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Abstract

The theory of weakly nonlinear resonant wave interactions was applied in the 1960s to water waves. It is not often recognized that much more dramatic instabilities can occur in the presence of a shear flow, because all modes can amplify by extracting energy from the basic mean flow. In this talk, the foregoing idea will be employed to propose a mechanism for generating subcritical nonlinear critical layer modes; i.e., neutral modes that cannot be explained by linear theory because their viscous counterparts would be damped. The problem of Rossby waves propagating in a mixing layer with velocity profile $\bar{u}(y)$ will be utilized to illustrate the theory. The beta parameter, which is a measure of the stabilizing Coriolis force, is taken to be large enough so that linear instability cannot occur. Then, full numerical simulations are carried out to illustrate how nonlinear critical layer modes can be generated by resonant interaction with ordinary Rossby waves, even when the singular mode is absent initially.

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1 Introduction

Weakly nonlinear theories can be formulated systematically by employing a perturbation scheme in which the dependent variables are expanded in powers of ε, a small dimensionless amplitude parameter. We suppose that the system of governing equations can be expressed in the form

$$\mathcal{L}\mathbf{u} = \varepsilon \mathcal{N}\mathbf{u} \; , \tag{1}$$

and the linearized problem is obtained by setting $\varepsilon = 0$. If the linear problem admits dispersive wave solutions, these will be proportional to $A_i \exp\{i(\mathbf{k}\cdot\mathbf{x} - \omega t)\}$, where the frequency and wavenumber are related through the dispersion relation $\omega = W(\mathbf{k})$.

Let us now consider the interaction of a set of three such wavetrains by writing

$$u = \sum_{n=1}^{3} A_i(X,T) \exp\{i(\mathbf{k}\cdot\mathbf{x} - \omega t)\} + A_i^*(X,T) \exp\{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)\} \, , \tag{2}$$

where $X = \varepsilon x$ and $T = \varepsilon t$ are slow space and time scales. The nonlinear terms $\mathcal{N}$ in (1) are usually quadratic so that a sum or difference between two of the waves in (2) may be equal to the third member of the triad. In that case, resonance is possible; the resonance conditions are often written

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0 \text{ and } \omega_1 \pm \omega_2 \pm \omega_3 = 0 \; . \tag{3}$$

O. M. Philips(1960) is usually credited with first formulating a resonant interaction theory along the foregoing lines and during the next 20 years, this was a very active area of research. A nice survey of this work can be found in Philips(1981). However, the above necessary conditions for resonance are not sufficient for the case of water waves, the application that had motivated Philips. There, four waves are necessary, the development must be carried out to higher order and the time scale for the interaction is $\varepsilon^2 t$. Because we are most interested in the application to waves in shear flows, the work of L. G. McGoldrick, who had been a student of Philips, turns out to be most pertinent. McGoldrick's research was on capillary-gravity waves and it develops that interacting triads are obtained when the effect of surface tension is included.

A special case, that is important, can be best used to illustrate the theory and this case is termed second harmonic resonance. It has all the features of triad resonance, but is simpler because only two waves are involved. McGoldrick (1972) used the following model equation to illustrate the theory:

$$L\{u\} = u_{tt} - u_{xx} + u + \frac{1}{4}\, u_{xxxx} = 3\,\varepsilon\, u^2 \ . \tag{4}$$

Suppose that we try to find a solution of (4) by a straightforward power series in ε. At zeroth order, the solution of the linear problem can be written

$$u^{(0)} = A\ \exp\{i(k\,x - \omega t)\} + A^*\ \exp\{-i(k\,x - \omega t)\},\ \text{where}\ \omega(k) = \pm(1 + \frac{1}{2}\, k^2)$$

is the dispersion relation. The $O(\varepsilon)$ term in the expansion must satisfy $L\{u^{(1)}\} = 3\,(u^{(0)})^2$, whose solution can be written

$$u^{(1)} = \frac{3}{\{[\omega(2k)]^2 - 2[\omega(k)]^2\}}\{A^2 e^{2i(k\,x-\omega t)} + (A^*)^2 e^{-2i(k\,x-\omega t)}\} + 6\,A\,A^* \,.$$

Clearly, resonance occurs when $\omega(2k) = 2\,\omega(k)$.

Whether treating capillary-gravity waves or shear flows, the method of multiple scales provides a systematic framework to analyze resonant interactions. The relevant slow scales are $X = \varepsilon x$ and $T = \varepsilon t$, so that in the governing equations we transform derivatives according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\,\frac{\partial}{\partial \tau} \ \ \text{and}\ \ \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\,\frac{\partial}{\partial X} \ .$$

To deal with the case of second harmonic resonance, the basic perturbation must include both modes, so we write

$$u^{(0)} = \sum_{n=1}^{2} A_n(X,T)\, e^{i\,n(kx-\omega t)} + A_n^*(X,T)\, e^{-i\,n(kx-\omega t)} \ . \tag{5}$$

To separate variables now in the $O(\varepsilon)$ problem, it is found that $u^{(1)}$ must be of the form

$$u^{(1)} = A_2 A_1^*\, e^{i\,(kx-\omega t)} + A_1^2\, e^{2i(kx-\omega t)} + \text{complex conjugates} + \text{nonsecular terms}\ .$$

In order for the expansion of u to be well ordered, the so-called secular terms must be eliminated and this requires the amplitudes to satisfy the following evolution equations:

$$\frac{\partial A_1}{\partial \tau} + \omega' \frac{\partial A_1}{\partial X} = i\,\gamma_1\, A_2 A_1^* \quad \text{and} \quad \frac{\partial A_2}{\partial \tau} + \omega' \frac{\partial A_2}{\partial X} = i\,\gamma_2\, A_1^2 \,. \tag{6}$$

It is informative to derive an energy integral from Eqs. (6) by neglecting the variation in X and forming expressions for $\frac{d|A_i|^2}{d\tau}$. These expressions, for $i = 1$ and 2, can be combined and integrated with respect to τ to obtain

$$|A_1|^2 + \frac{\gamma_1}{\gamma_2} |A_2|^2 = E \,. \tag{7}$$

In a conservative system, γ_1 and γ_2 will both have the same sign, so that the total energy is shared by the two waves. For example, in the model Eq. (4), $\gamma_1 = \frac{3}{\omega}$ and $\gamma_2 = \frac{3}{4\,\omega}$.

What is significant (and not generally realized) in shear flow stability problems is that it is possible for γ_1 and γ_2 to have opposite signs, in which case, *both* waves can amplify. While this seems counter-intuitive, the explanation is simply that both waves can amplify by extracting energy from the mean flow. To take into account the mean flow energy, it is necessary to go one step further in the perturbation expansion. The monograph by Craik (1985) treats this in detail (see, in particular, Sections 17.2 and 26.1, where it is explained that the energy is transferred to the perturbations in the vicinity of the critical layer).

Much of Craik's own research dealt with a special triad configuration that is possible in both boundary layers and mixing layers. Specifically, a plane wave is employed along with a pair of subharmonic oblique waves inclined at equal and opposite angles (slightly less than 60°) to the flow direction. The frequency of the oblique waves is half that of the plane wave and, because the conditions for resonance are satisfied exactly, all modes share a common critical layer. This sort of triad was found by Liu & Maslowe (1999) to be extremely effective in the case of an adverse pressure gradient boundary layer.

2 Resonance of two modes in a stratified mixing layer

As an application of the foregoing theory, we return to Hølmboe's mixing layer model. The stability boundary is given by $J_0 = \alpha(1 - \alpha)$ and because of the symmetry of the profiles, $c = 0$ along this boundary. The conditions for second harmonic resonance are satisfied at a Richardson number $J_0 = \frac{2}{9}$ for the two modes with wavenumbers $\alpha = \frac{1}{3}$ and $\alpha = \frac{2}{3}$.

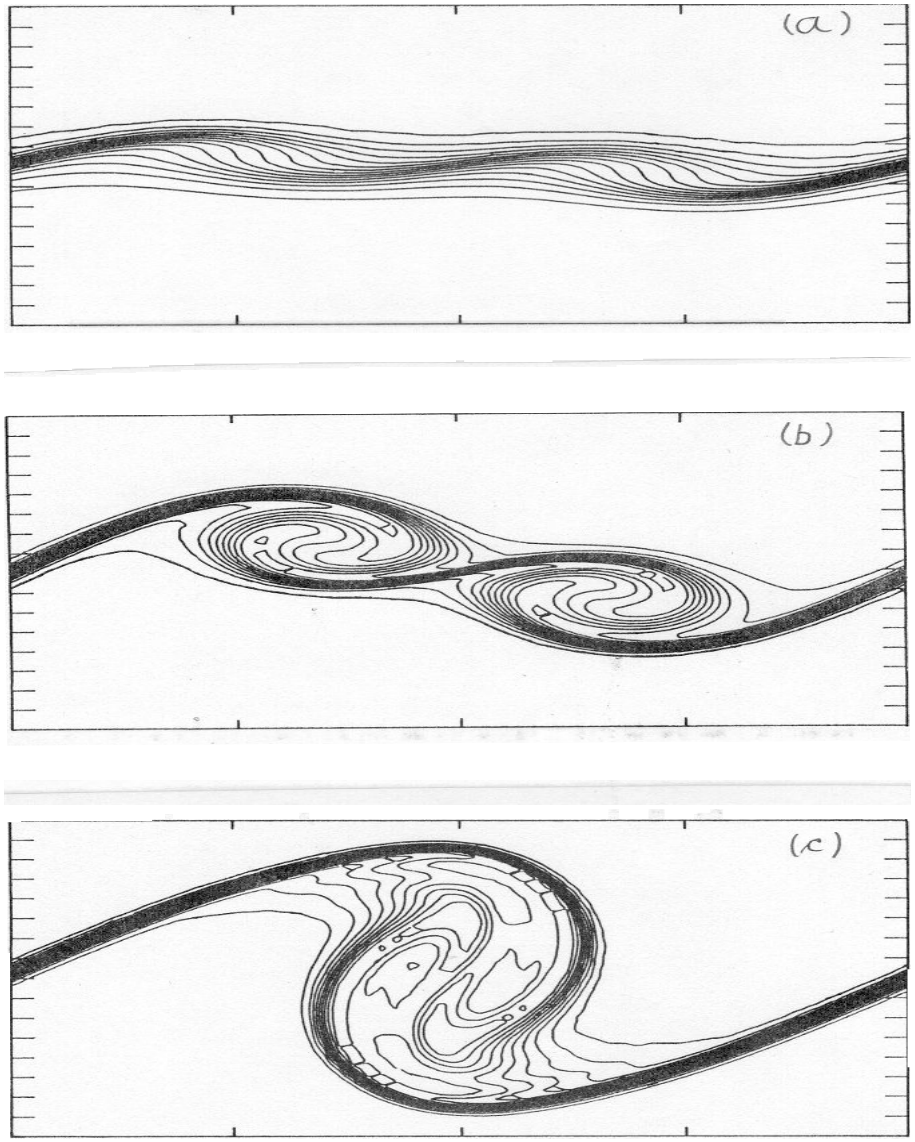

Figure 1: Evolution of constant-density contours for two waves, $\alpha = 0.215$ and 0.43, with $J_0 = 0.07$ and $Re = 200$: the times are (a) $t = 16$, (b) $t = 32$ and (c) $t = 48$.

The results shown in Fig. 1 above are from the paper by Collins & Maslowe (1988). As discussed therein, the amplitude equations (6) can be generalized to include weakly amplified modes by adding a term proportional to A on the right hand side. Because the phase speed $c_r = 0$, even for unstable modes, any two waves for which $\alpha_2 = 2\,\alpha_1$ will interact resonantly. In the paper by Collins and the author, results are reported primarily for $0.07 \leq J_c \leq 0.174$. For Richardson numbers in the lower part of this range, the streamlines can be said to depict the phenomenon of vortex pairing, familiar in homogeneous mixing layers. As J_c increases, the pairing is less energetic and at $J_c = 0.14$ a sort of limiting case is reached in which the vortex on the left rises only slightly and the one on the right descends a small amount. There is nonetheless a strong interaction because the equilibrium amplitude is 35 times as large as that attained by the single most amplified wave when the subharmonic is absent.

3 Resonant interactions in zonal shear flows

Atmospheric observations of the instability of zonal currents have motivated numerous studies of the barotropic stability characteristics of such flows. The theory is relevant to the oceans, as well, with a number of investigations motivated by Gulf Stream phenomena. We consider here perturbations to the zonal shear flow $\bar{u} = \tanh y$. The basic flow is to the east (x-direction), y is the north-south coordinate, and variations in the vertical are neglected. Kuo (1973) in a comprehensive survey article presents observational data for wind profiles over both the Atlantic and Pacific oceans which are well approximated by the hyperbolic tangent function.

The governing equation of the linear, inviscid theory is the Rayleigh-Kuo equation

$$(\bar{u} - c)(\phi'' - \alpha^2 \phi) + (\beta - \bar{u}'')\phi = 0 \quad . \tag{8}$$

This equation is the analogue of the Taylor-Goldstein equation in the sense that it is Rayleigh's equation with an additional term representing a stabilizing influence. Here, it is the Coriolis force resulting from planetary rotation rather than buoyancy; this influence is modelled in (8) by a linearization about some mean latitude and β is the derivative of the Coriolis parameter (assumed constant). The properties of (8) are well-known, the most significant being the generalization of Rayleigh's inflection point theorem stating that the quantity $(\beta - \bar{u}'')$ must change sign at some value of y for instability to occur.

Neutral mode solutions of (8) are usually regular due to the vanishing of the absolute vorticity $(\beta - \bar{u}'')$ at the critical point y_c, where $\bar{u} = c$. However, unlike the case $\beta = 0$, it is possible to have neutral "radiating modes"; for an unbounded, monotonic velocity profile such modes are singular and they decay exponentially on one side of the shear layer, but are oscillatory on the other side, where a boundary condition is imposed on the group velocity to ensure outward energy propagation.

We restrict attention here though to the more conventional "trapped modes" whose eigenfunctions decay exponentially to zero as $|y| \to \infty$. The linear, neutral solution was obtained in closed form by Howard & Drazin (1964). The eigenvalue condition relating the phase speed, wavenumber and beta parameter is given by

$$c^2 = 1 - \alpha^2 \quad \text{and} \quad \beta = -2c(1 - c^2). \tag{9}$$

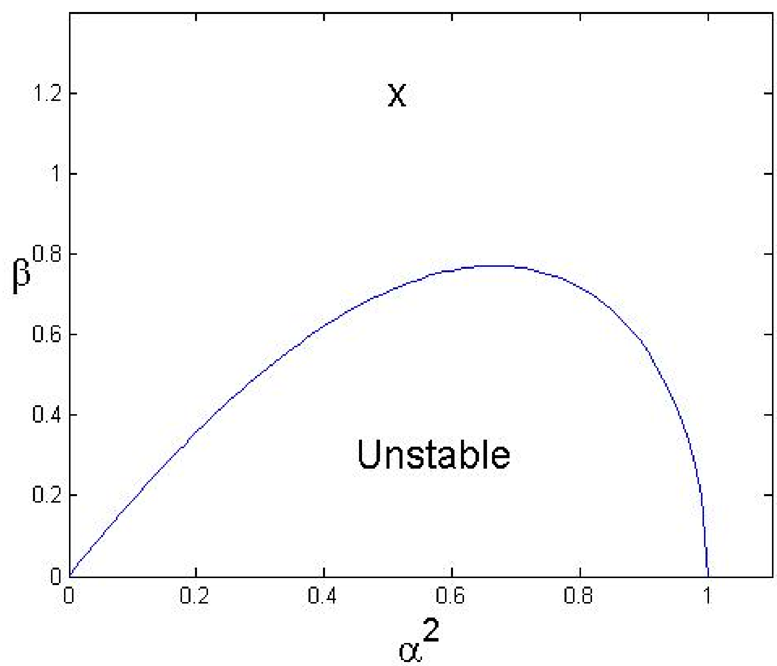

Figure 2: Stability diagram for the zonal shear flow $\bar{u} = \tanh y$.

As shown in Fig. 2, the primary effect of rotation is stabilizing and the range of unstable wavenumbers decreases with increasing β until the critical value $\beta = 4/3^{3/2}$ is reached;

above this value the flow is stable on a linear basis. We should add that the solution (9) applies also to the velocity profile $\bar{u} = -\tanh y$; in that case, which corresponds to the observations cited by Kuo, the positive root would be taken for c and β would be negative.

We return now to the subject of subcritical neutral modes with nonlinear critical layers. Within the framework of linear theory, such singular modes cannot exist for $\beta > 4/3^{3/2}$ (i.e., at a point such as X in Fig. 2). The reason is that the boundary conditions are not compatible with a jump in the Reynolds stress that would be the outcome of a phase change across the critical point y_c. It was shown, however, by Maslowe & Clarke (2002) that singular neutral modes can be obtained when there is no phase change and dispersion curves were computed for $\beta = 3$. The critical point singularity in the Rayleigh-Kuo equation (8) is of the same form as that for the Rayleigh equation. Therefore, in solving the eigenvalue problem, we simply write $\log(y - y_c) = \log|y - y_c|$ and integrate numerically from a small distance on either side of y_c to the boundaries.

Having shown that nonlinear neutral modes are possible mathematically, a mechanism for generating them must be found if they are to be of physical significance. The mechanism we employ here is that of resonant interaction with Rossby waves. The latter are nonsingular and they are modified only quantitatively by the shear; dispersion curves for each mode are computed easily by solving (8) numerically. The simulation is initiated using only the two Rossby waves, each of which in the case $\bar{u} = \tanh y$ belongs to a different mode. The frequencies and wavenumbers are chosen so that the triad resonance conditions (3) are satisfied with the third member being a singular neutral mode. Dispersion curves with the resonant values of ω and α are illustrated in Fig. 1 of Maslowe & Clarke.

One motivation for this approach is the proof by Becker & Grimshaw (1993) that in the similar problem of internal gravity wave interaction in a stratified flow, a necessary condition for explosive instability is that at least one mode must have a critical layer. This idea has been pursued in the present context by Vanneste (1998), who formulated a finite amplitude theory. In his analysis, the singular mode is represented by the continuous spectrum, a procedure quite different from that used in the numerical simulations of Maslowe & Clarke.

In fig. 3, the total vorticity contours are shown for a numerical simulation of the inviscid, nonlinear barotropic vorticity equation. Initially, only two Rossby waves are present, but at later times the nonlinear critical layer mode is generated, as is clear from the propagating (blue) cats-eyes pattern.

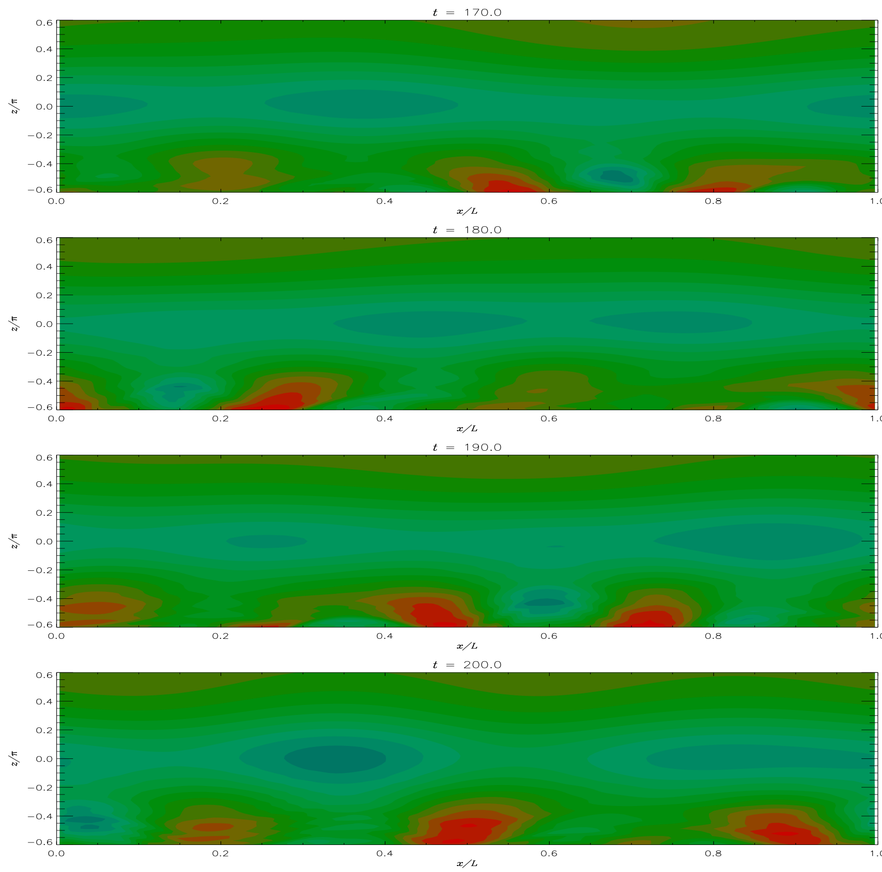

Figure 3: Formation of a nonlinear singular mode with $\alpha = 1.2$ as a result of resonant interaction with Rossby modes having wavenumbers $\alpha = 0.6$ and $\alpha = 1.8$. The basic zonal shear flow is the mixing layer $\bar{u} = \tanh y$ and the Coriolis parameter $\beta = 3$.

One important effect that is absent in the theory, but was very noticeable in the numerical simulations is a strong variation in the mean flow during the triad evolution. Very rapid oscillations were observed in the $\alpha = 0$ component of our pseudospectral computation. The linear shear profile $\bar{u} = y$ was also investigated, but the nonlinear critical layer mode generation was less evident for that case.

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