then $M = 8t$. Since $\sigma(\Lambda) = M - 8k = 0$, $k = t$. Therefore

$c_k = (-1)^t\, 2^{4t} = (-1)^t\, 2^{p-1} \equiv (-1)^t \pmod p$.
(2)If $p = 4t-1$, then $M = 8(2t-1)$. Since $\sigma(\Lambda) = M - 8k = 0$, $k = 2t-1$. Therefore
$c_k = (-1)^{2t-1}\, 2^{8t-4} = -2^{2(p-1)} \equiv -1 \pmod p$.

The facts mentioned above prove the following proposition.

Proposition 2.1. *Let p be an odd prime. Define $M = 4\varphi(p)$. There is an even unimodular lattice Λ of rank M such that*

$$\Theta_\Lambda = \theta_3^M + c_1\theta_3^{M-8}\Delta_8 + \cdots + c_{M/8}\Delta_8^{M/8} \equiv 1 \pmod p$$

where $c_i \in \mathbb{Z}, c_{M/8} \not\equiv 0 \pmod p$ and $M/8 \in \mathbb{N}$.

If $p = 4n+1$, the rank of Λ is $2\varphi(p)$ in Theorem 2.1. Then $\Theta_{\Lambda\oplus\Lambda} = {\Theta_\Lambda}^2 \equiv 1 \pmod p$ and $\Lambda \oplus \Lambda$ is an even unimodular lattice and rank $4\varphi(p)$.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we prove the following two results. Theorem 3.1 is independent of the existence of the lattice.

Theorem 3.1. *Let n be any integer which is not divisible by 8. For any odd prime p, there* ***does not*** *exist $\{a_0, a_1, \cdots, a_{[n/8]}\}$ such that*

$$\sum_{i=0}^{[n/8]} a_i\, \theta_3^{n-8i}\Delta_8^i \equiv 1 \pmod p$$

where $a_0 = 1$ and a_i are integers.

Proof. Assume the contrary. Then there exists $\{a_1, \cdots, a_{[n/8]}\}$ such that

$$\Theta_\Lambda = \theta_3^n + a_1\,\theta_3^{n-8}\Delta_8 + \cdots + a_{[n/8]}\,\theta_3^{n-8[n/8]}\Delta_8^{[n/8]} \equiv 1 \pmod p.$$

Since n is not divisible by 8, $n - 8[n/8] \neq 0$.

By Proposition 2.1 there exists an even unimodular lattice Λ' with rank M such that

$$\Theta_{\Lambda'} = \theta_3^M + c_1\,\theta_3^{M-8}\Delta_8 + \cdots + c_{M/8}\,\Delta_8^{M/8} \equiv 1 \pmod p$$

where $c_i \in \mathbb{Z}$ and $c_{M/8} \not\equiv 0 \pmod p$.

$${\Theta_{\Lambda'}}^n - {\Theta_\Lambda}^M = b_1\,\theta_3^{Mn-8}\Delta_8 + b_2\,\theta_3^{Mn-16}\Delta_8^2 + \cdots + b_{Mn/8}\,\Delta_8^{Mn/8} \equiv 0 \pmod p$$

where $b_i \in \mathbb{Z}$. Since $\theta_3^{Mn-8i}\Delta_8^i = q^i\,(1+\cdots)$, then $b_1 \equiv 0, b_2 \equiv 0, \cdots$ and $b_{Mn/8} \equiv 0 \pmod p$. However this contradicts $b_{Mn/8} = (c_{M/8})^n \not\equiv 0 \pmod p$. □

Theorem 3.2. *Let n be any integer which is divisible by 8. For any odd unimodular lattice Λ of rank n and for any odd prime p,*

$$\Theta_\Lambda = \sum_{i=0}^{k} a_i\,\theta_3^{n-8i}\Delta_8^i \not\equiv 1 \pmod p$$

where $a_0 = 1$, a_i are integers and $k = \frac{1}{8}(n - \sigma(\Lambda))$.

Proof. We have $\sigma(\Lambda) = 0$ if and only if Λ is even. Therefore $\sigma(\Lambda) = n - 8k \neq 0$. Remains of proof is similar to proof of theorem 3.1. □

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