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実モーメント・アングル多様体におけるカップ積とキャップ積

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Abstract

In this thesis, the algebraic topology of *real moment-angle manifolds* is studied, with emphasis on the cup and cap products in their (co)homology. Here a real moment-angle manifold refers to a real moment-angle complex which is a topological manifold.

The condition to characterize a real moment-angle manifold is discussed in Chapter 3. As a result, a necessary and sufficient condition for a *moment-angle complex* to be a topological manifold is obtained. Also, the well-known cochain algebra of a moment-angle complex given by V. Buchstaber, T. Panov and I. Baskakov is deduced from that of the associated real moment-angle complex, using the construction due to A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler.

The cup and cap products are discussed in Chapter 2, based on the (co)chain complex established in Chapter 1; by cap products a new proof of the simplicial Alexander duality in a generalized homology sphere is obtained.

In Appendix A, the (co)chain equivalence between the singular (co)chain complex of a real moment-angle complex and the (co)chain complex established in Chapter 1 is proved in detail, with cup and cap products involved.

Contents

Introduction

An *abstract simplicial complex* K with ground set V is a collection of subsets of V, such that

(1) $v \in K$, for each element $v \in V$, and
(2) if $\sigma \in K$, then $\tau \in K$ for all $\tau \subset \sigma$; $\emptyset \in K$.

An element in K is called a simplex and $|K|$ refers to the geometric realization of K.

Let m be a positive integer and K be an abstract simplicial complex with ground set $[m] := \{1, 2, \ldots, m\}$. The associated *real moment-angle complex* $(D^1, S^0)^K$ is defined as

$$(D^1, S^0)^K = \bigcup_{\sigma \in K} \{(x_i)_{i=1}^m \in \mathbb{R}^m \mid |x_i| \leq 1, \forall i;\ |x_i| = 1 \text{ if } i \notin \sigma\}.\text{[1]} \tag{1}$$

Clearly from the definition, $(D^1, S^0)^K$ has a *CW* decomposition with each cell being a cube of suitable dimension. It is well-known that the topology of $(D^1, S^0)^K$ is deeply related to the combinatorics of K. For instance, Bahri, Bendersky, Cohen and Gitler [**BBCG10a**] showed that the suspension $\Sigma(D^1, S^0)^K$ is homotopy equivalent to the wedge sum over the double suspensions of all full subcomplexes of $|K|$. Davis [**Dav83**] proved that $(D^1, S^0)^K$ is *aspherical* if and only if K is a *flag complex* (i.e., any finite set of vertices that are pairwise connected by edges spans a simplex of K), with $\pi_1((D^1, S^0)^K)$ isomorphic to the commutator subgroup of a right-angled Coxeter group, whose Coxeter diagram (with all edges labeled by ∞) is isomorphic to the one dimensional non-faces of K.

Throughout this thesis, coefficients in (co)homology groups are assumed to be integers.

The first main topic is to establish the cup and cap products in a real moment-angle complex, which can be calculated explicitly and effectively. This is done in Chapter 1 and Chapter 2. The approach here is different that from [**BBCG12**], the latter is more general, while lack of explicit calculations. It is well-known that the difficulty to make calculations for cup and cap products with certain cellular (co)chain complex comes from the diagonal approximations at the (co)chain level. To overcome this, we use H. Whitney's formulae for cup and cap products in a Cartesian product of compact polyhedra (see formulae (16), (17) for details); however, $(D^1, S^0)^K$ is embedded as a *proper subset* in the m-fold product of the 1-disks $[-1, 1]$, with each cell being a product of simplices. In Appendix A we shall prove that Whitney's formulae also work in this situation (see Theorem A.20, where Alexander-Whitney chain maps (68) are used for diagonal approximations).

[1] It is not difficult to see that by the definition here, $(D^1, S^0)^K$ is always connected. If points $i \in [m]$ with $\{i\} \notin K$ are allowed, by which we can factor out copies of $\{-1, 1\}$ in $(D^1, S^0)^K$, being a product, until we get a connected space $(D^1, S^0)^{K'}$, where the ground set of K' is smaller, with each of its points a vertex of K'.

$(D^1, S^0)^K$ is called a *real moment-angle manifold* if it is a topological manifold. The characterizations of real moment-angle manifolds were given by Davis (see [**Dav08**, Theorem 10.6.1, p. 197], with the assumption that K is a flag complex): $(D^1, S^0)^K$ is a homology n-manifold (resp. PL n-manifold) if and only if $|K|$ is a generalized homology $(n-1)$-sphere (resp. PL $(n-1)$-sphere); it is a topological n-manifold if and only if $|K|$ is a generalized homology $(n-1)$-sphere, which is simply connected when $n > 3$ (see Theorem 3.7).

We will give an alternative proof for this theorem in Section 3.2, without assuming that K is a flag complex.

When $|K|$ is the boundary complex of a convex polytope, $(D^1, S^0)^K$ is called *polytopal* and can be smoothed to a link, i.e. a transverse intersection of real quadrics with the unit sphere in $\mathbb{R}^m$ (see [**BM06**]). With respect to such a link, there is a special class of *open book constructions*, which is introduced in [**GL13**] by López de Medrano and Gitler (see also [**BLV13**]), and by which we can obtain a smooth manifold homeomorphic to the *moment-angle complex* $(D^2, S^1)^K$, called a *polytopal moment-angle manifold*. Bosio and Meersseman [**BM06**] showed that every even-dimensional polytopal moment-angle manifold admits a complex structure to be an *LV-M manifold*, which is non-Kähler in general.

Independently in [**BBCG10b**], Bahri, Bendersky, Cohen and Gitler introduced the *KJ construction*: for a sequence of m positive integers $J = (n_i)_{i=1}^m$, the resulting complex KJ can be constructed by a sequence of *simplicial wedge constructions*, such that $(D^1, S^0)^{KJ}$ is homeomorphic to the *polyhedral product* $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$. It turns out that, topologically the operations from $(D^1, S^0)^K$ to $(D^1, S^0)^{KJ}$ coincide with the associated open book constructions indicated above. For instance, when J is constant with integers 2, we get the moment-angle complex $(D^2, S^1)^K$ (see Definition 3.3, Lemma 3.4 for more details).

Based on their approaches and Davis's characterization theorem, in Section 3.1 we will prove that $(D^2, S^1)^K$ is a topological manifold if and only if $|K|$ is a generalized homology sphere (see Theorem 3.16). Moreover, there is a cochain algebra $R_K^* J$ associated to each polyhedral product $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$, as an analogue of the cochain algebra for $(D^2, S^1)^K$ given by Buchstaber, Panov and Baskakov, such that $H^*(R_K^* J) \cong H^*\left(((D^{n_i}, S^{n_i-1})_{i=1}^m)^K\right)$ as rings (see Definition 3.32 and Corollary 3.35). $R_K^* J$ has $2m$ generators, and is *graded commutative* if and only if $n_i > 1$, for all $i = 1, 2, \ldots, m$. This approach also follows the spirit of [**BBCG12**] and [**BBCG10b**], with emphasis on direct calculations at the cochain level.

Here is a list of other results.

In Chapter 1 we construct a (co)chain complex for $(D^1, S^0)^K$, which is isomorphic to the direct sum of the augmented simplicial (co)chain complexes of all full subcomplexes of K, with degrees shifted by 1. From this we obtain a well-known decomposition of the (co)homology of $(D^1, S^0)^K$.

In Chapter 2, we use Whitney's formulae for this (co)chain complex, yielding cup and cap products in (co)homology. When $(D^1, S^0)^K$ is a homology manifold, the Poincaré duality induced by cap products on the orientation class of $(D^1, S^0)^K$ implies the simplicial Alexander duality in $|K|$, a generalized homology sphere (see Section 2.3).

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CHAPTER 1

The cellular chain complex

In Section 1.1 we introduce a chain complex $\big(C_*((D^1, S^0)^K), \partial_C\big)$, which is chain-homotopy equivalent to the singular chain complex of $(D^1, S^0)^K$, through a cubical cell-decomposition. A change of basis gives a degree-shifted *chain isomorphism* between $\big(C_*((D^1, S^0)^K), \partial_C\big)$ and the direct sum of the *augmented simplicial chain complexes* of all full subcomplexes of K, providing a decomposition of $H_*((D^1, S^0)^K)$. This is done in Section 1.2. In Section 1.3, we perform a similar treatment for the cohomology $H^*((D^1, S^0)^K)$.

The (co)chain complex constructed in this chapter, together with its relation with the augmented simplicial (co)chain complexes of full subcomplexes of K, shall be used throughout this thesis.

In what follows, a cell (resp. simplex) may refer to either a space or an oriented cellular (resp. simplicial) chain, whose orientation will be illustrated depending on the situation.

1.1. A cellular decomposition

Let u be the 1-cell of the interval $I = [-1, 1]$ connecting the two 0-cells $\underline{t}$ and $\bar{t}$ at both ends, i.e. points $\{-1\}$ and $\{1\}$, respectively, such that

$$\partial u = \bar{t} - \underline{t}.$$

For the m-fold product I^m, products of cells give rise to a cellular chain complex $C_*(I^m) = \bigotimes_{i=1}^m C_*(I_i)$, in which the subgroup of p-chains is given by

$$C_p(I^m) = \bigoplus_{\sum_{i=1}^m p_i = p} \bigotimes_{i=1}^m C_{p_i}(I_i).$$

More explicitly, $C_*(I^m)$ is generated by cells of the form

$$u_\sigma \bar{t}_\tau \underline{t}_\gamma := \otimes_{i=1}^m e_i, \quad e_i = \begin{cases} u_i & \text{if } i \in \sigma, \\ \bar{t}_i \text{ (resp. } \underline{t}_i) & \text{if } i \in \tau \text{ (resp. } i \in \gamma), \end{cases} \tag{2}$$

where σ, τ and γ are pairwise disjoint subsets with their union $[m]$, and $\deg(u_\sigma \bar{t}_\tau \underline{t}_\gamma) = \operatorname{card}(\sigma)$. The boundary operator $\partial_C \colon C_*(I^m) \to C_*(I^m)$ shifts the degrees down by one, such that

$$\partial_C(u_\sigma \bar{t}_\tau \underline{t}_\gamma) = \sum_{j \in \sigma} (-1)^{\kappa(j, \sigma \backslash \{j\})} (u_{\sigma \backslash \{j\}} \bar{t}_{\tau \cup \{j\}} \underline{t}_\gamma - u_{\sigma \backslash \{j\}} \bar{t}_\tau \underline{t}_{\gamma \cup \{j\}}) \tag{3}$$

on each generator, where $\kappa(j, \sigma \backslash \{j\}) = \operatorname{card}(\{i \in \sigma \backslash \{j\} \mid i < j\})$. Let $\big(C_*((D^1, S^0)^K), \partial_C\big)$ be the restriction of $(C_*(I^m), \partial_C)$ to $(D^1, S^0)^K$ (see (1) for definition), which is closed under ∂_C. It can be checked that the cellular chain $u_\sigma \bar{t}_\tau \underline{t}_\gamma$ of form (2) belongs to $C_*((D^1, S^0)^K)$ if and only if $\sigma \in K$.

PROPOSITION 1.1. *There is a chain map*

$$\iota\colon \ \big(C_*((D^1,S^0)^K),\partial_C\big) \longrightarrow S_*((D^1,S^0)^K), \tag{4}$$

inducing a chain equivalence between $\big(C_*((D^1,S^0)^K),\partial_C\big)$ *and the singular chain complex of* $(D^1,S^0)^K$*, which implies isomorphisms*

$$H_*(C((D^1,S^0)^K),\partial_C) \xrightarrow[\cong]{\iota} H_*((D^1,S^0)^K)$$

between respective homology groups in all dimensions. After taking Hom *on* (4)*, we have isomorphisms*

$$H^*(C((D^1,S^0)^K),\delta_C) \xleftarrow[\cong]{\iota^*} H^*((D^1,S^0)^K)$$

between cohomology groups, where δ_C *is the coboundary operator dual to* ∂_C*.*

PROOF. This is a direct consequence of Proposition A.12 in Appendix A. $\square$

1.2. A change of basis

To understand the (co)homology of $(D^1,S^0)^K$ further, it is convenient to use a new basis for $\big(C_*((D^1,S^0)^K),\partial_C\big)$: for the three generators u, $\bar{t}$ and $\underline{t}$ in $(C_*(I),\partial)$, let ∂u be the chain $\bar{t}-\underline{t}$ and denote $\underline{t}$ by t. Then u, ∂u and t form a new basis of $C_*(I)$. Correspondingly, $\big(C_*((D^1,S^0)^K),\partial_C\big)$ is generated by

$$u_\sigma t_\gamma := u_\sigma t_\gamma(\partial u)_\tau = \otimes_i c_i \in \bigotimes_{i=1}^m C_*(I_i),\quad c_i = \begin{cases} u_i & \text{if } i\in\sigma\in K,\\ t_i & \text{if } i\in\gamma,\\ \partial u_i & \text{if } i\in\tau,\end{cases} \tag{5}$$

in which σ, γ and τ is a partition of $[m]$ (any two of the three may be empty). When σ and γ are both empty, we shall write the word $\oslash$ instead of the void.

REMARK 1.2. Notice that in each word expressing a basis element above, we omit the part $(\partial u)_\tau$. Actually it means a sum of $2^{\mathrm{card}(\tau)}$ cells with signs. For instance, we have

$$\oslash = (\partial u)_{[m]} = \sum_{\tau\subset[m]} (-1)^{\mathrm{card}(\tau)}\underline{t}_\tau \bar{t}_{[m]\setminus\tau},$$

a cellular chain with 2^m 0-cells involved.

From (5) we have

$$u_\sigma \bar{t}_\tau \underline{t}_\gamma = u_\sigma\prod_{i\in\tau}(\partial u_i + t_i)t_\gamma = u_\sigma\prod_{i\in\tau}(1+t_i)t_\gamma = \sum_{\tau'\subset\tau} u_\sigma t_{\tau'}t_\gamma = \sum_{\tau'\subset\tau} u_\sigma t_{\tau'\cup\gamma}, \tag{6}$$

where formally we use the notation of multiplication, with ∂u_i replaced by "1", and $t_{\tau'}t_\gamma = t_{\tau'\cup\gamma}$.

EXAMPLE 1.3. Let $K\subset 2^{[2]}$ be two discrete points $\{1\}$ and $\{2\}$. Hence $(D^1,S^0)^K$ is a simplicial 1-sphere with four 1-cells. Figure 1 illustrates a comparison of the two basis. It is easy to check that $H_0((D^1,S^0)^K)$ is generated by $t_{1,2}$, and $H_1((D^1,S^0)^K)$ is generated by u_2-u_1.

In what follows, suppose $\big(C_*((D^1,S^0)^K),\partial_C\big)$ is endowed with the basis (5).

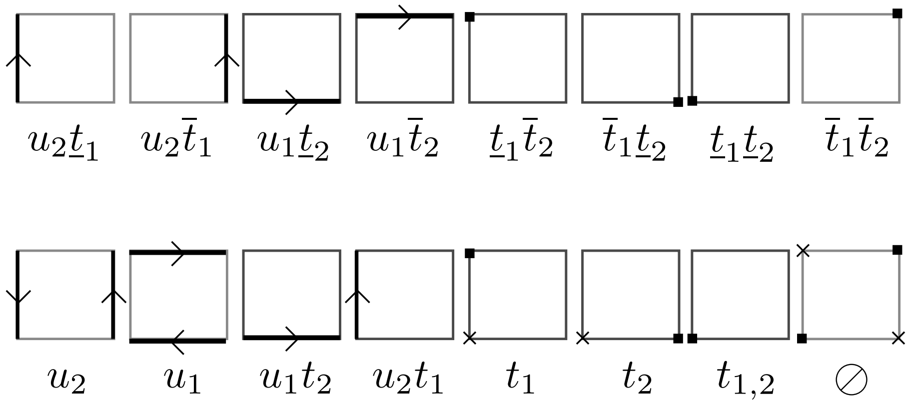

Figure 1. A comparison of the two basis

For $\omega \subset [m]$, let K_ω be the *full subcomplex* of K with ground set ω, namely

(7) $$K_\omega = \{\sigma \in K \mid \sigma \subset \omega\}.$$

Let $C_*((D^1, S^0)^K)|_\omega$ be the subgroup of $C_*((D^1, S^0)^K)$ generated by all chains of the form

(8) $$u_\sigma t_{[m]\setminus\omega} \ (\sigma \in K_\omega),$$

where the union of the subscripts of u and ∂u is ω.

Due to the basis-change, differential rule (3) becomes neater:

(9)
$$\partial_C(u_\sigma t_{[m]\setminus\omega}) = \sum_{j\in\sigma}(-1)^{\kappa(j,\sigma\setminus\{j\})}u_{\sigma\setminus\{j\}}t_{[m]\setminus\omega} \quad (\kappa(j, \sigma\setminus\{j\}) = \mathrm{card}(\{i \in \sigma\setminus\{j\} \mid i < j\}))\,.$$

We find $C_*((D^1, S^0)^K)|_\omega$ is closed under ∂_C. Observe that we have the decomposition

(10) $$\big(C_*((D^1, S^0)^K), \partial_C\big) = \bigoplus_{\omega\subset[m]} \big(C_*((D^1, S^0)^K)|_\omega, \partial_C\big)\,.$$

For instance, $C_*((D^1, S^0)^K)|_\emptyset$ is generated by a single chain $t_{[m]}$, and $C_*((D^1, S^0)^K)|_{[m]}$ is generated by basis elements of the form u_σ ($\sigma \in K$), meaning $u_\sigma(\partial u)_{[m]\setminus\sigma}$.

Denote by $(\widetilde{C}_*(K), \partial')$ the *augmented simplicial chain complex* of K, and suppose $[m]$ induces a natural ordering on the vertices of K. As a free Abelian group, $\widetilde{C}_*(K) = \bigoplus_{p\geq -1} C_p(K)$: for $p \geq 0$, $C_p(K)$ is generated by oriented simplices of the form $\sigma = [i_0, i_1, \dots, i_p]$, such that $i_0 < i_1 < \cdots < i_p$ (the interchanging of two indices yields a minus sign, we write σ as a subset of $[m]$ with the given ordering). The augmentation $C_{-1}(K)$

is generated the empty set. The boundary operator $\partial'\colon \widetilde{C}_*(K) \to \widetilde{C}_*(K)$ satisfies

$$\partial'\sigma = \begin{cases} \sum_{j\in\sigma}(-1)^{\kappa(j,\sigma\setminus\{j\})}\,(\sigma\setminus\{j\}) & \text{if } \sigma \in C_p(K),\ \text{for } p \geq 0,\\ 0 & \text{if } \sigma = \emptyset \in C_{-1}(K). \end{cases} \tag{11}$$

By definition, the reduced (co)homology groups of K_ω of dimension -1 is trivial when ω is non-empty, while $\widetilde{H}_{-1}(K_\emptyset)$ and $\widetilde{H}^{-1}(K_\emptyset)$ are both isomorphic to $\mathbb{Z}$. The (co)homology groups of $K_\emptyset$ vanish in other dimensions.

Theorem 1.4. *The map*

$$\eta\colon \bigoplus_{\omega\subset[m]} \widetilde{C}_*(K_\omega) \to C_*((D^1,S^0)^K)$$

defined via

$$\begin{aligned} \eta|_\omega\colon\ \widetilde{C}_p(K_\omega) &\longrightarrow C_{p+1}((D^1,S^0)^K)|_\omega \\ \sigma &\longmapsto u_\sigma t_{[m]\setminus\omega}, \end{aligned} \tag{12}$$

for each $\omega \subset [m]$ with $p \geq -1$, is a chain isomorphism shifting the degrees up by one. Consequently, we have isomorphisms

$$\bigoplus_{\omega\subset[m]} \widetilde{H}_p(K_\omega) \xrightarrow{\iota\circ\eta} H_{p+1}((D^1,S^0)^K);$$

in particular, $\widetilde{H}_{-1}(K_\emptyset)$ corresponds to $H_0((D^1,S^0)^K)$.

Proof. By definition, $\eta|_\omega$ is one-one onto, sending $\emptyset \in K_\omega$ to $t_{[m]\setminus\omega}$. A comparison of (9) and (11) shows that it preserves boundary operators on both sides. Together with decomposition (10) and Proposition 1.1, the second statement follows. $\square$

Remark 1.5. Note that when ω varies, η sends the empty sets in K_ω to different elements in $C_*(D^1,S^0)^K$.

1.3. Cohomology

Note that if we dualize the basis used in the previous section, the relation between the cochains of $(D^1,S^0)^K$ and those of full subcomplexes of K cannot be obtained directly, as what we have done for homology groups. Therefore we use another basis for the cochain complex $C^*((D^1,S^0)^K)$; the relation between $C^*((D^1,S^0)^K)$ and $C_*((D^1,S^0)^K)$ will be discussed in Section 2.2.

Let u^*, $\underline{t}^*$ and $\bar{t}^*$ be the basis in $(C^*(I),\delta)$ dual to u, $\underline{t}$ and $\bar{t}$ described in Section 1, respectively, thus

$$-\delta\underline{t}^* = \delta\bar{t}^* = u^*, \quad \delta u^* = 0.$$

Definition 1.6. Denote by $\imath^*$ the sum $\underline{t}^* + \bar{t}^*$ and set $t^* = \bar{t}^*$, and let $(C^*(I),\delta)$ be endowed with the basis u^*, $\imath^*$ and t^*. Similar to (5), a basis element of

$$C^*((D^1,S^0)^K) = \mathrm{Hom}(C_*((D^1,S^0)^K),\mathbb{Z})$$

is denoted by

$$(13)\qquad u^\sigma t^\tau := u^\sigma t^\tau \mathbf{1}^\gamma = \otimes_i c_i^* \in \bigotimes_{i=1}^m C^*(I_i), \quad c_i^* = \begin{cases} u_i^* & \text{if } i\in\sigma\in K,\\ t_i^* & \text{if } i\in\tau,\\ \mathbf{1}_i^* & \text{if } i\in\gamma,\end{cases}$$

where σ, τ and γ are pairwise disjoint subsets with their union $[m]$.

Note that $\delta\mathbf{1}^*=0$ and $\delta t^* = u^*$, thus the coboundary operator δ_C dual to ∂_C satisfies

$$(14)\qquad \delta_C(u^\sigma t^\tau) = \sum_{\substack{j\in\tau\\(\sigma\cup\{j\})\in K}} (-1)^{\kappa(j,\sigma)} u^{\sigma\cup\{j\}} t^{\tau\setminus\{j\}}, \quad \kappa(j,\sigma) = \mathrm{card}(\{i\in\sigma \mid i<j\}).$$

It can be easily checked that $\delta_C\circ\delta_C = 0$.

For each $\omega\subset[m]$, denote by $C^*((D^1,S^0)^K)|_\omega$ the subcomplex generated by

$$u^\sigma t^{\omega\setminus\sigma}, \quad \sigma\in K_\omega;$$

one can check that it is closed under δ_C, by (14). Similarly we have the decomposition

$$(C^*((D^1,S^0)^K),\delta_C) = \bigoplus_{\omega\subset[m]} (C^*((D^1,S^0)^K)|_\omega, \delta_C),$$

and for each $p\geq -1$, we have the cochain isomorphisms

$$(15)\qquad \begin{aligned} \mu\colon\ \textstyle\bigoplus_{\omega\subset[m]}(\widetilde{C}^p(K_\omega),\delta') &\longrightarrow (C^{p+1}((D^1,S^0)^K),\delta_C)\\ (\sigma^*,\omega) &\longmapsto u^\sigma t^{\omega\setminus\sigma},\end{aligned}$$

shifting the degrees up by one, where $(\widetilde{C}^p(K_\omega),\delta')$ is the augmented simplicial cochain complex dual to $(\widetilde{C}_p(K_\omega),\partial')$ (see (11)), with (σ^*,ω) a p-cochain such that

$$\delta'(\sigma^*,\omega) = \sum_{\substack{j\in\omega\setminus\sigma\\(\sigma\cup\{j\})\in K}} (-1)^{\kappa(j,\sigma)}(\sigma\cup\{j\})^*.$$

Passing to cohomology, it follows that

THEOREM 1.7. *We have additive isomorphisms*

$$\bigoplus_{\omega\subset[m]} \widetilde{H}^p(K_\omega) \xrightarrow{(\iota^*)^{-1}\circ\mu} H^{p+1}((D^1,S^0)^K)$$

in all dimensions $p\geq -1$.

EXAMPLE 1.8. Let $K\subset 2^{[4]}$ be the tetragon with maximal simplices $\{i,i+1\}$, $i=1,2,3,4 \bmod 4$. All full subcomplexes of K having non-trivial (co)homology groups are $K_\emptyset$, $K_{1,3}$, $K_{2,4}$ and $K_{[4]}$. By Theorems 1.4 and 1.7, $H_0((D^1,S^0)^K)$ and $H^0((D^1,S^0)^K)$ are generated by $[t_{1,2,3,4}]$ and $[\mathbf{1}^{1,2,3,4}]$, corresponding to the point with constant coordinates -1 and the sum of the 16 dual points with coordinates ±1, respectively. $H_1((D^1,S^0)^K)$ is generated by $\alpha_1 = [(u_1-u_3)t_{2,4}]$ with $\omega=\{1,3\}$, and $\alpha_2 = [(u_2-u_4)t_{1,3}]$ with $\omega=\{2,4\}$. They are the orientation classes of circles

$$S^1_{\alpha_1} = \{(x_i)_{i=1}^4 \in (D^1,S^0)^K \mid x_2=x_4=-1\},\ S^1_{\alpha_2} = \{(x_i)_{i=1}^4\in(D^1,S^0)^K \mid x_1=x_3=-1\},$$

respectively. $H^1((D^1,S^0)^K)$ is generated by $[u^2t^4]$ and $[u^1t^3]$. It can be checked that $(D^1,S^0)^K$ is a torus, in which tubular neighborhoods of $S^1_{\alpha_1}$ and $S^1_{\alpha_2}$ form a plumbing.

REMARK. Originally, $(D^1,S^0)^K$ was constructed by Davis and Januszkiewicz [**DJ91**], with the name universal Abelian cover, under the assumption that $|K|$ is the (polar) dual of a simple convex polytope P.

Explicitly, let G be the group $(\mathbb{Z}_2)^m$ acting on $\mathbb{R}^m$, generated by g_i changing the sign of the i-th coordinate $(i=1,2,\dots,m)$, and let X be the intersection of $(D^1,S^0)^K$ with the first orthant of $\mathbb{R}^m$. Suppose ∂X is the union $\bigcup_{i=1}^m X_i$ with X_i the subspace fixed by g_i. As a *manifold with faces*, X is homeomorphic to P. Then there is a piecewise linear homeomorphism between $(D^1,S^0)^K$ and the space

$$(G\times X)/\sim,$$

sending $g(x)$ to (g,x), $x\in X$ and $g\in G$, where $(g,x)\sim(g',x')$ if and only if $x=x'$ and $g^{-1}g'\in\langle g_i\rangle_{x\in X_i}$.

Davis [**Dav83**] proved that for each $\omega\subset[m]$, the chain map

$$\begin{aligned}\vartheta|_\omega\colon\ S_*(X,X^\omega) &\longrightarrow S_*((D^1,S^0)^K)\\ z &\longmapsto \textstyle\sum_{\tau\subset\omega}(-1)^{\mathrm{card}(\tau)}\prod_{i\in\omega}g_i\prod_{j\in\tau}g_j(z),\end{aligned}$$

induces a splitting in homology, where $X^\omega=\bigcup_{i\in\omega}X_i$ and z is a relative cycle in $S_*(X,X^\omega)$. This yields an isomorphism

$$\bigoplus_{\omega\subset[m]}H_*(X,X^\omega)\ \xrightarrow[\cong]{\vartheta}\ H_*((D^1,S^0)^K).$$

Later similar results were obtained by López de Medrano [**LdM89**], based on an earlier work of Wall [**Wal80**].

The (co)homological decomposition of real moment-angle complexes can also be obtained from [**BBCG10a**]: the suspension of $(D^1,S^0)^K$ is homotopy equivalent to the wedge sum over the double suspensions of all full subcomplexes of $|K|$.

On cup products in cohomology, a general approach based on homotopy theory was given [**BBCG12**]. In Chapter 3 we shall give an answer to a question proposed in [**BBCG12**, p. 462] about the relation between $H^*((D^1,S^0)^K)$ and $H^*((D^2,S^1)^K)$. The cellular cochain algebra constructed in Chapter 1 follows the spirit of Baskakov-Buchstaber-Panov [**BBP04**], [**Pan08**] for $H^*((D^2,S^1)^K)$.

CHAPTER 2

Cup and cap products

This chapter is devoted to the formulae for cup and cap products in the (co)homology of $(D^1, S^0)^K$, using the (co)chain complexes with basis (5) and (13), respectively.

The idea here is as follows, due to Whitney [**Whi38**]. Let $X = \prod_{i=1}^m |K_i|$ be a product of compact polyhedra, in which every K_i is a finite simplicial complex. Then there is a natural cochain equivalence between $S^*(X)$ and the tensor product $\bigotimes_{i=1}^m C^*(K_i)$ over simplicial cochain complexes $C^*(K_i)$, which preserves the cup products, up to cochain homotopy. Here cup products in $\bigotimes_{i=1}^m C^*(K_i)$ is defined in this way: suppose that $c^p = \otimes_{i=1}^m c^{p_i}$ is a p-cochain with p_i-cochains $c^{p_i} \in C^{p_i}(K_i)$, $c^q = \otimes_{i=1}^m c^{q_i}$ a q-cochain with q_i-chains $c^{q_i} \in C^{q_i}(K_i)$, respectively, then we have

$$(16) \qquad c^p \smile c^q = (-1)^\kappa (c^{p_1} \smile c^{q_1}) \otimes \cdots \otimes (c^{p_m} \smile c^{q_m}), \quad \kappa = \sum_{i=1}^m q_i (\sum_{j>i} p_j),$$

being a cochain of degree $p+q$. Likewise, for cap products, assume $c_r = \otimes_{i=1}^m c_{r_i} \in \bigotimes_{i=1}^m C_*(K_i)$ an r-chain with r_i-chains $c_{r_i} \in C_{r_i}(K_i)$, then

$$(17) \qquad c^p \frown c_r = (-1)^\nu (c^{p_1} \frown c_{r_1}) \otimes \cdots \otimes (c^{p_m} \frown c_{r_m}), \quad \nu = \sum_{i=1}^m p_i (\sum_{j>i} (r_i - p_i)),$$

which coincides with the cap product using singular (co)chains for X when passing to (co)homology.

In Appendix A we shall extend the property above to the situation here, namely $(D^1, S^0)^K$ is embedded in I^m as a proper subcomplex with each cell a product of simplices (see Theorem A.20).

The cup product will be discussed in Section 2.1, where we treat $\big(C^*((D^1, S^0)^K), \delta_C\big)$ as a differential graded algebra (R_K^*, d), which is *not* commutative in any sense (while its cohomology is graded commutative). In Section 2.2 the cap product is formulated as an R_K^*-module structure on the chain complex $(C_*((D^1, S^0)^K), \partial_C)$. Once $(D^1, S^0)^K$ is a homology manifold, the Poincaré duality coincides with the Alexander duality in $|K|$. This will be discussed in Section 2.3.

2.1. The algebra R_K^* and the cup product

Recall that the *simplicial cup product* in an abstract simplicial complex K is defined as follows: choose a partial ordering on the vertex set of K inducing a total ordering on each simplex, then

$$C^p(K) \bigotimes C^q(K) \xrightarrow{\smile} C^{p+q}(K)$$

is defined by

$$(c^p \smile c^q)([v_{i_0}, \dots, v_{i_{p+q}}]) = c^p([v_{i_0}, \dots, v_{i_p}])c^q([v_{i_p}, \dots, v_{i_{p+q}}]) \tag{18}$$

where $[v_{i_0}, \dots, v_{i_{p+q}}]$ is a simplex in $C^{p+q}(K)$ with $v_{i_0} < \cdots < v_{i_{p+q}}$ in the given ordering.

For instance, with the induced ordering of $\mathbb{R}$, cup products in $C^*(I)$ (see Definition 1.6) are listed below:

$$\begin{aligned} &\mathbf{1}^* \smile u^* = u^* \smile \mathbf{1}^* = u^*, \quad \mathbf{1}^* \smile t^* = t^* \smile \mathbf{1}^* = t^*, \quad \mathbf{1}^* \smile \mathbf{1}^* = \mathbf{1}^*; \\ &t^* \smile t^* = t^*, \quad t^* \smile u^* = 0, \quad u^* \smile t^* = u^* \quad u^* \smile u^* = 0. \end{aligned}$$

Definition 2.1. Denote by (R^*, d) the differential graded algebra with the properties below:

- R^* is the quotient of the free $\mathbb{Z}$-algebra with degree-one generators u^1, u^2, …, u^m and degree-zero generators t^1, t^2, …, t^m, subject to relations

(19)
$$u^it^i = u^i, \quad t^iu^i = 0, \quad u^it^j = t^ju^i, \quad t^it^i = t^i, \quad u^iu^i = 0, \quad u^iu^j = -u^ju^i, \quad t^it^j = t^jt^i,$$

  for $i, j = 1, 2, \dots, m$ with $i \neq j$. We say that a monomial in R^* is *reduced* if it is written in the square-free form $u^\sigma t^\tau = x^1 \dots x^m$, where σ and τ are disjoint subsets of $[m]$, with $x^i = u^i$ for $i \in \sigma$, $x^i = t^i$ for $i \in \tau$, and $x^i = 1$ (the identity with degree zero) otherwise.
- The differential d satisfies

$$\mathrm{d}(xy) = (\mathrm{d}x)y + (-1)^{\deg(x)}x(\mathrm{d}y) \tag{20}$$

  for homogeneous elements $x, y \in R^*$, with

$$\mathrm{d}u^i = 0, \quad \mathrm{d}t^i = u^i \ \text{ and } \ \mathrm{d}1 = 0.$$

For a simplicial complex K with ground set $[m]$, the corresponding *Stanley-Reisner ideal* $\mathcal{I}_K$ in R^* is generated by all square-free monomials of the form u^τ, where τ is *not* a simplex of K.

The differential graded algebra (R^*_K, d) is defined to be the quotient $R^*/\mathcal{I}_K$ endowed with the differential d above. A reduced monomial in R^*_K is the (non-trivial) image of a reduced monomial in R^* under the quotient homomorphism.

By definition, every monomial of R^*_K can be uniquely written in a reduced form, i.e. of the form $u^\sigma t^\tau$ where σ and τ are disjoint subsets of $[m]$, such that $\sigma \in K$.

It can be checked that the differential d is compatible with the algebraic structure. For instance, $\mathcal{I}_K$ is closed under d, and

$$u^i = \mathrm{d}t^i = \mathrm{d}(t^it^i) = u^it^i + t^iu^i = u^i.$$

Example 2.2. Let K be the pentagon with ground set $[5]$ with the set of maximal simplices

$$\{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}.$$

It can be checked that the Stanley-Reisner ideal is generated by

$$u^{1,3}, \quad u^{1,4}, \quad u^{2,4}, \quad u^{2,5}, \quad u^{3,5},$$

therefore u^1t^3 is a cocycle in R_K^1, since

$$\mathrm{d}(u^1t^3) = (\mathrm{d}u^1)t^3 - u^1(\mathrm{d}t^3) = -u^{1,3} = 0.$$

In the same way $u^3t^{1,4} + u^4t^{1,3}$ is a cocycle in R_K^1, whose difference with $\mathrm{d}(t^{1,3,4})$ gives the cocycle $u^1t^{3,4}$.

For any subset $\omega \subset [m]$, let $R_K^*|_\omega$ be the submodule generated by reduced monomials of the form $u^\sigma t^{\omega\setminus\sigma}$, where σ runs through subsets of ω. Clearly by (20), $R_K^*|_\omega$ is closed under d.

LEMMA 2.3. *The map*

$$\begin{aligned}\phi\colon\ (C^*((D^1,S^0)^K),\delta_C) &\longrightarrow (R_K^*,\mathrm{d}) \\ u^\sigma t^\tau &\longmapsto u^\sigma t^\tau,\end{aligned} \tag{21}$$

is a cochain isomorphism preserving products: the cup product in $C^*((D^1,S^0)^K)$ on the left-hand side and the multiplication in R_K^* on the right.

PROOF. Clearly as a homomorphism between $\mathbb{Z}$-modules, ϕ is one-one onto, preserving differentials on both sides, by formulae (14) and (20). By Theorem A.20, the simplicial cup product in $C^*((D^1,S^0)^K)$ follows (16), hence ϕ preserves products, since $u^iu^j = -u^ju^i$. □

With the lemma above, we can associate each reduced monomial $u^\sigma t^\tau$ in R_K^* to the dual simplex σ^* in $\widetilde{C}^*(K_{\sigma\cup\tau})$ ($K_{\sigma\cup\tau}$ is the full subcomplex), by the composition $\phi\circ\mu$, in which μ is defined by (15).

DEFINITION 2.4. The $\bar{*}$-*product* in the second row of the diagram

$$\begin{tikzcd} C^*((D^1,S^0)^K)\bigotimes C^*((D^1,S^0)^K) \arrow[r, "\smile"] & C^*((D^1,S^0)^K) \\ \Big(\bigoplus\limits_{\omega\subset[m]}\widetilde{C}^*(K_\omega)\Big)\bigotimes\Big(\bigoplus\limits_{\omega\subset[m]}\widetilde{C}^*(K_\omega)\Big) \arrow[u, "\cong"', "(\mu,\mu)"] \arrow[r, "\bar{*}"] & \bigoplus\limits_{\omega\subset[m]}\widetilde{C}^*(K_\omega) \arrow[u, "\cong"', "\mu"] \end{tikzcd} \tag{22}$$

is defined via its commutativity. More explicitly, for two dual simplices $\sigma_p^* = \sigma^p \in \widetilde{C}^p(K_\omega)$ and $\sigma_{p'}^* = \sigma^{p'} \in \widetilde{C}^{p'}(K_{\omega'})$, we have

$$\begin{aligned}\sigma^p\bar{*}\sigma^{p'} &= \mu^{-1}\left(\mu(\sigma^p)\smile\mu(\sigma^{p'})\right) = \mu^{-1}\left(u^{\sigma_p}t^{\omega\setminus\sigma_p}\smile u^{\sigma_{p'}}t^{\omega'\setminus\sigma_{p'}}\right) \\ &= \begin{cases}(-1)^{\kappa(\sigma_p,\sigma_{p'})}\left(\sigma_p\cup\sigma_{p'}\right)^* & \text{if } \sigma_{p'}\cap\omega=\emptyset,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where $\kappa(\sigma_p,\sigma_{p'}) = \sum_{j\in\sigma_{p'}}\mathrm{card}(\{i\in\sigma_p \mid i>j\})$, and $(\sigma_p\cup\sigma_{p'})^*$ is the dual of $(\sigma_p\cup\sigma_{p'}) \in \widetilde{C}_{p+p'+1}(K_{\omega\cup\omega'})$.

From the cochain isomorphism (21) and the differential rule (20), it turns out that the $\bar{*}$-product satisfies

$$\delta'(c^p\bar{*}c^{p'}) = (\delta'c^p)\bar{*}c^{p'} + (-1)^{p+1}c^p\bar{*}(\delta'c^{p'}), \tag{23}$$

for $c^p \in \widetilde{C}^p(K_\omega)$ and $c^{p'} \in \widetilde{C}^{p'}(K_{\omega'})$. Therefore it gives rise to a product structure for $\bigoplus_{\omega\subset[m]} \widetilde{H}^*(K_\omega)$. The following theorem is based on Lemma 2.3, Theorem 1.7 and Proposition 1.1.

Theorem 2.5. *We have the following isomorphisms between algebras:*

$$\bigoplus_{\omega}\widetilde{H}^*(K_\omega) \xrightarrow[\cong]{(\iota^*)^{-1}\circ\mu} H^*((D^1,S^0)^K) \xrightarrow[\cong]{\phi\circ\iota^*} H^*(R_K,\mathrm{d}),$$

where the first is endowed with the $\bar{*}$-product, the second with the cup product and the third with the multiplication in R_K^*.

Proof. The only thing we need to prove now is that the additive isomorphism

$$\iota^*\colon\ H^*((D^1,S^0)^K) \to H^*(C((D^1,S^0)^K),\delta_C)$$

appearing in Proposition 1.1 preserves product structures. This follows from Theorem A.20. □

Example 2.6. Let K be the pentagon in Example 2.2. Here we describe its cohomology by Theorem 2.5. It can be easily checked that all full subcomplexes of K with non-vanishing (co)homology are: $K_{i,i+2}$ with $i=1,2,3$, $K_{i,i+3}$ with $i=1,2$, $K_{i,i+2,i+3}$ with $i=1,2,\dots,5$, (we are using mod 5 integers i), together with $K_\emptyset$ and $K_{[5]}$. Correspondingly, we can choose a basis for $H^*(R_K)$. For $H^1(R_K)$ we have ten basis elements,

$$\alpha_1=[u^1t^3],\ \alpha_2=[u^1t^4],\ \alpha_3=[u^2t^4],\ \alpha_4=[u^2t^5],\ \alpha_5=[u^3t^5], \tag{24}$$

and

$$\begin{aligned}&\beta_1=[u^1t^3(1-t^4)],\ \beta_2=[u^2t^4(1-t^5)],\ \beta_3=[u^3t^5(1-t^1)],\ \beta_4=[u^4t^1(1-t^2)],\\ &\beta_5=[u^5t^2(1-t^3)].\end{aligned} \tag{25}$$

For $H^0(R_K)$, we choose the identity $[1]\in R_K^0$. For $H^2(R_K)$, a basis can be $\gamma=[u^{i,i+1}t^{i+2,i+3,i+4}]$ with i any mod 5 integer (for instance, $u^{1,2}t^{3,4,5}-u^{2,3}t^{1,4,5}=\mathrm{d}(u^2t^{1,3,4,5})$). By Theorem 3.7, $(D^1,S^0)^K$ is a closed manifold, thus it is an orientable surface of genus 5. It can be checked that

$$\gamma=-\alpha_1\beta_2=\alpha_2\beta_5=-\alpha_3\beta_3=\alpha_4\beta_1=-\alpha_5\beta_4$$

presents all non-trivial products in $H^*(R_K)$, since products between any two α-elements from (24) or any two β-elements from (25) vanish.

2.2. R_K^*-module $C_*((D^1,S^0)^K)$ via the cap product

Recall that in a simplicial complex K, the *simplicial cap product*

$$\frown\colon\ C^*(K)\bigotimes C_*(K)\to C_*(K)$$

is defined as follows. Choose a partial ordering on the vertex set of K which gives a total ordering on each simplex, then

$$c^p\frown[i_0,\dots,i_r]=c^p([i_{r-p},\dots,i_r])[i_0,\dots,i_{r-p}]\quad(c^p([i_{r-p},\dots,i_r])\in\mathbb{Z}), \tag{26}$$

for $c^p \in C^p(K)$ and $[i_0,\dots,i_r]$ a simplex in $C_r(K)$ $(r \geq p)$ with $i_0 < \dots < i_r$ in the given ordering.

For instance, the cap products in the simplicial (co)chain complex of $[-1,1]$ (i.e. $C_*(I)$ and $C^*(I)$), with the basis given in Sections 1.2 and 1.3, respectively, are listed below (note that t^* is the dual of the cell $\{1\}$ rather than that of $t = \{-1\}$):

(27)
$$t^* \frown \partial u = \partial u + t,\ t^* \frown u = u,\ t^* \frown t = 0, \quad \imath^* \frown \partial u = \partial u,\ \imath^* \frown u = u,\ \imath^* \frown t = t;$$
$$u^* \frown u = t,\ u^* \frown \partial u = 0,\ u^* \frown t = 0.$$

Remark. Here other basis for $C_*(I)$ and $C^*(I)$ also works. The basis presented above, as we have already seen in the previous chapter, illustrates the correspondence between the (co)chains of $(D^1,S^0)^K$ and those of full subcomplexes of K in an explicit way. Another reason is that, no extra minus signs come out from these products.

Using the cochain isomorphism $\phi\colon C^*((D^1,S^0)^K) \to R_K^*$ (see (21)), $C_*((D^1,S^0)^K)$ can be endowed with an R_K-module structure, by formula (17) and list (27):

Definition 2.7. The cap product $\frown\colon R_K^* \otimes C_*((D^1,S^0)^K) \to C_*((D^1,S^0)^K)$ is defined via

$$u^\sigma t^\tau \frown u_{\sigma'} t_{\tau'} = \begin{cases} (-1)^{\nu(\sigma,\sigma')} u_{\sigma'\setminus\sigma} t_\sigma \prod_{i\in\tau\setminus\sigma'}(1+t_i) t_{\tau'} & \text{if } \sigma \subset \sigma' \text{ and } \tau\cap\tau' = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

here $\nu(\sigma,\sigma') = \sum_{j\in\sigma} \mathrm{card}(\{i \in \sigma'\setminus\sigma \mid i > j\})$, and $\prod_{i\in\tau\setminus\sigma'}(1+t_i)$ is a formal notation ("1" means ∂u_i in each bracket), i.e.

$$u_{\sigma'\setminus\sigma} t_\sigma \prod_{i\in\tau\setminus\sigma'} (1+t_i) t_{\tau'} = \sum_{\gamma\subset(\tau\setminus\sigma')} u_{\sigma'\setminus\sigma} t_{\sigma\cup\gamma\cup\tau'}.$$

The proposition below follows directly from [**Whi38**] (see also (77)):

Proposition 2.8. *Let α^p and $\alpha^{p'}$ be two homogeneous elements in R_K^p and $R_K^{p'}$, respectively, and let c_r be an r-chain in $C_r((D^1,S^0)^K)$ $(r \geq p+p')$. Then we have*

$$\partial_C(\alpha^p \frown c_r) = (-1)^{r-p}(\mathrm{d}\alpha^p) \frown c_r + \alpha^p \frown \partial_C c_r \tag{29}$$

and

$$\alpha^{p'} \frown (\alpha^p \frown c_r) = (\alpha^{p'}\alpha^p) \frown c_r. \tag{30}$$

Theorem 2.9. *Passing to (co)homology, the cap product defined by* (28) *coincides with the one for the singular (co)chain complex of $(D^1,S^0)^K$.*

Proof. This follows from Theorem A.20. □

The following example was studied by López de Medrano in full detail, with a more geometric argument (see [**LdM89**], [**BLV13**]). As a comparison, here we consider it with the products established so far.

Example 2.10. Let P be the polytope defined as the *Gale transform* of the 7-tuple $\lambda = (\lambda_i)_{i=1}^7$ in $\mathbb{R}^2$, consisting of the real and imaginary parts of the solutions of the

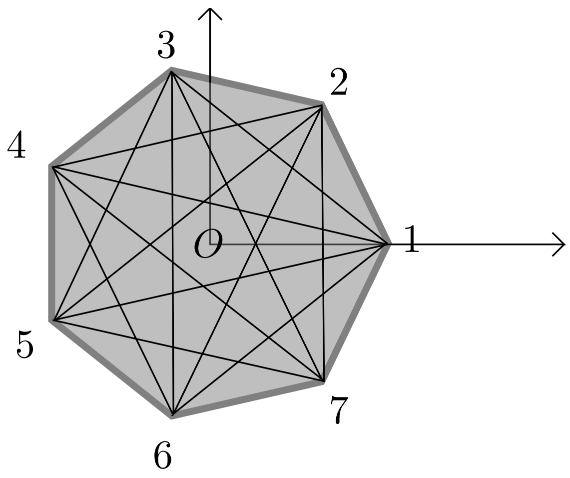

Figure 1. The Heptagon

equation $z^7 = 1$. Here by Gale transform we mean the transpose of a basis of (real) solutions of the equation

$$\begin{cases} \sum_{i=1}^{7} x_i \lambda_i = 0, \\ \sum_{i=1}^{n} x_i = 1, \end{cases} \tag{31}$$

namely if we write a basis as a (4×7)-matrix of rank 4, $A = (A_i)_{i=1}^{7}$, whose row vectors satisfies (31), then the Gale transform of λ is the column vectors of A. Let $P \subset \mathbb{R}^4$ be the convex hull $\operatorname{conv}(A_i)_{i\in[7]}$. Actually the combinatorial type of P is independent of the chosen basis: by the fundamental property of Gale transforms, for any $\sigma \subset [7]$, $\operatorname{conv}(A_i)_{i\in\sigma}$ is a face of P if and only if the origin of $\mathbb{R}^2$ is in the relative interior of $\operatorname{conv}(\lambda_i)_{i\in[7]\setminus\sigma}$. Therefore all faces of P are determined by the configuration given by λ (see Figure 1). For instance, let K be the boundary complex of P, then

(i) any subset of [7] with cardinality 2 is a simplex of K (i.e. P is a neighborly 4-polytope), and
(ii) any subset of [7] not in K must contain three consecutive points of the form $\{i, i+1, i+2\}$, $i = 1, 2, \ldots, 7 \bmod 7$.

From (i), $(D^1, S^0)^K$ is a simply connected 4-manifold (see [**Dav08**, Chapter 1, p. 12]). Using Theorem 1.4, we can write down the orientation class of $(D^1, S^0)^K$:

$$\begin{aligned} \Gamma = & [u_{1,2,4,5} - u_{1,2,4,6} + u_{1,2,5,6} + u_{1,3,4,6} - u_{1,3,4,7} - u_{1,3,5,6} + u_{1,3,5,7} - u_{1,4,5,7} \\ & + u_{2,3,5,6} - u_{2,3,5,7} + u_{2,3,6,7} + u_{2,4,5,7} - u_{2,4,6,7} + u_{3,4,6,7}], \end{aligned} \tag{32}$$

which is also a list of all 14 codimension-one faces of P. Since K is a simplicial 3-sphere, by Alexander duality we have

$$\widetilde{H}^{3-i-1}(K_\omega) \cong \widetilde{H}_i(K_{[7]\setminus\omega}).$$

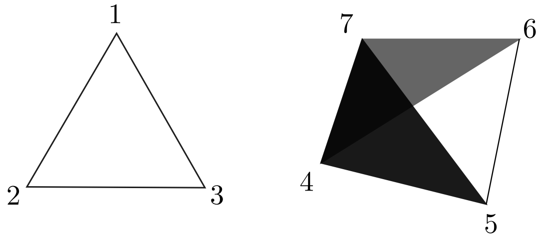

Figure 2. Full subcomplexes $K_{1,2,3}$ and $K_{4,5,6,7}$

Together with (i), it follows that K_ω has non-trivial reduced (co)homology only when $\mathrm{card}(\omega) = 0, 3, 4, 7$, in which $\widetilde{H}_{-1}(K_\emptyset)$ and $\widetilde{H}_3(K_{[7]})$ (resp. $\widetilde{H}^{-1}(K_\emptyset)$ and $\widetilde{H}^3(K_{[7]})$) correspond to $H_0((D^1,S^0)^K)$ and $H_4(D^1,S^0)^K)$ (resp. $H^0((D^1,S^0)^K)$ and $H^4(D^1,S^0)^K)$), respectively. By (ii) it follows that other (co)homology groups of dimension 2 arise from full subcomplexes of the form $\{i,i+1,i+2\}$ and their complements in $[7]$, as shown in Figure 2.

As a conclusion, a basis for $H^2(R_K)$ can be chosen and divided into two groups $B_1 = \{[u^{i,i+1}t^{i+2}]\}_{i=1}^7$ and $B_2 = \{[u^{j+1,j+2}t^{j,j+3}]\}_{j=1}^7$ (i,j are mod 7 integers) according to $\mathrm{card}(\omega)=3$ and $\mathrm{card}(\omega)=4$, respectively. It is not difficult to check that all products are trivial between basis elements in B_1, and each basis element in B_1 has a unique pairing in B_2, such that their product generates $H^4(R_K)$. For instance, $[u^{1,2}t^3u^{5,6}t^{4,7}] = [u^{1,2,5,6}t^{3,4,7}]$. But notice that elements in B_2 have non-trivial products: $[u^{1,2}t^{3,7}u^{5,6}t^{4,7}] = [u^{1,2,5,6}t^{3,4,7}]$. To remedy this, we can use another basis

$$B_2' = \{[u^{j+1,j+2}t^j(1-t^{j+3})] = [u^{j+1,j+2}t^j - u^{j+1,j+2}t^{j,j+3}]\}_{j=1}^7$$

instead of B_2, and a straightforward calculation shows that all products between elements from B_2' vanish. With basis elements from B_1 and B_2', we see that $(D^1,S^0)^K$ and $\sharp_7 S^2\times S^2$ have isomorphic cohomology, hence they are homeomorphic by the classification theorem of Freedman [**Fre82**].

To illustrate the intersection of submanifolds through cup products, by Poincaré duality, we proceed with (28). For instance,

$$[u^{1,2}t^3 \frown \Gamma] = \underbrace{[(u_{4,5}-u_{4,6}+u_{5,6})t_{1,2}(1+t_3)]}_{K_{4,5,6,7}\sqcup K_{3,4,5,6,7}} = \underbrace{[(u_{4,5}-u_{4,6}+u_{5,6})t_{1,2,3}]}_{K_{4,5,6,7}}, \tag{33}$$

because $K_{3,4,5,6,7}$ is acyclic. Here the geometric meaning of the class $[(u_{4,5}-u_{4,6}+u_{5,6})t_{1,2,3}]$ is as follows: consider the sphere

$$\alpha = \{(x_i)_{i=1}^7 \in (D^1,S^0)^K \mid x_1 = x_2 = x_3 = -1, x_7 = 1\}$$

with suitable orientation, and let $s_7\colon \mathbb{R}^7 \to \mathbb{R}^7$ be the reflection changing the sign of the last coordinate, then $[(u_{4,5}-u_{4,6}+u_{5,6})t_{1,2,3}]$ corresponds to the class $[\alpha]-s_7[\alpha]$ (see (5) for definition). Therefore if we use the representative $[u^{1,2}t^3(1-t^7)]$ in (33) instead of $[u^{1,2}t^3]$, we will get the class $-s_7[\alpha]$, which is represented by a submanifold whose intersection with the sphere

$$\alpha' = \{(x_i)_{i=1}^7 \in (D^1, S^0)^K \mid x_4 = x_5 = x_6 = x_7 = -1\}$$

is the point with constant coordinates -1. In the same way, a calculation of $[u^{5,6}t^4(1-t^7) \frown \Gamma]$ shows that it coincides with suitable orientation class of α'. From the Poincaré duality we can read the plumbing of spheres $s_7(\alpha)$ and α', which can also be checked directly.

REMARK 2.11. Gitler and López de Medrano [**GL13**] gave a full topological classification of $(D^1, S^1)^K$ with $|K|$ dual to a neighborly simple 4-polytope: each of them is homeomorphic to a connected sum of copies of $S^2 \times S^2$. In general, however, a direct *diffeomorphim* between them is still missing in current literature. In the example presented above, as a special case, Gutiérrez and López de Medrano [**GL13'**] illustrated such a diffeomorphism.

2.3. Alexander Duality

Recall that a locally compact topological space X is an unbounded homology n-manifold, if for each $x_0 \in X$, the local homology group $H_i(X, X \setminus \{x_0\})$ vanishes if $i \neq n$ and is infinite cyclic if $i = n$. Davis [**Dav08**] proved that $(D^1, S^0)^K$ is a homology n-manifold if and only if $|K|$ is a generalized homology sphere of dimension $n-1$, namely $|K|$ has the homology of an $(n-1)$-sphere, as a homology $(n-1)$-manifold (see Theorem 3.7 for details).

Now we define another combinatorial product, whose relation with the cap product (see Definition 2.7) is analogous to that between the $\bar{*}$-product (see Definition 2.4) and the cup product. As we will see below, by these combinatorial products the (co)homology of K can be understood in another way.

DEFINITION 2.12. The $\sqcap$-product in the second row of the diagram

$$\begin{array}{ccc} R_K^* \bigotimes C_*((D^1,S^0)^K) & \xrightarrow{\frown} & C_*((D^1,S^0)^K) \\ \cong \Big\uparrow (\phi\circ\mu,\eta) & & \cong \Big\uparrow \eta \\ \left(\bigoplus_{\omega\subset[m]} \widetilde{C}^*(K_\omega)\right) \bigotimes \left(\bigoplus_{\omega\subset[m]} \widetilde{C}_*(K_\omega)\right) & \xrightarrow{\sqcap} & \bigoplus_{\omega\subset[m]} \widetilde{C}_*(K_\omega), \end{array} \tag{34}$$

is defined by the commutativity, where the (degree-shifted) (co)chain isomorphisms ϕ, η and μ are defined by (21), (12) and (15), respectively.

By Definition 2.7 and Proposition 2.8, the following properties are straightforward:

PROPOSITION 2.13. *The generator of* $\widetilde{C}^{-1}(K_{[m]})$, $\emptyset = (\phi \circ \mu)^{-1}(1)$, *is the unique identity such that* $\emptyset \sqcap c_r = c_r$, *for every* $c_r \in \widetilde{C}_r(K_{\omega'})$. *For* $\alpha^p \in \widetilde{C}^p(K_{\omega_1})$, *we have*

$$\partial'(\alpha^p \sqcap c_r) = (-1)^{r-p}\delta'\alpha^p \sqcap c_r + \alpha^p \sqcap \partial' c_r,$$

where ∂' (resp. δ') is the simplicial boundary (resp. coboundary) operator. Additionally, assume that $\alpha^q \in \widetilde{C}^q(K_{\omega_2})$, then

$$\alpha^q \sqcap (\alpha^p \sqcap c_r) = (\alpha^q \bar{*} \alpha^p) \sqcap c_r.$$

LEMMA 2.14. *Choose a p-cochain $c^p \in R^p_K|_\omega$ and an r-chain $c_r \in C_r((D^1,S^0)^K)|_{[m]}$, we have*

$$c^p \frown c_r \in \bigoplus_{\tau\subset\omega} \widetilde{C}_{r-p}(K_{[m]\setminus\tau}).$$

Moreover, for each $\tau \subset \omega$, there is a p-chain $\widetilde{c}^p_\tau \in R^p_K|_\tau$ such that

$$(c^p - \widetilde{c}^p_\tau) \frown c_r \in \bigoplus_{\tau\subsetneq\gamma\subset\omega} \widetilde{C}_{r-p}(K_{[m]\setminus\gamma}).$$

PROOF. Without loss of generality, we can write c^p as a finite sum,

$$c^p = \sum_{\substack{\sigma\subset\omega \\ \mathrm{card}(\sigma)=p}} k_\sigma u^\sigma t^{\omega\setminus\sigma},$$

where k_σ are integers as coefficients. Then by Definition 2.7 and a direct calculation, it follows that

$$c^p \frown c_r = \underbrace{(\dots)t_\omega}_{\in\widetilde{C}_{r-p}(K_{[m]\setminus\omega})} + \sum_{\tau\subsetneq\omega} \underbrace{(\dots)t_\tau}_{\in\widetilde{C}_{r-p}(K_{[m]\setminus\tau})}, \tag{35}$$

from which the first statement follows. For the second one, we claim that if $(\dots)t_\tau$ appears in the resulting $c^p \frown c_r$ as the sum of all terms in $\widetilde{C}_{r-p}(K_{[m]\setminus\tau})$, then $(\dots)t_\tau$ must also appear as the corresponding term in $\widetilde{c}^p_\tau \frown c_r$, where

$$\widetilde{c}^p_\tau = \sum_{\substack{\sigma\subset\tau \\ \mathrm{card}(\sigma)=p}} k_\sigma u^\sigma t^{\tau\setminus\sigma} \in R^p_K|_\tau$$

is obtained by changing associated terms in c^p. To see this, observe that $u^\sigma t^{\omega\setminus\sigma} \frown u_{\sigma'}$ (since $c_r \in C_r((D^1,S^0)^K)|_{[m]}$ by assumption, see (8)) is non-trivial only when $\sigma \subset \sigma'$ (see (28)), which is

$$(-1)^{\nu(\sigma,\sigma')} u_{\sigma'\setminus\sigma} t_\sigma \prod_{i\in\omega\setminus\sigma'} (1+t_i) = (-1)^{\nu(\sigma,\sigma')} \sum_{\xi\subset\omega\setminus\sigma'} u_{\sigma'\setminus\sigma} t_{\sigma\cup\xi}.$$

Then for any fixed $\gamma = \sigma\cup\xi$ with $\gamma \subset \tau$, replacing ω by τ yields the same summand $(-1)^{\nu(\sigma,\sigma')} u_{\sigma'\setminus\sigma} t_\gamma$, thus by summing up these terms with their coefficients k_σ in c^p, which is exactly $\widetilde{c}^p_\tau$, the claim follows. $\square$

Actually, in the order that $\mathrm{card}(\tau)$ decreases, we can eliminate every extra term of the form $(\dots)t_\tau$ in (35), where $\tau \subsetneq \omega$, by repeating the construction above:

COROLLARY 2.15. *Assume that $c^p \in R^p_K|_\omega$ and $c_r \in C_r((D^1,S^0)^K)|_{[m]}$, then there exists $\widetilde{c}^p \in \bigoplus_{\tau\subsetneq\omega} R^p_K|_\tau$, such that $(c^p - \widetilde{c}^p) \frown c_r$ belongs to $\widetilde{C}_{r-p}(K_{[m]\setminus\omega})$.*

Assume that $|K|$ is a generalized homology $(n-1)$-sphere, thus $(D^1,S^0)^K$ is a homology n-manifold, which is orientable by Theorem 1.4 with its orientation class given by the orientation class of $|K|$. Let $\Gamma_K \in \widetilde{C}^{n-1}(K)$ be the orientation class, and define

$$\mathtt{AD}\colon \bigoplus_{\omega\subset[m]} \widetilde{C}^*(K_\omega) \to \bigoplus_{\omega\subset[m]} \widetilde{C}_*(K_\omega)$$

via

$$\mathtt{AD}_\omega\colon \widetilde{C}^*(K_\omega) \xrightarrow{\sqcap\Gamma_K} \bigoplus_{\omega\subset[m]} \widetilde{C}_*(K_\omega) \xrightarrow{\pi_\omega} \widetilde{C}_*(K_{[m]\setminus\omega}), \tag{36}$$

where π_ω is the projection onto the direct summand. An explicit formula for $\mathtt{AD}$ is as follows. Suppose that $\sigma_p^* \in \widetilde{C}^p(K_\omega)$ is a dual p-simplex and $\sigma' \in \widetilde{C}_{n-1}(K)$ is a simplex appearing in the class Γ_K (as a finite sum of simplices of degree $n-1$, with coefficients ±1), then

$$\pi_\omega(\sigma_p^* \sqcap \sigma') = \begin{cases} (-1)^{\nu(\sigma_p,\sigma')}(\sigma' \setminus \sigma_p) & \text{if } \sigma_p = \sigma' \cap \omega, \\ 0 & \text{otherwise,} \end{cases}$$

where $\nu(\sigma_p, \sigma') = \sum_{j\in\sigma_p} \mathrm{card}(\{i \in \sigma' \setminus \sigma_p \mid i > j\})$ and $\mathtt{AD}(\sigma_p^*)$ is the sum with terms of above form, multiplied with their coefficients in Γ_K.

Example 2.16. Let Γ be the orientation class given in (32), as shown in Example 2.10. Interpreting the calculation (33) in the language of $\sqcap$-product, it becomes (see Figure (2))

$$\begin{aligned}\pi_{4,5,6,7}(\underbrace{\{1,2\}^*}_{K_{1,2,3}}\sqcap\Gamma) &= \pi_{4,5,6,7}(\underbrace{\{4,5\}-\{4,6\}+\{5,6\}}_{K_{3,4,5,6,7}} + \underbrace{\{4,5\}-\{4,6\}+\{5,6\}}_{K_{4,5,6,7}}) \\ &= \underbrace{\{4,5\}-\{4,6\}+\{5,6\}}_{K_{4,5,6,7}}.\end{aligned}$$

Theorem 2.17. *Let $|K|$ be a generalized homology $(n-1)$-sphere with the orientation class Γ_K. Then for each $\omega \subset [m]$, $\mathtt{AD}$ induces isomorphisms*

$$\widetilde{H}^i(K_\omega) \xrightarrow[\cong]{\mathtt{AD}_\omega} \widetilde{H}_{n-i-2}(K_{[m]\setminus\omega}) \tag{37}$$

with $-1 \le i \le n-1$.

Proof. By Theorem 1.4 and diagram (34), the diagram

$$\begin{array}{ccc} R_K^i & \xrightarrow{\frown\eta(\Gamma_K)} & C_{n-i}((D^1,S^0)^K) \\ {\scriptstyle\phi\circ\mu}\uparrow{\scriptstyle\cong} & & {\scriptstyle\eta}\uparrow{\scriptstyle\cong} \\ \bigoplus_{\omega\subset[m]} \widetilde{C}^{i-1}(K_\omega) & \xrightarrow{\sqcap\Gamma_K} & \bigoplus_{\omega\subset[m]} \widetilde{C}_{n-i-1}(K_{[m]\setminus\omega}) \end{array}$$

commutes for every i with $0 \le i \le n$, where $(D^1, S^0)^K$ is now a compact homology manifold with the orientation class $\eta(\Gamma_K)$. The Poincaré duality in $(D^1, S^0)^K$ (see [**Mun84**, Section 67, p. 394–397]) implies that the bottom row is an isomorphism when passing to (co)homology. It remains to show the isomorphism still holds after projections π_ω.

This is clear when ω is empty, by Proposition 2.13. In what follows we assume $\omega \neq \emptyset$.

(Injectivity of $\mathtt{AD}_\omega$.) If there exists an i-cochain $c^i \in \widetilde{C}^i(K_\omega)$, such that $[c^i] \in \ker(\mathtt{AD}_\omega)$, then by Corollary 2.15, we have

$$(c^i - \widetilde{c}^i) \sqcap \Gamma_K = \mathtt{AD}(c^i), \text{ for some } \widetilde{c}^i \in \bigoplus_{\tau \subsetneq \omega} \widetilde{C}^i(K_\tau),$$

hence $[c^i - \widetilde{c}^i]$ is in the kernel of the isomorphism induced by $\sqcap\Gamma_K$, and it follows that it must be a coboundary, so must be c^i, since $\bigoplus_{\tau \subsetneq \omega} \widetilde{C}^*(K_\tau)$ is closed under δ'.

(Surjectivity of $\mathtt{AD}_\omega$.) Let $[c_{n-2-q}] \in \widetilde{H}_{n-2-q}(K_{[m]\setminus\omega})$ be any class such that

$$c_{n-2-q} = c^q \sqcap \Gamma_K. \tag{38}$$

It remains to prove that c^q can be chosen from $\widetilde{C}^q(K_\omega)$. First notice that if c^q has a summand $c^q_\tau \in \widetilde{C}^q(K_\tau)$ with $\tau \subsetneq \omega$, then by Lemma 2.14, $c^q_\tau \cap \Gamma_K$ has no contributions to $\widetilde{C}_{n-2-q}(K_{[m]\setminus\omega})$, thus it can be ignored in the image $\mathtt{AD}_\omega(c^q)$. Therefore, without loss of generality, suppose that $c^q \in \bigoplus_{\omega\subset\gamma} \widetilde{C}^q(K_\gamma)$. For any non-trivial summand of c^q not contained in $\widetilde{C}^q(K_\omega)$, say $c^q_\gamma \in \widetilde{C}^q(K_\gamma)$ with $\omega \subsetneq \gamma$, the proof of Lemma 2.14 implies that there is a q-cochain $c^q_{\gamma'} \in \widetilde{C}^q(K_\omega)$ such that

$$\mathtt{AD}_\omega(c^q_{\gamma'}) = \mathtt{AD}_\omega(c^q_\gamma),$$

thus we can replace the summand c^q_γ by $c^q_{\gamma'}$. After using this trick for all summands of this form, the proof will be completed. □

CHAPTER 3

Polyhedral products as real moment-angle complexes

With respect to a sequence $J = (n_i)_{i=1}^m$ of positive integers, Bahri, Bendersky, Cohen and Gitler [**BBCG10b**] introduced an operation on a simplicial complex K (whose ground set is $[m]$), such that with the resulting complex KJ (with the original notation $K(J)$ in [**BBCG10b**]) and pairs $(D^{n_i}, S^{n_i-1})_{i=1}^m$, the associated polyhedral product $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ is homeomorphic to the real moment-angle complex $(D^1, S^0)^{KJ}$ (see Lemma 3.4). Also, they developed a local construction that arose in earlier works with the name *simplicial wedge construction* (see [**PB80**]), such that KJ can be obtained by using a sequence of the local constructions consecutively (i.e. Lemma 3.13).

On the other hand, in the language of simple polytopes, simplicial wedge constructions appeared in the work of Wall [**Wal80**], and later they were intensely used in [**LdM89**] and [**GL13**] by López de Medrano and Gitler. In these works, this construction (also named as the Buchstaber construction) is closely related to a class of open book constructions on certain transverse intersections of real quadrics with the unit spheres: they are smooth manifolds homeomorphic to the polyhedral products indicated above.

Based on the characterization work of Davis for real moment-angle manifolds (see Theorem 3.7), we shall proceed with homology manifolds, and consider the infulence of a simplicial wedge construction on associated manifold. Consequently, in Section 3.1, a necessary and sufficient condition that a moment-angle complex $(D^2, S^1)^K$ is a topological manifold is obtained (see Theorem 3.16). In Section 3.2, we give an alternative proof of Theorem 3.7, which is an adaptation of the approaches in [**BP02**] and [**Dav08**]. In Section 3.3, we analyse the topological open book construction by cochains, and illustrate that the augmented simplicial cochain complexes of all full subcomplexes of K are sufficient to give the cohomology of the real moment-angle complex $(D^1, S^0)^{KJ}$, in the form of the cohomology of a differential graded algebra R_K^*J (see Corollary 3.35). This idea follows that of Baskakov-Buchstaber-Panov [**BBP04**], [**Pan08**] for $H^*((D^2, S^1)^K)$.

Throughout this chapter, let K be an abstract simplicial complex with ground set $[m]$; for $\omega \subset [m]$, the full subcomplex K_ω will be denoted as $K|_\omega$.

3.1. KJ and simplicial wedge constructions

Definition 3.1 (see [**BBCG10a**]). Let $\mathfrak{K}$ be the category induced by K, in which objects are simplices of K, and morphisms are inclusions. Let $\mathfrak{T}$ be the category of topological spaces and suppose $(X_i, A_i)_{i=1}^m$ are m pairs chosen from $\mathfrak{T}$. The *polyhedral product functor* determined by $(X_i, A_i)_{i=1}^m$ and K is given by

$$D\colon \mathfrak{K} \longrightarrow \mathfrak{T}$$

sending each object $\sigma \in \mathfrak{K}$ to the space

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{otherwise.} \end{cases}$$

The colimit of D (see Definition A.1) is referred to as the corresponding *polyhedral product* and will be denoted as $((X_i, A_i)_{i=1}^m)^K$. If all m pairs $(X_i, A_i)_{i=1}^m$ are homeomorphic to a given one, (X, A), then $((X_i, A_i)_{i=1}^m)^K$ will be abbreviated as $(X, A)^K$.

It can be checked that the polyhedral product $((X_i, A_i)_{i=1}^m)^K$ is homeomorphic to the union $\bigcup_{\sigma \in \mathfrak{K}} D(\sigma)$. For instance, a real moment-angle complex $(D^1, S^0)^K$ is the polyhedral product with m pairs (D^1, S^0).

Definition 3.2. A subset of $[m]$ not contained in K is called a *missing face* of K, if all of its proper subsets are simplices of K.

For abstract simplicial complexes K_1 and K_2 with disjoint ground sets V_1 and V_2, respectively, their *join* is defined as

$$K_1 * K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_1 \in K_1, \sigma_2 \in K_2\},$$

with the new ground set $V_1 \cup V_2$. Let $v \in K$ be a vertex (i.e. $\mathrm{card}(v) = 1$), then the *link* of v in K is given by

$$\mathrm{Link}(v, K) = \{\sigma \in K \mid (v \cup \sigma) \in K, v \cap \sigma = \emptyset\};$$

the link of a vertex in $|K|$ refers to the geometric realization of the associated link with the form above.

Clearly, each $\tau \subset [m]$ not contained in K must contain certain missing face of K, and vice versa. Thus with a given ground set, K is completely determined by its missing faces. For example, the $(m-1)$-simplex Δ^{m-1} consisting of all subsets of $[m]$ has no missing faces, and any $K \subset 2^{[m]}$ with no missing faces must be Δ^{m-1}.

Definition 3.3 (see [**BBCG10b**]). Let $J = (n_i)_{i=1}^m$ be a sequence of positive integers with $N_i = \sum_{j=1}^{i} n_j$, $i = 1, 2, \ldots, m$. For each missing face $\tau = \{i_1, i_2, \ldots, i_l\}$ of K, let $\tau(J) = \{B_{i_1}, B_{i_2}, \ldots, B_{i_l}\}$ be a subset of $[N_m]$, such that B_{i_j} is a block of consecutive indices from $N_{i_j-1} + 1$ to $N_{i_j-1} + n_{i_j}$ (here we set $N_0 = 0$), $j = 1, 2, \ldots, l$. Let KJ the abstract simplicial complex with ground set $[N_m]$, given by its missing faces: as τ runs through the missing faces of K, every $\tau(J)$ is a missing face of KJ, and all missing faces of KJ are of this form.

Note that by definition, we have $\mathrm{card}(\tau(J)) = \sum_{i \in \tau} n_i$ for each missing face τ of K. $KJ = K$ if and only if $n_i = 1$ for all $i = 1, 2, \ldots, m$.

For the sequence $J = (n_i)_{i=1}^m$, let D^{n_i} be the unit n_i-disk with its boundary S^{n_i-1}. Observe that (D^{n_i}, S^{n_i-1}) is homeomorphic to a pair of polyhedral products

$$(I^{n_i}, \partial I^{n_i}) = ((D^1, S^0)^{\Delta^{n_i-1}}, (D^1, S^0)^{\partial \Delta^{n_i-1}}).$$

Lemma 3.4 (see [**BBCG10b**]). *The polyhedral product $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ is homeomorphic to the real moment-angle complex $(D^1, S^0)^{KJ}$.*

PROOF. Let $f_i\colon (I^{n_i}, \partial I^{n_i}) \to (D^{n_i}, S^{n_i-1})$ be m chosen homeomorphisms a priori, $i = 1, 2, \ldots, m$, and let $f_J = (f_1, f_2, \ldots, f_m)\colon I^{N_m} \to \prod_{i=1}^m D^{n_i}$ be their product. Clearly f_J is a homeomorphism.

Denote by $\mathfrak{K}J$ the category induced by KJ, and suppose $D_J\colon \mathfrak{K}J \to \mathfrak{T}$ is the polyhedral product functor defining $(D^1, S^0)^{KJ}$. Now we define a map

$$\begin{array}{rccc} F_J\colon & (D^1, S^0)^{KJ} & \longrightarrow & \prod_{i=1}^m D^{n_i} \\ & D_J(\sigma') & \longmapsto & f_J(D_J(\sigma')). \end{array} \tag{39}$$

Note that by passing each $\sigma' \in \mathfrak{K}(J)$ to the colimit, F_J is well-defined and continuous.

We claim that with its image restricted to $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$, F_J is a bijection. Since $(D^1, S^0)^{KJ}$ is compact and $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ is Hausdorff, if follows from the claim that F_J is closed thus it induces a desired homeomorphism.

To prove the claim, we check the image of F_J. Choose a point $x = (x_i)_{i=1}^m \in \prod_{i=1}^m D^{n_i}$ with $x_i \in D^{n_i}$, $i = 1, 2, \ldots, m$, by definition, $x \in ((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ if and only if

$$\sigma_x := \{i \in [m] \mid x_i \in D^{n_i} \setminus S^{n_i-1}\}$$

does not contain any missing faces of K (see Definition 3.1). This happens if and only if

$$\sigma_{f_J^{-1}(x)}(J) := \{B_i \mid f_i^{-1}(x_i) \in I^{n_i} \setminus \partial I^{n_i}\}$$

does not contain any missing faces of KJ, i.e., $\sigma_{f_J^{-1}(x)}(J) \in KJ$ (where B_i is a block of consecutive indices

$$N_{i-1} + 1, N_{i-1} + 2, \ldots, N_{i-1} + n_i,$$

with $N_{i-1} = \sum_{j=1}^{i-1} n_j$). Therefore the claim follows. □

REMARK 3.5. As mentioned in Section 2.3, a topological space X is a homology n-manifold (without boundary) if it has the same local homology groups with $\mathbb{R}^n$. Using the long exact sequence of relative homology groups, it turns out that (see for example, [**Mun84**, Theorem 63.2, p. 375; Exercise 1, p. 377]) a polyhedron X (see Definition 3.21) is a homology n-manifold if and only if, with any given triangulation of X, the link of each vertex has the homology of an $(n-1)$-sphere.

DEFINITION 3.6. Let n be a positive integer, and let $|K|$ be the geometric realization of K.

(1) $|K|$ is a *generalized homology $(n-1)$-sphere* (abbr. GHS^{n-1}), if $|K|$ is a homology manifold and has the homology of an $(n-1)$-sphere.
(2) $|K|$ is a PL *$(n-1)$-sphere* (abbr. PLS^{n-1}), if $|K|$ is PL homeomorphic to the boundary of an n-simplex (see Definition 3.21 for details).
(3) $|K|$ is a *polytopal $(n-1)$-sphere* (abbr. PS^{n-1}), if $|K|$ is simplicially homeomorphic to the boundary of a convex n-polytope $P^n \subset \mathbb{R}^n$.

We say that the corresponding real moment-angle complex $(D^1, S^0)^K$ is *polytopal*, if $|K|$ is a polytopal sphere.

THEOREM 3.7 (see [**Dav08**, Theorem 10.6.1, p. 197], [**Dav83**, Section 17]). *Let $(D^1, S^0)^K$ be a real moment-angle complex. Then*

(1) $(D^1, S^0)^K$ is a homology n-manifold if and only if $|K|$ is a GHS^{n-1}.

(2) It is a topological n-manifold if and only if $|K|$ is a GHS^{n-1}*, which is simply connected when $n-1 \neq 0, 1$.*
(3) It is a PL *n-manifold if and only if $|K|$ is a* PLS^{n-1}*.*
(4) It is homeomorphic to a smooth manifold if $|K|$ is polytopal.

REMARK 3.8. Panov and Ustinovsky [**PU12**] proved that the last statement in Theorem 3.7 can be generalized to the case when K is induced from a *complete simplicial fan* (which is equivalent to the condition that K is *star-shaped*). The last statement can be strengthened that, there is a *piecewise differentiable* homeomorphism from $(D^1, S^0)^K$ to a smooth manifold embedded in $\mathbb{R}^m$ (see, for example, [**Cai14**, Lemma 6.3]).

CONVENTION 3.9. In what follows, "$|K|$ is a sphere" means that it is either a GHS, a PLS or a PS. Correspondingly, a manifold may refer to either a homology manifold, a topological manifold, a PL manifold or a smooth manifold. If we say that a polyhedral product $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ is a smooth manifold, *then K shall be implicitly assumed to be polytopal.*[1]

PROPOSITION 3.10. *Let $J = (n_i)_{i=1}^m$ be a sequence of positive integers with $N_m = \sum_{i=1}^m n_i$. Assume that $|K|$ is a sphere of dimension $n-1$, then $|KJ|$ will be a sphere of the same type and of dimension $N_m - m + n - 1$. Moreover, $|KJ|$ is simply connected, provided that $N_m > m$ and $n \neq 1$.*

COROLLARY 3.11. *Suppose $(D^1, S^0)^K$ is a manifold of dimension n, then the polyhedral product $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ is a manifold of the same type and of dimension $N_m - m + n$. Moreover, $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ is always a topological manifold whenever $N_m > m$.*

To prove Proposition 3.10 and Corollary 3.11, we need to understand KJ in more detail.

DEFINITION 3.12 (see [**BBCG10b**], [**PB80**, p. 578]). For $i = 1, 2, \ldots, m$, let $v_i = \{i\}$ be the i-th vertex of K. The *simplicial wedge* of K on v_i, being an abstract simplicial complex with ground set $[m+1]$, is defined as

$$Kv_i := (\{i, i+1\} * \mathrm{Link}(\{i\}, K)) \bigcup \left(\{i\} * K|_{[m]\setminus\{i\}}\right) \bigcup \left(\{i+1\} * K|_{[m]\setminus\{i\}}\right), \tag{40}$$

such that for $j = i+1, i+2, \ldots, m$, the original label of the j-th vertex of K is shifted to $j+1$ (see Figure 1 for an illustration). Other labels are preserved.

It is easy to check that if K is the boundary complex of an $(m-1)$-simplex Δ^{m-1}, then Kv_i is the boundary complex of the m-simplex Δ^m, $i = 1, 2, \ldots, m$. This observation can be generalized as follows.

LEMMA 3.13 (see [**BBCG10b**]). *Kv_i is simplicially isomorphic to KJ_i (see Definition 3.3), with*

$$J_i = (1, \ldots, 1, 2, 1, \ldots, 1),$$

[1]S. Choi and H. Park [**CP13**] had already considered the *star-shaped* cases, which we will not discuss here.

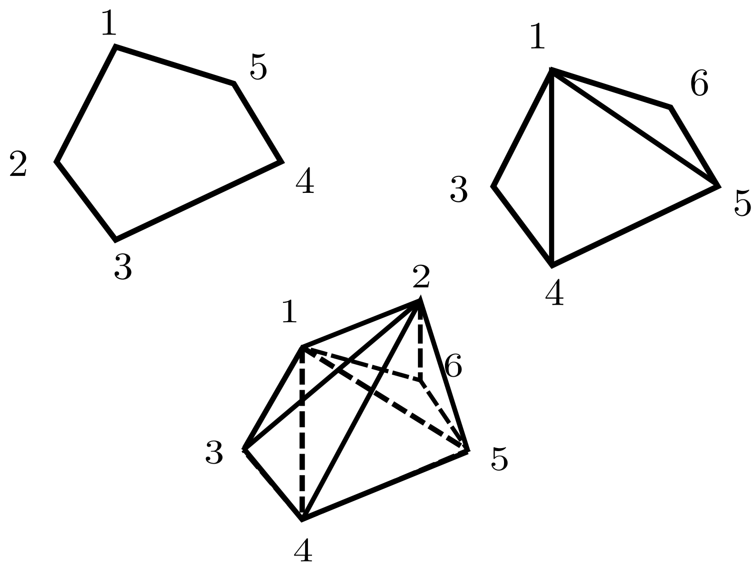

Figure 1. From K to Kv_1

whose i-th entry is 2 and other entries are 1. Consequently, simplicial wedge constructions are commutative in the following sense: for $i \in [m]$ and $j \in [m+1]$,

$$Kv_iv_j = \begin{cases} Kv_{j-1}v_i & \text{if } j \geq i+1, \\ Kv_jv_{i+1} & \text{otherwise}, \end{cases}$$

and KJ can be obtained by $N_m - m$ simplicial wedge constructions consecutively on K.

Proof. By their definitions, Kv_i and KJ_i are both simplicial complexes with ground set $[m+1]$, thus it suffices to check their missing faces. Let $\tau \subset [m]$ be a missing face of K. According to (40), (a) if $i \notin \tau$, then it will be a missing face of Kv_i (whose labels may be shifted); (b) otherwise by the observation above, $\tau \cup \{i+1\}$ is a (label-shifted) missing face of Kv_i. Conversely, it can be checked that any missing face of Kv_i comes from either (a) or (b), namely in the form $\tau(J_i)$ in Definition 3.3. As τ runs through missing faces of K, we find that all missing faces of Kv_i coincides with those of KJ_i. The second statement follows from the definition directly. $\square$

The following two lemmas are well-known:

Lemma 3.14 ([**CP13**, Proposition 2.2, p. 8]). *$|Kv_1|$ is* PL *homeomorphic to the suspension $S|K| = |S^0 * K|$ (here S^0 consists of two points not contained in K).*

Proof. It suffices to show that, after a subdivision, Kv_1 is simplicially heomeomorphic to $S^0 * K$ (see [**RS72**]). The subdivision is constructed as follows: first add an extra vertex labeled by $m+2$ to the middle point of the edge $\{1,2\}$, then generate other simplices in subcomplexes $\{1, m+2\} * \mathrm{Link}(\{1\}, K)$ and $\{2, m+2\} * \mathrm{Link}(\{1\}, K)$, respectively. This is well-defined by (40). To see its relation with $S^0 * K$, we relabel the vertices of K: label the first vertex $\{1\}$ by $m+2$, and then shift the label i to $i+1$ for each $i = 2, 3, \ldots, m$ (see Figure 2); after this we take joins $\{1\} * K$ and $\{2\} * K$, with $\{1\}$ and $\{2\}$ being considered as two external points, whose union along the common part K

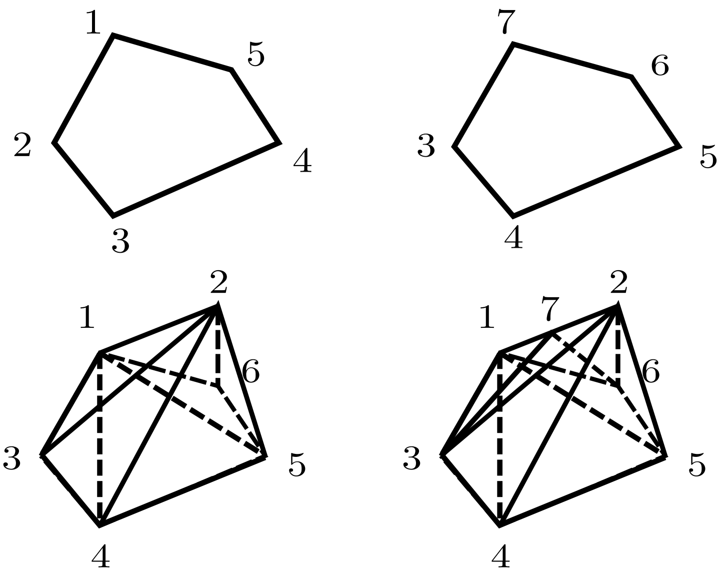

Figure 2. The suspension

givens $S^0 * K$. By sending vertices to those with the same labels, $|S^0 * K|$ is simplicially homeomorphic to $|Kv_1|$. $\square$

LEMMA 3.15. *$|K|$ is a* PLS^{n-1} *(resp.* GHS^{n-1}*) if and only if $|S^0 * K|$ is a* PLS^n *(resp.* GHS^n*).*

PROOF. (The case when $|K|$ is a PLS^{n-1}.) For the "if" part, we use the fact that, if a PL homeomorphism $(A, a) \to (B, b)$ between polyhedra sending point a to b, then $|\mathrm{Link}(a, A)|$ is PL homeomorphic to $\mathrm{Link}(b, B)$, for any triangulations on A and B with a and b respective vertices (see [**RS72**, Lemma 2.19, p. 21]). Since the link of a vertex in the boundary of an $(n+1)$-simplex bounds its opposite face, i.e. an n-simplex, and the link of a point of S^0 in $|S^0 * K|$ is $|K|$, the conclusion follows from the fact above. The "only if " part follows from the fact that the suspension of a PL $(n-1)$-sphere is a PL n-sphere (see [**RS72**, Proposition 2.23, pp. 23–24]).

(The case when $|K|$ is a GHS^{n-1}.) Now the "if" part follows from Proposition 3.22, since K is a link in $S^0 * K$. For the "only if" part, a standard argument using Mayer-Vietoris exact sequence shows that

$$H_i\left(|S^0 * K|\right) \cong H_{i-1}\left(|K|\right),$$

thus $|S^0 * K|$ has the homology of an n-sphere. It remains to show that $|\mathrm{Link}(v, S^0 * K)|$ has the homology of an $(n-1)$-sphere. This is clear when $v \in S^0$. Otherwise suppose $v \in K$, we have

$$\mathrm{Link}(v, S^0 * K) = S^0 * \mathrm{Link}(v, K)$$

where $|\mathrm{Link}(v, K)|$ has the homology of an $(n-2)$-sphere. Then we can repeat the previous argument to complete the proof. $\square$

Proof of Proposition 3.10. By Lemma 3.14, every simplicial wedge construction will increase the dimension by one, therefore the dimension of $|KJ|$ can be deduced from Lemma 3.13, which is $N_m - m + n - 1$.

Without loss of generality, suppose $KJ = Kv_1$.

(The case when $|K|$ is a GHS^{n-1}.) Since (local) homology groups are topological invariants, by Lemma 3.14, it suffices to consider the suspension $S|K|$, which is proven in Lemma 3.15.

(The case when $|K|$ is a PLS^{n-1}.) By Lemma 3.14, $|Kv_1|$ is PL homeomorphic to $S|K|$, thus again we can use Lemma 3.15.

(The case when $|K|$ is a PS^{n-1}.) This is well-known and we omit the proof here (see [**CP13**, Proposition 2.2, p. 8]).

The second statement follows from the van Kampen theorem. □

It turns out that Corollary 3.11 follows from Proposition 3.10 and Theorem 3.7, except for the case $n = 1$, i.e. K consists of two points ($m = 2$). But in this case $|K|$ is polytopal, hence $(D^1, S^0)^{KJ}$ is a topological manifold.

Theorem 3.16. *Let $J = (n_i)_{i=1}^m$ be a sequence with constant integers $n_i = 2$. Then $(D^1, S^0)^{KJ}$ is a topological $(n+m)$-manifold if and only if $|K|$ is a* GHS^{n-1}. *Respectively, it is a* PL *$(n+m)$-manifold if and only if $|K|$ is a* PLS^{n-1}.

Proof. In either case, the "if" part follows from Corollary 3.11 and Theorem 3.7, respectively. For the "only if" parts, we claim that $|Kv_1|$ is either a GHS^n or PLS^n, then $|K|$ must be of the same type. By Theorem 3.7, starting from KJ, as a GHS or a PLS obtained from K via m simplicial wedge constructions consecutively, the statement follows by using the claim for m times.

Notice that by the PL homeomorphism between $|Kv_1|$ and $|S^0 * K|$, it suffices to prove claim for $|S^0 * K|$. This is done in Lemma 3.15. □

Remark 3.17. By Lemma 3.4, the theorem above gives a necessary and sufficient condition for a moment-angle complex $(D^2, S^1)^K$ to be a topological manifold. But for the PL case, we can only say that $(D^2, S^1)^K$ is homeomorphic to a PL manifold, since it does not have an obvious PL structure a priori.

By Proposition 3.22, $|K|$ is a GHS^{n-1} if and only if K is *Gorenstein** over $\mathbb{Z}$ (see [**BP02**, Theorem 3.38, p. 46]).

3.2. An alternative proof of Theorem 3.7

Since the language Davis used to prove Theorem 3.7 is not standard for most toric topologists, this section is devoted to a more direct proof, which is adapted from [**Dav08**, Chapter 10].

The approach is as follows. First we triangulate $(D^1, S^0)^K$, by a homeomorphism $u_{\varphi'}\colon \mathcal{U}(G, |K'_+|) \to (D^1, S^0)^K$ with $\mathcal{U}(G, |K'_+|)$ a simplicial complex, such that $u_{\varphi'}$ is linear when restricted to a simplex in $\mathcal{U}(G, |K'_+|)$ (see (46) for more details). This part is adapted from [**BP02**, Chapter 4, pp. 53–55]. With this triangulation at hand, the barycentric subdivision of $|K|$ appears as the link of a vertex in $(D^1, S^0)^K$, from which we get the necessary conditions for $(D^1, S^0)^K$ to be a manifold of certain type (see Convention 3.9). The main task is to show that these conditions are also sufficient.

In what follows, let K' be the *derived complex* of K, in which a simplex is in the form of a dimension-increasing sequence of simplices in K, such that a simplex of lower dimension is included in those of higher dimensions. The augmentation K'_+ means the first simplex in each sequence of K' can be the empty set. Geometrically, $|K'|$ is the barycentric subdivision of $|K|$. $|K'_+|$ is a cone over $|K'|$ with the collapsed end point corresponding to the sequence $(\emptyset)$.

The *star* of a simplex σ in K is defined as

$$\mathrm{Star}(\sigma, K) = \{\tau \in K \mid \sigma \subset \tau\}. \tag{41}$$

From Definition 3.2, immediately we have $\sigma * \mathrm{Link}(\sigma, K) = \mathrm{Star}(\sigma, K)$.

Definition 3.18 (The Basic Construction [**Dav08**, Chapter 5]). Suppose $(X, \mathcal{F}_X)$ is a topological space with a system of subspaces $\mathcal{F}_X = \{X_1, \ldots, X_m\}$, and let (G, S) be a group finitely generated by $S = \{s_1, s_2, \ldots, s_m\}$. Let X_I be the intersection of all X_i with $i \in I \subset [m]$, and for a subset $T \subset S$, G_T refers to the subgroup generated by T (it is convenient to set $X_\emptyset = X$ and $G_\emptyset$ as the identity). The associated *Basic Construction* $\mathcal{U}(G, X)$ is a G-space

$$\mathcal{U}(G, X) = (G \times X)/\sim,$$

where $(g, x) \sim (g', x')$ if

$$x = x' \in X_{I_x} \text{ and } g^{-1}g' \in G_{\{s_i\}_{i \in I_x}},$$

where $I_x := \{I \subset [m] \mid x \in X_i\}$, and the G-action follows

$$g'[g, x] = [g'g, x]$$

for $[g, x] \in \mathcal{U}(G, X)$ and $g' \in G$.

It is easy to check that this G-action is well defined.

Throughout this section, let G be the group $(\mathbb{Z}_2)^m$. Consider the canonical action of G on $\mathbb{R}^m$ generated by

$$s_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_m), \tag{42}$$

$i = 1, 2, \ldots, m$. It can be checked that the quotient space $(D^1, S^0)^K/G$ of the induced action on $(D^1, S^0)^K$ is

$$X_K := \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in [0,1]^m \mid x_j = 1 \text{ if } j \notin \sigma\}. \tag{43}$$

Proposition 3.19. *X_K is simplicially homeomorphic to $|K'_+|$. More precisely, let $\varphi\colon K \to \mathbb{R}^m$ be the map given by*

$$\varphi(\sigma) = (x_1, \ldots, x_m), \quad x_i = \begin{cases} 0 & \textit{if } i \in \sigma, \\ 1 & \textit{otherwise}, \end{cases} \tag{44}$$

for $\sigma \in K$, and denote by $\varphi'\colon |K'_+| \to X_K$ the simplicial map derived from φ, i.e. for $\sigma' \in K'_+$ being a dimension-increasing sequence $(\sigma_{i_0}, \sigma_{i_1}, \ldots, \sigma_{i_l})$ with $\sigma_{i_j} \subset \sigma_{i_{j+1}}$, $j =$

$0, 1, \ldots, l-1$,

$$\varphi'(|\sigma'|) = l(\varphi(\sigma_{i_0}), \varphi(\sigma_{i_1}), \ldots, \varphi(\sigma_{i_l})), \tag{45}$$

where the object on the right-hand side is the simplex linearly spanned by corresponding vertices. Then φ' is a homeomorphism.

This is a direct consequence of the following lemma, on the decomposition of cubical complexes (see [**BP02**, Chapter 4, p. 53]).

LEMMA 3.20. *For the cube $C_\sigma := \{(x_1, \ldots, x_m) \in [0,1]^m \mid x_j = 1 \text{ if } j \notin \sigma\}$ where $\sigma \subset [m]$ with $\operatorname{card}(\sigma) = k$, there is a simplicial decomposition*

$$C_\sigma = \bigcup_{\sigma' \in (2^\sigma)'_+} \varphi'(|\sigma'|)$$

where σ' runs through $(2^\sigma)'_+$ as dimension-increasing sequences. (As the power set of σ, 2^σ is an abstract simplicial complex of dimension $k-1$.)

PROOF. Consider those simplices $\sigma' \in (2^\sigma)'_+$ of maximal dimension k: each is a sequence of the form

$$\sigma' = (\sigma_0, \sigma_1, \ldots, \sigma_k),$$

where $\sigma_0 = \emptyset$, $\sigma_k = \sigma$ and $\operatorname{card}(\sigma_i) = i$, $i = 0, 1, \ldots, k$. $(\varphi(\sigma_i))_{i=1}^k$ gives rise to a path with $k+1$ nodes ordered by $0, 1, \ldots, k$, from $\varphi(\emptyset) = (1, 1, \ldots, 1)$ to $\varphi(\sigma)$, such that each node has coordinates with values 0 or 1, and adjacent nodes $(x_i)_{i=1}^m$ and $(x'_i)_{i=1}^m$ in order have Euclidean distance 1, with $x_i \geq x'_i$ for $i = 1, 2, \ldots, m$. It turns out that the nodes on each path linearly spans a simplex of dimension k, such that the interiors of these $k!$ different simplices and those of their boundaries give rise to a partition of C_σ (see [**ES52**, p. 68]). □

Now we use the basic construction to triangulate $(D^1, S^0)^K$. Denote by $\varphi'\colon |K'_+| \to (D^1, S^0)^K$ the composition of $\varphi'\colon |K'_+| \to X_K$ and the obvious inclusion $X_K \to (D^1, S^0)^K$. Let $|K'_+|$ be equipped with the system $\mathcal{F}_{|K'_+|} = \{F_1, F_2, \ldots, F_m\}$, such that a simplex $|\sigma'|$ belongs to F_i if and only if $\varphi'(|\sigma'|)$ is fixed by s_i, i.e., points with i-th coordinate 0. Clearly σ' is a sequence with each simplex containing $i \in [m]$. It follows that $F_i = |\mathrm{Star}\,((\{i\}), K')\,|$. Let $\mathcal{U}(G, |K'_+|)$ be the resulting space of the basic construction, and let $u_{\varphi'}$ be the G-map defined in the bottom row of the diagram

$$\begin{tikzcd} & {|K'_+|} \arrow[dl, "\mathrm{i}"'] \arrow[dr, "\varphi'"] & \\ u_{\varphi'}\colon \mathcal{U}(G, |K'_+|) \arrow[rr] & & (D^1, S^0)^K \\ {[g, x]} \arrow[rr, mapsto] & & g\varphi'(x), \end{tikzcd} \tag{46}$$

in which $\mathrm{i}(x) = [1, x]$. We see that $\mathcal{U}(G, |K'_+|)$ is a simplicial complex with each simplex of the form $[g, |\sigma'|]$, $g \in G$ and $\sigma' \in K'_+$, and $u_{\varphi'}$ is a homeomorphism preserving the G-action, which is linear when restricted to a simplex in $\mathcal{U}(G, |K'_+|)$ (see Proposition 3.19).

Next we turn to the proof of Theorem 3.7. Before that we need some preparations.

DEFINITION 3.21. A subspace $X \subset \mathbb{R}^m$ is called a *polyhedron* if each point $x \in X$ has a neighborhood of the form $x * L_x = \{tx + (1-t)l \mid l \in L_x, t \in [0,1]\} \subset X$, with L_x compact. A (continuous) map $f\colon X \to Y$ between two polyhedra X and Y is *piecewise linear* (abbr. PL) if for all $y \in Y$ with a given neighborhood U_y, there is a neighborhood $x * L_x$ for each $x \in f^{-1}(y)$, such that $f(x * L_x) \subset U_y$ with $f(tx + (1-t)l) = ty + (1-t)f(l)$, for all $l \in L_x$ and $t \in [0,1]$.

A *triangulation* of X is a PL homeomorphism $f\colon |K_X| \to X$ such that the restriction of f to each simplex in $|K_X|$ is linear. X is an unbounded PL n-manifold if $|\mathrm{Link}(v, K_X)|$ is a PLS^{n-1} (see Definition 3.6), for any vertex $v \in K_X$.

Note that every polyhedron admits a triangulation, which is unique up to PL homeomorphism (see [RS72, Theorems 2.11, 2.14, pp. 16–17]). Up to PL homeomorphism, the link of each vertex in a PL manifold (without boundary) is independent of the triangulation (see [RS72, Lemma 2.19, p. 21]).

Using the cone construction (see [RS72, Lemma 1.10, p. 8]), the cone over a PLS^{n-1} is PL homeomorphic to an n-simplex Δ^n, by extending the defining homeomorphism $\mathrm{PLS}^{n-1} \to \partial\Delta^n$ linearly to the collapsing point (this is a crucial difference between PL and smooth categories).

In what follows, a PL (resp. *homology*) n-disk refers to the cone over a PLS^{n-1} (resp. GHS^{n-1}); if $n = 0$, PLS^{-1} and GHS^{-1} are both empty, and the cone over which is a point. All manifolds mentioned are assumed to have no boundaries.

The following proposition is well-known, and we shall omit the proof here (see [RS72, Exercise 2.24(3), p. 24], [Mun84, Section 63, Exercise 2, p. 377], respectively):

PROPOSITION 3.22. *Suppose that a polyhedron X is a* PL *(resp. homology) n-manifold with a given triangulation $f\colon |K| \to X$. Let $\sigma \in K$ be a k-simplex with $k \geq 0$, then* $|\mathrm{Link}(\sigma, K)|$ *is a* PLS^{n-k-1} *(resp.* GHS^{n-k-1}*) in $|K|$ (which is empty when $k = n$).*

For a simplex σ in K, denote by $L'(\sigma)$ the subcomplex of K' consisting of all sequences with σ the *first simplex*, namely a simplex in $L'(\sigma)$ is a sequence of the form $(\sigma, \ldots)$. For example, if σ is a vertex, then we have $L'(\sigma) = \mathrm{Star}\,((\sigma), K')$.

COROLLARY 3.23. *Let $|K|$ be a* PL *(resp. homology) $(n-1)$-manifold and let $|K'|$ be its barycentric subdivision. Then for a k-simplex σ in K with $k \geq 0$, $|L'(\sigma)|$ is a* PL *(resp. homology) $(n-k-1)$-disk in $|K'|$.*

PROOF. Let $\sigma'_k = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ be any sequence in K' with $\sigma_k = \sigma$, such that the dimension of $|\sigma_i|$ is i, $i = 0, 1, \ldots, k$. It can be checked that

$$L'(\sigma) = \mathrm{Star}\,((\sigma), \mathrm{Link}(\sigma'_k, K'))\,. \tag{47}$$

Then the statement follows from Proposition 3.22, since $|K|$ is PL homeomorphic to $|K'|$. □

Now we prove Theorem 3.7. Let $(D^1, S^0)^K$ be the polyhedron equipped with the triangulation $u_{\varphi'}\colon \mathcal{U}(G, |K'_+|) \to (D^1, S^0)^K$, as in (46).

First assume that $(D^1, S^0)^K$ is a PL (resp. homology) manifold. Let $x_+ \in (D^1, S^0)^K$ be the point with constant coordinates 1. By definition, $\mathrm{Link}\,\big(u_{\varphi'}^{-1}(x_+), \mathcal{U}(G, |K'_+|)\big)$ coincides with Link $\big((\varphi')^{-1}(x_+), |K'_+|\big)$, which is $|K'|$ (see Proposition 3.19). Therefore $|K|$ is a PLS (resp. GHS) by Proposition 3.22. This gives the necessary condition for $(D^1, S^0)^K$ to be a PL (resp. homology) manifold. For the condition that it is a topological manifold, we need the following famous theorem (see [**Dav08**, Theorem 10.4.10, p. 194]):

THEOREM 3.24 (Edwards [**Edw78**], Freedman [**Fre82**]). *For $n \geq 3$, a polyhedral homology n-manifold is a topological manifold if and only if the link of each of its vertices is simply connected.*

When $(D^1, S^0)^K$ is an n-manifold with $n = 1, 2$, $|K|$ is S^0 or an m-gon, thus is polytopal.

THEOREM 3.25 ([**Dav83**, Section 17, pp. 321–323]). *$(D^1, S^0)^K$ is homeomorphic to a smooth n-manifold, provided that $|K|$ is a* PS^{n-1}.

It remains to prove the sufficient conditions in other cases. For a k-simplex $\sigma \in K$ ($k \geq 0$), denote by $S'_+(\sigma)$ the subcomplex consisting of all sequences in K'_+, such that each simplex appearing in the sequence is a *proper face* of σ. That is to say, every simplex of maximal dimension in $S'_+(\sigma)$ has the form $\sigma' = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ with $\mathrm{card}(\sigma_i) = i$, $i = 0, 1, \ldots, k$. Hence $(\varphi(\sigma_i))_{i=0}^k$ is a path with $k+1$ nodes in the cube

$$C_\sigma = \{(x_i)_{i=1}^m \in [0,1]^m \mid x_j = 1 \text{ if } j \notin \sigma\},$$

such that the ending node has distance 1 from $\varphi(\sigma)$ (see Proposition 3.19, Lemma 3.20 and the proof for details). It follows that

$$u_{\varphi'}\left(\bigcup_{g \in G_{\{s_i\}_{i\in\sigma}}} g|S'_+(\sigma)|\right) = \bigcup_{g \in G_{\{s_i\}_{i\in\sigma}}} g\varphi'\left(|S'_+(\sigma)|\right) \tag{48}$$

is the PL k-sphere bounding the $(k+1)$-cube $C'_\sigma := \{(x_i)_{i=1}^m \in [-1,1]^m \mid x_i = 1 \text{ if } i \notin \sigma\}$ (s_i is the reflection changing the sign of the i-th coordinate, see (42)).

Recall that, for two polyhedra X and Y embedded in $\mathbb{R}^N$, if for all points $x \in X$, $y \in Y$ and $t \in [0,1]$, we have

$$tx + (1-t)y = t'x' + (1-t')y'$$

only when $t = t' = 0$ with $y = y'$, or $t = t' = 1$ with $x = x'$, then we say that their *external join exists*, which is defined by

$$X * Y = \{tx + (1-t)y \mid x \in X, y \in Y, t \in [0,1]\}.$$

If either X or Y is empty, say $Y = \emptyset$, then $X * Y$ always exists and is defined to be X. It can be checked that if $|K_1|$ and $|K_2|$ are embedded in $\mathbb{R}^N$ such that $|K_1| * |K_2|$ exists, then it is a geometric realization of the join $K_1 * K_2$, and in this case we write

$$|K_1| * |K_2| = |K_1 * K_2|.$$

For three polyhedra X, Y and Z such that $X * Y$ and $(X * Y) * Z$ exists, we have

$$X * Y = Y * X \quad \text{and} \quad (X * Y) * Z = X * (Y * Z) \tag{49}$$

up to PL homeomorphism (see [**Mun84**, Lemma 62.4, p. 371]).

LEMMA 3.26. *Suppose that* $|K|$ *is a* PLS^{n-1} *(resp.* GHS^{n-1}*) with* $n \geq 3$*, and let* σ *be a* k*-simplex of* K *with* $k \geq 0$*. Then*

$$N(\sigma) = \bigcup_{g \in G_{\{s_i\}_{i\in\sigma}}} g\varphi'\left(|S'_+(\sigma) * L'(\sigma)|\right) = \left(\bigcup_{g \in G_{\{s_i\}_{i\in\sigma}}} g\varphi'\left(|S'_+(\sigma)|\right)\right) * \varphi'(|L'(\sigma)|) \tag{50}$$

is a star of the point $\varphi'(|(\sigma)|) = \varphi(\sigma)$ *in* $(D^1, S^0)^K$*, in which the link of* $\varphi'(|(\sigma)|)$ *is a simply connected* PLS^{n-1} *(resp.* GHS^{n-1}*).*

PROOF. As a sequence of simplices of K (may start with $\emptyset$), each simplex in $S'_+(\sigma)$ ends with a proper face of σ, and each simplex in $L'(\sigma)$ starts with a simplex containing σ, the join $S'_+(\sigma) * L'(\sigma)$ is well-defined: the join of two simplices is the new sequence connecting them. With the same reason the external join $\varphi'\left(|S'_+(\sigma)|\right) * \varphi'\left(|L'(\sigma)|\right)$ exists. Note that each simplex in $\varphi'(|L'(\sigma)|)$ is linearly spanned by those points with i-th coordinate 0, for all $i \in \sigma$ (see Proposition 3.19), hence $\varphi'(|L'(\sigma)|)$ is fixed under $G_{\{s_i\}_{i\in\sigma}}$, by which (50) holds.

We have shown that

$$S^k_{PL} := \bigcup_{g \in G_{\{s_i\}_{i\in\sigma}}} g\varphi'\left(|S'_+(\sigma)|\right)$$

is a PL k-sphere in both cases (see (48)), hence by Lemma 3.15, it has the form

$$S^k_{PL} = \underbrace{|S^0| * |S^0| * \cdots * |S^0|}_{k+1},$$

up to PL homeomorphism. On the other hand, by (47), $\varphi'(|L'(\sigma)|)$ is a cone $\varphi(\sigma) * S^{n-k-2}_\sigma$ with S^{n-k-2}_σ a PLS^{n-k-2} (resp. GHS^{n-k-2}), then by (49) we have

$$\begin{aligned} N(\sigma) &= S^k_{PL} * \left(\varphi'(|(\sigma)|) * S^{n-k-2}_\sigma\right) = \left(S^k_{PL} * S^{n-k-2}_\sigma\right) * \varphi(\sigma) \\ &= \left(\underbrace{|S^0| * |S^0| * \cdots * |S^0|}_{k+1} * S^{n-k-2}_\sigma\right) * \varphi(\sigma). \end{aligned}$$

Therefore the statement follows from Lemma 3.15. The simple connectivity can be deduced from the van Kampen theorem, with the assumption $n \geq 3$. □

PROPOSITION 3.27. *If* $|K|$ *is a* PLS^{n-1} *(resp.* GHS^{n-1}*) with* $n \geq 3$*, then* $(D^1, S^0)^K$ *is a PL (resp. homology) manifold of dimension* n*. Moreover, it is a topological manifold, provided that* $|K|$ *is simply connected.*

PROOF. Let $(D^1, S^0)^K$ be triangulated by $u_{\varphi'} \colon \mathcal{U}(G, |K'_+|) \to (D^1, S^0)^K$. We need to check the links of all vertices. Since G acts on $(D^1, S^0)^K$ simplicially, it suffices to consider the vertices contained in the image of φ', each of the form $\varphi(\sigma)$ with $\sigma \in K$.

If the dimension of σ is non-negative, then by Lemma 3.26, $N(\sigma)$ contains the desired link, which is always simply connected. Otherwise suppose $\sigma = \emptyset$. By the definition of $u_{\varphi'}$, the link of $\varphi(\emptyset)$ in $(D^1, S^0)^K$ is $|K'|$, which is PL homeomorphic to $|K|$. Then the whole proof is completed by Edwards-Freedman Theorem 3.24. □

3.3. Cochains and the topological open book constructions

CONVENTION 3.28. In what follows, suppose that $|K|$ is a GHS^{n-1}, and the ground set of K is $[m]$.

- Let (X, X_+, X_0) be a triple of polyhedra in $\mathbb{R}^m$, such that

$$X = (D^1, S^0)^K,\ X_+ = X \bigcap \{(x_i)_{i=1}^m \in \mathbb{R}^m \mid x_1 \geq 0\},\ X_0 = X \bigcap \{(x_i)_{i=1}^m \in \mathbb{R}^m \mid x_1 = 0\}.$$

- Respectively, let (X', X'_+, X'_0) be a triple of polyhedra in $\mathbb{R}^{m+1}$, such that

$$X' = (D^1, S^0)^{Kv_1} \bigcap \{(x_i)_{i=1}^{m+1} \in \mathbb{R}^{m+1} \mid x_1^2 + x_2^2 \leq 1\},$$
$$X'_+ = X' \bigcap \{(x_i)_{i=1}^{m+1} \in \mathbb{R}^{m+1} \mid x_1 \geq 0\}, \quad X'_0 = X' \bigcap \{(x_i)_{i=1}^{m+1} \in \mathbb{R}^{m+1} \mid x_1 = 0\},$$

  in which Kv_1 is the simplicial wedge on the first vertex (see (40) for definition).

Observe that from the convention above, X is the union of X_+ with its image under the reflection in $\mathbb{R}^m$ changing the sign of the first coordinate, along their common part X_0. Moreover, X_0 is a disjoint union of 2^{m-k} copies of the polyhedral product $(D^1, S^0)^{\mathrm{Link}(\{1\},K)}$, here k is the number of vertices in Star $(\{1\}, K)$. For the same reason, X can be embedded in X'_+ as its boundary X'_0, since Link $(\{1\}, Kv_1)$ coincides with K, after a label-shifting (see (40), together with Figure 1 for an illustration).

REMARK 3.29. It is not difficult to show that (X_+, X_0) and (X'_+, X) are pairs of homology manifolds. if $|K|$ is a PLS, it follows from Lemma 3.15 that they are two pairs of PL manifolds. If $|K|$ is a PS, it can be shown that these pairs are homeomorphic to pairs of smooth manifolds (see [**GL13**]).

Notice that X_0 may not be a topological manifold even if X is: a link in a simply connected GHS can be non-simply connected (consider the suspension of a Poincaré homology sphere).

Note that X' is the polyhedral product associated to K, with the first pair (D^2, S^1) and the other pairs (D^1, S^0), which is homeomorphic to $(D^1, S^0)^{Kv_1}$. A key ingredient to consider X' is that it can be endowed with the S^1-action

$$\begin{array}{ccc} S^1 \times X' & \longrightarrow & X' \\ \left(e^{\sqrt{-1}\theta}, (x_1, x_2, x_3, \ldots, x_m)\right) & \longmapsto & (x_1 \cos\theta, x_2 \sin\theta, x_3, \ldots, x_m), \end{array} \tag{51}$$

such that the quotient space can be identified with X_+:

$$\begin{array}{ccc} X' & \longrightarrow & X_+ \\ (x_1, x_2, x_3, \ldots, x_m) & \longmapsto & (\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_m). \end{array}$$

Consider the projection

$$\begin{array}{ccc} X' & \longrightarrow & [0,1] \\ (x_1, x_2, x_3, \ldots, x_m) & \longmapsto & \sqrt{x_1^2 + x_2^2}; \end{array}$$

the two parts $p^{-1}([0, 1/2])$ and $p^{-1}([1/2, 1])$ decompose X' into the union

$$X_+ \times S^1 \bigcup \partial X_+ \times D^2 = \partial\left(X_+ \times D^2\right),$$

which is compatible with the S^1-action (51) (here $\partial X_+ = X_0$). Therefore, X' can be constructed as an *open book*, where X_+ is the *leaf* and X_0 the *binding*, such that the *holonomy* is trivial.

The remaining part of this section is devoted to understanding these open book constructions at the cochain level. Recall that the cohomology of $(D^1, S^0)^K$ is isomorphic to that of the differential graded algebra R_K^* (see Definition 2.1, Lemma 2.3); R_K^* can be decomposed as a direct sum $\bigoplus_{\omega\subset[m]} R_K^*|_\omega$, where $R_K^*|_\omega$ is closed under the differential d, and it is generated by those reduced monomials of the form $u^\sigma t^{\omega\setminus\sigma}$ ($\sigma \in K$) such that $\sigma \subset \omega$.

Definition 3.30. Let Kv_i be the simplicial wedge of K on the i-th vertex. Denote by $\varpi_i\colon R_K^* \to R_{Kv_i}^*$ the additive homomorphism such that for each reduced monomial $u^\sigma t^{\omega\setminus\sigma} \in R_K^p|_\omega$ with $\operatorname{card}(\sigma) = p$, we have

$$\varpi_i(u^\sigma t^{\omega\setminus\sigma}) = \begin{cases} u^{\varepsilon_i(\sigma)} t^{\varepsilon_i(\omega)\setminus\varepsilon_i(\sigma)} \in R_{Kv_i}^p|_{\varepsilon_i(\omega)} & \text{if } i \notin \omega, \\ u^{\varepsilon_i(\sigma)} t^{\varepsilon_i(\omega)\setminus\varepsilon_i(\sigma)} u^{i+1} \in R_{Kv_i}^{p+1}|_{\varepsilon_i(\omega)\cup\{i+1\}} & \text{otherwise,} \end{cases} \tag{52}$$

where $\varepsilon_i\colon 2^{[m]} \to 2^{[m+1]}$ is a label-shifting map sending each $\tau = \{j_k\}_{k=0}^l$ to $\varepsilon_i(\tau) = \{j'_k\}_{k=0}^l$, in which $j'_k = j_k$ if $j_k \le i$ and $j'_k = j_k + 1$ otherwise (thus the label $i+1$ is skipped in the image), and $\varepsilon_i(\emptyset) = \emptyset$.

Theorem 3.31. *The homomorphism $\varpi_i\colon R_K^* \to R_{Kv_i}^*$ preserves the differential* d *on both sides, inducing an additive isomorphism $H^*(R_K) \cong H^*(R_{Kv_i})$. More precisely, between*

$$H^*(R_K) = \bigoplus_{\omega\subset[m]} H^*(R_K|_\omega) \quad \textit{and} \quad H^*(R_{Kv_i}) = \bigoplus_{\omega'\subset[m+1]} H^*(R_{Kv_i}|_{\omega'}),$$

ϖ_i induces isomorphisms

$$H^p(R_K|_\omega) \cong H^{p+1}(R_{Kv_i}|_{\varepsilon_i(\omega)\cup\{i+1\}}) \tag{53}$$

if $i \in \omega$, and isomorphisms

$$H^p(R_K|_\omega) \cong H^p(R_{Kv_i}|_{\varepsilon_i(\omega)}) \tag{54}$$

otherwise, for all $p \ge 0$.

Proof. By their definitions, it is straightforward to check that for every $\omega \subset [m]$, the restriction of ϖ_i to $R_K^*|_\omega$, with its image in $R_{Kv_i}^*|_{\varepsilon_i(\omega)}$ ($i \notin \omega$) or $R_{Kv_i}^*|_{\varepsilon_i(\omega)\cup\{i+1\}}$ ($i \in \omega$), is a monomorphism preserving the differential d on both sides. Since when $i \notin \omega$, the full subcomplex $K|_\omega$ is simplicially isomorphic to $Kv_i|_{\varepsilon_i(\omega)}$, (54) follows from Theorem 2.5 (notice that $H^0(R_{Kv_i}) \cong H^0(R_{Kv_i}|_\emptyset)$).

It remains to prove (53). Suppose $i \in \omega$. By definition, we have $i \in \varepsilon_i(\omega)$ while $(i+1) \notin \varepsilon_i(\omega)$. For each $p \ge 0$, consider the exact sequence

$$0 \longrightarrow R_K^p|_\omega \xrightarrow{\varpi_i} R_{Kv_i}^{p+1}|_{\varepsilon_i(\omega)\cup\{i+1\}} \longrightarrow R_{Kv_i}^{p+1}|_{\varepsilon_i(\omega)\cup\{i+1\}} \Big/ R_K^p|_\omega u^{i+1} \longrightarrow 0,$$

in which $R_K^p|_\omega u^{i+1}$ is denoted as the image of $R_K^p|_\omega$ under ϖ_i. We claim that each relative cohomology group $H^{p+1}\left(R_{K(\{i\})}|_{\varepsilon_i(\omega)\cup\{i+1\}}, R_K|_\omega u^{i+1}\right)$ vanishes. Then (53) follows by using the long exact sequence. To prove the claim, consider those cochains in

$R^{p+1}_{K(\{i\})}|_{\varepsilon_i(\omega)\cup\{i+1\}}/R^p_K|_\omega u^{i+1}$: they are generated by monomials of the form

$$\left(u^\sigma t^{\varepsilon_i(\omega)\setminus(\sigma\cup\{i\})}u^i\right)t^{i+1} \tag{55}$$

with $\sigma\subset\varepsilon_i(\omega)\setminus\{i\}$ such that $\sigma\in K$ and $\mathrm{card}(\sigma)=p$, or by those of the form

$$\left(u^{\sigma'}t^{\varepsilon_i(\omega)\setminus\sigma'}\right)t^{i+1} \tag{56}$$

with $\sigma'\subset\varepsilon_i(\omega)\setminus\{i\}$ such that $\sigma'\in K$ and $\mathrm{card}(\sigma')=p+1$. We find that those terms appearing in the parentheses of (55) and (56) are in one-one correspondence with the simplicial $(p+1)$-cochains of the full subcomplex $Kv_i|_{\varepsilon_i(\omega)}$, i.e., $\mathrm{Star}\left(\{i\},Kv_i|_{\varepsilon_i(\omega)}\right)$ (see (40)), by sending each dual simplex σ^* to $u^\sigma t^{\varepsilon_i(\omega)\setminus\sigma}$. Thus we can extend it to be a cochain map

$$\widetilde{C}^p(Kv_i|_{\varepsilon_i(\omega)})\to R^{p+1}_{K(\{i\})}|_{\varepsilon_i(\omega)\cup\{i+1\}}/R^p_K|_\omega u^{i+1}$$

by multiplying t^{i+1} from the right, because

$$\mathrm{d}\left((\dots)t^{i+1}\right)=\left(\mathrm{d}(\dots)\right)t^{i+1}+(\dots)\mathrm{d}t^{i+1},$$

where the second summand vanishes in the quotient group (recall that $\mathrm{d}t^{i+1}=u^{i+1}$). Then the acyclicity of $\mathrm{Star}\left(\{i\},Kv_i|_{\varepsilon_i(\omega)}\right)$ implies that the claim holds. □

Let KJ be the construction given in Definition 3.3. By Lemma 3.13, KJ can be obtained by a sequence of consecutive simplicial wedge constructions, and here we specify the following one (the order is from left to right):

$$\begin{aligned}KJ=K\,&\underbrace{v_mv_{m+1}\cdots v_{m+n_m-2}}_{n_m-1}v_{m-1}v_m\cdots v_{m-1+n_{m-1}-2}\cdots\\&v_{m-i+1+n_{m-i+1}-2}\underbrace{v_{m-i}v_{m-i+1}\cdots v_{m-i+n_{m-i}-2}}_{n_{m-i}-1}\\&v_{m-i-1}\cdots v_{2+n_2-2}\underbrace{v_1v_2\cdots v_{1+n_1-2}}_{n_1-1},\end{aligned}\tag{57}$$

where if $n_i=1$, the block marked by n_i-1 shall be deleted.

Let R^*_{KJ} be the algebra associated to KJ. With respect to (57) we define compositions

$$\varpi_{n_k}:=\varpi_{k+n_k-2}\circ\cdots\circ\varpi_{k+1}\circ\varpi_k,$$

for $k=1,2,\dots,m$, and consider the cochain map

$$\varpi'_J:=\varpi_{n_1}\circ\varpi_{n_2}\circ\cdots\circ\varpi_{n_m}$$

sending each monomial $u^\sigma t^{\omega\setminus\sigma}\in R^*_K$ to $\varpi'_J(u^\sigma t^{\omega\setminus\sigma})\in R^*_{KJ}$. Suppose $\omega=\{i_1,i_2,\dots,i_l\}$ with $i_1<i_2<\ldots<i_l$ and

$$u^\sigma t^{\omega\setminus\sigma}=x^{i_1}x^{i_2}\dots x^{i_l}\quad(x^{i_j}=u^{i_j}\text{ or }t^{i_j}),$$

then by (52), when $k\geq i_l$, ϖ_{n_k} will not change the form of $u^\sigma t^{\omega\setminus\sigma}$ until $k=i_l$, i.e.,

$$\varpi_{n_{i_l}}\varpi_{n_{i_l+1}}\cdots\varpi_{n_m}(u^\sigma t^{\omega\setminus\sigma})=\varpi_{n_{i_l}}(u^\sigma t^{\omega\setminus\sigma})=x^{i_1}x^{i_2}\dots x^{i_l}u^{\widetilde{B}_{i_l}},$$

where

$$\widetilde{B}_k=\{k+1,k+2,\dots,k+n_k-1\},$$

which is empty when $n_k = 1$, $k = 1, 2, \dots, m$. As k decreases, next non-trivial term will not appear until $k = i_{l-1}$. That is to say,

(58)
$$\begin{aligned}\varpi_{n_{i_{l-1}}}\varpi_{n_{i_{l-1}+1}}\cdots\varpi_{n_m}(u^\sigma t^{\omega\setminus\sigma}) &= \varpi_{n_{l-1}}\varpi_{n_{i_l}}(u^\sigma t^{\omega\setminus\sigma})\\ &= x^{i_1}x^{i_2}\dots x^{i_{l-1}}x^{i'_l}u^{\widetilde{B}'_{i_l}}u^{\widetilde{B}_{i_{l-1}}}\\ &= (-1)^{(n_{i_{l-1}}-1)(n_{i_l}-1+\deg(x^{i_l}))}x^{i_1}x^{i_2}\dots x^{i_{l-1}}u^{\widetilde{B}_{i_{l-1}}}x^{i'_l}u^{\widetilde{B}'_{i_l}},\end{aligned}$$

where $i'_l = i_l + \sum_{j=i_{l-1}}^{i_l-1}(n_j-1)$ and

$$\widetilde{B}'_{i_l} = \{i_l + 1 + \sum_{j=i_{l-1}}^{i_l-1}(n_j-1), i_l + 2 + \sum_{j=i_{l-1}}^{i_l-1}(n_j-1), \dots, i_l + n_l - 1 + \sum_{j=i_{l-1}}^{i_l-1}(n_j-1)\},$$

due to the label-shifting. This calculation implies that we can set blocks

$$\widetilde{u}^i = t^i u^{\widetilde{B}_i} \quad\text{and}\quad \widetilde{v}^i = u^i u^{\widetilde{B}_i}$$

to simplify the image of ϖ'_J:

DEFINITION 3.32. Suppose that $J = (n_i)_{i=1}^m$ is a sequence of positive integers, and $(D^{n_i}, S^{n_i-1})_{i=1}^m$ are corresponding pairs of disks and spheres. Let R^*J be the differential graded algebra given as the quotient of the free $\mathbb{Z}$-algebra, generated by $2m$ generators $\widetilde{v}_i$ and $\widetilde{u}_i$, $i = 1, 2, \dots, m$, with $\deg(\widetilde{v}_i) = n_i$ and $\deg(\widetilde{u}_i) = n_i - 1$, respectively, subject to relations

$$x^i x^j = (-1)^{\deg(x^i)\deg(x^j)}x^j x^i, \quad (x^i)^2 = \begin{cases} x^i & \text{if } \deg(x^i) = 0,\\ 0 & \text{otherwise,}\end{cases} \tag{59}$$

where x^i is $\widetilde{u}^i$ or $\widetilde{v}^i$, for *distinct* $i, j = 1, 2, \dots, m$, together with

$$\widetilde{u}^i\widetilde{v}^i = 0, \quad \widetilde{v}^i\widetilde{u}^i = \begin{cases} v^i & \text{if } \deg(u^i) = 0,\\ 0 & \text{otherwise,}\end{cases}$$

for all $i = 1, 2, \dots, m$.

We say that a monomial in R^*J is *reduced* if it is written in the square-free form $\widetilde{v}^\sigma\widetilde{u}^\tau = x^1 \dots x^m$, where σ and τ are disjoint subsets of $[m]$, with $x^i = \widetilde{v}^i$ for $i \in \sigma$, $x^i = \widetilde{u}^i$ for $i \in \tau$, and $x^i = 1$ (the identity with degree zero) otherwise.

The differential d satisfies

$$\mathrm{d}(xy) = (\mathrm{d}x)y + (-1)^{\deg(x)}x(\mathrm{d}y) \tag{60}$$

for homogeneous elements $x, y \in R^*J$, with

$$\mathrm{d}\widetilde{v}^i = 0, \quad \mathrm{d}\widetilde{u}^i = \widetilde{v}^i \;\text{ and }\; \mathrm{d}1 = 0.$$

For a simplicial complex K with ground set $[m]$, the corresponding *Stanley-Reisner ideal* $\mathcal{I}_K J \subset R^*J$ is generated by all square-free monomials of the form $\widetilde{v}^\tau$, where τ is *not* a simplex of K.

The differential graded algebra $(R^*_K J, \mathrm{d})$ is defined to be the quotient $R^*J/\mathcal{I}_K J$ endowed with the differential d above. A reduced monomial in $R^*_K J$ is the (non-trivial) image of a reduced monomial in R^*J under the quotient homomorphism.

Note that R^*_{KJ} has $2\sum_{i=1}^m n_i$ generators while $R^*_K J$ has $2m$.

Definition 3.33. Let $J = (n_i)_{i=1}^m$ be a sequence of positive integers, with $N_i = \sum_{j=1}^i n_j$. Denote by $\varpi_J \colon R^*_K \to R^*_K J$ the homomorphism sending each reduced monomial $u^\sigma t^{\omega\setminus\sigma} = x^1 x^2 \dots x^m$ to

$$\varpi_J(u^\sigma t^{\omega\setminus\sigma}) = (-1)^{\nu'(\sigma,\omega)} \widetilde{v}^\sigma \widetilde{u}^{\omega\setminus\sigma} = (-1)^{\nu'(\sigma,\omega)} y^1 y^2 \dots y^m, \tag{61}$$

where $y^1 y^2 \dots y^m$ is the full form of the reduced monomial $\widetilde{v}^\sigma \widetilde{u}^{\omega\setminus\sigma}$, and as a mod 2 integer,

$$\nu'(\sigma,\omega) = \sum_{i=1}^m \big(\deg(y^i) - \deg(x^i)\big) \sum_{j>i} \deg(y^j) = \sum_{i\in\omega} (n_i - 1)\mathrm{card}\left(\{k \in \sigma \mid k > i\}\right) + c_\omega, \tag{62}$$

where

$$c_\omega = \sum_{i\in\omega} (n_i - 1) \sum_{\substack{j\in\omega \\ j>i}} (n_j - 1)$$

is a constant about ω.

Lemma 3.34. *Let $\varpi'_J \colon R^*_K \to R^*_{KJ}$ be the homomorphism by composing the sequences of $\varpi_i \colon R^*_K \to R^*_{Kv_i}$ with respect to (57). Then ϖ'_J coincides with ϖ_J, if we set*

$$u^i u^{\widetilde{B}_i} = \widetilde{v}^i \quad \textit{and} \quad t^i u^{\widetilde{B}_i} = \widetilde{u}^i.$$

Here $\widetilde{B}_i = \{i+1, i+2, \dots, i+n_i-1\}$ (which is empty when $n_i = 1$) $i = 1, 2, \dots, m$. Consequently, the homomorphism ϖ_J is a cochain map preserving the differential d *on both sides.*

Proof. This follows from the definitions and the calculation (58). Here we check the cochain map (61) directly. First note that from (62), we have

$$\nu'(\sigma \cup \{i\}, \omega) - \nu'(\sigma, \omega) = \sum_{k\in\omega, k<i} (n_k - 1)$$

for any $i \in \omega \setminus \sigma$. By taking d on both sides of (61), it turns out that (in what follows suppose $k \in \omega$)

$$\begin{aligned}
\varpi_J\left(\mathrm{d}(u^\sigma t^{\omega\setminus\sigma})\right) &= \sum_{\substack{i\in\omega\setminus\sigma \\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\mathrm{card}(\{k\in\sigma | k<i\})} \varpi_J\left(u^{\sigma\cup\{i\}} t^{\omega\setminus(\sigma\cup\{i\})}\right) \\
&= \sum_{\substack{i\in\omega\setminus\sigma \\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\mathrm{card}(\{k\in\sigma | k<i\}) + \nu'(\sigma\cup\{i\},\omega)} \widetilde{v}^{\sigma\cup\{i\}} \widetilde{u}^{\omega\setminus(\sigma\cup\{i\})} \\
&= \sum_{\substack{i\in\omega\setminus\sigma \\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\mathrm{card}(\{k\in\sigma | k<i\}) + \sum_{k<i}(n_k-1) + \nu'(\sigma,\omega)} \widetilde{v}^{\sigma\cup\{i\}} \widetilde{u}^{\omega\setminus(\sigma\cup\{i\})} \\
&= \sum_{\substack{i\in\omega\setminus\sigma \\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\sum_{k<i, k\in\sigma} n_k + \sum_{k<i, k\in\omega\setminus\sigma}(n_k-1) + \nu'(\sigma,\omega)} \widetilde{v}^{\sigma\cup\{i\}} \widetilde{u}^{\omega\setminus(\sigma\cup\{i\})}
\end{aligned}$$

$$= \mathrm{d}\left((-1)^{\nu'(\sigma,\omega)}\widetilde{v}^{\sigma}\widetilde{u}^{\omega\setminus\sigma}\right),$$

where the last equation holds by (60), with $\deg(\widetilde{v}^k) = n_k$ and $\deg(\widetilde{u}^k) = n_k - 1$. □

We shall end this section with the corollary below, which follows directly from Theorem 2.5, Lemma 3.4, Lemma 3.13, Theorem 3.31 and Lemma 3.34:

COROLLARY 3.35. *The cohomology of the polyhedral product* $((D^{n_i}, S^{n_i-1})_{i=1}^m)^K$ *is isomorphic to that of* $R_K^* J$*, with degrees and products preserved. Moreover, there is a degree-shifting cochain map*

$$\begin{aligned} \bigoplus\nolimits_{\omega\subset[m]} \widetilde{C}^*(K|_\omega) &\longrightarrow R_K^* J \\ \sigma^* &\longmapsto (-1)^{\nu'(\sigma,\omega)}\widetilde{v}^{\sigma}\widetilde{u}^{\omega\setminus\sigma}, \end{aligned}$$

in which $\sigma \in K|_\omega$ *and* $\nu'(\sigma,\omega)$ *is defined by* (62)*, such that it induces an additive isomorphism when passing to cohomology.*

APPENDIX A

Colimits and Chain equivalences

This part is devoted to the proofs of several properties we have used so far.

A standard description of the *colimit* that we have used in the definition of polyhedral products (see Definition 3.1) is given in Section A.1 (see [**May99**]).

Section A.2 is based on the related materials from [**Mun84**], in which we treat the chain complex $C_*\left((D^1, S^0)^K\right)$ defined in Section 1.1 as a colimit $\mathrm{colim} C_*|_{\mathfrak{A}}$ (see Diagram (71)), which is chain equivalent to the singular chain complex $S_*\left((D^1, S^0)^K\right)$ (see Proposition A.12).

Following [**Spa66**, Section 6, Chapter 5], Section A.3 is devoted to extending Whitney's formulae (16) and (17) (see Theorem A.20). Then we obtain the desired formulae for cup and cap products in a real moment-angle complex (see Chapter 2).

A.1. Colimits

Definition A.1. Let $F\colon \mathfrak{A} \to \mathfrak{C}$ be a functor between categories $\mathfrak{A}$ and $\mathfrak{C}$, where $\mathfrak{A}$ is small (i.e. $\mathfrak{A}$ has a set of objects). Any object C of $\mathfrak{C}$ determines the constant functor $\underline{C}$ that sends each object of $\mathfrak{A}$ to C, and sends each morphism of $\mathfrak{A}$ to the identity morphism of C.

Denoted by $\mathrm{colim} F$ the *colimit* of F, which is an object of $\mathfrak{C}$, together with a natural transformation $\tau\colon F \to \underline{\mathrm{colim} F}$, such that for any natural transformation $\bar{\tau}\colon F \to \underline{C}$ with C arbitrary, there is a unique morphism $\varphi_{\bar{\tau}}\colon \mathrm{colim} F \to C$ in $\mathfrak{C}$, with the property $\varphi_{\bar{\tau}} \circ \tau = \bar{\tau}$. That is to say, for each morphism $f\colon a \to a'$ in $\mathfrak{A}$, we have a commutative diagram

$$\begin{tikzcd}
F(a) \arrow[rr, "F(f)"] \arrow[dr, "\tau_a"] \arrow[ddr, "\bar{\tau}_a"'] & & F(a') \arrow[dl, "\tau_{a'}"'] \arrow[ddl, "\bar{\tau}_{a'}"] \\
& \mathrm{colim} F \arrow[d, "\varphi_{\bar{\tau}}"] & \\
& C. &
\end{tikzcd}$$

Notice that the uniqueness of $\mathrm{colim} F$ can be deduced from that of each morphism $\varphi_{\bar{\tau}}$.

For instance, let K be an abstract simplicial complex with ground set $[m]$ and $\mathfrak{T}$ be the category of topological spaces. We treat K as a category $\mathfrak{K}$, in which objects are simplices and morphisms are inclusions, then the *geometric realization functor*

$$G\colon \mathfrak{K} \longrightarrow \mathfrak{T}$$

sends each $\sigma = \{i_0, i_1, \ldots, i_p\}$ to the linear simplex spanned by $\{e_{i_0}, e_{i_1}, \ldots, e_{i_p}\} \subset \mathbb{R}^m$, where e_i is the i-th canonical basis element in $\mathbb{R}^m$. We call $\mathrm{colim} G$ the *geometric realization* of K and denote it by $|K|$.

A.2. Chain equivalences

DEFINITION A.2. Let $\mathfrak{C}$ be the category of chain complexes endowed with chain maps as morphisms. A chain complex refers to a graded Abelian group endowed with a boundary operator shifting the degrees by one,[1] and chain maps are homomorphisms preserving degrees and boundary operators. A *chain homotopy* between chain maps $f_1, f_2\colon (C_1, \partial_1) \to (C_2, \partial_2)$ is a homomorphism $D\colon C_1 \to C_2$ shifting the degree up by one, such that

$$f_2 - f_1 = D \circ \partial_1 + \partial_2 \circ D.$$

A *chain equivalence* between (C_1, ∂_1) and (C_2, ∂_2) refers to a chain map $f\colon C_1 \to C_2$ together with its chain-homotopy inverse $g\colon C_2 \to C_1$, such that there is a chain homotopy between $g \circ f$ and the identity morphism of C_1, together with a chain homotopy between $f \circ g$ and the identity morphism of C_2.

The dual of an object (resp. a morphism) in $\mathfrak{C}$ is the corresponding object (resp. morphism) with respect to the Hom-functor.

It is straightforward to check that if there is a chain homotopy between f_1 and f_2, then the dual of the chain homotopy gives a cochain homotopy between their duals f_1^* and f_2^*, respectively; moreover, f_1 and f_2 (resp. f_1^* and f_2^*) induces the same homomorphisms between corresponding homology groups (resp. cohomology groups).

Let $S_*\colon \mathfrak{T} \to \mathfrak{C}$ be the singular chain functor sending each space X to its singular chain complex $(S_*(X), \partial)$, in which the subgroup of p-chains shall be denoted as $S_p(X)$.

DEFINITION A.3. Let $X = \prod_{i=1}^m |K_i|$ be the product space of m compact simplicial complexes $|K_i|$, $i = 1, 2, \ldots, m$. Assume that X is endowed with a cell structure to be a CW complex, with each cell being a product of m simplices

$$|\sigma_1| \times |\sigma_2| \times \ldots \times |\sigma_m|, \quad \sigma_i \in K_i, \quad i = 1, 2, \cdots, m, \tag{63}$$

and suppose $A \subset X$ is a subcomplex with respect to this cell structure. Let $\mathfrak{A}$ be the subcategory of $\mathfrak{T}$ with objects as cells contained in A and morphisms as inclusions, and let $S_*|_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{C}$ be the restriction of S_* to $\mathfrak{A}$, sending each cell $e_A \in A$ to $S_*(e_A)$.

REMARK A.4. In what follows we shall implicitly use the fact that $\mathfrak{C}$ is *cocomplete*, namely colim$F$ *always exists as a suitable object* of $\mathfrak{C}$, for any functor F with $\mathfrak{C}$ as the target category, whose source category is small (see [**May99**]).

By Definition A.1 there is a unique chain map

$$\mathrm{i}_A\colon (\mathrm{colim} S_*|_{\mathfrak{A}}, \partial) \longrightarrow (S_*(A), \partial) \tag{64}$$

induced by chain inclusions $S_*(e_A) \to S_*(A)$ for each $e_A \in \mathfrak{A}$. Here the boundary operator of colim$S_*|_{\mathfrak{A}}$ is defined by composing each $\partial\colon S_*(e_A) \to S_*(e_A)$ and $\tau_{e_A}\colon S_*(e_A) \to \mathrm{colim} S_*|_{\mathfrak{A}}$, and then passing to colimit.

REMARK A.5. It turns out that colim$S_*|_{\mathfrak{A}}$ is generated by those singular simplices whose image lies in some cell $e_A \in A$. With the fact that in a CW complex, each cell is a

[1] if the boundary operator in a chain complex increases the degree by one, we shall call it a *cochain complex* with associated *coboundary operator*

deformation retract of its certain neighborhood, it can be proved that i_A in (64) induces a chain equivalence (see [**Mun84**, pp. 179–180]).

DEFINITION A.6. Let $\mathfrak{T}^m$ be the m-fold product category of topological spaces, whose morphisms are m-tuples of continuous maps, and let $S'_*\colon \mathfrak{T}^m \to \mathfrak{C}$ be the functor sending $\prod_{i=1}^m X_i$ to the chain complex $\bigotimes_{i=1}^m S_*(X_i)$, such that the boundary operator is generated by

$$\partial_{S'}(\sigma_1^s \otimes \ldots \otimes \sigma_m^s) = \sum_{i=1}^m (-1)^{\sum_{j<i} \deg(\sigma_j^s)} \sigma_1^s \otimes \ldots \otimes \partial\sigma_i^s \otimes \ldots \otimes \sigma_m^s, \tag{65}$$

in which $\sigma_i^s \in S_*(X_i)$ is a singular simplex, $i = 1, 2, \ldots, m$. Here the subgroup of p-chains is given by

$$S'_p(X) := \bigoplus_{\sum_{i=1}^m p_i = p} \bigotimes_{i=1}^m S_{p_i}(X_i).$$

To proceed, we need the following fact:

THEOREM A.7 (Eilenberg-Zilber [**EZ53**]). *There is a natural transformation T between S_* and S'_*, as functors from $\mathfrak{T}^m$ to $\mathfrak{C}$, inducing a degree-preserving chain equivalence between them. Moreover, T is unique up to natural chain homotopy.*

That is to say, for such a natural transformation T there is a natural transformation $T^{-1}\colon S'_* \to S_*$, together with natural transformations $D_S\colon S_* \to S_*$ and $D_{S'}\colon S'_* \to S'_*$ shifting the degree up by one, such that

$$T^{-1} \circ T - \mathrm{id}_{S_*} = \partial \circ D_S + D_S \circ \partial \tag{66}$$

and

$$T \circ T^{-1} - \mathrm{id}_{S'_*} = \partial_{S'} \circ D_{S'} + D_{S'} \circ \partial_{S'}, \tag{67}$$

where id_{S_*} (resp. $\mathrm{id}_{S'_*}$) is the identity of S_* (resp. S'_*). Moreover, if we have two such natural transformations T and T' from S_* to S'_*, then there is a natural transformation $D'\colon S_* \to S'_*$, such that

$$T' - T = \partial_{S'} \circ D' + D' \circ \partial.$$

The following construction provides an explicit natural transformation in Eilenberg-Zilber theorem A.7:

DEFINITION A.8. Let $X = \prod_{i=1}^m X_i$ be an object in $\mathfrak{T}^m$. Suppose that for each non-negative integer p, $\Delta^p \subset \mathbb{R}^p$ is the simplex spanned by vertices $e_0 = O$ (the origin in $\mathbb{R}^p$), e_1, e_2, …, e_p, with labels 0, 1, …, p, respectively, where e_i is the i-th canonical basis vector in $\mathbb{R}^p$. We treat each singular p-simplex in X as a continuous map $\Delta^p \to X$. The *Alexander-Whitney chain map* $T_X^{AW}\colon S_*(X) \to S'_*(X)$ is defined by sending each singular simplex $f(\Delta^p)$ to the sum

(68)

$$T_X^{AW}(f)(\Delta^p) = \sum_{0=i_0 \le i_1 \le \ldots \le i_{m-1} \le i_m = p} f_1(\Delta^p|_{[i_0,i_1]}) \otimes f_2(\Delta^p|_{[i_1,i_2]}) \otimes \ldots \otimes f_m(\Delta^p|_{[i_{m-1},i_m]}),$$

where $f = (f_1, f_2, \dots, f_m)$, such that $f_{j+1}(\Delta^p|_{[i_j,i_{j+1}]})\colon \Delta^{i_{j+1}-i_j} \to X_{j+1}$ is defined by composing $f_{j+1}\colon \Delta^p \to X_{j+1}$, and $\Delta^{i_{j+1}-i_j} \to \Delta^p$ that is linearly spanned by mapping vertices of $\Delta^{i_{j+1}-i_j}$ to those of Δ^p labeled from i_j to i_{j+1}, respectively, with the order preserved, $j = 0, 1, \dots, m-1$.

It can be checked directly that T_X^{AW} is a desired chain map for each given $X \in \mathfrak{T}^m$, which is natural with respect to morphisms in $\mathfrak{T}^m$.

Theorem A.9. *Let T be the natural transformation in the Eilenberg-Zilber theorem and let (X, A) be the CW pair given in Definition A.3. Denote by $S'_*|_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{C}$ the restriction of S'_* to $\mathfrak{A}$, then we have the commutative diagram*

$$\begin{array}{ccc} (S_*(X), \partial) & \xrightarrow{T_X} & (\bigotimes_{i=1}^m S_*(|K_i|), \partial_{S'}) \\ \mathrm{j}_A \uparrow & & \uparrow \mathrm{j}'_A \\ (\mathrm{colim} S_*|_{\mathfrak{A}}, \partial) & \xrightarrow{T_A} & (\mathrm{colim} S'_*|_{\mathfrak{A}}, \partial_{S'}), \end{array} \tag{69}$$

in which T_A induces a chain equivalence.

Proof. By Definition A.1, j_A and j'_A are the unique morphisms defined via each inclusion $e_A \to X$, with e_A running through cells in A, and T_A can be defined by passing compositions $T_{e_A}\colon S_*(e_A) \to S'_*(e_A)$ and $\tau'_{e_A}\colon S'_*(e_A) \to \mathrm{colim} S'_*|_{\mathfrak{A}}$ to colimit. Boundary operator $\partial_{S'}$ on $\mathrm{colim} S'_*|_{\mathfrak{A}}$ can also be given in this way.

Likewise, $T_A^{-1}\colon \mathrm{colim} S'_*|_{\mathfrak{A}} \to \mathrm{colim} S|_{\mathfrak{A}}$ is defined by passing the compositions of chain-homotopy inverse $T^{-1}_{S'_*(e_A)}\colon S'_*(e_A) \to S_*(e_A)$ and $\tau_{e_A}\colon S_*(e_A) \to \mathrm{colim} S_*|_{\mathfrak{A}}$ to colimit. At last, natural transformations D_S and $D_{S'}$ in (66) and (67) can be extended over colimits $\mathrm{colim} S_*|_{\mathfrak{A}}$ and $\mathrm{colim} S'_*|_A$, respectively, due to their definitions on each piece $S_*(e_A)$ and $S'_*(e_A)$, and the naturality. Then the statement follows. ☐

Next we turn to the relation between the simplicial and singular chain complexes. The following theorem is fundamental (see, for example, [**Mun84**]):

Theorem A.10. *Let K be an abstract simplicial complex with the simplicial chain complex $(C_*(K), \partial')$. With a partial ordering of the vertices of K that induces a total ordering on each simplex, the chain map*

$$\begin{array}{rcl} \varsigma\colon C_*(K) & \longrightarrow & S_*(|K|) \\ \sigma = [v_{i_0}, v_{i_1}, \dots, v_{i_p}] & \longmapsto & l(v_{i_0}, v_{i_1}, \dots, v_{i_p}), \end{array} \tag{70}$$

induces a chain equivalence, such that its chain-homotopy inverse ς^{-1} satisfies

$$\varsigma^{-1}(l(v_{i_0}, v_{i_1}, \dots, v_{i_p})) = [v_{i_0}, v_{i_1}, \dots, v_{i_p}].$$

Here $v_{i_0} < v_{i_1} < \dots < v_{i_p}$ is in the given ordering and $l(v_{i_0}, v_{i_1}, \dots, v_{i_p})\colon \Delta^p \to |\sigma|$ is linearly spanned by corresponding vertices.

Let $(\bigotimes_{i=1}^m C_*(K_i), \partial_C)$ be the tensor product of simplicial chain complexes $C_*(K_i)$, with each K_i given an ordering a priori. Suppose ∂_C follows (65) with $\partial_{S'}$ replaced by ∂_C and singular simplices replaced by simplicial ones, respectively. A well-known construction implies that the chain equivalence (70) can be extended over tensor products, as in the following lemma.

LEMMA A.11. *The chain map*

$$\begin{aligned}\widetilde{\varsigma}\colon\ (\textstyle\bigotimes_{i=1}^{m} C_*(K_i), \partial_C) &\longrightarrow (\textstyle\bigotimes_{i=1}^{m} S_*(|K_i|), \partial_{S'})\\ \sigma_1\otimes\ldots\otimes\sigma_m &\longmapsto \varsigma_1(\sigma_1)\otimes\ldots\otimes\varsigma_m(\sigma_m)\end{aligned}$$

induces a chain equivalence, where $\varsigma_i\colon C_*(K_i)\to S_*(|K_i|)$ *is defined by* (70) *for* $i=1,2,\ldots,m$. *Moreover,* $\widetilde{\varsigma}$ *is natural with respect to* m*-tuples of simplicial maps.*

PROOF. Clearly $\widetilde{\varsigma}$ is a well-defined chain map. It suffices to show that $\widetilde{\varsigma}^{-1}=\varsigma_1^{-1}\otimes\varsigma_2^{-1}\otimes\ldots\otimes\varsigma_m^{-1}$ is the chain-homotopy inverse of $\widetilde{\varsigma}$, where ς_i^{-1} is the chain-homotopy inverse of ς_i with chain homotopies $D_i\colon C_*(K_i)\to C_*(K_i)$, such that

$$\varsigma^{-1}\circ\varsigma-\mathrm{id}_i=\partial'\circ D_i+D_i\circ\partial'$$

with id_i the identity of $C_*(K_i)$, $i=1,2,\ldots,m$. With a direct calculation, it turns out that $D_C\colon\ \bigotimes_{i=1}^{m} C_*(K_i)\to\bigotimes_{i=1}^{m} C_*(K_i)$ defined by

$$\begin{aligned}D_C(\sigma_1\otimes\cdots\otimes\sigma_m)=\sum_{i=1}^{m}&(-1)^{\sum_{j<i}\deg(\sigma_j)}\varsigma_1^{-1}\circ\varsigma_1(\sigma_1)\otimes\varsigma_2^{-1}\circ\varsigma_2(\sigma_2)\otimes\cdots\\ &\otimes\varsigma_{i-1}^{-1}\circ\varsigma_{i-1}(\sigma_{i-1})\otimes D_i\sigma_i\otimes\sigma_{i+1}\otimes\cdots\otimes\sigma_m\end{aligned}$$

satisfies

$$\widetilde{\varsigma}^{-1}\circ\widetilde{\varsigma}-\mathrm{id}_{C_*}=\partial_C\circ D_C+D_C\circ\partial_C,$$

in which id_{C_*} is the identity of $(\bigotimes_{i=1}^{m} C_*(K_i),\partial_C)$. Similarly, we can construct a chain homotopy for $(\bigotimes_{i=1}^{m} S_*(|K_i|),\partial_{S'})$, thus the first statement follows. By the construction above, the second statement follows from the naturality of each ς_i, $i=1,2,\ldots,m$. □

Now let (X,A) be the CW pair given in Definition A.3, and let $C_*|_{\mathfrak{A}}\colon\ \mathfrak{A}\to\mathfrak{C}$ be the functor sending each object $e_A=\prod_{i=1}^{m}|\sigma_i|$ to $\bigotimes_{i=1}^{m} C_*(\sigma_i)$. Since morphisms (i.e. inclusions) in $\mathfrak{A}$ are simplicial, $\mathrm{colim} C_*|_{\mathfrak{A}}$ is well-defined. Moreover, from each chain inclusion $\bigotimes_{i=1}^{m} C_*(\sigma_i)\to\bigotimes_{i=1}^{m} C_*(K_i)$ with $e_A=\prod_{i=1}^{m}|\sigma_i|$ running through all cells of A, we have a unique chain map $\mathrm{j}''_A\colon\ \mathrm{colim} C_*|_{\mathfrak{A}}\to\bigotimes_{i=1}^{m} C_*(K_i)$.

With a proof similar to that of Theorem A.9, it can be shown that diagram

$$\begin{array}{ccc}(\bigotimes_{i=1}^{m} C_*(K_i),\partial_C) & \xrightarrow{\widetilde{\varsigma}} & (\bigotimes_{i=1}^{m} S_*(|K_i|),\partial_{S'})\\ {\scriptstyle \mathrm{j}''_A}\big\uparrow & & {\scriptstyle \mathrm{j}'_A}\big\uparrow\\ (\mathrm{colim} C_*|_{\mathfrak{A}},\partial_C) & \xrightarrow{\widetilde{\varsigma}_A} & (\mathrm{colim} S'_*|_{\mathfrak{A}},\partial_{S'})\end{array}\tag{71}$$

commutes, in which $\widetilde{\varsigma}_A$ induces a chain equivalence.

Here we make a conclusion to end this section.

PROPOSITION A.12. *Let* (X,A) *be the* CW *pair given in Definition A.3. Then* $\mathrm{colim} C_*|_{\mathfrak{A}}$ *is generated by chains of the form* $\sigma_1\otimes\ldots\otimes\sigma_m$*, where* $\prod_{i=1}^{m}|\sigma_i|$ *is a cell of* A*, and the boundary operator* ∂_C *satisfies*

$$\partial_C(\sigma_1\otimes\ldots\otimes\sigma_m)=\sum_{i=1}^{m}(-1)^{\sum_{j<i}\deg(\sigma_j)}\sigma_1\otimes\ldots\otimes\sigma_{i-1}\otimes\partial'\sigma_i\otimes\sigma_{i+1}\otimes\ldots\otimes\sigma_m$$

on each generator. Then the sequence

$$S_*(A) \xrightarrow{(\mathrm{i}_A)^{-1}} \mathrm{colim} S_*|_{\mathfrak{A}} \xrightarrow{T_A} \mathrm{colim} S'_*|_{\mathfrak{A}} \xrightarrow{(\widetilde{\varsigma}_A)^{-1}} \mathrm{colim} C_*|_{\mathfrak{A}} \tag{72}$$

induces a chain equivalence between $(S_*(A), \partial)$ *and* $(\mathrm{colim} C_*|_{\mathfrak{A}}, \partial_C)$*, yielding an isomorphism* $H_*(A) \cong H_*(\mathrm{colim} C_*|_{\mathfrak{A}}, \partial_C)$*.*

Moreover, as the dual of (72), the sequence

$$S^*(A) \xleftarrow{(\mathrm{i}^*_A)^{-1}} \mathrm{Hom}(\mathrm{colim} S_*|_{\mathfrak{A}}, \mathbb{Z}) \xleftarrow{(T_A)^*} \mathrm{Hom}(\mathrm{colim} S'_*|_{\mathfrak{A}}, \mathbb{Z}) \xleftarrow{(\widetilde{\varsigma}^*_A)^{-1}} \mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z})$$

induces a cochain equivalence between $(S^*(A), \delta)$ *and* $(\mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}), \delta_C)$*, where* δ *and* δ_C *are the duals of* ∂ *and* ∂_C*, respectively, and hence we have the isomorphism* $H^*(A) \cong H_*(\mathrm{colim} C_*|_{\mathfrak{A}}, \delta_C)$*.*

A.3. On cup and cap products

Let $S^*\colon \mathfrak{T} \to \mathfrak{C}$ be the functor of singular cochains, sending each space X to $S^*(X) = \mathrm{Hom}(S_*(X), \mathbb{Z})$. Let $\theta_2\colon S^*(X) \bigotimes S^*(X) \to \mathrm{Hom}(S_*(X) \bigotimes S_*(X), \mathbb{Z})$ be the cochain map given by

$$\theta_2(c^{p_1} \otimes c^{p_2})(c_{q_1} \otimes c_{q_2}) = c^{p_1}(c_{q_1})c^{p_2}(c_{q_2}), \tag{73}$$

for each $c^{p_1} \otimes c^{p_2} \in S^{p_1}(X) \bigotimes S^{p_2}(X)$ and $c_{q_1} \otimes c_{q_2} \in S_{q_1}(X) \bigotimes S_{q_2}(X)$, respectively. Note that $c^{p_1}(c_{q_1})c^{p_2}(c_{q_2})$ vanishes if either $p_1 \neq q_1$ or $p_2 \neq q_2$.

Consider the sequence

$$S_*(X) \xrightarrow{d_*} S_*(X \times X) \xrightarrow{T_{X\times X}} S_*(X) \bigotimes S_*(X), \tag{74}$$

in which d_* is induced by the diagonal map $d\colon X \to X \times X$, and T is the natural transformation in the Eilenberg-Zilber theorem A.7, with $m = 2$. To define the cap product, we need a homomorphism

$$\begin{aligned} h\colon S^*(X) \bigotimes S_*(X) \bigotimes S_*(X) &\longrightarrow S_*(X) \\ (c^p, c_{r_1} \otimes c_{r_2}) &\longmapsto c^p(c_{r_2})c_{r_1}. \end{aligned} \tag{75}$$

With a straightforward calculation, it can be checked that h satisfies the following property: for $c^p \in S^p(X)$ and $c_r \in S'_r(X \times X) = \bigoplus_{r_1+r_2=r} S_{r_1}(X) \otimes S_{r_2}(X)$ (see Definition A.6),

$$\partial \circ h(c^p, c_r) = (-1)^{r-p} h(\delta c^p, c_r) + h(c^p, \partial_{S'} c_r). \tag{76}$$

Definition A.13. The cup product $\smile\colon S^*(X) \bigotimes S^*(X) \to S^*(X)$ is a homomorphism defined as the composition $d^* \circ T^*_{X\times X} \circ \theta_2$, where d^* and $T^*_{X\times X}$ are the dual maps of d_* and $T_{X\times X}$, respectively (see (74)).

The cap product $\frown\colon S^*(X) \bigotimes S_*(X) \to S_*(X)$ is a homomorphism defined as

$$c^p \frown c_r := h(c^p, T_{X\times X} \circ d_*(c_r)).$$

Immediately from the definition above, formula (76) can be interpreted as

$$\partial(c^p \frown c_r) = (-1)^{r-p} \delta c^p \frown c_r + c^p \frown \partial c_r. \tag{77}$$

Notice that we have already defined the simplicial cup and cap products by (18) and (26), respectively. The following (well-known) lemma shows that how they are related with Definition A.13:

Lemma A.14. *Suppose that K is a simplicial complex with a partial ordering on vertices, such that it gives a total ordering on each simplex. Let $T^{AW}_{|K|\times|K|}\colon S_*(|K|\times|K|)\to S_*(|K|)\bigotimes S_*(|K|)$ be the Alexander-Whitney map with $m=2$ (see (68)), then the bottom rows of the commutative diagrams*

$$\begin{array}{ccc} S^*(|K|)\bigotimes S^*(|K|) & \xrightarrow{\smile} & S^*(|K|) \\ \Big\downarrow{\scriptstyle \varsigma^*\otimes\varsigma^*} & & \Big\downarrow{\scriptstyle \varsigma^*} \\ C^*(K)\bigotimes C^*(K) & \longrightarrow & C^*(K) \end{array}$$

and

$$\begin{array}{ccc} S^*(|K|)\bigotimes S_*(|K|) & \xrightarrow{\frown} & S_*(|K|) \\ \Big\downarrow{\scriptstyle \varsigma^*\otimes\varsigma^{-1}} & & \Big\downarrow{\scriptstyle \varsigma^{-1}} \\ C^*(K)\bigotimes C_*(K) & \longrightarrow & C_*(K) \end{array}$$

give the simplicial cup and cap products, respectively, where ς induces the chain equivalence between the simplicial and singular chain complexes (see (70)), and ς^* is the dual of ς.

Proof. Without loss of generality, we may assume that $[v_0,v_1,\ldots,v_{p+q}]$ is an oriented simplex in $C_{p+q}(K)$ (i.e., $v_0<v_1<\ldots<v_{p+q}$ is in the given ordering) with $p+q=r$. Choose $c^p\in S^p(|K|)$ and $c^q\in S^q(|K|)$, respectively, then a direct calculation shows that

$$\begin{aligned} & \varsigma^*(c^p)\smile\varsigma^*(c^q)([v_0,v_1,\ldots,v_{p+q}]) \\ =& c^p\smile c^q(\varsigma([v_0,v_1,\ldots,v_{p+q}])) \\ =& c^p\smile c^q(l(v_0,v_1,\ldots,v_{p+q})) \\ =& \theta_2(c^p\otimes c^q)\left(T^{AW}_{|K|\times|K|}\circ d_*\left(l(v_0,v_1,\ldots,v_{p+q})\right)\right) \\ =& \theta_2(c^p\otimes c^q)\left\{\sum_{i=0}^{p+q} l(v_0,v_1,\ldots,v_i)\otimes l(v_i,v_{i+1},\ldots,v_{p+q})\right\} \qquad \text{due to (68)} \\ =& c^p\left(l(v_0,v_1,\ldots,v_p)\right)c^q\left(l(v_p,v_{p+1},\ldots,v_{p+q})\right), \end{aligned}$$

yielding (18) (see Theorem A.10). As for the cap product, similarly we have

$$\begin{aligned} & \varsigma^*(c^p)\frown[v_0,v_1,\ldots,v_{p+q}] \\ =& \varsigma^{-1}\Big(c^p\frown\varsigma([v_0,v_1,\ldots,v_{p+q}])\Big) \\ =& \varsigma^{-1}\Big(c^p\frown l(v_0,v_1,\ldots,v_{p+q})\Big) \\ =& \varsigma^{-1}\Big(h\left(c^p,T^{AW}_{|K|\times|K|}\circ d_*\left(l(v_0,v_1,\ldots,v_{p+q})\right)\right)\Big) \\ =& \varsigma^{-1}\Big(\sum_{i=0}^{p+q} h\left(c^p,l(v_0,v_1,\ldots,v_i)\otimes l(v_i,v_{i+1},\ldots,v_{p+q})\right)\Big) \\ =& \varsigma^{-1}\Big(c^p\left(l(v_q,v_{q+1},\ldots,v_{q+p})\right)l(v_0,v_1,\ldots,v_q)\Big) \qquad \text{by Theorem A.10} \end{aligned}$$

$$=c^p\left(l(v_q, v_{q+1}, \ldots, v_{q+p})\right)[v_0, v_1, \cdots, v_q], \tag{78}$$

from which (26) follows. □

CONVENTION A.15. In what follows, we shall use the notation $T^m\colon\ S_* \to S'_*$ to denote the natural transformation in the Eilenberg-Zilber theorem A.7. In particular, we need T^2 to define the cup and cap products (see Definition A.13).

Now we consider functors from $\mathfrak{T}^m$ to $\mathfrak{C}$. Let $X = \prod_{i=1}^m X_i$ be an object in $\mathfrak{T}^m$ with diagonal maps $d_i\colon\ X_i \to X_i \times X_i$, $i = 1, 2, \ldots, m$, then we have a diagram

$$\begin{tikzcd} S_*(X) \arrow[r, "d_*"] \arrow[ddd, "T^m_X"] & S_*(X \times X) \arrow[r, "T^2_{X\times X}"] & S_*(X) \bigotimes S_*(X) \arrow[dl, "T^m_X \otimes T^m_X"'] \\ & \bigotimes_{i=1}^m S_*(X_i) \bigotimes \bigotimes_{i=1}^m S_*(X_i) & \\ & & \\ \bigotimes_{i=1}^m S_*(X_i) \arrow[rr, "\otimes_{i=1}^m T^2_{X_i\times X_i} \circ (d_i)_*"] & & \bigotimes_{i=1}^m (S_*(X_i) \bigotimes S_*(X_i)), \arrow[uul, "\Omega"'] \end{tikzcd} \tag{79}$$

in which the homomorphism Ω is defined as follows: let $\otimes_{i=1}^m c_{p_i} \otimes c_{q_i}$ be a chain from $\bigotimes_{i=1}^m S_{p_i}(X_i) \bigotimes S_{q_i}(X_i)$, then

$$\Omega\left(\otimes_{i=1}^m c_{p_i} \otimes c_{q_i}\right) = (-1)^\kappa \left(\otimes_{i=1}^m c_{p_i}\right) \otimes \left(\otimes_{i=1}^m c_{q_i}\right), \quad \kappa = \sum_{i=1}^m q_i \sum_{j>i} p_j, \tag{80}$$

namely the mod 2 integer κ is generated by the rule that, every time an interchange of the positions of two homogenous elements gives rise to a summand of κ, which is the product of their degrees.

PROPOSITION A.16. *As homomorphisms from $S_*(X)$ to $\bigotimes_{i=1}^m S_*(X_i) \bigotimes \bigotimes_{i=1}^m S_*(X_i)$,*

$$\Omega \circ (\otimes_{i=1}^m T^2_{X_i\times X_i} \circ (d_i)_*) \circ T^m_X \quad \textit{and} \quad (T^m_X \otimes T^m_X) \circ T^2_{X\times X} \circ d_*$$

are both chain maps, as natural transformations between functors from $\mathfrak{T}^m$ to $\mathfrak{C}$. Moreover, the two chain maps above are differed by a natural chain homotopy, i.e. a natural transformation between the two functors above, inducing a chain homotopy when the space is specified.

As a conclusion, Diagram (79) commutes, up to natural chain homotopy.

PROOF. The first statement, i.e. Ω is a chain map, can be checked by a direct calculation. The second one follows by the Acyclic Model Theorem (see [**EM53**]), with models as $2m$-product of simplices (see [**Spa66**, Section 6, Chapter 5, p. 252] for the details of the case $m = 2$, and the proof for the general case is similar). □

Now we define a canonical cochain map (sometimes called an evaluation map) analogous to (73), namely

$$\theta\colon\ \bigotimes_{i=1}^m S^*(X_i) \to \mathrm{Hom}(\bigotimes_{i=1}^m S_*(X_i), \mathbb{Z})$$

such that

$$\theta(\otimes_{i=1}^m c^{p_i})(\otimes_{i=1}^m c_{r_i}) = \prod_{i=1}^m c^{p_i}(c_{r_i})$$

for $\otimes_{i=1}^m c^{p_i} \in \bigotimes_{i=1}^m S^{p_i}(X_i)$ and $\otimes_{i=1}^m c_{r_i} \in \bigotimes_{i=1}^m S_{r_i}(X_i)$. Observe that in the bottom row of the diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(S_*(X),\mathbb{Z}) & \bigotimes & S_*(X) & \bigotimes & S_*(X) & \xrightarrow{h} & S_*(X) \\ {\scriptstyle (T_X^m)^*}\big\uparrow & & {\scriptstyle T_X^m}\big\downarrow & & {\scriptstyle T_X^m}\big\downarrow & & {\scriptstyle T_X^m}\big\downarrow \\ \mathrm{Hom}(S'_*(X),\mathbb{Z}) & \bigotimes & S'_*(X) & \bigotimes & S'_*(X) & \xrightarrow{h} & S'_*(X), \end{array}$$

we can define the homomorphism h by replacing $S_*(X)$ and $S^*(X)$ with $S'_*(X)$ and $\mathrm{Hom}(S'_*(X),\mathbb{Z})$ in (75), respectively, such that the diagram above is commutative in the following sense:

LEMMA A.17. *Assume that $c^p \in \bigotimes_{i=1}^m S^*(X_i)$ is a p-cochain and $c_{r_1}, c_{r_2} \in S_*(X)$ are two chains of degrees r_1 and r_2, respectively, we have*

$$T_X^m \circ h((T_X^m)^* \circ \theta(c^p), c_{r_1} \otimes c_{r_2}) = h(\theta(c^p), T_X^m(c_{r_1}) \otimes T_X^m(c_{r_2})). \tag{81}$$

Henceforth in this section, let (X, A) be the CW pair given by Definition A.3. and occasionally we shall identify $S^*(A)$ with $\mathrm{Hom}(\mathrm{colim}S_*|_{\mathfrak{A}},\mathbb{Z})$, as discussed in Remark A.5. Recall that in Theorem A.9, we have defined the unique chain map $T_A^m\colon \mathrm{colim}S_*|_{\mathfrak{A}} \to \mathrm{colim}S'_*|_{\mathfrak{A}}$, by passing sequences

$$S_*(e_A) \xrightarrow{T_{e_A}} S'_*(e_A) \xrightarrow{\tau'_{e_A}} \mathrm{colim}S'_*|_{\mathfrak{A}},$$

to colimit, $e_A \in A$. Let $\mathrm{j}_A\colon \mathrm{colim}S_*|_{\mathfrak{U}} \to S_*(X)$ and $\mathrm{j}'_A\colon \mathrm{colim}S'_*|_{\mathfrak{U}} \to S'_*(X)$ be the chain maps induced by inclusions $e_A \to X$, with their duals j_A^* and $(\mathrm{j}'_A)^*$, respectively.

LEMMA A.18. *The diagram*

$$\begin{array}{ccc} \mathrm{colim}S_*|_{\mathfrak{A}} & \xrightarrow{T^2_{A\times A}\circ d_*} & \mathrm{colim}S_*|_{\mathfrak{A}} \bigotimes \mathrm{colim}S_*|_{\mathfrak{A}} \\ {\scriptstyle T_A^m}\big\downarrow & & {\scriptstyle T_A^m\otimes T_A^m}\big\downarrow \\ \mathrm{colim}S'_*|_{\mathfrak{A}} & \xrightarrow{\Omega\circ(\otimes_{i=1}^m T^2_{|K_i|\times|K_i|}\circ(d_i)_*)} & \mathrm{colim}S'_*|_{\mathfrak{A}} \bigotimes \mathrm{colim}S'_*|_{\mathfrak{A}} \end{array} \tag{82}$$

commutes, up to chain homotopy.

PROOF. By Proposition A.16, the statement holds if A is an m-fold product of simplices.

Suppose that in the top row of Diagram (79), X is replaced by e_A. With e_A running through all cells of A, we see that the composition $T^2_{A\times A} \circ d_*$ is well-defined by passing $T^2_{e_A\times e_A} \circ d_*$ to colimit. Similarly, $\Omega \circ (\otimes_{i=1}^m T^2_{|K_i|\times|K_i|} \circ (d_i)_*)$ can be defined through each $S'_*(e_A)$. The chain homotopy $\mathrm{colim}S_*|_{\mathfrak{A}} \to \mathrm{colim}S'_*|_{\mathfrak{A}} \bigotimes \mathrm{colim}S'_*|_{\mathfrak{A}}$ can also be obtained from all pieces $e_A \in A$. ☐

PROPOSITION A.19. *Assume that $c^p = \otimes_{i=1}^m c^{p_i} \in \bigotimes_{i=1}^m S^{p_i}(|K_i|)$ and $c^q = \otimes_{i=1}^m c^{q_i} \in \bigotimes_{i=1}^m S^{q_i}(|K_i|)$ are two cochains of degree p and q, respectively, and $c_r = \otimes_{i=1}^m c_{r_i}$ is an r-chain from $\mathrm{colim}S'_*|_{\mathfrak{A}}$, where $e_A = \prod_{i=1}^m |\sigma_i|$ is a cell of A. Then by a diagram chasing* *on*

(83)
$$
\begin{tikzcd}
S_*(X) \arrow[rrr, "T^2_{X\times X}\circ d_*"] \arrow[ddd, "T^m_X"'] & & & S_*(X)\bigotimes S_*(X) \arrow[ddd, "T^m_X\otimes T^m_X"] \\
& \mathrm{colim} S_*|_{\mathfrak{A}} \arrow[ul, "\mathrm{j}_A"] \arrow[r, "T^2_{A\times A}\circ d_*"] \arrow[d, "T^m_A"'] & \mathrm{colim} S_*|_{\mathfrak{A}} \bigotimes \mathrm{colim} S_*|_{\mathfrak{A}} \arrow[ur, "\mathrm{j}_A\otimes \mathrm{j}_A"] \arrow[d, "T^m_A\otimes T^m_A"] & \\
& \mathrm{colim} S'_*|_{\mathfrak{A}} \arrow[r, "\Omega\circ(\otimes^m_{i=1}T^2_{|K_i|\times|K_i|}\circ(d_i)_*)"] \arrow[dl, "\mathrm{j}'_A"] & \mathrm{colim} S'_*|_{\mathfrak{A}} \bigotimes \mathrm{colim} S'_*|_{\mathfrak{A}} \arrow[dr, "\mathrm{j}'_A\otimes \mathrm{j}'_A"] & \\
\bigotimes^m_{i=1} S_*(|K_i|) \arrow[rrr, "\Omega\circ(\otimes^m_{i=1}T^2_{|K_i|\times|K_i|}\circ(d_i)_*)"'] & & & \bigotimes^m_{i=1} S_*(|K_i|)\bigotimes\bigotimes^m_{i=1} S_*(|K_i|)
\end{tikzcd}
$$

and its dual diagram, the following properties holds:

(1) *let κ be the* mod 2 *integer $\sum_{j=1}^m q_j \sum_{i>j} p_i$, we have*

(84) $$(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^p)\smile(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^q)=(-1)^\kappa(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(\otimes^m_{i=1}c^{p_i}\smile c^{q_i}),$$

up to cochain homotopy from $\bigotimes^m_{i=1} S_*(|K_i|)\bigotimes\bigotimes^m_{i=1} S_*(|K_i|)$ to $\mathrm{colim} S_*|_{\mathfrak{A}}$, *where the cup product on the left-hand side is the one in $S^*(A)$, $(T^m_A)^*$ the dual of T^m_A;*

(2) *let ν be the* mod 2 *integer $\sum_{j=1}^m p_j \sum_{i>j}(r_i-p_i)$, we have*

$$
\begin{aligned}
(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^p)\frown(T^m_A)^{-1}(c_r) = (-1)^\nu(T^m_X)^{-1}(\otimes^m_{i=1}c^{p_i}\frown c_{r_i})+ \\
(T^m_A)^{-1}\circ h\big((\mathrm{j}'_A)^*\circ\theta(c^p),\partial_{S'\times S'}\circ D(c_r)+D\circ\partial_{S'}(c_r)\big),
\end{aligned}
\tag{85}
$$

in which we treat $\mathrm{j}'_A(c_r)=c_r$ since they have the same form, and

$$D\colon\ \mathrm{colim} S'_*|_{\mathfrak{A}}\to\mathrm{colim} S'_*|_{\mathfrak{A}}\bigotimes\mathrm{colim} S'_*|_{\mathfrak{A}}$$

is a chain homotopy, $\partial_{S'\times S'}$ the boundary operator in $\mathrm{colim} S'_*|_{\mathfrak{A}}\bigotimes\mathrm{colim} S'_*|_{\mathfrak{A}}$.

PROOF. Clearly Diagram (83) is a combination of Diagrams (79) and (82). For the first statement, we proceed as follows:

$$
\begin{aligned}
&(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^p)\smile(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^q)\\
=&d^*\circ(T^2_{A\times A})^*\circ\theta_2\Big((T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^p)\otimes(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(c^q)\Big) && \text{by Definition A.13}\\
=&\mathrm{j}^*_A\circ d^*\circ(T^2_{X\times X})^*\circ\theta_2\Big((T^m_X)^*\circ\theta(c^p)\otimes(T^m_X)^*\circ\theta(c^q)\Big)\\
=&\mathrm{j}^*_A\circ(T^m_X)^*\circ(\otimes^m_{i=1}d^*_i\circ(T^2_{|K_i|\times|K_i|})^*)\circ\Omega^*\circ\theta_2(\theta(c^p)\otimes\theta(c^q)) && \text{by Proposition A.16}\\
=&(-1)^\kappa\mathrm{j}^*_A\circ(T^m_X)^*\circ(\otimes^m_{i=1}d^*_i\circ(T^2_{|K_i|\times|K_i|})^*)\circ\theta\big(\otimes^m_{i=1}\theta_2(c^{p_i}\otimes c^{q_i})\big) && \text{by (80)}\\
=&(-1)^\kappa\mathrm{j}^*_A\circ(T^m_X)^*\circ\theta(\otimes^m_{i=1}c^{p_i}\smile c^{q_i})\\
=&(-1)^\kappa(T^m_A)^*\circ(\mathrm{j}'_A)^*\circ\theta(\otimes^m_{i=1}c^{p_i}\smile c^{q_i}),
\end{aligned}
$$

thus (84) holds.

Now we prove (85). First we can do the expansion (the subscript at the bottom of a chain means its degree)

$$\Omega \circ \left(\otimes_{i=1}^{m} T^{2}_{|K_i|\times|K_i|} \circ (d_i)_*\right)(c_r) = \Omega\Big(\otimes_{i=1}^{m} \big(c_{r_i-p_i} \otimes \bar{c}_{p_i} + \sum_{q_i\neq p_i} c_{r_i-q_i} \otimes c_{q_i}\big)\Big)$$
$$= (-1)^{\nu} \left(\otimes_{i=1}^{m} c_{r_i-p_i}\right) \otimes \left(\otimes_{i=1}^{m} \bar{c}_{p_i}\right) + L,$$

where the last term $L \in \bigotimes_{i=1}^{m} S_*(|K_i|) \bigotimes \bigotimes_{i=1}^{m} S_*(|K_i|)$ is a sum with each summand of the form $\alpha \otimes \beta$, in which $\beta = \otimes_{i=1}^{m} c_{t_i}$ such that at least one t_i is not p_i. Therefore, $h\left((\mathrm{j}'_A)^* \circ \theta(c^p), L\right)$ vanishes since degrees do not agree (see (75)). On the other hand,

$$h\Big((\mathrm{j}'_A)^* \circ \theta(c^p), (\otimes_{i=1}^{m} c_{r_i-p_i}) \otimes (\otimes_{i=1}^{m} \bar{c}_{p_i})\Big) = \prod_{i=1}^{m} c^{p_i}(\bar{c}_{p_i})\,(\otimes_{i=1}^{m} c_{r_i-p_i})$$
$$= \otimes_{i=1}^{m} h\Big(c^{p_i}, T^{2}_{|K_i|\times|K_i|} \circ (d_i)_*(c_{r_i})\Big) = \otimes_{i=1}^{m} c^{p_i} \frown c_{r_i},$$

where the last term is in $S_*(A)$. It follows that

$$\begin{aligned}
&(T^m_X)^* \circ (\mathrm{j}'_A)^* \circ \theta(c^p) \frown (T^m_A)^{-1}(c_r)\\
=&h\big((T^m_X)^* \circ (\mathrm{j}'_A)^* \circ \theta(c^p), T^2_{A\times A} \circ d_* \circ (T^m_A)^{-1}(c_r)\big)\\
=&(T^m_A)^{-1} \circ h\Big((\mathrm{j}'_A)^* \circ \theta(c^p), \Omega \circ (\otimes_{i=1}^{m} T^2_{|K_i|\times|K_i|} \circ (d_i)_*)(c_r) + \partial_{S'\times S'} \circ D(c_r) + D \circ \partial_{S'}(c_r)\Big)\\
=&(-1)^{\nu}(T^m_A)^{-1}\Big(\otimes_{i=1}^{m} c^{p_i} \frown c_{r_i}\Big) + (T^m_A)^{-1} \circ h\big((\mathrm{j}'_A)^* \circ \theta(c^p), \partial_{S'\times S'} \circ D(c_r) + D \circ \partial_{S'}(c_r)\big),
\end{aligned}$$

in which from the second line to the third, we have used (81), and the chain homotopy D to make Diagram (83) commute (see Lemma A.18). $\square$

Now we turn to the simplicial cup and cap products. Recall that $(\mathrm{colim} C_*|_{\mathfrak{A}}, \partial_C)$ coincides with the cellular chain complex of A, such that each cell e_A is a product of simplices $\prod_{i=1}^{m} |\sigma_i|$ (see Proposition A.12), and $\mathrm{colim} S'_*|_{\mathfrak{A}}$ is generated by all tensor products of the form $\otimes_{i=1}^{m} \sigma_i^s$, such that there exists certain cell $\prod_{i=1}^{m} |\sigma_i|$ of A, with σ_i^s a singular simplex whose image lies in $|\sigma_i|$, $i = 1, 2, \dots, m$. It follows that j'_A and j''_A are monic (i.e. they are chain inclusions), hence their duals $(\mathrm{j}'_A)^*$ and $(\mathrm{j}''_A)^*$ are epic, because $\mathrm{colim} C_*|_{\mathfrak{A}}$ and $\mathrm{colim} S'_*|_{\mathfrak{A}}$ are both free Abelian groups.

The finiteness of each K_i guarantees that $\theta\colon \bigotimes_{i=1}^{m} C^*(K_i) \to \mathrm{Hom}(\bigotimes_{i=1}^{m} C_*(K_i), \mathbb{Z})$ is an isomorphism, therefore we have a sequence of epimorphisms

$$\bigotimes_{i=1}^{m} C^*(K_i) \xrightarrow[\cong]{\theta} \mathrm{Hom}(\bigotimes_{i=1}^{m} C_*(K_i), \mathbb{Z}) \xrightarrow{(\mathrm{j}'_A)^*} \mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}). \tag{86}$$

We shall end this section with the following theorem, in which the same notations in Proposition A.12 will be used.

Theorem A.20. *Assume that $c^p = \otimes_{i=1}^{m} c^{p_i} \in \bigotimes_{i=1}^{m} C^{p_i}(K_i)$ is a p-cochain, $c^q = \otimes_{i=1}^{m} c^{q_i} \in \bigotimes_{i=1}^{m} C^{q_i}(K_i)$ a q-cochain, and $c_r = \otimes_{i=1}^{m} c^{r_i} \in \bigotimes_{i=1}^{m} C_{r_i}(\sigma_i)$ an r-chain such that $\prod_{i=1}^{m} |\sigma_i|$ is a cell of A. Then we have*

$$\begin{aligned}
&(\mathrm{i}^*_A)^{-1} \circ (T^m_A)^* \circ (\widetilde{\varsigma}^*_A)^{-1} \circ (\mathrm{j}''_A)^* \circ \theta(c^p) \smile (\mathrm{i}^*_A)^{-1} \circ (T^m_A)^* \circ (\widetilde{\varsigma}^*_A)^{-1} \circ (\mathrm{j}''_A)^* \circ \theta(c^q)\\
=&(\mathrm{i}^*_A)^{-1} \circ (T^m_A)^* \circ (\widetilde{\varsigma}^*_A)^{-1} \circ (\mathrm{j}''_A)^* \circ \theta\left((-1)^{\kappa} \otimes_{i=1}^{m} c^{p_i} \smile c^{q_i}\right),
\end{aligned} \tag{87}$$

up to cochain homotopy, where $\kappa = \sum_{i=1}^{m} q_i \sum_{j>i} p_j$ *and*

$$(\mathrm{i}_A^*)^{-1} \circ (T_A^m)^* \circ (\widetilde{\varsigma}_A^*)^{-1} \colon \mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}) \to S^*(A)$$

is the cochain equivalence, together with

$$\begin{aligned}
&(\mathrm{i}_A^*)^{-1} \circ (T_A^m)^* \circ (\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(c^p) \frown \mathrm{i}_A \circ (T_A^m)^{-1} \circ \widetilde{\varsigma}_A(c_r) \\
=&\mathrm{i}_A \circ (T_A^m)^{-1} \circ \widetilde{\varsigma}_A \left((-1)^{\nu} \otimes_{i=1}^{m} c^{p_i} \frown c_{r_i}\right) + \\
&\mathrm{i}_A \circ (T_A^m)^{-1} \circ h\big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(c^p), \partial_{S'\times S'} \circ D \circ \widetilde{\varsigma}_A(c_r) + D \circ \partial_{S'} \circ \widetilde{\varsigma}_A(c_r)\big),
\end{aligned} \tag{88}$$

with $\nu = \sum_{j=1}^{m} p_j \sum_{i>j} (r_i - p_i)$.

Moreover, if we give $\mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z})$ *the cup-product structure via the commutativity of the diagram*

$$\begin{tikzcd}
\bigotimes_{i=1}^{m} C^*(K_i) \bigotimes \bigotimes_{i=1}^{m} C^*(K_i) \arrow[r, "\smile"] \arrow[d, "(\mathrm{j}_A'')^* \otimes (\mathrm{j}_A'')^*"'] & \bigotimes_{i=1}^{m} C^*(K_i) \arrow[d, "(\mathrm{j}_A'')^*"'] \\
\mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}) \bigotimes \mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}) \arrow[r] & \mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}),
\end{tikzcd} \tag{89}$$

then by formula (87)*, after passing to cohomology, we have a ring isomorphism* $H^*(A) \cong H^*(\mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}))$.

Likewise, if the cap product on $\mathrm{colim} C_*|_{\mathfrak{A}}$ *and its dual is given by the bottom row of the diagram*

$$\begin{tikzcd}
\bigotimes_{i=1}^{m} C^*(K_i) \bigotimes \arrow[d, "(\mathrm{j}_A'')^*"'] & \bigotimes_{i=1}^{m} C_*(K_i) \arrow[r, "\frown"] & \bigotimes_{i=1}^{m} C_*(K_i) \\
\mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z}) \bigotimes & \mathrm{colim} C_*|_{\mathfrak{A}} \arrow[u, "\mathrm{j}_A''"] \arrow[r] & \mathrm{colim} C_*|_{\mathfrak{A}} \arrow[u, "\mathrm{j}_A''"']
\end{tikzcd} \tag{90}$$

via its commutativity, then by formula (88) *it coincides with the one* $\frown\colon H^*(A) \otimes H_*(A) \to H_*(A)$ *using singular (co)homolgy.*

Proof. By Diagram (71), we have

$$(\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta = (\mathrm{j}_A')^* \circ (\widetilde{\varsigma}^*)^{-1} \circ \theta,$$

thus (87) (resp. (88)) follows from (84) (resp. (85)) and Lemma A.14.

The second statement follows from the epicness of $(\mathrm{j}_A'')^* \circ \theta$, as illustrated in (86).

For the last statement, it remains to show that the last summand in (88) vanishes when passing to (co)homology. Let $\widetilde{c}^p$ be a p-cocycle in $\mathrm{Hom}(\mathrm{colim} C_*|_{\mathfrak{A}}, \mathbb{Z})$ and $\widetilde{c}_r$ an r-cycle in $(\mathrm{colim} C_*|_{\mathfrak{A}}, \partial_C)$, it follows that

$$\begin{aligned}
&h\big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\widetilde{c}^p), \partial_{S'\times S'} \circ D \circ \widetilde{\varsigma}_A(\widetilde{c}_r) + D \circ \partial_{S'} \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\big) \\
=&h\big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\widetilde{c}^p), \partial_{S'\times S'} \circ D \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\big) \\
=&\partial_{S'} \circ h\Big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\widetilde{c}^p), D \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\Big) \pm h\Big(\delta_{S'} \circ (\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\widetilde{c}^p), D \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\Big) \\
=&\partial_{S'} \circ h\Big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\widetilde{c}^p), D \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\Big) \pm h\Big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\delta_C \widetilde{c}^p), D \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\Big) \\
=&\partial_{S'} \circ h\Big((\widetilde{\varsigma}_A^*)^{-1} \circ (\mathrm{j}_A'')^* \circ \theta(\widetilde{c}^p), D \circ \widetilde{\varsigma}_A(\widetilde{c}_r)\Big),
\end{aligned}$$

where the third line follows from (76). Since i_A and $(T_A^m)^{-1}$ are chain maps, the last statement holds from the calculation above. $\square$

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