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出版情報：COE Lecture Note. 36, pp.26-35, 2012-01-27. 九州大学マス・フォア・インダストリ研究所

Proceedings of the Czech–Japanese Seminar in Applied Mathematics 2010
Czech Technical University in Prague, August 30 - September 4, 2010
pp. 26–35

FINITE DIFFERENCE SCHEME FOR THE ERICKSEN-LESLIE EQUATION

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Abstract. Ericksen Leslie equation describes the time evolution of a spin vector and velocity in liquid crystals. This equation has following property:
(i) the length preserving of a spin vector,
(ii) the energy conservation or the dissipation property,
(iii) the incompressibility of a velocity vector.
In physics papers, the fourth order Runge-Kutta's method is used for numerical analysis of some types of the liquid crystal model(ex. [6] etc.). However, it abandons the properties (i), (ii). Some schemes which have already been proposed as the mathematical study inherit (ii) and (iii). By these schemes, the property (i) is obtained approximately. For example, these are based on the penalization method(ex. [3]). In this paper, we construct the new implicit scheme for Ericksen-Leslie equation which is based on the MAC method and inherits above three properties. Especially, this scheme inherits the property (i) directly.

Key words. Finite difference method, structure-preserving method, Ericksen-Leslie equation

AMS subject classifications. 65N06, 35K55, 76A15, 35Q35

1. Introduction.

The Ericksen-Leslie model describes the time evolution of a spin and a velocity vector in liquid crystals and is stated as the following nonlinear system:

[**EL1**]

$$\begin{cases} \dfrac{\partial s}{\partial t} + (u\cdot\nabla)\, s - \gamma\left(\Delta s + |\nabla s|^2 s\right) \;=\; 0, & x\in\Omega,\ t\in(0,T], \quad [\mathbf{H1}] \\ \dfrac{\partial u}{\partial t} + (u\cdot\nabla)\, u - \nu\Delta u + \nabla p + \lambda\nabla\cdot(\nabla s\odot\nabla s) \;=\; 0, & x\in\Omega,\ t\in(0,T], \quad [\mathbf{NS1}] \\ \mathrm{div}\ u \;=\; 0 & x\in\Omega,\ t\in(0,T], \quad [\mathbf{IN1}] \\ \dfrac{\partial s}{\partial n} \;=\; 0, \quad u \;=\; 0, & x\in\partial\Omega,\ t\in[0,T], \quad [\mathbf{BC1}] \\ s(x,0)=s_0(x),\ \ |s_0(x)| \;=\; 1,\ \ u(x,0) \;=\; u_0(x),\ \ \mathrm{div}\, u_0(x) \;=\; 0, & x\in\Omega, \quad [\mathbf{IC1}] \end{cases}$$

where Ω is a bounded and simply connected domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$ and γ, ν and λ are positive constants. The vector $s(x,t)\in\mathbb{R}^K$ $(K=2,3)$ describes the directions of the molecules of the liquid crystal. The vector $u(x,t)\in\mathbb{R}^M$ $(M=2,3)$ and the scalar $p(x,t)\in\mathbb{R}$ describe the velocity and the pressure of the fluid, respectively. Each component of the stress term $\lambda\nabla\cdot(\nabla s\odot\nabla s)$ in [NS1] is given by $(\nabla\cdot(\nabla a\odot\nabla b))_i = \sum_{j=1}^{M}\frac{\partial}{\partial x_j}\left(\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_j}\right)$ for $i=1,...,M$ and $a,b\in\mathbb{R}^M$.

We here note two important properties of the solutions. One is the length preserving property of the directions vector s and the other is the energy law.

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Proposition 1.1. *The solutions of* [EL1] *satisfy the length-preserving property:*

$$|s(x,t)| = 1 \qquad x \in \Omega, \quad t \in (0,T], \tag{1.1}$$

and the energy law:

$$\frac{1}{2}\frac{d}{dt}\left(\lambda \|\nabla s(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Omega)}\right) \\ +\lambda\gamma \left\|s(t)\dot{\times}\Delta s(t)\right\|^2_{L^2(\Omega)} + \nu \|\nabla u(t)\|^2_{L^2(\Omega)} = 0, \quad t \in (0,T]. \tag{1.2}$$

Here we use $\dot{\times}$ *as follows:* $a\dot{\times}b = a_1b_2 - a_2b_1$ *for* $a, b \in \mathbb{R}^2$ *and* $a\dot{\times}b = a \times b$ *for* $a, b \in \mathbb{R}^3$*.*

We can also obtain the following equivalent form of [EL1].

$$[\mathbf{EL2}]\begin{cases} \dfrac{\partial s}{\partial t} - s \otimes \left(s\dot{\times}(u\cdot\nabla)s\right) + \gamma s \otimes \left(s\dot{\times}\Delta s\right) \;=\; 0, \quad x \in \Omega,\; t \in (0,T], \qquad [\mathbf{H2}] \\ \dfrac{\partial u_i}{\partial t} + (u\cdot\nabla)u_i - \nu\Delta u_i + \partial_i\left(p + \dfrac{\lambda}{2}|\nabla s|^2\right) \\ -\lambda\Delta s\cdot\left(s\otimes\left(s\dot{\times}\partial_i s\right)\right) \;=\; 0, \quad i = 1,...,M,\; x \in \Omega,\; t \in (0,T], \qquad [\mathbf{NS2}] \\ [\mathbf{IN1}],\; [\mathbf{BC1}],\; [\mathbf{IC1}]. \end{cases}$$

Here $\otimes$ is defined by $a \otimes c = (a_2c,\ -a_1c)^t$ for $a \in \mathbb{R}^2$, $c \in \mathbb{R}$ and $a \otimes b = a \times b$ for $a, b \in \mathbb{R}^3$. Note that $\otimes$ and $\dot{\times}$ are the same operator as the vector product $\times$ in R^3.

In this paper we propose structure-preserving finite difference scheme for the Ericksen-Leslie model in the case where Ω is rectangle and $K = M = 2$ for simplicity. The paper is organized as follows: In the next section, we prepare some notation and definitions. In Section 3, we construct a finite difference scheme for [EL2] and show that finite difference solution of our proposed scheme inherits the important properties in proposition 1.1. In Section 4, we show unique solvability of the scheme since the scheme is implicit and nonlinear. In Section 5, we show numerical experiments. We finally remark that the convergence will be discussed in a forthcoming paper.

2. Preliminaries. Let $\Omega = (0, L_1) \times (0, L_2)$. We set $\Delta t = T/J$ and $\Delta x_k = L/N_k$(for $k = 1, 2$) as the time mesh size and the spatial mesh size, respectively, where J the number of time steps and N_k is the number of spatial meshes. In the next section, we propose a finite difference scheme for [EL2]. Let us prepare several mesh domains and boundaries:

First, we introduce the mesh for the spin:

$$\Omega_{1,\Delta} = \{n_1\}_{n_1=1}^{N_1-1}, \qquad \Omega_{2,\Delta} = \{n_2\}_{n_2=1}^{N_2-1}, \qquad \Omega_\Delta = \Omega_{1,\Delta} \times \Omega_{2,\Delta}.$$

and the mesh of the expanded domains:

$$\overline{\Omega}_{1,\Delta} = \{n_1\}_{n_1=0}^{N_1}, \qquad \check{\Omega}_{1,\Delta} = \{n_1\}_{n_1=-1}^{N_1+1}.$$

We define $\overline{\Omega}_{2,\Delta}$, $\check{\Omega}_{2,\Delta}$, $\overline{\Omega}_\Delta$ and $\check{\Omega}_\Delta$ in the same manner as the above.

Next, we introduce the mesh for the staggered grid:

$$\Omega_{-,1,\Delta} = \{n_1\}_{n_1=0}^{N_1-1}, \qquad \Omega_{-,2,\Delta} = \{n_2\}_{n_2=0}^{N_2-1}, \qquad \Omega_{-,\Delta} = \Omega_{-,1,\Delta} \times \Omega_{-,2,\Delta},$$

and the mesh of the expanded domains and the interior domains:

$$\mathring{\Omega}_{-,1,\Delta} = \{n_1\}_{n_1=1}^{N_1-2}, \qquad \overline{\Omega}_{-,1,\Delta} = \{n_1\}_{n_1=-1}^{N_1}.$$

We define $\mathring{\Omega}_{-,2,\Delta}$, $\overline{\Omega}_{-,2,\Delta}$, $\mathring{\Omega}_{-,\Delta}$ and $\overline{\Omega}_{-,\Delta}$ in the same manner as the above. By these meshes, the domains for the discrete functions are defined as follows:

$$\Omega^P_\Delta = \overline{\Omega}_\Delta \setminus \{(0,0),(0,N_2),(N_1,0),(N_1,N_2)\},$$
$$\check{\Omega}^S_\Delta = \check{\Omega}_\Delta \setminus \{(-1,-1),(-1,N_2+1),(N_1+1,-1),(N_1+1,N_2+1)\},$$

$$\overline{\Omega}^{U_1}_\Delta = \left(\overline{\Omega}_{-,1,\Delta} \times \overline{\Omega}_{2,\Delta}\right), \quad \overline{\Omega}^{\Phi}_\Delta = \overline{\Omega}_{-,\Delta},$$
$$\overline{\Omega}^{U_1,h}_\Delta = \left\{n_1+1/2 | n_1 \in \overline{\Omega}_{-,1,\Delta}\right\} \times \overline{\Omega}_{2,\Delta}, \quad \overline{\Omega}^{\Phi,h}_\Delta = \overline{\Omega}_{-,\Delta}, \quad \partial\Omega_\Delta = \overline{\Omega}_\Delta \setminus \Omega_\Delta,$$
$$\partial\Omega^{U_1}_\Delta = \left(\{0,N_1\} \times \overline{\Omega}_{2,\Delta}\right) \cup \left(\Omega_{-,1,\Delta} \times \{0,N_2\}\right), \quad \partial\overline{\Omega}^{U_1}_\Delta = \overline{\Omega}^{U_1}_\Delta \setminus \left(\mathring{\Omega}_{-,1,\Delta} \times \Omega_{2,\Delta}\right).$$

$\overline{\Omega}^{U_2}_\Delta$, $\overline{\Omega}^{U_2,h}_\Delta$, $\partial\Omega^{U_2}_\Delta$ and $\partial\overline{\Omega}^{U_2}_\Delta$ are also similarly defined.

Next we introduce shift and difference operators:

$$\tau_1^+ X_{n_1,*} = X_{n_1+1,*}, \quad D_1^+ X_{n_1,*} = \frac{\tau_1^+ X_{n_1,*} - X_{n_1,*}}{\Delta x_1}, \quad n_1 \in \overline{\Omega}_{-,1,\Delta},$$
$$\tau_1^- X_{n_1,*} = X_{n_1-1,*}, \quad D_1^- X_{n_1,*} = \frac{X_{n_1,*} - \tau_1^- X_{n_1,*}}{\Delta x_1}, \quad n_1 \in \tau_1^+ \overline{\Omega}_{-,1,\Delta},$$
$$\tilde{D}_1 X_n = \frac{D_1^+ X_n + D_1^- X_n}{2}, \quad n \in \overline{\Omega}_{1,\Delta} \times \check{\Omega}_{2,\Delta}.$$

For X' on $\overline{\Omega}_{1,\Delta}$ ($\tau_1^+ \overline{\Omega}_{1,\Delta}$) $\in \mathbf{R}^{\overline{\Omega}^{U_1}_\Delta}$, τ_1^+, D_1^+ (τ_1^-, D_1^-) are also defined. τ_2^+, τ_2^-, D_2^+, D_2^- and $\tilde{D}_2$ are similarly defined.

$$\tilde{\Delta} X_n = D_1^- D_1^+ X_n + D_2^- D_2^+ X_n, \quad n \in \overline{\Omega}_\Delta, \qquad \tilde{D} X_n = \left(\tilde{D}_1 X_n, \tilde{D}_2 X_n\right), \quad n \in \overline{\Omega}_\Delta,$$
$$\tilde{\operatorname{div}}(X)_n = D_1^+ X'_{n_1-1/2,n_2} + D_2^+ X''_{n_1,n_2-1/2}, \qquad n \in \overline{\Omega}_\Delta.$$

where $X = (X', X'') \in \mathbf{R}^{\overline{\Omega}^{U_1}_\Delta} \times \mathbf{R}^{\overline{\Omega}^{U_2}_\Delta}$. For X' on $\Omega_{-,1,\Delta} \times \Omega_{2,\Delta}$ and X'' on $\Omega_{1,\Delta} \times \Omega_{-,2,\Delta}$, $\tilde{\Delta}$ is also defined.

$$D_t Z^j_* = \frac{Z^{j+1}_* - Z^j_*}{\Delta t}, \quad Z^{(j+1,j)}_* = \frac{Z^{j+1}_* + Z^j_*}{2}, \quad j = 0,1,...,J-1.$$

$$\frac{\tilde{D}}{\tilde{D}n} X_n = \begin{cases} -(-1)^{n_2/N_2} \tilde{D}_2 X_n, & n \in \overline{\Omega}_{1,\Delta} \times \{0,N_2\}, \\ -(-1)^{n_1/N_1} \tilde{D}_1 X_n, & n \in \{0,N_1\} \times \overline{\Omega}_{2,\Delta}, \end{cases} \quad (\text{for } n \in \partial\Omega_\Delta).$$

We denote the following boundary operator.

$$\mathcal{B}_1 (X')_n = \begin{cases} \frac{X'_{n_1+1/2,n_2} + X'_{n_1-1/2,n_2}}{2}, & n \in \{0,N_1\} \times \overline{\Omega}_{2,\Delta}, \\ X'_{n_1+1/2,n_2}, & n \in \Omega_{-,1,\Delta} \times \{0,N_2\}, \end{cases} \quad \left(\text{for } n \in \partial\Omega^{U_1}_\Delta\right).$$

$\mathcal{B}_2 (X'')_n$ is similarly defined.

Finally we introduce the following notation on summations, inner products and norms. Here, we denote the vector (X', X'') by $\tilde{X}$.

$$\sum_{k=0}^{N}{}'' X_k = \frac{X_0}{2} + \sum_{k=1}^{N-1} X_k + \frac{X_N}{2}, \quad \langle X, Y \rangle = \sum_{n_1=0}^{N_1}{}'' \sum_{n_2=0}^{N_2}{}'' \Delta x_1 \Delta x_2 X_n Y_n,$$
$$\langle X', Y' \rangle^{(-,1)} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=1}^{N_2-1} \Delta x_1 \Delta x_2 X'_{n_1+1/2,n_2} Y'_{n_1+1/2,n_2}.$$

$\langle X'', Y''\rangle^{(-,2)}$ is similarly defined.

$$\|X\|_2 = \sqrt{\langle X, X\rangle}, \qquad \left\|\tilde{X}\right\|_2^{(-,\mathrm{vec})} = \sqrt{\langle X', X'\rangle^{(-,1)} + \langle X'', X''\rangle^{(-,2)}},$$

$$\left\|D^+X\right\|_{I,2} = \left(\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2}{}'' \Delta x_1 \Delta x_2 \left|D_1^+X_n\right|^2 + \sum_{n_1=0}^{N_1}{}''\sum_{n_2=0}^{N_2-1} \Delta x_1 \Delta x_2 \left|D_2^+X_n\right|^2\right)^{1/2},$$

$$\begin{aligned}
&\left\|D^+\tilde{X}\right\|_2^{(\mathrm{vec})} \\
&= \left(\sum_{n_1=0}^{N_1}{}''\sum_{n_2=1}^{N_2-1} \Delta x_1 \Delta x_2 \left|D_1^+X'_{n_1-1/2,n_2}\right|^2 + \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} \Delta x_1 \Delta x_2 \left|D_2^+X'_{n_1+1/2,n_2}\right|^2\right. \\
&\quad \left. + \sum_{n_1=1}^{N_1-1}\sum_{n_2=0}^{N_2}{}'' \Delta x_1 \Delta x_2 \left|D_2^+X''_{n_1,n_2-1/2}\right|^2 + \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} \Delta x_1 \Delta x_2 \left|D_1^+X''_{n_1,n_2+1/2}\right|^2\right)^{1/2}.
\end{aligned}$$

3. The proposed scheme. We propose the following finite difference scheme which inherits the principal properties [EL1]. Here, the approximation for $s(x,t)$, $u_1(x,t)$, $u_2(x,t)$ are denoted by S_n^j, $U_{1,n_1+1/2,n_2}^j$, $U_{2,n_1,n_2+1/2}^j$. Note that U_1 and U_2 are defined on the staggered grid.

[DEL1]

$$\left.\begin{aligned}
&D_t S_n^j - S_n^{(j+1,j)} \otimes \left(S_n^{(j+1,j)} \dot{\times} \left(\overline{U}_{1,n}^{(j+1,j)} \tilde{D}_1 + \overline{\overline{U}}_{2,n}^{(j+1,j)} \tilde{D}_2\right) S_n^{(j+1,j)}\right) \\
&= -\gamma S_n^{(j+1,j)} \otimes \left(S_n^{(j+1,j)} \dot{\times} \tilde{\Delta} S_n^{(j+1,j)}\right), \qquad n \in \overline{\Omega}_\Delta, j \in \{j'\}_{j'=0}^{J-1},
\end{aligned}\right\} \textbf{[DH1]}$$

$$\left.\begin{aligned}
&D_t U_{1,n_1+1/2,n_2}^j + Q_{con,1}\left(\overline{U}_1^{(j+1,j)}, U_2^{(j+1,j)}\right)_{n_1+1/2,n_2} - \nu \tilde{\Delta} U_{1,n_1+1/2,n_2}^{(j+1,j)} \\
&-D_1^+\left(-P_n^{j+1/2} + \lambda Q_{st,I}\left(S^{(j+1,j)}\right)_n\right) - \lambda Q_{st,II,1}\left(S^{(j+1,j)}\right)_{n_1+1/2,n_2} \\
&= 0, \qquad n \in \Omega_{-,1,\Delta} \times \Omega_{2,\Delta}, j \in \{j'\}_{j'=0}^{J-1}, \\
&D_t U_{2,n_1,n_2+1/2}^j + Q_{con,2}\left(\overline{\overline{U}}_2^{(j+1,j)}, U_1^{(j+1,j)}\right)_{n_1,n_2+1/2} - \nu \tilde{\Delta} U_{2,n_1,n_2+1/2}^{(j+1,j)} \\
&-D_2^+\left(-P_n^{j+1/2} + \lambda Q_{st,I}\left(S^{(j+1,j)}\right)_n\right) - \lambda Q_{st,II,2}\left(S^{(j+1,j)}\right)_{n_1,n_2+1/2} \\
&= 0, \qquad n \in \Omega_{1,\Delta} \times \Omega_{-,2,\Delta}, j \in \{j'\}_{j'=0}^{J-1},
\end{aligned}\right\} \textbf{[DNS1]}$$

$$\tilde{\mathrm{div}}\left(U^j\right)_n = 0, \qquad n \in \overline{\Omega}_\Delta, j \in \{j'\}_{j'=0}^{J}, \quad \textbf{[DIN1]}$$

$$\left.\begin{aligned}
&\frac{\tilde{D}}{\tilde{D}n} S_n^j = 0, && n \in \partial\Omega_\Delta, j \in \{j'\}_{j'=0}^{J}, \\
&\mathcal{B}_1\left(U_1^j\right)_n = 0, && n \in \partial\Omega_\Delta^{U_1}, \quad j \in \{j'\}_{j'=0}^{J}, \\
&\mathcal{B}_2\left(U_2^j\right)_n = 0, && n \in \partial\Omega_\Delta^{U_2}, \quad j \in \{j'\}_{j'=0}^{J},
\end{aligned}\right\} \textbf{[DBC1]}$$

$$\left.\begin{aligned}
&S_n^0 = s_0(n_1\Delta x_1, n_2\Delta x_2), \quad |s_0(n_1\Delta x_1, n_2\Delta x_2)| = 1, && n \in \overline{\Omega}_\Delta \\
&U_{n_1+1/2,n_2}^0 = D_2^+\Phi_{n_1+1/2,n_2-1/2}^0, && n \in \Omega_{-,1,\Delta} \times \Omega_{2,\Delta}, \\
&U_{n_1,n_2+1/2}^0 = -D_1^+\Phi_{n_1-1/2,n_2+1/2}^0, && n \in \Omega_{1,\Delta} \times \Omega_{-,2,\Delta},
\end{aligned}\right\} \textbf{[DIC1]}$$

$$S : \check{\Omega}^{S}_{\Delta} \times \{j'\}_{j'=0}^{J} \to \mathbb{R}^2, \qquad P : \Omega^{P}_{\Delta} \times \{j' + 1/2\}_{j'=0}^{J-1} \to \mathbb{R},$$
$$U_1 : \overline{\Omega}^{U_1,h}_{\Delta} \times \{j'\}_{j'=0}^{J} \to \mathbb{R}, \qquad U_2 : \overline{\Omega}^{U_2,h}_{\Delta} \times \{j'\}_{j'=0}^{J} \to \mathbb{R}, \quad \Phi^0 : \overline{\Omega}^{\Phi,h}_{\Delta} \to \mathbb{R}$$

and the vector of (U_1^j, U_2^j) is denoted by U^j. The function Φ^0 is defined by

$$\Phi^0_{n_1+1/2,n_2+1/2} = \begin{cases} \phi_0\left((n_1 + 1/2)\,\Delta x_1, (n_2 + 1/2)\,\Delta x_2\right), & n \in \overset{\circ}{\Omega}_{-,\Delta}, \\ 0, & n \in \overline{\Omega}^{\Phi}_{\Delta} \setminus \overset{\circ}{\Omega}_{-,\Delta}, \end{cases}$$

where ϕ_0 is the scalar potential function that is given by incompressibility: div $u_0 = 0$([IC1]) and defined by: $\phi_0(x) = \int_0^{x_2} u_{0,1}(x_1, x_2')\, dx_2' - \int_0^{x_1} u_{0,2}(x_1', 0)\, dx_1'$, $x \in \overline{\Omega}$.

The notation and terms in [DEL1] are defined as follows:

$$\overline{X}_n = \frac{X_{n_1+1/2,n_2} + X_{n_1-1/2,n_2}}{2}, \quad \overline{\overline{X}}_n = \frac{X_{n_1,n_2+1/2} + X_{n_1,n_2-1/2}}{2}, \qquad n \in \overline{\Omega}_{\Delta},$$

$$Q_{con,1}\left(X, Y''\right)_{n_1+1/2,n_2} = Q_{con,I,1}\left(X\right)_{n_1+1/2,n_2} + Q_{con,II,1}\left(X, Y''\right)_{n_1+1/2,n_2}, \qquad n \in \Omega_{-,1,\Delta} \times \overline{\Omega}_{2,\Delta},$$
$$Q_{con,I,1}\left(X\right)_{n_1+1/2,n_2} = \frac{1}{2} D_1^+\left(X_n^2\right), \qquad n \in \Omega_{-,1,\Delta} \times \overline{\Omega}_{2,\Delta},$$
$$Q_{con,II,1}\left(X, Y''\right)_{n_1+1/2,n_2} = \frac{1}{4}\left(Y''_{n_1,n_2-1/2} D_2^+ X_{n_1,n_2-1} + Y''_{n_1,n_2+1/2} D_2^+ X_{n_1,n_2}\right)$$
$$+ \frac{1}{4}\left(Y''_{n_1+1,n_2-1/2} D_2^+ X_{n_1+1,n_2-1} + Y''_{n_1+1,n_2+1/2} D_2^+ X_{n_1+1,n_2}\right), \quad n \in \Omega_{-,1,\Delta} \times \overline{\Omega}_{2,\Delta},$$

$$Q_{st,I}\left(Y\right)_n = -\frac{1}{2}\left(\left|\tilde{D}_1 Y_n\right|^2 + \left|\tilde{D}_2 Y_n\right|^2\right), \qquad n \in \overline{\Omega}_{\Delta},$$
$$Q_{st,II,1}\left(Y\right)_{n_1+1/2,n_2} = \frac{\tilde{Q}_{st,II,1}\left(Y\right)_{n_1+1,n_2} + \tilde{Q}_{st,II,1}\left(Y\right)_n}{2}, \qquad n \in \Omega_{-,1,\Delta} \times \overline{\Omega}_{2,\Delta},$$
$$\tilde{Q}_{st,II,1}\left(Y\right)_n = \tilde{\Delta} Y_n \cdot \left(Y_n \otimes \left(Y_n \dot{\times} \tilde{D}_1 Y_n\right)\right), \qquad n \in \overline{\Omega}_{\Delta}.$$

$Q_{con,2}$, $Q_{con,I,2}$, $Q_{con,II,2}$, $Q_{st,II,2}$, $\tilde{Q}_{st,II,2}$ are similarly defined.

The next theorem shows that the proposed scheme inherits length-preserving and energy structures.

THEOREM 3.1. *In* [DEL1], *we have*

$$|S_n^j| = 1, \qquad n \in \overline{\Omega}_{\Delta}, j = 0, 1, 2, ..., J. \tag{3.1}$$

and

$$\frac{1}{2} D_t\left[\lambda \left\|D^+ S^j\right\|_{I,2}^2 + \left\|U^j\right\|_2^{(-,\mathrm{vec})^2}\right] + \lambda\gamma \left\|S^{(j+1,j)} \dot{\times} \tilde{\Delta} S^{(j+1,j)}\right\|_2^2 + \nu \left\|D^+ U^{(j+1,j)}\right\|_2^{(\mathrm{vec})^2} = 0, \qquad j = 0, 1, 2, ..., J-1. \tag{3.2}$$

Proof. (3.1) is verified by considering the inner product between [DH1] and $S_n^{(j+1,j)}$. By considering [DBC1] and the inner product between [DH1] and $\tilde{\Delta} S_n^{(j+1,j)}$, we obtain

$$\frac{1}{2}D_t\left[\lambda\left\|D^+S^j\right\|_{I,2}^2\right]+\lambda\gamma\left\|S^{(j+1,j)}\dot{\times}\tilde{\Delta}S^{(j+1,j)}\right\|_2^2$$
$$=-\lambda\left\langle\overline{U}_1^{(j+1,j)},\tilde{Q}_{st,I,1}\left(S^{(j+1,j)}\right)\right\rangle-\lambda\left\langle\overline{\overline{U}}_2^{(j+1,j)},\tilde{Q}_{st,I,2}\left(S^{(j+1,j)}\right)\right\rangle. \quad (3.3)$$

Note that the discretized convection term equals to 0. From [DNS1] and [DBC1] and by using the summation by parts, we obtain

$$\frac{1}{2}D_t\left\|U^j\right\|_2^{(-,\mathrm{vec})^2}+\nu\left\|D^+U^{(j+1,j)}\right\|_2^{(\mathrm{vec})^2}=\lambda\sum_{k=1}^{2}\left\langle U_k^{(j+1,j)},Q_{st,II,k}\left(S^{(j+1,j)}\right)\right\rangle^{(-,k)}$$
$$-\sum_{n_1=0}^{N_1}{}''\sum_{n_2=0}^{N_2}{}''\Delta x_1\Delta x_2\,\tilde{\operatorname{div}}\left(U^{(j+1,j)}\right)_n\left(-P_n^{j+1/2}+\lambda Q_{st,I}\left(S^{(j+1,j)}\right)_n\right). \quad (3.4)$$

From [DIN1], (3.3) and (3.4), we have (3.2). □

Remark 1. *We can propose a finite difference scheme for the three dimensional case by the same concept of the discretization and also discuss the uniqueness and stability of the numerical solution as in the next section.*

4. Unique solvability and stability of the proposed scheme.

4.1. Uniqueness. Here, we mention the uniqueness of the solution to the proposed scheme. Fix $j\in\{j'\}_{j'=0}^{J-1}$. By replacing S^{j+1} and U^{j+1} in [DEL1] with Θ^j and V^j respectively, we obtain a new problem [DEL2]. Here, V^j stands for $\left(V_1^j,V_2^j\right)$.

Theorem 4.1. (Uniqueness) *If*

$$\Delta t\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)\leq\frac{\beta_{\mathrm{uni,I}}}{\frac{1}{\sqrt{2}}\left\|U^0\right\|_2^{(-,\mathrm{vec})}+\frac{1}{8}+4\gamma}, \quad (4.1)$$
$$\Delta t\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)\leq\min\left\{\frac{\beta_{\mathrm{uni,II}}}{5\sqrt{2}\left\|U^0\right\|_2^{(-,\mathrm{vec})}},\left(\frac{\beta_{\mathrm{uni,II}}\left(1-\beta_{\mathrm{uni,I}}\right)}{64\lambda^2}\right)^{1/3}\right\}, \quad (4.2)$$

for $\beta_{\mathrm{uni,I}},\beta_{\mathrm{uni,II}}\in(0,1)$, *then* $\left(\Theta^j,V^j\right)$ *which satisfies* [DEL2] *is unique.*

Lemma 4.2. *If* $\left(\Theta^j,V^j\right)$ *satisfies* [DEL2], *then we have*

$$\left|\Theta_n^j\right|=1,\qquad n\in\overline{\Omega}_\Delta,\ j\in\{j'\}_{j'=0}^{J-1}, \quad (4.3)$$

and

$$\frac{1}{2\Delta t}\left[\left(\lambda\left\|D^+\Theta^j\right\|_{I,2}^2+\left\|V^j\right\|_2^{(-,\mathrm{vec})^2}\right)-\left(\lambda\left\|D^+S^j\right\|_{I,2}^2+\left\|U^j\right\|_2^{(-,\mathrm{vec})^2}\right)\right]$$
$$+\lambda\gamma\left\|\frac{\Theta^j+S^j}{2}\dot{\times}\tilde{\Delta}\frac{\Theta^j+S^j}{2}\right\|_2^2+\nu\left\|D^+\frac{V^j+U^j}{2}\right\|_2^{(\mathrm{vec})^2}=0,\ \ j\in\{j'\}_{j'=0}^{J-1}. \quad (4.4)$$

Proof. By replacing U^{j+1} with V^j in proof of Theorem 3.1, we can obtain (4.3) and (4.4). □

Proof of Theorem 4.1. Assume that $\left(\Theta^{j,(1)}, V^{j,(1)}\right)$ and $\left(\Theta^{j,(2)}, V^{j,(2)}\right)$ satisfy [DEL2]. By the equation, the boundary and initial condition for Θ^j in [DEL2] and Theorem 3.1, we obtain

$$\left\|\Theta^{j,(2)} - \Theta^{j,(1)}\right\|_2^2 \leq \Delta t \left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2}\right)\left[\frac{1}{\sqrt{2}} \left\|U^0\right\|_2^{(-,\mathrm{vec})} + \frac{1}{8} + 4\gamma\right]\left\|\Theta^{j,(2)} - \Theta^{j,(1)}\right\|_2^2 + \frac{\Delta t}{2} \left\|V^{j,(2)} - V^{j,(1)}\right\|_2^{(-,\mathrm{vec})\,2}. \tag{4.5}$$

Next, we consider the velocity. By the equation, the boundary and initial condition for V^j in [DEL2] and Theorem 3.1, we have

$$\left\|V^{j,(2)} - V^{j,(1)}\right\|_2^{(-,\mathrm{vec})2} \leq \frac{5}{\sqrt{2}} \Delta t \left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2}\right) \left\|U^0\right\|_2^{(-,\mathrm{vec})} \left\|V^{j,(2)} - V^{j,(1)}\right\|_2^{(-,\mathrm{vec})\,2} + 64\Delta t^2 \lambda^2 \left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2}\right)^3 \left\|\Theta^{j,(2)} - \Theta^{j,(1)}\right\|_2^2. \tag{4.6}$$

From (4.5) and (4.6), the uniqueness is verified.
□

4.2. Stability and solvability. Our scheme which calculates $\left(S^{j+1}, U^{j+1}\right)$ from $\left(S^j, U^j\right)$ is implicit and nonlinear. In this section, we establish the unique solvability of this method. In this paper, we use the following iteration to get $\left(S^j, U^j\right)$. We set the following space

$$\begin{aligned}\mathcal{X} = \Big\{ (\Theta, V)\,|\Theta \in \left(\mathrm{R}^3\right)^{\breve{\Omega}_\Delta^S}, \ \frac{D}{Dn}\Theta_n = 0, \ n \in \partial\Omega_\Delta, \\ V \in \mathrm{R}^{\overline{\Omega}_\Delta^{U_1}} \times \mathrm{R}^{\overline{\Omega}_\Delta^{U_2}}, \ \operatorname{div}(V)_n = 0, \ n \in \overline{\Omega}_\Delta, \\ \mathcal{B}_1 (V_1)_n = 0, \ n \in \partial\Omega_\Delta^{U_1}, \ \mathcal{B}_2 (V_2)_n = 0, \ n \in \partial\Omega_\Delta^{U_2} \Big\}\end{aligned} \tag{4.7}$$

which is equipped with $\|(\Theta, V)\|_{\mathcal{X}} = \left(\|\Theta\|_2^2 + \|V\|_2^{(-,\mathrm{vec})\,2}\right)^{1/2}$. For some $\xi > 0$, we consider the mapping $\rho^j : \mathcal{X} \to \mathcal{X}$ is defined by:

[CEL1]

$$\left.\begin{aligned}&\frac{\hat{\Theta}_n - S_n^j}{\Delta t} - \frac{\hat{\Theta}_n + S_n^j}{2} \otimes \left(\frac{\Theta_n + S_n^j}{2} \dot{\times} \left(\frac{\overline{V}_{1,n} + \overline{U}_{1,n}^j}{2} \tilde{D}_1 + \frac{\overline{\overline{V}}_{2,n} + \overline{\overline{U}}_{2,n}^j}{2} \tilde{D}_2\right) \frac{\Theta_n + S_n^j}{2}\right) \\ &= -\gamma \frac{\hat{\Theta}_n + S_n^j}{2} \otimes \left(\frac{\Theta_n + S_n^j}{2} \dot{\times} \tilde{\Delta} \frac{\Theta_n + S_n^j}{2}\right), \qquad n \in \overline{\Omega}_\Delta,\end{aligned}\right\}\textbf{[CH1]}$$

$$\frac{\tilde{D}}{\tilde{D}n}\hat{\Theta}_n = 0, \qquad n \in \partial\Omega_\Delta, \ \textbf{[CHBC1]}$$

$$\left.\begin{aligned}-\tilde{\Delta}\hat{\Pi}_n = \tilde{\operatorname{div}}\left(\left(Q_{con,1}\left(\frac{\overline{V}_1 + \overline{U}_1^j}{2}, \frac{V_2 + U_2^j}{2}\right), Q_{con,2}\left(\frac{\overline{\overline{V}}_2 + \overline{\overline{U}}_2^j}{2}, \frac{V_1 + U_1^j}{2}\right)\right)\right)_n \\ -\lambda\, \tilde{\Delta} Q_{st,I}\left(\frac{\hat{\Theta} + S^j}{2}\right)_n - \lambda \tilde{\operatorname{div}}\left(Q_{st,II}\left(\frac{\hat{\Theta} + S^j}{2}\right)\right)_n, \qquad n \in \Omega_\Delta,\end{aligned}\right\}\textbf{[CP1]}$$

$$\left.\begin{aligned}&D_1^+\hat{\Pi}_n = -Q_{con,1}\left(\frac{\overline{V}_1 + \overline{U}_1^j}{2}, \frac{V_2 + U_2^j}{2}\right)_{n_1+1/2, n_2} + \nu\tilde{\Delta}\frac{V_{1,n_1+1/2,n_2} + U_{1,n_1+1/2,n_2}^j}{2} \\ &+\lambda D_1^+\left(Q_{st,I}\left(\frac{\hat{\Theta} + S^j}{2}\right)_n\right) + \lambda Q_{st,II,1}\left(\frac{\hat{\Theta} + S^j}{2}\right)_{n_1+1/2, n_2}, \\ &\qquad n \in \{0, N_1 - 1\} \times \Omega_{2,\Delta}, \\ &D_2^+\hat{\Pi}_n = \text{the analogy of the 1st component}, \ldots\end{aligned}\right\}\textbf{[CPBC1]}$$

$$\left.\begin{array}{l}\frac{\hat{V}_{1,n_1+1/2,n_2}-U^j_{1,n_1+1/2,n_2}}{\Delta t}+Q_{con,1}\left(\frac{\overline{V}_1+\overline{U}^j_1}{2},\frac{V_2+U^j_2}{2}\right)_{n_1+1/2,n_2}\\ -\nu\tilde{\Delta}\frac{V_{1,n_1+1/2,n_2}+U^j_{1,n_1+1/2,n_2}}{2}-D^+_1\left(-\hat{\Pi}_n+\lambda Q_{st,I}\left(\frac{\hat{\Theta}+S^j}{2}\right)_n\right)\\ -\lambda Q_{st,II,1}\left(\frac{\hat{\Theta}+S^j}{2}\right)_{n_1+1/2,n_2}=0,\quad n\in\Omega_{-,1,\Delta}\times\Omega_{2,\Delta},\\ \frac{\hat{V}_{2,n_1,n_2+1/2}-U^j_{2,n_1,n_2+1/2}}{\Delta t}+\text{ the analogy of the 1st component}=0,\ldots\end{array}\right\}\textbf{[CNS1]}$$

$$\mathcal{B}_1\left(\hat{V}_1\right)_n=0,\qquad n\in\partial\Omega^{U_1}_\Delta,\qquad \mathcal{B}_2\left(\hat{V}_2\right)_n=0,\qquad n\in\partial\Omega^{U_2}_\Delta,\qquad\textbf{[CBC1]}$$

for $(\Theta,V)\in\mathcal{X}$.

Setting $\left(\Theta^{j,0},V^{j,0}\right)=\left(S^j,U^j\right)$, we use the iteration : $\left(\Theta^{j,m+1},V^{j,m+1}\right)=\rho^j\left(\Theta^{j,m},V^{j,m}\right)$. Here, we fix $j\in\{j'\}_{j'=0}^{J-1}$.

For discussing the stability, we take some $\xi>0$ and restrict the domain of the mapping ρ^j. We set the following closed subset:

$$\mathcal{S}_\xi=\left\{(\Theta,V)\in\mathcal{X}\,|\,|\Theta_n|=1,\ n\in\check{\Omega}^S_\Delta,\ \|V\|_2^{(-,\mathrm{vec})}\le\xi\ \right\}.\tag{4.8}$$

We consider the mapping $\rho^j:\mathcal{S}_\xi\to\mathcal{X}$

Lemma 4.3. *For $\hat{\Theta}$ in the definition of ρ^j, we have $\left|\hat{\Theta}_n\right|=\left|S^j_n\right|$ for $n\in\overline{\Omega}^S_\Delta$.*

Proof. From [CH1], it is easily verified. □

We define β_{bou} by

$$\beta_{\mathrm{bou}}=\min\left\{\frac{\eta^{3/2}\left[\frac{\sqrt{2}}{4}\left(2\left\|U^0\right\|_2^{(-,\mathrm{vec})}+\eta\right)^2+2\nu\left(2\left\|U^0\right\|_2^{(-,\mathrm{vec})}+\eta\right)\right]^{3/2}}{2\sqrt{2}\,T'^{1/2}},\frac{\eta}{8\lambda L_1^{1/2}L_2^{1/2}}\right\},$$

for any $\eta>0$ and any $T'\in(0,T]$. Hereinafter, we fix $\eta>0$ and $T'\in(0,T]$ arbitrarily.

Theorem 4.4. *If $\Delta t\le T'$ and $\Delta t\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)^{3/2}\le\beta_{\mathrm{bou}}$, then we obtain $\rho^j:\mathcal{S}_{\xi^*}\to\mathcal{S}_{\xi^*}$ where $\xi^*=\left\|U^0\right\|_2^{(-,\mathrm{vec})}+\eta$.*

Proof. By [CP1], [CPBC1], [CBC1], [CNS1], Lemma 4.3 and (3.2) in Theorem 3.1, we can conclude that $\rho^j(\Theta,V)\in\mathcal{S}_{\xi^*}$ for $(\Theta,V)\in\mathcal{S}_{\xi^*}$. □

Theorem 4.5. *If $\Delta t\le T'$, $\Delta t\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)^{3/2}\le\beta_{\mathrm{bou}}$,*

$$\frac{3}{4}\Delta t\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)\le\beta_{\mathrm{con,I}},\tag{4.9}$$

$$\Delta t\le\frac{4\left(1-\beta_{\mathrm{con,I}}\right)\beta_{\mathrm{con,II}}}{3},\tag{4.10}$$

$$\Delta t\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)^{4/3}$$
$$\le \min\left\{\left(\frac{3\left(1-\beta_{\mathrm{con,I}}\right)^3\beta_{\mathrm{con,II}}^3}{2^2\left(\frac{1}{\sqrt{2}}\beta_{\mathrm{con,III}}+4\gamma\right)^8}\right)^{1/3},\left(\frac{9\left(1-\beta_{\mathrm{con,I}}\right)\beta_{\mathrm{con,II}}}{2048\lambda^2\left(\frac{1}{\sqrt{2}}\beta_{\mathrm{con,III}}+4\gamma\right)^2}\right)^{1/3},\right.$$
$$\left.\left(\frac{\beta_{\mathrm{con,II}}}{3\cdot 256\left(1-\beta_{\mathrm{con,I}}\right)\left(\beta_{\mathrm{con,III}}^2+1\right)^2}\right)^{1/3},\left(\frac{3\left(1-\beta_{\mathrm{con,I}}\right)\beta_{\mathrm{con,II}}}{\lambda^8 2^{36+6}}\right)^{1/9}\right\}, \quad (4.11)$$

where $\beta_{\mathrm{con,I}},\beta_{\mathrm{con,II}}\in(0,1)$ *and* $\beta_{\mathrm{con,III}}=\frac{\xi^*+\left\|U^0\right\|_2^{(-,\mathrm{vec})}}{2}$ *, then* ρ^j *is contraction mapping.*

Proof. We set $\left(\Theta^{(\alpha)},V^{(\alpha)}\right)\in\mathcal{S}_{\xi^*}$ and $\left(\hat{\Theta}^{(\alpha)},\hat{V}^{(\alpha)}\right)=\rho^j\left(\Theta^{(\alpha)},V^{(\alpha)}\right)$ for $\alpha=1,2$. By (4.9), [CH1], and Lemma 4.3, we obtain

$$\frac{1}{\Delta t}\left\|\hat{\Theta}^{(2)}-\hat{\Theta}^{(1)}\right\|_2^2$$
$$\le\frac{1}{2}\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)\left[\frac{1}{2\sqrt{2}}\sum_{k=1}^{2}\left\|\frac{V^{(k)}+U^j}{2}\right\|_2^{(-,\mathrm{vec})}+4\gamma\right]^2\left\|\Theta^{(2)}-\Theta^{(1)}\right\|_2^2$$
$$+\frac{3}{4}\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)\left\|\hat{\Theta}^{(2)}-\hat{\Theta}^{(1)}\right\|_2^2+\frac{1}{4}\left\|V^{(2)}-V^{(1)}\right\|_2^{(-,\mathrm{vec})\,2}. \quad (4.12)$$

Next, we can consider the velocity field. By [DBC2], Lemma 4.3, Theorem 4.4, (4.12), we have

$$\left\|\hat{V}^{(2)}-\hat{V}^{(1)}\right\|_2^{(-,\mathrm{vec})\,2}$$
$$\le\frac{1024\lambda^2\Delta t^3}{9\left(1-\beta_{\mathrm{con,I}}\right)}\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)^4\left[\frac{1}{2\sqrt{2}}\sum_{k=1}^{2}\left\|\frac{V^{(k)}+U^j}{2}\right\|_2^{(-,\mathrm{vec})}+4\gamma\right]^2\left\|\Theta^{(2)}-\Theta^{(1)}\right\|_2^2$$
$$+\left[8\Delta t^2\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)^2\left[\left(\frac{1}{4}\left\|\frac{V^{(1)}+U^j}{2}\right\|_2^{(-,\mathrm{vec})}+\frac{3}{4}\left\|\frac{V^{(2)}+U^j}{2}\right\|_2^{(-,\mathrm{vec})}\right)^2+2\right]\right.$$
$$\left.+\frac{512\lambda^2\Delta t^3}{9\left(1-\beta_{\mathrm{con,I}}\right)}\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta x_2^2}\right)^3\right]\left\|V^{(2)}-V^{(1)}\right\|_2^{(-,\mathrm{vec})\,2}. \quad (4.13)$$

Here, $V^{(\alpha)},\hat{V}^{(\alpha)}\in\mathcal{S}_{\xi^*}$ $(\alpha=1,2)$. Since (4.12), (4.13) hold and (4.10), (4.11) are assumed, it is verified that $\rho^j:\mathcal{S}_{\xi^*}\to\mathcal{S}_{\xi^*}$ is contraction. □

5. Numerical experiments. We show the numerical example for the following initial data which are expanded from ([3]): $s_0(x)=\left(\sin\left(\theta_\alpha(x)\right),\cos\left(\theta_\alpha(x)\right)\right)$, $u_0(x)=0$, $x\in\overline{\Omega}$, where $\theta_\alpha(x)=\cos\left(\frac{2\pi\alpha_2 x_2}{L_2}\right)-\cos\left(\frac{2\pi\alpha_1 x_1}{L_1}\right)$, $x\in\overline{\Omega}$ with $\alpha=(\alpha_1,\alpha_2)$ for $\alpha_1,\alpha_2\in\{0,\pm1,\pm2,\pm3,...\}$. We set as $\gamma=0.1$, $\lambda=0.1$ and $\nu=0.02$ and take $\alpha=(4,4)$. We show the result of a numerical calculation for $\left(S,\overline{U},\overline{\overline{U}}\right)$ on $\overline{\Omega}$ with $\Delta x_1=\Delta x_2=\frac{1}{20}$, $\Delta t=\frac{1}{5000}$, $L_1=L_2=2.0$ and $T=1.0$.

We compare the proposed scheme with the standard scheme that is discretized by the fourth order Runge-Kutta's method in time with standard finite difference discretization in space.

The comparison of two methods is in Table 5.1. Here, we look into the length of spin and the energy of numerical solution which is defined by:

$$E(S,U)^j = \begin{cases} \frac{1}{2}\left[\lambda \left\|D^+ S^j\right\|_{I,2}^2 + \left\|U^j\right\|_2^{(-,\mathrm{vec})^2}\right], & j=0, \\ \frac{1}{2}\left[\lambda \left\|D^+ S^j\right\|_{I,2}^2 + \left\|U^j\right\|_2^{(-,\mathrm{vec})^2}\right] + \sum_{j'=0}^{j-1}\left[\lambda\gamma \left\|S^{(j'+1,j')} \dot{\times} \tilde{\Delta} S^{(j'+1,j')}\right\|_2^2 \right. \\ \qquad \left. + \nu \left\|D^+ U^{(j'+1,j')}\right\|_2^{(\mathrm{vec})^2}\right], & j=1,2,\ldots,J-1. \end{cases}$$

It is verified that our proposed scheme gives good results on preserving length and energy law.

	standard scheme	proposed scheme
$\max_{0\leq j\leq J} \max_{n\in\overline{\Omega}_\Delta} \left\| S_n^j \right\| - 1 \right\|$	$\simeq 10^{-3}$	$\simeq 10^{-16}$
$\max_{0\leq j\leq J} \left\| E(S,U)^j - E(S,U)^0 \right\| / E(S,U)^0$	$\simeq 10^{-6}$	$\simeq 10^{-16}$

TABLE 5.1
Comparison between the standard scheme and the proposed scheme

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