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Huang, Chien Chung
Department of Computer Science and Engineering, Chalmers University of Technology

Kakimura, Naonori
Graduate School of Arts and Sciences, University of Tokyo

Kamiyama, Naoyuki
Institute of Mathematics for Industry, Kyushu University

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Chien-Chung Huang,
Naonori Kakimura
& Naoyuki Kamiyama

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Institute of Mathematics for Industry
Graduate School of Mathematics
Kyushu University
Fukuoka, JAPAN

Exact and Approximation Algorithms for Weighted Matroid Intersection

Chien-Chung Huang[*] Naonori Kakimura[†] Naoyuki Kamiyama[‡]

Abstract

We present exact and approximation algorithms for the weighted matroid intersection problems. Our exact algorithms are faster than previous algorithms when the largest weight is relatively small. Our approximation algorithms deliver a $(1-\epsilon)$-approximate solution with running times significantly faster than known exact algorithms.

The core of our algorithms is a decomposition technique: we decompose the weighted version of the problem into a set of unweighted matroid intersection problems. The computational advantage of this approach is that we can then make use of fast unweighted matroid intersection algorithms as a black box for designing algorithms. To be precise, we show that we can find an exact solution via solving W unweighted matroid intersections problems, where W is the largest given weight. Furthermore, we can find a $(1-\epsilon)$-approximate solution via solving $O(\epsilon^{-1}\log r)$ unweighted matroid intersection problems, where r is the smallest rank of the given two matroids.

Our algorithms are simple and flexible: they can be adapted to specific matroid intersection problems, making use of specialized unweighted matroid intersection algorithms. In this paper, we show the following results.

1. Given two general matroids, using Cunningham's algorithm, we solve the problem exactly in $O(\tau W n r^{1.5})$ time and $(1-\epsilon)$-approximately in $O(\tau\epsilon^{-1}nr^{1.5}\log r)$ time, where τ is the time complexity of an independence oracle call.
2. Given two graphic matroids, using the algorithm of Gabow and Xu, we solve the problem exactly in $O(W\sqrt{r}n\log r)$ time and $(1-\epsilon)$-approximately in $O(\epsilon^{-1}\sqrt{r}n\log^2 r)$ time.
3. Given two linear matroids (in the form of two r-by-n matrices), using the algorithm of Cheung, Kwok and Lau, we solve the problem exactly in $O(nr\log r_* + Wnr_*^{\omega-1})$ time and $(1-\epsilon)$-approximately in $O(nr\log r_* + \epsilon^{-1}nr_*^{\omega-1}\log r_*)$ time, where r_* is the maximum size of a common independent set.

1 Introduction

Suppose that a pair of matroids $\mathbf{M}_1 = (S, \mathcal{I}_1), \mathbf{M}_2 = (S, \mathcal{I}_2)$, and a weight function $w\colon S \to \mathbb{Z}_{\geq 0}$ are given. The goal is to find a maximum-weight common independent set I of $\mathbf{M}_1$ and $\mathbf{M}_2$, i.e., a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $\sum_{e\in I} w(e)$ being maximized. This is the classical *weighted matroid intersection problem*, introduced by Edmonds [9, 11] and solved by Edmonds [9, 11] and others [1, 23, 29, 30] in 1970s. This problem is a generalization of various combinatorial optimization problems such

[*]Department of Computer Science and Engineering, Chalmers University of Technology. huangch@chalmers.se.
Supported by an STINT initiation grant between Chalmers and University of Tokyo (the project number 3720461).
[†]Graduate School of Arts and Sciences, University of Tokyo. kakimura@global.c.u-tokyo.ac.jp.
Supported by JST, ERATO, Kawarabayashi Large Graph Project, and by JSPS KAKENHI Grant Numbers 25730001, 24106002.
[‡]Institute of Mathematics for Industry, Kyushu University. kamiyama@imi.kyushu-u.ac.jp.
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as bipartite matching, packing spanning trees, arborescences in a directed graph, and has many applications, e.g., in electric circuit theory [34], rigidity theory [37], network coding [7]. The fact that two matroids capture the underlying common structures behind a large class of polynomially solvable problems has been impressive and motivated substantial follow-up research (see, e.g., [14, 38]). The techniques and theorems developed surrounding this problem have become canon in contemporary combinatorial optimization literature.

Since 1970s, quite a few algorithms have been proposed for the matroid intersection problem, e.g., [2, 6, 12, 15, 39], with better running time and/or simpler proofs. See Table 1 for a summary. Throughout the paper, n is the size of the ground set, r is the smallest rank of the two matroids and W is the largest given weight. The oracle to check the independence of a given set has the running time of τ.

Table 1: Matroid intersection algorithms for general matroids. See also [9, 11, 35].

Algorithm	Weight	Time complexity
Aigner–Dowling [1]	Unweighted	$O(\tau n r^2)$
Cunningham [6]	Unweighted	$O(\tau n r^{1.5})$
Lawler [29, 30], Iri–Tomizawa [23]	Weighted	$O((n+\tau) n r^2)$
Frank [12]	Weighted	$O(\tau n^2 r)$
Brezovec–Cornuéjols–Glover [2]	Weighted	$O(\tau n r^2)$
Fujishige–Zhang [15], Shigeno–Iwata [39][1]	Weighted	$O(\tau n^2 \sqrt{r} \log rW)$
This paper	Weighted	$O(\tau W n r^{1.5})$
(approximation)	Weighted	$O(\tau \epsilon^{-1} n r^{1.5} \log r)$

1.1 Our Contribution

We present both exact and approximation algorithms for the weighted matroid intersection problem. Our exact algorithms are faster than known algorithms when the largest given weight W is relatively small. Our approximation algorithms deliver a $(1-\epsilon)$-approximate solution for any fixed $\epsilon > 0$ in times substantially faster than known exact algorithms in almost all cases. Our algorithms and their analysis are surprisingly simple. Moreover, these algorithms can be specialized for particular classes of matroids.

The core of our algorithms is a decomposition technique. We show that the weighted matroid intersection problem can be decomposed into a set of unweighted versions of the same problem. More precisely, we can solve the weighted problem exactly by solving W unweighted ones. Furthermore, we can solve the weighted problem $(1-\epsilon)$-approximately by solving $O(\epsilon^{-1}\log r)$ unweighted ones.

Our decomposition technique not only establishes a hitherto unclear connection between the weighted and unweighted problems, but also leads to computational advantages: the known unweighted matroid intersection algorithms are significantly faster than their weighted counterparts. We can then make use of the former to design faster algorithms. It may be expected that in the future, there will be even more efficient unweighted matroid intersection algorithms, and that would imply our algorithms will become faster as well.

[1] The Shigeno–Iwata algorithm uses a (co-)circuit oracle, which is replaced with n independence oracle calls.

We summarize the complexity of our exact algorithms below. For comparison of our algorithms with previous results, see Tables 1–3.

1. Given two general matroids, using the unweighted matroid intersection algorithm of Cunningham [6], we solve the weighted problem in $O(\tau W n r^{1.5})$ time, where τ is the time complexity of an oracle call. This algorithm is faster than all known algorithms when $W = o(\min\{\sqrt{r}, \frac{n \log r}{r}\})$. A slightly different analysis shows that the same algorithm has the complexity[2] of $O(\tau(\sum_{e \in S} w(e)) r^{1.5})$.

2. Given two graphic matroids, using the unweighted graphic matroid intersection algorithm of Gabow and Xu [17], we solve the weighted problem in $O(W\sqrt{r}n \log r)$ time. This is faster than the current fastest algorithm when $W = o(\log r \log(rW))$. In the case that the graph is relatively dense, i.e., $n = \Omega(r^{1.5} \log r)$, then we can use the algorithm of Gabow and Stallman [16] to solve the problem in $O(W\sqrt{r}n)$ time.

3. Given two linear matroids (in the form of two r-by-n matrices), using the unweighted linear matroid intersection algorithm of Cheung, Kwok and Lau [3], we solve the weighted problem in $O(nr \log r_* + W n r_*^{\omega-1})$ time, where ω is the exponent of the matrix multiplication time and $r_* \leq r$ is the maximum size of a common independent set. This is faster than all known algorithms when $W = o(r^{\frac{\omega^2 - 7\omega + 12}{5-\omega}})$. (If $\omega \approx 2.37$ [5, 19, 42], it is $W = o(r^{0.41})$)

A recent trend in research is to design fast approximation algorithms for fundamental optimization problems, even if they are polynomially solvable. Examples include maximum weight matching [8], shortest paths [41], and maximum flow [4, 27, 32]. Using the algorithms of [3, 6, 17], our decomposition technique delivers a $(1-\epsilon)$-approximate solution in (1) $O(\tau\epsilon^{-1} n r^{1.5} \log r)$ time with two general matroids, (2) $O(\epsilon^{-1}\sqrt{r}n \log^2 r)$ time with two graphic matroids, and (3) $O(nr \log r_* + \epsilon^{-1} n r_*^{\omega-1} \log r_*)$ time with two linear matroids.

Our approximation algorithms are significantly faster than all known exact algorithms except for the case of two general matroids with $r = \Theta(n)$. To our knowledge, prior to our results, there are no such algorithms with both good approximation guarantee and fast running time, except a simple greedy $\frac{1}{2}$-approximation algorithm [24, 28] dated in 1970s. It is worth mentioning that for a generalization of the matroid intersection, called the *matroid matching problem* (which is known to be intractable in an independence oracle model [25, 33]), there are PTASs for the unweighted case [31] and a special class of the weighted case [40].

Table 2: Matroid intersection algorithms for graphic matroids.

Algorithm	Weight	Time complexity
Gabow–Stallman [16]	Unweighted	$O(\sqrt{r}n)$ if $n = \Omega(r^{3/2} \log r)$
	Unweighted	$O(rn^{2/3} \log^{1/3} r)$ if $n = \Omega(r \log r)$ & $n = O(r^{3/2} \log r)$
	Unweighted	$O(r^{4/3} n^{1/3} \log^{2/3} r)$ if $n = O(r \log r)$
Gabow–Xu [17]	Unweighted	$O(\sqrt{r}n \log r)$
Gabow–Xu [17]	Weighted	$O(\sqrt{r}n \log^2 r \log(rW))$
This paper	Weighted	$O(W\sqrt{r}n \log r)$
(approximation)	Weighted	$O(\epsilon^{-1}\sqrt{r}n \log^2 r)$

[2] This complexity is superior to the previous one only when the given weights are very "unbalanced."

Table 3: Linear matroid intersection algorithms.

Algorithm	Weight	Time complexity
Cunningham [6]	Unweighted	$O(nr^2 \log r)$
Gabow–Xu [18]	Unweighted	$O(nr^{\frac{5-\omega}{4-\omega}} \log r)$
Harvey [21]	Unweighted	$O(nr^{\omega-1})$
Cheung, et al. [3]	Unweighted	$O(nr \log r_* + nr_*^{\omega-1})$
Gabow–Xu [18]	Weighted	$O(nr^{\frac{7-\omega}{5-\omega}} \log^{\frac{\omega-1}{5-\omega}} r \log nW)$
Harvey [20]	Weighted	$\tilde{O}(W^{1+\epsilon} nr^{\omega-1})$
This paper	Weighted	$O(nr \log r_* + Wnr_*^{\omega-1})$
(approximation)	Weighted	$O(nr \log r_* + \epsilon^{-1} nr_*^{\omega-1} \log r_*)$

1.2 Our Technique

The idea of reducing a weighted optimization problem into unweighted ones has been successfully applied in the context of maximum-weight matching in bipartite graphs [26] and in general graphs [22, 36]. Roughly speaking, these matching algorithms proceed iteratively as follows: in each round, in a subgraph with only edges of the largest (updated) weights, a maximum-cardinality matching and its corresponding optimal dual are computed; the latter is then used to update the edge weights. The correctness of the solution is shown via the complementary slackness condition.

The difficulty of extending this approach to the matroid intersection setting lies in the dual part. In the matching problem, the dual variables have a clear graph-theoretic interpretation: they correspond to the potential of the vertices and the odd sets. This makes manipulating the interaction between the primal and the dual problems relatively easy. However, in the more general and abstract matroid intersection setting, the dual variables are harder to reason with and to control in subsequent iterations.

To overcome the aforementioned difficulty,[3] we make use of Frank's weight-splitting approach [12, 13]. He shows that the dual variables used in primal-dual schema can be replaced by a much simpler weight-splitting $w = w_1 + w_2$ of the element weights. The complementary slackness condition for optimality can also be replaced by weight-optimality in w_1 and w_2. Harvey [20] also makes use of the weight splitting to solve the weighted linear matroid intersection in an algebraic way.

Our main insight is that the splitted weights w_1 and w_2 can also be used to re-define two new matroids for subsequent operations. This is analogous to using the dual optimal solution to update the edge weights in the maximum-weight matching [22, 26, 36].

Our exact algorithms can be briefly summarized as follows. In each round, (1) a pair of new matroids are defined based on the current weight splitting w_1 and w_2. (2) A maximum-cardinality common independent set of the two new matroids is computed using the previously found independent set. (3) Based on the computed independent set, the weights w_1 and w_2 are re-adjusted. The correctness of our algorithms boils down to arguing that the maintained common independent set always satisfies a relaxed optimality condition, called (w_1, w_2)-near-optimality (see Definition 3.1 in Section 3.1), during the iteration.

Another technical obstacle in the above approach is the second step: we need to find a maximum-cardinality common independent set satisfying the (w_1, w_2)-near-optimality. This has to be done

[3] Here we assume the readers are familiar with matroid literature. Readers unfamiliar with the technical terms in the following discussion can find their formal definitions in Section 2.

without resorting to reduction to weighted matroid intersection (that would defeat the entire purpose). As we show in Section 5, this step is in fact not too difficult: If the previous common independent set is already (w_1, w_2)-near-optimal, we can compute a maximum-cardinality one by augmentation-type unweighted matroid intersection algorithms. For the linear matroid case, we can use a faster algebraic algorithm [3, 21] with slight modification.

It may be worthwhile contrasting our exact algorithms with Frank's algorithm [12]. His algorithm is designed for two general matroids, using a modified auxiliary graph. The weights w_1 and w_2 are used to "suppress" some edges in the original auxiliary graph. It can be shown that the modified auxiliary graph in his algorithm would be identical to the auxiliary graph of our matroids defined in each round. He augments the current independent set I repeatedly in the modified auxiliary graph, preserving the condition that I is a maximum-weight common independent set with size $|I|$. On the other hand, our algorithm only maintains the relaxed optimality condition, and dramatically augments I with the aid of unweighted matroid intersection algorithms.

Our approximation algorithms use a scaling technique of Duan and Pettie [8] for approximating maximum-weight matching. Again, we exploit the weight splitting w_1 and w_2 as dual variables. In each phase, w is rounded to multiples of a parameter δ. We then apply the three steps in our exact algorithms, with the difference that the amount of weights adjusted is δ. We repeat this while changing δ (in fact halved in each phase). Throughout the algorithm, we maintain (w_1, w_2)-near-optimality, while the weights w_1 and w_2 only approximate the original weight w. To maintain (w_1, w_2)-near-optimality, we need to take some extra weight adjustment when the phase transitions.

2 Preliminaries

2.1 Matroid

The pair $\mathbf{M} = (S, \mathcal{I})$ of a finite set S and a family $\mathcal{I}$ of subsets of S is called a *matroid*, if $\mathcal{I} \neq \emptyset$ and it satisfies the following two conditions.

1. If $I \subseteq J$ and $J \in \mathcal{I}$, then $I \in \mathcal{I}$.
2. For every $I, J \in \mathcal{I}$ with $|I| < |J|$, there exists $e \in J \setminus I$ such that $I + e \in \mathcal{I}$.[4]

A set in $\mathcal{I}$ is said to be *independent*, and a maximal independent set is called a *base*. A minimal non-independent subset C of S is called a *circuit*.

Let $\mathbf{M} = (S, \mathcal{I})$ be a matroid and X a subset of S. The *restriction* of $\mathbf{M}$ to X is defined by $\mathbf{M}|X = (X, \mathcal{I}|X)$ with $\mathcal{I}|X = \{I \in \mathcal{I} \mid I \subseteq X\}$. The *contraction* of $\mathbf{M}$ with respect to X is defined as $\mathbf{M}/X = (S \setminus X, \mathcal{I}/X)$ with $\mathcal{I}/X = \{I \subseteq S \setminus X \mid I \cup B \in \mathcal{I} \text{ for some base } B \text{ of } \mathbf{M}|X\}$. The *direct sum* of matroids $\mathbf{M}_1 = (S_1, \mathcal{I}_1)$ and $\mathbf{M}_2 = (S_2, \mathcal{I}_2)$, denoted by $\mathbf{M}_1 \oplus \mathbf{M}_2$, is defined to be $(S_1 \cup S_2, \mathcal{I}')$, where $\mathcal{I}' = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$.

Given a matroid $\mathbf{M} = (S, \mathcal{I})$ and a weight function $w\colon S \to \mathbb{Z}_{\geq 0}$, $I \in \mathcal{I}$ is said to be *w-maximum*, if its weight $\sum_{e \in I} w(e)$ is maximum among all independent sets in $\mathcal{I}$. A base is called a *w-maximum base*, if its weight is maximum among all bases. Using the family of w-maximum bases of a matroid $\mathbf{M} = (S, \mathcal{I})$, one can define a new matroid $\mathbf{M}^w = (S, \mathcal{I}^w)$, where

$$\mathcal{I}^w = \{I \subseteq S \mid I \subseteq B \text{ for some } w\text{-maximum base } B \text{ of } \mathbf{M}\}.$$

The fact that $\mathbf{M}^w$ is a matroid is well known, see e.g., [10].

The following lemma states some important properties of such a derived matroid $\mathbf{M}^w$.

[4] We use the shorthand $I + e$ and $I - e$ to stand for $I \cup \{e\}$ and $I \setminus \{e\}$, respectively.

Lemma 1. *Assume that we are given a matroid* $\mathbf{M} = (S, \mathcal{I})$ *and a weight function* $w \colon S \to \{0, 1, \ldots, W\}$*. Furthermore, define* $Z(t) = \{e \in S \mid w(e) \geq t\}$ *for an integer* $t \geq 1$*.*

(i) *Using the notation*

$$\overline{\mathbf{M}} = \bigoplus_{t=0}^{W} (\mathbf{M}|Z(t))/Z(t+1),$$

we have $\mathbf{M}^w = \overline{\mathbf{M}}$*.*

(ii) $I \in \mathcal{I}$ *is* w*-maximum if and only if* $I \cap Z(t)$ *is a base of* $\mathbf{M}|Z(t)$ *for every* $t = 1, 2, \ldots, W$*.*

(iii) *Suppose that* $I \in \mathcal{I}$ *satisfies the condition that* $I \cap Z(t)$ *is a base in* $\mathbf{M}|Z(t)$ *for* $(\min_{e \in S} w(e)) + 1 \leq t \leq W$*, and* $I + e$ *contains a circuit* C' *of* $\mathbf{M}^w$*. Then, each element in* C' *has weight equal to* $w(e)$*. Furthermore, there exists a circuit* $C \supseteq C'$ *in* $I + e$ *with respect to* $\mathbf{M}$*, and each element in* $C \setminus C'$ *has weight greater than* $w(e)$*.*

Proof. **(i)** It suffices to prove that the family of bases of $\mathbf{M}^w$ and that of $\overline{\mathbf{M}}$ are the same. Let B be a base of $\mathbf{M}^w$, and assume that B is not a base of $\overline{\mathbf{M}}$. Let B' be a base of $\overline{\mathbf{M}}$. Then, it follows from the definition of contraction that $B' \cap Z(t)$ is a base of $\mathbf{M}|Z(t)$, i.e., $|B \cap Z(t)| \leq |B' \cap Z(t)|$ for any $t \geq 1$. Since B is not a base of $\overline{\mathbf{M}}$, there exists an integer t' such that $|B \cap Z(t')| < |B' \cap Z(t')|$. From these facts, we see that there exists a bijection $\varphi \colon B \to B'$ such that $w(e) \leq w(\varphi(e))$ for every element e in B and $w(f) < w(\varphi(f))$ for some element f in B. This contradicts the fact that B is w-maximum base of $\mathbf{M}$. Thus, B is a base of $\overline{\mathbf{M}}$. Using similar ideas as above, we can show that every base of $\overline{\mathbf{M}}$ is also a base of $\mathbf{M}^w$.

(ii) The sufficiency direction is straightforward. For the necessity direction, observe that a w-maximum independent set I can be extended to a w-maximum base B. It is well known that a greedy algorithm finds a w-maximum base, and moreover, there exists a (non-increasing) order of elements such that the greedy algorithm returns B. This means that $B \cap Z(t)$ is a base of $\mathbf{M}|Z(t)$ for every $t = 0, 1, \ldots, W$. As $e \in B \setminus I$ has $w(e) = 0$, $I \cap Z(t)$ is a base of $\mathbf{M}|Z(t)$ for every $t = 1, 2, \ldots, W$.

(iii) It follows from (i) that $\mathbf{M}^w$ is equivalent to $\overline{\mathbf{M}}$. Now (iii) follows from the definitions of restriction, contraction, and direct sum operations. □

2.2 Matroid Intersection

Suppose that we are given two matroids $\mathbf{M}_1 = (S, \mathcal{I}_1)$ and $\mathbf{M}_2 = (S, \mathcal{I}_2)$ on the same ground set S. A subset I of S is a *common independent set*, if I belongs to $\mathcal{I}_1 \cap \mathcal{I}_2$. The goal of the *matroid intersection problem* is to find a maximum-cardinality common independent set. For a nonnegative weight $w(e)$ on each element e, the *weighted matroid intersection problem* is to find a common independent set with maximum weight.

For two matroids $\mathbf{M}_1$ and $\mathbf{M}_2$, the *auxiliary graph* is a directed graph $G_{\mathbf{M}_1,\mathbf{M}_2}(I) = (S, E_1 \cup E_2)$, where

$$E_1 = \{ef \mid I + e \notin \mathcal{I}_1,\ I + e - f \in \mathcal{I}_1\}, \quad E_2 = \{fe \mid I + e \notin \mathcal{I}_2,\ I + e - f \in \mathcal{I}_2\}.$$

In the auxiliary graph, we also define

$$X_1 = \{e \in S \setminus I \mid I + e \in \mathcal{I}_1\}, \quad X_2 = \{e \in S \setminus I \mid I + e \in \mathcal{I}_2\}.$$

In the auxiliary graph, a directed path from X_2 to X_1 is an *augmenting path*. Let P be a shortest augmenting path. Let $I \triangle P = (I \setminus P) \cup (P \setminus I)$. Then, $I \triangle P$ is another common independent set, whose size is one larger than I. If there is no augmenting path in the auxiliary graph, then I is already a maximum-cardinality common independent set.

3 Exact Algorithm

In this section, we present an exact algorithm for the weighted matroid intersection. Let $W = \max_{e\in S} w(e)$.

Our algorithm runs in W rounds. For descriptional convenience, we start from Round W and down to Round 1. In Round i, the subset $S' \subseteq S$ of elements e with $w(e) \geq i$ is the ground set of the two matroids.

We maintain a pair of weight functions w_1 and w_2 as a weight splitting of the original weight w, that is, $w = w_1 + w_2$. We define a new pair of matroids $\mathbf{M}'_1$ and $\mathbf{M}'_2$ as the restriction of $\mathbf{M}_1^{w_1}$ and $\mathbf{M}_2^{w_2}$ to S'. In each round, the algorithm finds a maximum-cardinality common independent set I between $\mathbf{M}'_1$ and $\mathbf{M}'_2$ using I', where I' is the common independent set found in the previous round, and update w_1, w_2 based on the auxiliary graph $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I)$. Below we first present the algorithm and then elaborate the details.

Step 1. Set $i = W$. Define $w_1 = 0$ and $w_2 = w$, and $I' = \emptyset$.

Step 2. While $i > 0$ do the following steps.

- **(2-1)** Define $S' = \{e \in S \mid w_2(e) \geq i\}$.
- **(2-2)** Define $\mathbf{M}'_\ell = (S', \mathcal{I}'_\ell)$ as $\mathbf{M}_\ell^{w_\ell}|S'$ for $\ell = 1, 2$.
- **(2-3)** Unweighted_Matroid_Intersection
  Construct I using[5] the previous set I' so that
  - (i) I is a maximum-cardinality common independent set of $\mathbf{M}'_1$ and $\mathbf{M}'_2$, and
  - (ii) I is (w_1, w_2)-near-optimal in S'.
- **(2-4)** Update_Weight
  - **(2-4-1)** Let $T \subseteq S'$ be the set of elements reachable from X_2 in the auxiliary graph $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I)$.
  - **(2-4-2)** For any $e \in T$, let $w_1(e) := w_1(e) + 1$, $w_2(e) := w_2(e) - 1$.
- **(2-5)** Set $i := i - 1$ and $I' := I$.

Step 3. Return I.

3.1 Analysis

The final goal of our algorithm is to find a common independent set that is w_1-maximum in $\mathbf{M}_1$ and w_2-maximum in $\mathbf{M}_2$, which would imply that I is w-maximum if $w = w_1 + w_2$. For an integer t, let $Z_1(t) = \{e \in S \mid w_1(e) \geq t\}$ and $Z_2(t) = \{e \in S \mid w_2(e) \geq t\}$. It follows from Lemma 1(ii) that I being w_1-maximum in $\mathbf{M}_1$ and w_2-maximum in $\mathbf{M}_2$ is equivalent to

1. $I \cap Z_1(t)$ is a base of $\mathbf{M}_1|Z_1(t)$ for any integer $t \geq 1$, and

2. $I \cap Z_2(t)$ is a base of $\mathbf{M}_2|Z_2(t)$ for any integer $t \geq 1$.

We call such a common independent set I of $\mathbf{M}_1$ and $\mathbf{M}_2$ (w_1, w_2)-*optimal.*

We relax the above condition as follows. Here we define $Z'_\ell(t) = Z_\ell(t) \cap S'$ for a subset $S' \subseteq S$ and $\ell = 1, 2$.

Definition 3.1. A common independent set I of $\mathbf{M}_1$ and $\mathbf{M}_2$ is (w_1, w_2)-*near-optimal in a subset* $S' \subseteq S$ if

[5] The term "using" is purposely chosen to be vague. The implementation details are deferred to Section 5.

1. $I \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$ for any integer $t \geq 1$, and
2. $I \cap Z'_2(t)$ is a base of $\mathbf{M}_2|Z'_2(t)$ for any integer $t \geq \alpha + 1$, where $\alpha = \min_{e \in S'} w_2(e)$.

Note that if $\alpha = 0$ and $S' = S$, a (w_1, w_2)-near-optimal common independent set in S' is (w_1, w_2)-optimal.

We will see that, during the execution of our algorithm, the current set I is always (w_1, w_2)-near-optimal in S'. To prove this, in the following, we analyze the two procedures Unweighted_Matroid_Intersection and Update_Weight used in Steps (2-3) and (2-4).

In Step (2-3), in the procedure Unweighted_Matroid_Intersection, if we only want a maximum-cardinality common independent set I of $\mathbf{M}'_1$ and $\mathbf{M}'_2$, the step is trivial. The difficulty is how to guarantee that I is also (w_1, w_2)-near-optimal in S' *without* resorting to weighted matroid intersection. We show that, if the previous common independent set I' is (w_1, w_2)-near-optimal in S', then we can construct I satisfying the two stated conditions in Step (2-3) using unweighted matroid intersection algorithms. How Step (2-3) is implemented depends on the given matroids and their specialized algorithms. The details are deferred to Section 5. We use a lemma to summarize the outcome of Step (2-3). Recall we denote $\mathbf{M}'_\ell = \mathbf{M}^{w_\ell}_\ell|S'$ for $\ell = 1, 2$.

Lemma 2. *Suppose that I' is (w_1, w_2)-near-optimal in a subset S'. Then we can construct another common independent set I, using known unweighted matroid intersection algorithms, that is simultaneously (i) a maximum-cardinality common independent set of $\mathbf{M}'_1$ and $\mathbf{M}'_2$, and (ii) (w_1, w_2)-near-optimal in S'.*

We next prove that, if the maximum-cardinality common independent set I of $\mathbf{M}'_1$ and $\mathbf{M}'_2$ is (w_1, w_2)-near-optimal in S', then we can modify w_1 and w_2 at Step (2-4) so that I is still (w_1, w_2)-near-optimal in S'.

Lemma 3. *Suppose that all weights of w_1 and w_2 are nonnegative integers, where $w_1(e) \leq p_1$ and $w_2(e) \geq p_2$ for any $e \in S'$ for some integers p_1 and p_2. Suppose that I is (i) a maximum-cardinality common independent set of $\mathbf{M}'_1$ and $\mathbf{M}'_2$, and (ii) (w_1, w_2)-near-optimal in S'. Then after the procedure Update_Weight, we have*

(1) *$I \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$ for any integer t with $1 \leq t \leq p_1 + 1$, and*

(2) *$I \cap Z'_2(t)$ is a base of $\mathbf{M}_2|Z'_2(t)$ for any integer $t \geq p_2$.*

Notice that the lemma implies that after Step (2-4), I is still (w_1, w_2)-near-optimal in S', since then $\max_{e \in S'} w_1(e) \leq p_1 + 1$ and $\min_{e \in S'} w_2(e) \geq p_2 - 1$.

Proof. We only prove (1), since (2) follows symmetrically. To avoid confusion, let $\tilde{Z}'_1(t)$ denote the set $Z'_1(t)$ *after* the weights w_1 and w_2 are updated. Observe that, for an integer t with $1 \leq t \leq p_1 + 1$,

$$\tilde{Z}'_1(t) = Z'_1(t) \cup ((Z'_1(t-1) \setminus Z'_1(t)) \cap T),$$

where we note that $Z'_1(p_1 + 1) = \emptyset$ and $Z'_1(0) = S'$.

As $I \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$, we argue that given an element $e \in ((Z'_1(t-1) \setminus Z'_1(t)) \cap T) \setminus I$:

$$(*)\ I + e \notin \mathcal{I}_1|S', \text{ and } (**) \text{ the circuit of } I + e \text{ in } \mathbf{M}_1|S' \text{ is contained in } \tilde{Z}'_1(t).$$

This will establish that $I \cap \tilde{Z}'_1(t)$ is a base of $\mathbf{M}_1|\tilde{Z}'_1(t)$ for $t = 1, 2, \ldots, p_1 + 1$.

To see $(*)$, observe that in the auxiliary graph $G_{\mathbf{M}'_1, \mathbf{M}'_2}(I)$, e is not part of X_1 (otherwise, there would be an augmenting path, contradicting to the assumption that I is a maximum-cardinality common independent set in $\mathbf{M}'_1$ and $\mathbf{M}'_2$). So, $I+e$ contains a circuit C' in $\mathbf{M}'_1$, and by Lemma 1(iii) applied to $\mathbf{M}_1|S'$ (as the assumption is that $I \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$ for $t = 1, 2, \ldots, p_1$), $I + e$ also has a circuit $C \supseteq C'$ in $\mathbf{M}_1|S'$. Thus, $(*)$ is proved.

For $(**)$, consider an element e' in $C'-e$. Then, e' is contained in $Z'_1(t-1) \setminus Z'_1(t)$ by Lemma 1(iii). Then, in the auxiliary graph $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I)$, there exists an arc from e to e'. Then e' is part of T, implying that the circuit C' is in $(Z'_1(t-1) \setminus Z'_1(t)) \cap T$, which in turn, by Lemma 1(iii), implies that the circuit $C \supseteq C'$ in $I+e$ with respect to $\mathbf{M}_1|S'$ is part of $Z'_1(t) \cup ((Z'_1(t-1) \setminus Z'_1(t)) \cap T) = \tilde{Z}'_1(t)$. Then, we prove $(**)$. □

Lemma 4. *In Round i with $1 \le i \le W$, the following holds.*

(1) $w = w_1 + w_2$,

(2) *After Step (2-4), $I \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$ for $1 \le t \le W - i + 1$, and*

(3) *After Step (2-4), $I \cap Z'_2(t)$ is a base of $\mathbf{M}_2|Z'_2(t)$ for $i \le t \le W$.*

Proof. (1) can be easily seen. We prove (2) and (3) by induction on i.

For the base case of $i = W$, as $Z'_1(1) = \emptyset$ and $Z'_2(W+1) = \emptyset$, $I' = \emptyset$ is (w_1, w_2)-near-optimal in S', and thus Lemma 2 implies that we can obtain a maximum-cardinality common independent set I of $\mathbf{M}'_1$ and $\mathbf{M}'_2$ satisfying the condition that $I \cap Z'_1(1)$ is a base of $\mathbf{M}_1|Z'_1(1)$ and $I \cap Z'_2(W+1)$ is a base of $\mathbf{M}_2|Z'_2(W+1)$. Now applying Lemma 3 (with $p_1 = 0$ and $p_2 = W$), we have that $I \cap Z'_1(1)$ is a base of $\mathbf{M}_1|Z'_1(1)$ and $I \cap Z'_2(W)$ is a base of $\mathbf{M}_2|Z'_2(W)$.

For the induction step $i < W$, let I' be the common independent set obtained in Round $i+1$. By induction hypothesis, $I' \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$ for $1 \le t \le W - i$ and $I' \cap Z'_2(t)$ is a base of $\mathbf{M}_2|Z'_2(t)$ for $i+1 \le t \le W$. Notice that when Round i begins, only elements e with $w_1(e) = 0$ and $w_2(e) = i$ are added to S'. Hence the two conditions remain true after Step (2-1).

By these facts, as $w_2(e) \ge i$ for $e \in S'$, Step (2-3) can be correctly applied by Lemma 2, and we obtain the new independent set I satisfying the two conditions stated in Step (2-3). The proof now follows by applying Lemma 3 (with $p_1 = W - i$ and $p_2 = i$). □

Theorem 1. *The common independent set I returned by the exact algorithm is a maximum-weight common independent set of $\mathbf{M}_1$ and $\mathbf{M}_2$.*

Proof. By Lemma 4, after the last round when $i = 1$, as $S' = S$, it holds that $I \cap Z_1(t)$ is a base of $\mathbf{M}_1|Z_1(t)$ for $t = 1, 2, \ldots, W$, and $I \cap Z_2(t)$ is a base of $\mathbf{M}_2|Z_2(t)$ for $t = 1, 2, \ldots, W$. It follows from Lemma 1(ii) that I is w_ℓ-maximum in $\mathbf{M}_\ell$ for $\ell = 1, 2$. Then, for every common independent set J, we have

$$w(J) = w_1(J) + w_2(J) \le w_1(I) + w_2(I) = w(I).$$

Thus, I is a maximum-weight common independent set. □

The algorithm clearly runs in $O(W(T_{\mathrm{u}} + T_{\mathrm{d}}))$ time, where T_{u} and T_{d} are the running times for executing Unweighted_Matroid_Intersection and Update_Weight, respectively. Note that T_{u} and T_{d} depend on the representation of the given matroids. Their complexities are discussed in Section 5.

4 Approximation Algorithm

In this section, we design a $(1 - \epsilon)$-approximation algorithm for the weighted matroid intersection. Let W be the maximum weight. First of all, we show that we can round weights to small integers, and bound W from above.

Lemma 5. *We can reduce a given instance of the weighted matroid intersection problem to one with integral weights whose maximum weight is at most $2r_*/\epsilon$, where $r_* \le r$ is the maximum size of a common independent set.*

Proof. Let r_* be the maximum size of a common independent set. Set $\eta = \epsilon W/2r_*$, and define $w'(e) = \lfloor w(e)/\eta \rfloor$ for any $e \in S$. Then a $(1-\epsilon/2)$-approximate solution I' for the weight w' is a $(1-\epsilon)$-approximate solution for the weight w. Indeed, since $w(e) - 1 \le \eta w'(e) \le w(e)$ for any $e \in S$, we have

$$
\begin{aligned}
w(I') &\ge \eta w'(I') \\
&\ge \eta(1-\epsilon/2)w'(I'_{\mathrm{opt}}) \quad (I'_{\mathrm{opt}} \text{ is an optimal solution for } w') \\
&\ge \eta(1-\epsilon/2)w'(I_{\mathrm{opt}}) \quad (I_{\mathrm{opt}} \text{ is an optimal solution for } w) \\
&\ge (1-\epsilon/2)(w(I_{\mathrm{opt}}) - \eta|I_{\mathrm{opt}}|) \\
&\ge (1-\epsilon/2)(w(I_{\mathrm{opt}}) - \eta r_*) = (1-\epsilon/2)(w(I_{\mathrm{opt}}) - \epsilon W/2) \\
&\ge (1-\epsilon)w(I_{\mathrm{opt}}),
\end{aligned}
$$

where the last inequality follows because we may assume that a matroid has no loop, and thus $w(I_{\mathrm{opt}}) \ge W$. Thus it suffices to solve the problem for w', whose max weight is at most $W/\eta \le 2r_*/\epsilon$. □

During the algorithm, the weight w is split so that $w \approx w_1 + w_2$; furthermore, we will guarantee that all weights of w_1 and w_2 are nonnegative multiples of some integer $\delta > 0$, where δ may change in different phases of the algorithm. At the end, we find a common independent set that is w_1-maximum in $\mathbf{M}_1$ and w_2-maximum in $\mathbf{M}_2$, which would imply that I is a $(1-\epsilon)$-approximate solution if $w \le w_1 + w_2 \le (1+\epsilon)w$.

For simplicity, we assume that the bound W and ϵ are both powers of 2. Our algorithm runs in $1+\log_2 \epsilon W$ phases. In every phase, we apply a number (roughly $O(\epsilon^{-1})$) of Unweighted_Matroid_Intersection and Update_Weight operations. Note that $\log_2 \epsilon W = O(\log r)$ by Lemma 5.

Let $\delta_0 = \epsilon W$ and define $\delta_i = \delta_0/2^i$ for $1 \le i \le \log_2 \epsilon W$. The term δ_i will be the amount of change in the weights w_1 and w_2 during Phase i every time Update_Weight is invoked. For each element $e \in S$ and each integer i with $0 \le i \le \log_2 \epsilon W$, define $w^i(e)$ to be the truncated weight of element e in Phase i, that is, $w^i(e) = \lfloor \frac{w(e)}{\delta_i} \rfloor \delta_i$. Note that $w^{i+1}(e) = w^i(e)$ or $w^{i+1}(e) = w^i(e) + \delta_{i+1}$. The algorithm is presented below.

Step 1. Set $i = 0$. Define $w_1 = 0$, $w_2 = w^0$, and $I' = \emptyset$. Let $h = W$.

Step 2. While $i \le \log_2 \epsilon W$, do the following steps.

- **(2-0)** Set $L = \frac{W}{2^{i+1}}$ if $i < \log_2 \epsilon W$, and $L = 1$ if $i = \log_2 \epsilon W$.
- **(2-1)** While $h \ge L$
  - **(2-1-1)** Define $S' = \{e \mid w_2(e) \ge h\}$.
  - **(2-1-2)** Define $\mathbf{M}'_\ell = (S', \mathcal{I}'_\ell)$ as $\mathbf{M}_\ell^{w_\ell}|S'$ for $\ell = 1, 2$.
  - **(2-1-3)** Unweighted_Matroid_Intersection
    Construct I using the previous set I' so that
    (i) I is a maximum-cardinality common independent set of $\mathbf{M}'_1$ and $\mathbf{M}'_2$, and
    (ii) I is (w_1, w_2)-near-optimal in S'.
  - **(2-1-4)** Update_Weight
    - Let $T \subseteq S'$ be the set of elements reachable from X_2 in the auxiliary graph $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I)$.

- For any $e \in T$, let $w_1(e) := w_1(e) + \delta_i$, $w_2(e) := w_2(e) - \delta_i$.

(2-1-5) Set $h := h - \delta_i$ and $I' := I$.

(2-2) Weight Adjustment: If $i < \log_2 \epsilon W$ do the following.

(2-2-1) $\forall e \in I'$, let $w_2(e) = w_2(e) + \delta_{i+1}$.

(2-2-2) $\forall e \in S \setminus I'$ where $w^{i+1}(e) = w^i(e) + \delta_{i+1}$, let $w_2(e) = w_2(e) + \delta_{i+1}$.

(2-2-3) Set $h := h + \delta_{i+1}$.

(2-3) Set $i := i + 1$.

Step 3. Return I.

The outer loop Step 2 corresponds to a phase. We use a counter h to keep track of the progress of the algorithm. Initially $h = W$. In Phase i, the weights are always kept as nonnegative multiples of δ_i. In Step (2-1), the two matroids $\mathbf{M}'_1$ and $\mathbf{M}'_2$ are defined on the common ground set $S' = \{e \mid w_2(e) \geq h\}$, and the two procedures Unweighted_Matroid_Intersection and Update_Weight are invoked as was done in the exact algorithm in Section 3. The counter h is decreased by the amount of δ_i each time after Update_Weight is invoked in Step (2-1).

Each time h is halved, we make ready to move to the next phase, except in the last phase: in Phase $\log_2 \epsilon W$, we stop when h goes down to 0. The reason that we adjust the w_2-weights at Step (2-2) is that we want to ensure that in the beginning of the next phase, the weights w_1 and w_2 still approximate the next weight w^{i+1} (see Lemma 9). In particular, we increase the w_2-weights of all elements in the current common independent set I. This is done to make sure that I is still w_2-maximum in the beginning of the next phase (with respect to the newly-defined set S' in Step (2-1)).

4.1 Analysis

We first observe the number of iterations in the algorithm.

Lemma 6. (1) *During Phase i with $0 \leq i \leq \log_2 \epsilon W$, w_1 and w_2 are nonnegative multiples of δ_i, except in Step (2-2).*

(2) *Step (2-1) is executed at most $\frac{\epsilon^{-1}}{2}$ times in Phase i with $0 < i < \log_2 \epsilon W$. In the last phase, Step (2-1) is executed $\epsilon^{-1} + 1$ times.*

(3) *The total number of iterations in Step (2-1) is $O(\epsilon^{-1} \log r)$.*

Proof. (1) can be easily verified. For (2), observe that in Phase 0, Step (2-1) is executed $\frac{W - W/2}{\delta_0} = \frac{\epsilon^{-1}}{2}$ times. For Phase $i \geq 1$, in the beginning of that phase, $h = \frac{W}{2^i} - \delta_i$. Hence, if $i < \log_2 \epsilon W$, Step (2-1) is executed $\frac{(W/2^i - \delta_i) - W/2^{i+1}}{\delta_i} \leq \frac{\epsilon^{-1}}{2}$ times, and if $i = \log_2 \epsilon W$, Step (2-1) is executed $\frac{W/2^i - \delta_i}{\delta_i} \leq \epsilon^{-1}$ times. (3) now immediately follows from (2). □

We say an element $e \in S$ *joins* in Phase j if in Phase j, element e becomes part of the ground set S' in Step (2-1-1) the first time.

Lemma 7. *Suppose that an element $e \in S$ joins in Phase j for some $j < \log_2 \epsilon W$. Then the following holds.*

(1) $w^j(e) \geq \frac{W}{2^{j+1}} = \frac{\delta_j}{2\epsilon}$.

(2) *In all phases $i \geq j$, $w^i(e) \leq w_1(e) + w_2(e) \leq w^i(e) + 2\delta_j$.*

If $e \in S$ joins in the last phase $j = \log_2 \epsilon W$, then we have the following.

(3) $w_1(e) + w_2(e) = w^j(e)$.

Proof. We note that immediately before e joins in Phase j, we have $w_1(e) + w_2(e) = w^j(e)$. This follows from the observation that unless e is part of I when Step (2-2-1) is executed, the weight splitting $w_1(e)$ and $w_2(e)$ is *exact* with respect $w^{j'}(e)$ for $j' \leq j$. (3) follows easily from this observation. In the case that $j < \log_2 \epsilon W$, we have that $w^j(e) \geq w_2(e) \geq \frac{W}{2^{j+1}}$. Thus (1) is proved.

(2) follows from the fact the difference between the sum of $w_1(e)$ and $w_2(e)$ and the truncated weight $w^{j'}(e)$ grows larger only when Step (2-2-1) is executed in Phase $j' \geq j$ and e is part of the common independent set I in that step. Hence it holds that

$$w^i(e) \leq w_1(e) + w_2(e) \leq w^i(e) + \sum_{s=j}^{i} \delta_s \leq w^i(e) + 2\delta_j.$$

□

Since all weights of w_1 and w_2 are nonnegative multiples of δ_i and we modify w_1 and w_2 by δ_i at Update_Weight, we have the following lemma, which can be obtained similarly to Lemma 3 by dividing all the values by δ_i.

Lemma 8. *Suppose that all weights of w_1 and w_2 are nonnegative multiples of δ, where $w_1(e) \leq p_1$ and $w_2(e) \geq p_2$ for any $e \in S'$ for some integers p_1 and p_2. Suppose that I is (i) a maximum-cardinality common independent set of $\mathbf{M}_1'$ and $\mathbf{M}_2'$, and (ii) (w_1, w_2)-near-optimal in S'. Then after the procedure Update_Weight, we have*

(1) *$I \cap Z_1'(t)$ is a base of $\mathbf{M}_1|Z_1'(t)$ for any integer t with $1 \leq t \leq p_1 + \delta$, and*

(2) *$I \cap Z_2'(t)$ is a base of $\mathbf{M}_2|Z_2'(t)$ for any integer $t \geq p_2$.*

Note that the lemma implies that the current independent set I is still (w_1, w_2)-near-optimal in S' after Step (2-1-4).

Lemma 9. *In Phase i, after Step (2-1) terminates, we have the following.*

(1) *$I' \cap Z_1'(t)$ is a base of $\mathbf{M}_1|Z_1'(t)$ for any integer $t \geq 1$.*

(2) *$I' \cap Z_2'(t)$ is a base of $\mathbf{M}_2|Z_2'(t)$ for any integer $t \geq h + \delta_i$.*

Proof. We first prove the following claim.

Claim 1. *In each phase, if (1) and (2) hold before the first iteration of Step (2-1) starts, we have (1) and (2) after the final iteration of Step (2-1) terminates.*

Proof of Claim. We prove the claim by induction on the number of times Step (2-1) is invoked. For the base case, we have (1) and (2) in the beginning by the assumption.

Suppose that we have (1) and (2) for the previous set I' at the beginning of the current iteration in Step (2-1). At Step (2-1-1), some elements may be added into S'. However, all such elements have $w_1(e) = 0$ and $w_2(e) = h$. So I' still satisfies (1) and (2), and thus it is (w_1, w_2)-near-optimal in S' since $w_2(e) \geq h$ for any $e \in S'$. By Lemma 2, Step (2-1-3) can be correctly implemented, and we obtain a maximum-cardinality common independent set I of $\mathbf{M}_1'$ and $\mathbf{M}_2'$ that is (w_1, w_2)-near-optimal in S'. After Step (2-1-4), by Lemma 8 (by setting $\delta = \delta_i$, $p_1 = \max_{e \in S'} w_1(e)$, and $p_2 = h$), I satisfies (1) and that $I \cap Z_2'(t)$ is a base of $\mathbf{M}_2|Z_2'(t)$ for any integer $t \geq h$. Since h is decreased by δ_i in Step (2-1-5), we have (1) and (2) at the end of the current iteration. This proves the claim. □

We then prove the lemma by induction on the number of phases. For the base case, as in the beginning of the algorithm, $h = W$ and $I' = \emptyset$, the set I' is (w_1, w_2)-near-optimal in S'. This means that we have (1) and (2) for I', and hence Claim 1 implies that we have (1) and (2) after the iterations of Step (2-1) terminates in Phase 0.

For the induction step, suppose that currently the algorithm is in Phase i, and that (1) and (2) are satisfied after Step (2-1) are done. We argue that after the weight adjustment done in Step (2-2), I' still satisfies (1) and (2).

To avoid confusion, let $\tilde{Z}_\ell(t)\,(\ell = 1, 2)$ denote the sets *after* w_2-weights are modified in Steps (2-2-1) and (2-2-2), and let $\tilde{h}$ be the value of h after Step (2-2-3), i.e., $\tilde{h} = h + \delta_{i+1}$.

By Lemma 6(1), all w_1 and w_2 weights are multiples of δ_i in Phase i before Step (2-2). So no weight is of the form of $a\delta_i + \delta_{i+1}$ for some $a \geq 0$. Therefore, after Step (2-1), the fact that I' satisfies (2) implies

($\star$) $I' \cap Z'_2(t)$ is a base of $\mathbf{M}_2|Z'_2(t)$ for any integer $t \geq h + \delta_{i+1}$.

Note that I' satisfying (2) only guarantees this property for $t \geq h + \delta_i$. We can subtract δ_{i+1} further because there is no element with w_2-weight of the form $a\delta_i + \delta_{i+1}$ for some $a \geq 0$. Hence the range of t starts from $h + \delta_i - \delta_{i+1} = h + \delta_{i+1}$.

As we increase the w_2-weights of all elements in I' and a subset of elements in $S' \setminus I'$, while leaving the w_1-weights unchanged, we have

(i) $\tilde{Z}_1(t) = Z_1(t)$ for all $t \in \mathbb{Z}_{\geq 0}$.

(ii) $I \cap \tilde{Z}'_1(t)$ is a base of $\mathbf{M}_1|\tilde{Z}'_1(t)$ for any integer $t \geq 1$.

(iii) $I \cap \tilde{Z}'_2(t)$ is a base of $\mathbf{M}_2|\tilde{Z}'_2(t)$ for any integer $t \geq \tilde{h} + \delta_{i+1}$.

(i) and (ii) are easy to see, since w_1-weights are unchanged and (1) holds before Step (2-2). For (iii), consider any integer $t \geq \tilde{h} + \delta_{i+1} = h + 2\delta_{i+1}$. We have that $I' \cap \tilde{Z}'_2(t) = I' \cap Z'_2(t - \delta_{i+1})$, where the latter is a base of $\mathbf{M}_2|Z'_2(t-\delta_{i+1})$ by ($\star$). As $\tilde{Z}'_2(t) \subseteq Z'_2(t-\delta_{i+1})$, we infer that $I' \cap \tilde{Z}'_2(t)$ is still a base of $\mathbf{M}_2|\tilde{Z}'_2(t)$.

Therefore, at the beginning of Phase $i + 1$, we have (1) and (2), and hence the proof follows from Claim 1.

□

Lemma 10. *The common independent set I returned by the algorithm is a maximum-weight common independent set with the weight function $w_1 + w_2$ in the end.*

Proof. After the last time Step (2-1-5) is executed, by Lemma 9 and the fact that $S' = S$, $I \cap Z_1(t)$ is a base of $\mathbf{M}_1|Z_1(t)$ for all $t \geq 1$, and $I \cap Z_2(t)$ is a base of $\mathbf{M}_2|Z_2(t)$ for all $t \geq \delta_{\log_2 \epsilon W}$. Since $\delta_{\log_2 \epsilon W} = 1$, it follows from Lemma 1(ii) that I is w_1-maximum in $\mathbf{M}_1$ and w_2-maximum in $\mathbf{M}_2$. Therefore, for any common independent set J, we have

$$w_1(J) + w_2(J) \leq w_1(I) + w_2(I).$$

The proof follows. □

Theorem 2. *Let I be the common independent set returned by the algorithm. Then I is a $1 - 4\epsilon$ approximation.*

Proof. For each element $e \in S$, if it joins in Phase $j < \log_2 \epsilon W$, then by Lemma 7(2),

$$w^{\log_2 \epsilon W}(e) \leq w_1(e) + w_2(e) \leq w^{\log_2 \epsilon W}(e) + 2\delta_j \leq (1+4\epsilon) w^{\log_2 \epsilon W}(e),$$

where the last inequality holds since $\delta_j \leq 2\epsilon w^j(e) \leq 2\epsilon w^{\log_2 \epsilon W}(e)$ by Lemma 7(1). Moreover, if $j = \log_2 \epsilon W$, then $w^{\log_2 \epsilon W}(e) = w_1(e) + w_2(e)$ by Lemma 7(3). Since $w^{\log_2 \epsilon W}(e) = w(e)$, we conclude that

$$w(e) \leq w_1(e) + w_2(e) \leq (1+4\epsilon) w(e).$$

Therefore, letting I_{opt} be the maximum-weight common independent set, Lemma 10 implies that

$$w(I_{\text{opt}}) \leq w_1(I_{\text{opt}}) + w_2(I_{\text{opt}}) \leq w_1(I) + w_2(I) \leq (1+4\epsilon) w(I).$$

The proof follows. □

5 Implementation of Unweighted Matroid Intersection

In this section, we discuss how to implement the procedure Unweighted_Matroid_Intersection and the actual complexities of our algorithms for various weighted matroid intersection problems.

Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two matroids, and w_1 and w_2 be weights. Suppose that a common independent set I' of $\mathbf{M}'_1$ and $\mathbf{M}'_2$ is (w_1, w_2)-near-optimal in a subset $S' \subseteq S$ (recall that $\mathbf{M}'_\ell = \mathbf{M}_\ell^{w_\ell} | S'$ for $\ell = 1, 2$). We consider finding a maximum-cardinality common independent set I between $\mathbf{M}'_1$ and $\mathbf{M}'_2$ that is (w_1, w_2)-near-optimal in S'.

5.1 Two General Matroids

Cunningham [6] shows how to find a maximum-cardinality common independent set by repeatedly augmenting a common independent set in the auxiliary graph, using $O(nr^{1.5})$ independence oracle calls. We argue that if we apply his algorithm to $\mathbf{M}'_1$ and $\mathbf{M}'_2$ with I' as the initial common independent set, each new independent set resulted from augmentation will satisfy the same property as I'.

Claim 2. *Suppose that I' is (w_1, w_2)-near-optimal in S'. Let P be the shortest path from X_2 to X_1 in $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I')$. Then the set $I = I' \triangle P$ is also (w_1, w_2)-near-optimal.*

Proof. By Lemma 1(iii), in the auxiliary graph $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I')$, an element $e \in (Z_1(t) \setminus Z_1(t+1)) \setminus I'$ has outgoing arcs to only other elements in $(Z_1(t) \setminus Z_1(t+1))$ for every $t \geq 1$. Similarly, an element $e \in (Z_2(t) \setminus Z_2(t+1)) \cap I'$ has only outgoing arcs towards other elements in $(Z_2(t) \setminus Z_2(t+1)) \setminus I'$ for every $t \geq p+1$, where $p = \min w_2(e)$.

These two facts imply that along the augmenting path P in $G_{\mathbf{M}'_1,\mathbf{M}'_2}(I')$, the number of elements in $(Z_1(t) \setminus Z_1(t+1)) \setminus I'$ is the same as the number of elements in $(Z_1(t) \setminus Z_1(t+1)) \cap I'$ for every $t \geq 1$. Similarly, the number of elements in $(Z_2(t) \setminus Z_2(t+1)) \cap I'$ is the same as that in $(Z_2(t) \setminus Z_2(t+1)) \setminus I'$ for $t \geq p+1$. Thus, $|I \cap Z_1(t)| = |I' \cap Z_1(t)|$ for $t \geq 1$, and $|I \cap Z_2(t)| = |I' \cap Z_2(t)|$ for $t \geq p+1$. The proof follows. □

In Cunningham's algorithm, we need to have an independence oracle to test whether $I' + e \in \mathcal{I}'_\ell$ and whether $I' + e - f \in \mathcal{I}'_\ell$ (recall $\mathcal{I}'_\ell$ is the family of independent sets of matroid $\mathbf{M}'_\ell$) for an independent set I', $e \in S' \setminus I'$, and $f \in I'$ for $\ell = 1, 2$. Such an oracle can be implemented as follows. By Lemma 1(iii), if $I' + e \notin \mathcal{I}'_\ell$, then $I' + e - f \in \mathcal{I}'_\ell$ if and only if $I' + e - f \in \mathcal{I}_\ell$ and

$w_\ell(e) = w_\ell(f)$. Furthermore, $I' + e \in \mathcal{I}'_1$ if and only if $I' + e \in \mathcal{I}_1$ and $w_1(e) = 0$, and $I' + e \in \mathcal{I}'_2$ if and only if $I' + e \in \mathcal{I}_2$ and $w'_2(e) = \min w_2(e)$. Thus an independence oracle for $\mathbf{M}'_1$ and $\mathbf{M}'_2$ can be implemented using that for $\mathbf{M}_1$ and $\mathbf{M}_2$.

We can perform Update_Weight in $O(nr)$ independence oracle calls. Therefore, we have the following theorem for two general matroids.

Theorem 3. *For two general matroids, we can solve the weighted matroid intersection exactly in $O(\tau n r^{1.5})$ time, and $(1-\epsilon)$-approximately in $O(\tau \epsilon^{-1} n r^{1.5} \log r)$ time, where τ is the running time to check the independence of given matroids.*

For the exact algorithm, a slight sharpening in the running time is possible. Observe that in Round i, Cunningham's algorithm takes $O(\tau |S'| r^{1.5})$ time, where $S' = \{e \mid w(e) \geq i\}$. Since $\sum_{i=1}^{W} |\{e \mid w(e) \geq i\}| = \sum_{e \in S} w(e)$, the total running time is $O(\tau (\sum_{e \in S} w(e)) r^{1.5})$.

5.2 Two Graphic Matroids

Suppose that $\mathbf{M}_1$ and $\mathbf{M}_2$ are graphic matroids. That is, $\mathbf{M}_\ell = (S, \mathcal{I}_\ell)$ $(\ell = 1, 2)$ is represented by a graph $G_\ell = (V_\ell, S)$ so that $\mathcal{I}_\ell$ is the family of edge subsets in S that are forests in G_ℓ. Note that the number of edges in G_ℓ is $n = |S|$, and the number of vertices is $O(r)$, since we may assume that there is no isolated vertex. Gabow and Xu [17] designed algorithms that run in $O(\sqrt{r} n \log r)$ time for the unweighted graphic matroid intersection, and in $O(\sqrt{r} n \log^2 r \log(rW))$ time for the weighted case.

It is well known that, if $\mathbf{M}_\ell$ is graphic, then so is $\mathbf{M}'_\ell = \mathbf{M}_\ell^{w_\ell} | S'$ for a subset S' and $\ell = 1, 2$. Indeed, for a subset $X \subseteq S$, the restriction of G_ℓ to X (the subgraph induced by an edge subset X), denoted by $G_\ell | X$, represents $\mathbf{M}_\ell | X$. Moreover, the graph obtained from G_ℓ by contracting X, denoted by G_ℓ / X, represents $\mathbf{M}_\ell / X$. Then, by Lemma 1(i), $\mathbf{M}'_\ell = \mathbf{M}_\ell^{w_\ell} | S'$ has a graph representation $G'_\ell | S'$, where G'_ℓ is in the form of

$$G'_\ell = \bigoplus_{t=0}^{W} (G_\ell | Z_\ell(t)) / Z_\ell(t+1).$$

That is, G'_ℓ is the disjoint union of graphs $(G_\ell | Z_\ell(t)) / Z_\ell(t+1)$ obtained by restriction and contraction.

We apply Gabow and Xu's algorithm [17] for the unweighted problem to $\mathbf{M}'_1$ and $\mathbf{M}'_2$ with I' as the initial common independent set. Since their algorithm is an augmentation-type algorithm, it follows from Claim 2 that the obtained maximum-cardinality common independent set is (w_1, w_2)-near-optimal in S'. Note that the numbers of vertices and edges in G' are $O(r)$ and n, respectively. Since the reachable set T in the procedure Update_Weight can be found in the end of Gabow and Xu's algorithm, we have the following.

Theorem 4. *For two graphic matroids, we can solve the weighted matroid intersection exactly in $O(W \sqrt{r} n \log r)$ time, and $(1-\epsilon)$-approximately in $O(\epsilon^{-1} \sqrt{r} n \log^2 r)$ time.*

5.3 Two Linear Matroids

In the case that $\mathbf{M}_1$ and $\mathbf{M}_2$ are linear, we can use a faster algebraic algorithm by Harvey [21] instead of the augmentation algorithm. His algorithm is an algebraic one for finding a common base of two linear matroids. We reduce our instance to the problem of finding a common base, that corresponds to a (w_1, w_2)-near-optimal maximum-cardinality common independent set.

We first describe basic properties of a linear matroid $\mathbf{M} = (S, \mathcal{I})$ of rank r. We assume that $\mathbf{M}$ is represented by an $r \times n$ matrix A whose column set is S and row set is denoted by R. We denote by $A[I, J]$ the submatrix consisting of row set I and column set J. For a set X, we denote the complement by $\overline{X}$.

It is known that the restriction and contraction of the linear matroid $\mathbf{M}$ are both linear. Indeed, for a subset $X \subseteq S$, $\mathbf{M}|X$ has the matrix representation $A|X = A[R, X]$. Moreover, taking a nonsingular submatrix of maximum size in $A[R, X]$, denoted by $A[Y, Z]$, we have the matrix representation A/X of the contraction $\mathbf{M}/X$ in the form of

$$A/X = A[\overline{Y}, \overline{X}] - A[\overline{Y}, Z]A[Y, Z]^{-1}A[Y, \overline{X}].$$

The row set of A/X is $\overline{Y} = R \setminus Y$. See e.g., [20] for more details. The direct sum of linear matroids $\mathbf{M}_1$ and $\mathbf{M}_2$ is also linear, whose matrix representation is the block diagonal matrix arranging the two matrices for $\mathbf{M}_1$ and $\mathbf{M}_2$ on the diagonal.

Suppose that we are given a weight $w : S \to \{0, 1, \ldots, W\}$. Then, by Lemma 1(i), $\mathbf{M}^w$ is also linear, and its matrix representation A^w is in the form of

$$A^w = \bigoplus_{t=0}^{W} (A|Z(t))/Z(t+1),$$

where recall $Z(t) = \{e \in S \mid w(e) \geq t\}$ for $t = 0, \ldots, W + 1$. We denote by $Y(t)$ the set of the nonzero rows in $A^w[R, Z(t)]$ for $t = 0, \ldots, W$. Thus A^w is a block-diagonal matrix whose blocks are $A^w[Y(t) \setminus Y(t+1), Z(t) \setminus Z(t+1)]$ for $t = 0, \ldots, W$, where $Y(W + 1) = \emptyset$. Note that the size of A^w is the same as A; the ground set of $\mathbf{M}^w$ is S, and the row set of A^w is also R. Moreover, A^w can be computed in $O(nr^{\omega-1})$ time, since this can be obtained by Gaussian elimination (see [20]).

The following lemma is easily observed.

Lemma 11. *For an independent set I of a linear matroid $\mathbf{M}$ and a nonnegative integer p, $I \cap Z(t)$ is a base of $\mathbf{M}|Z(t)$ for $t = p, \ldots, W$ if and only if $A^w[Y(p), I \cap Z(p)]$ is nonsingular.*

Proof. Since A^w is a block-diagonal matrix, $A^w[Y(p), I \cap Z(p)]$ is nonsingular if and only if it is a block-diagonal square matrix each of whose block is nonsingular. This is equivalent to that $I \cap Z(t)$ is a base of $\mathbf{M}^w|Z(t)$ for $t = p, \ldots, W$, and hence a base of $\mathbf{M}|Z(t)$ for $t = p, \ldots, W$. □

We now go back to the weighted matroid intersection. For $\ell = 1, 2$, let $\mathbf{M}_\ell$ be a linear matroid of rank r_ℓ on S, whose matrix representation is given by A_ℓ with the same field. We also denote by R_ℓ the row set of A_ℓ for $\ell = 1, 2$. Given two weights w_1 and w_2, recall $Z_\ell(t) = \{e \in S \mid w_\ell(e) \geq t\}$ for $t = 0, \ldots, W + 1$, and let $Y_\ell(t)$ be the set of the nonzero rows in $A_\ell^{w_\ell}[R_\ell, Z_\ell(t)]$ for $t = 0, \ldots, W$. For a subset S', let $Z'_\ell(t) = Z_\ell(t) \cap S'$.

Lemma 12. *For two linear matroids, suppose that I' is (w_1, w_2)-near-optimal in a subset S'. Then we can construct another common independent set I, in $O(nr^{\omega-1})$ time, that is simultaneously (i) a maximum-cardinality common independent set of $\mathbf{M}'_1$ and $\mathbf{M}'_2$, and (ii) (w_1, w_2)-near-optimal in S'.*

Proof. We denote $A'_\ell = A_\ell^{w_\ell}[R_\ell, S']$, which is a matrix representation of $\mathbf{M}'_\ell|S'$, for $\ell = 1, 2$. Define the matrix

$$Q = \begin{pmatrix} O & A'_1 \\ A'^{\top}_2 & D \end{pmatrix},$$

where D is defined to be the diagonal matrix whose ith diagonal entry is a nonzero parameter d_i. We assume that the set of d_i's is algebraically independent. Then it is known [21] that the maximum size of common independent sets in $\mathbf{M}'_1$ and $\mathbf{M}'_2$ is equal to the rank of Q. The row set of Q is $R_1 \cup S'$, and the column set is $R_2 \cup S'$.

Define $N = -A'_1 D^{-1} {A'_2}^{\top}$. The matrix N has the row set R_1 and column set R_2. Note that N is known as the *Schur complement*, and it can be computed in $O(nr^{\omega-1})$ time (see [21]). It is not difficult to see that, for $U_1 \subseteq R_1$ and $U_2 \subseteq R_2$, the submatrix $N[U_1, U_2]$ is nonsingular if and only if $Q[U_1 \cup S, U_2 \cup S]$ is nonsingular. Hence the maximum size of common independent sets in $\mathbf{M}'_1$ and $\mathbf{M}'_2$ is equal to the rank of N plus $|S|$.

By the assumption, $I' \cap Z'_1(t)$ is a base of $\mathbf{M}_1|Z'_1(t)$ for every integer $t \geq 1$, and $I' \cap Z'_2(t)$ is a base of $\mathbf{M}'_2|Z_2(t)$ for every integer $t \geq p+1$, where $p = \min_{e \in S'} w_2(e)$. By Lemma 11, both $A'_1[Y_1(1), I' \cap Z'_1(1)]$ and $A'_2[Y_2(p+1), I' \cap Z_2(p+1)]$ are nonsingular. Then $Q[Y_1(1) \cup S', Y_2(p+1) \cup S']$ is nonsingular, and hence $N[Y_1(1), Y_2(p+1)]$ is nonsingular. We can take $U_1 \subseteq R_1$ and $U_2 \subseteq R_2$ of maximum size such that $N[U_1, U_2]$ is nonsingular and $Y_1(1) \subseteq U_1$ and $Y_2(p+1) \subseteq U_2$. Then $Q[U_1 \cup S', U_2 \cup S']$ is nonsingular, and hence the two linear matroids defined by $A_1^{w_1}[U_1, S']$ and $A_2^{w_2}[U_2, S']$ have a common base I. Since $A'_1[R_1 \setminus Y_1(1), Z_1(1)]$ and $A'_2[R_2 \setminus Y_2(p+1), Z_2(p+1)]$ are zero matrices, the common base I has nonsingular submatrices $A'_1[Y_1(1), I \cap Z_1(1)]$ and $A'_2[Y_2(p+1), I \cap Z_2(p+1)]$. Hence, by Lemma 11, the statement holds.

Note that we can find such subsets U_1 and U_2 in $O(r^{\omega})$ time from N, since the rank of N is at most r. Applying the Harvey's algorithm [21] to $A'_1[U_1, S']$ and $A'_2[U_2, S']$, we can find a common base of them in $O(nr^{\omega-1})$ time, which gives a desired maximum-cardinality common independent set. □

Since Update_Weight can be performed in $O(nr^{\omega-1})$ time, we can solve the weighted matroid intersection exactly in $O(Wnr^{\omega-1})$ time and approximately in $O(\epsilon^{-1}nr^{\omega-1}\log r)$ time.

Furthermore, using a preprocessing technique by Cheung, Kwok, and Lau [3], we improve the computational time. Given a positive integer k, their algorithm reduces an $r \times n$ matrix A to an $O(k) \times n$ matrix A' such that, if a column set in A' of size at most k is independent then it is independent in A with high probability. This can be done in $O(nr)$ time.

We simply use this algorithm where k is set to be the maximum size $r_* \leq r$ of a common independent set of $\mathbf{M}_1$ and $\mathbf{M}_2$. The size r_* can be computed in $O(nr\log r_* + nr_*^{\omega-1})$ time [3]. After we obtain two $O(r_*) \times n$ matrices by their method, apply our algorithm to obtain a maximum-weight common independent set. This takes $O(Wnr_*^{\omega-1})$ time for an exact algorithm and $O(\epsilon^{-1}nr_*^{\omega-1}\log r_*)$ time for an approximation algorithm.

Therefore, we have the following theorem.

Theorem 5. *For two linear matroids, we can solve the weighted matroid intersection exactly in $O(nr\log r_* + Wnr_*^{\omega-1})$ time and $(1-\epsilon)$-approximately in $O(nr\log r_* + \epsilon^{-1}nr_*^{\omega-1}\log r_*)$ time, where r_* is the size of a common independent set.*

It should be noted that our algorithm is simple in the sense that it involves only a constant matrix and does not need to manipulate a univariate-polynomial matrix.

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