MATHEMATICAL MODELLING OF FLOW OVER PERIODIC STRUCTURES

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Abstract. We examine the influence of roofs' shapes on the boundary layer of a simplified urban canopy by computing non-stationary Navier-Stokes flow over a periodic pattern. The solution is obtained by means of finite element method (FEM). We use non-conforming Crouzeix-Raviart elements for velocity and piecewise constant elements for pressure. The resulting linear system is solved by the multigrid method. We present computational studies of the problem.

Key words. Incompressible flow, finite element method, Crouzeix-Raviart elements, multigrid, Vanka type smoothers

AMS subject classifications. 35K60, 35K65, 65N06, 68U10

1. Introduction. This paper deals with the flow over periodic patterns. Our primary motivation is modelling of the urban canopy, though other applications exist, including the modelling of rough surfaces in biology, or in the construction of pipelines.

Let \( \Omega \subset \mathbb{R}^2 \) be a polygonal domain derived from a rectangle by substitution of the bottom edge by a piecewise linear line representing the terrain. The boundary of \( \Omega \) splits into four clearly defined parts, three of them being plain line segments. We will denote them by \( \Gamma_1 \) to \( \Gamma_4 \) in a counterclockwise fashion, or simply refer to them as the “terrain”, “outlet”, “upper”, and “inlet” parts; see Fig. 1.1.

![Computational domain](image)

Fig. 1.1. Computational domain

We consider a periodic pattern of buildings with the basic parameters given by Fig. 1.2, and we examine the properties of the boundary layer depending on the roofs’ shape.

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On \([0, T] \times \Omega\), we solve the incompressible Navier-Stokes equations for velocity \(\vec{u}\) and pressure \(p\), with the kinematic viscosity \(\nu = 1.5 \cdot 10^{-5}\)

\[
\frac{\partial \vec{u}(t, \vec{x})}{\partial t} + \vec{u}(t, \vec{x}) \cdot \nabla \vec{u}(t, \vec{x}) - \nu \Delta \vec{u}(t, \vec{x}) + \nabla p(t, \vec{x}) = 0, \\
\nabla \cdot \vec{u}(t, \vec{x}) = 0.
\] (1.1a)

The boundary conditions are of the Dirichlet and do-nothing type

\[
u \vec{n} + \nu (\nabla \vec{u}) \cdot \vec{n} = 0 \quad \vec{x} \in \Gamma_2, \\
-p + \nu \frac{\partial u_x}{\partial z} = 0, \quad u_z = 0 \quad \vec{x} \in \Gamma_3, \\
\vec{u} = \vec{u}_\text{in} = (u_\alpha, 0) \quad \vec{x} \in \Gamma_4.
\] (1.2a, 1.2b, 1.2c, 1.2d)

The inlet profile \(u_\alpha\) is given by the power law

\[u_\alpha(z) = \bar{u}_{\text{ref}} \left( \frac{z}{z_{\text{ref}}} \right)^\alpha,
\] (1.3)

where \(\bar{u}_{\text{ref}}\) is the average wind speed at some reference height \(z_{\text{ref}}\), and \(\alpha\) is the profile exponent.

As the initial condition \(\vec{u}(0, \vec{x})\), we take the solution of the stationary Stokes problem, obtained by omitting the first two terms in (1.1a) and setting \(\nu = 1.0\).

2. Weak formulation of Navier-Stokes equations. Let \(X = (H^1(\Omega))^2\), \(V(\vec{u}_\text{in}) = \{\vec{u} \in X, \text{satisfying (1.2)}\}\), and \(Q = L^2(\Omega)\). We set the following forms

\[a(\vec{u}, \vec{v}) = \nu \int_{\Omega} \sum_{i,j=1}^2 \left( \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right) \, dx \, dz, \quad b(\vec{u}, \vec{v}, \vec{w}) = \int_{\Omega} \sum_{i,j=1}^2 \left( u_j \frac{\partial v_i}{\partial x_j} w_i \right) \, dx \, dz.
\]

For time discretization, we use the semi-implicit Oseen scheme [2]. The time derivative is approximated by the backward Euler difference

\[
\frac{\partial \vec{u}^n(x)}{\partial t} \approx \frac{\vec{u}^n - \vec{u}^{n-1}}{\tau},
\] (2.1)

where \(t^n = n\tau\). The convective term is discretized by \(\vec{u}^{n-1} \cdot \nabla \vec{u}^n\), and the remaining terms are taken implicitly.

For each timestep \(t^n\), we seek \(\vec{u}^n \in V(\vec{u}_\text{in})\) and \(p^n \in Q\), such that \(\forall \vec{v} \in V(\vec{u})\) and \(\forall q \in Q\)

\[
(\vec{u}^n, \vec{v}) + \tau b(\vec{u}^{n-1}, \vec{u}^n, \vec{v}) + \tau a(\vec{u}^n, \vec{v}) - \tau (p^n, \nabla \cdot \vec{v}) = (\vec{u}^{n-1}, \vec{v}), \quad (\nabla \cdot \vec{u}^n, q) = 0.
\] (2.2, 2.3)
To deal with the nonlinear convective term, we employ the upwinding technique proposed by [3], and denote the respective approximation by $\tilde{b}(\tilde{u}_h^{n-1}, \tilde{w}_h^n, \tilde{v})$. The main idea of the method is to split the domain into lumped regions $R_l$ around each midpoint $Q_l$ of the original triangulation; see Fig. 2.1.

**2.1. Discrete formulation.** Let index $h$ denote the respective finite-dimensional set $V^h(\tilde{u}_n)$, space $Q^h$, and the corresponding discrete functions $\tilde{u}_h^n$, $p_h^n$. We introduce $\tilde{w}_h \in V^h(\tilde{u}_n)$ to represent the inhomogeneous Dirichlet data, and we denote $\tilde{v}_h = \tilde{u}_h - \tilde{w}_h \in V^h(\bar{0})$. The discrete problem for each timestep $t^n$ reads: Find $\tilde{v}_h^n \in V^h(\bar{0})$ and $p_h^n \in Q^h$, such that $\forall \tilde{v} \in V^h(\bar{0})$ and $\forall q \in Q^h$

$$
(\tilde{v}_h^n, \tilde{v}) + \tau \tilde{b}(\tilde{u}_h^{n-1}, \tilde{w}_h^n, \tilde{v}) + \tau \alpha(\tilde{w}_h^n, \tilde{v}) - \tau (p_h^n, \nabla \cdot \tilde{v}) = 0,
$$

and the corresponding discrete functions $\bar{\tilde{v}}_h^n, \bar{p}_h^n$ can then be expressed as

$$
\bar{\tilde{v}}_h^n = \sum_{j \in J^h} v_{h,j} \phi_j^n, \quad \bar{p}_h^n = \sum_{k \in K^h} p_{h,k} \psi_k^n. \tag{2.5}
$$

We introduce the matrices

$$
M_i = (M_{ij})_{i,j=1}^{J^h}, \quad A_i = (A_{ij})_{i,j=1}^{J^h}, \quad B_i = (B_{ij})_{i,j=1}^{J^h}, \quad N_i = (N_{ij})_{i,j=1}^{J^h},
$$

and the coefficient vectors

$$
\bar{v}_h^n = (v_{h,j}^n)_{j=1}^{J^h}, \quad \bar{p}_h^n = (p_{h,k}^n)_{k=1}^{K^h}. \tag{2.10}
$$

Taking $\tilde{v} = \phi_i$ for $i = 1, \ldots, J^h$ and $q = \psi_k$ for $k = 1, \ldots, K^h$, the discrete formulation (2.4) leads to a system of linear equations for each timestep $t^n$

$$
Mv_h^n + \tau N(\tilde{u}_h^{n-1})v_h^n + \tau Av_h^n + \tau B^T p_h^n = \tilde{f}, \tag{2.11a}
$$

$$
Bv_h^n = \tilde{g}. \tag{2.11b}
$$

**Fig. 2.1. Upwind - lumped region $R_l$ (grey) around the midpoint $Q_l$.**
where

\[
\begin{align*}
\tilde{f}_i &= (\bar{u}_h^{m-1}, \phi_i) - (\bar{w}_h^m, \phi_i) - \tau \bar{b}(\bar{w}_h^{m-1}, \bar{w}_h^m, \bar{\phi}_i) - \tau a(\bar{w}_h^m, \bar{\phi}_i), \\
\tilde{g}_k &= -(\nabla \cdot \bar{w}_h^m, \psi_k).
\end{align*}
\]  

(2.11c)  

(2.11d)

The matrices \(M\) and \(A\) are called the mass matrix and the stiffness matrix of the problem (2.11).

3. Numerical solution using FEM. We choose non-conforming Crouzeix-Raviart elements [4] to approximate the components of velocity, Fig. 3.1, and piecewise constant elements for pressure.

![Fig. 3.1. Crouzeix-Raviart element - scalar test function](image)

We use a hierarchy of uniformly refined structured meshes, Fig. 3.2, and employ a multigrid solver based on Vanka-type smoother [5] to solve the linear system (2.11). An extension for higher order elements can be found in [6].

![Fig. 3.2. Coarsest mesh](image)

4. Experimental order of convergence. To verify the numerical scheme, we compute the experimental orders of convergence (EOCs) using an artificial problem with prescribed velocity field

\[
\bar{u}_* = \cos(2\pi t) \begin{pmatrix} -x + 4y \\ -3x^2 + y \end{pmatrix}.
\]  

(4.1)

The problem is defined on a unit square with both the initial condition and the non-stationary Dirichlet boundary conditions given by (4.1). We evaluate the left-hand side of (1.1a) for \(\bar{u}_*\) and put it as an additional forcing term in the right-hand side of the system.
Solving the problem on two different grids $M_1$ and $M_2$ with space steps $h_1$ and $h_2$ respectively, we define the EOC for each time level as

$$EOC(t, M_1, M_2) = \log \left( \frac{\| \tilde{u}_{h_1}(t) - \tilde{u}_*(t) \|_{L^2(\Omega)}}{\| \tilde{u}_{h_2}(t) - \tilde{u}_*(t) \|_{L^2(\Omega)}} \right) / \log \left( \frac{h_1}{h_2} \right).$$

(4.2)

In our case, the coarsest mesh $M_0$ consists of four triangles obtained by cutting the square alongside both its diagonals. Through uniform refinement, we get $M_1$, consisting of 16 triangles, and so on. The actual computations were done for four consecutive grids $M_5$ to $M_8$, yielding three series of EOCs, which are shown in Fig. 4.1 for $t \in [0, 0.1]$.

![Fig. 4.1. EOC series for $t \in [0, 0.1]$](image)

The EOCs of approximately one are mostly attributed to the inaccurate representation of the boundary condition. Currently, the function $w_h$ is constructed as a combination of CR-type boundary elements.

5. Numerical results. We take a pattern consisting of eight buildings with square canyons between them and consider five different arrangements of roofs. The first one is a simple pattern with flat roofs. The next two configurations contain pitched roofs, one in the upwind direction and the other one in the downwind direction. The height of the pitched part is 0.5. We also examine an alternating pattern where the pairs of one higher building ($H = 1.5$) and one lower building ($H = 1.0$) periodically repeat. The Reynolds number is $10^5$ in these four cases.

In the last simulation, we use symmetric saddle roofs with a relative height of 0.2. The Reynolds number is $5 \times 10^4$ in this case, and also the numerical parameters slightly differ. The complete set of parameters for all five cases is given in tables below.

<table>
<thead>
<tr>
<th>PARAMETERS</th>
<th>cases (1), 2, ..., 4</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain size</td>
<td>$24 \times (8), 10.5$</td>
<td>velocity 5109056, pressure 1703936</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>$10^5$</td>
<td>cases 2, ..., 4</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.28</td>
<td>velocity 7000896, pressure 2334720</td>
</tr>
<tr>
<td>initial flow</td>
<td>Stokes</td>
<td>$\tau = 1/256$</td>
</tr>
<tr>
<td>time interval</td>
<td>$T = 60$</td>
<td>accuracy 1.0e-08</td>
</tr>
</tbody>
</table>
The results are shown in Fig. 5.2 and Fig. 5.1. We can observe a clear difference depending on the type of the roofs, which is in agreement with what we would expect. The aerodynamical shape of the second arrangement creates a very thin boundary layer, while the third and fourth arrangement create a strongly turbulent layer. The saddle roofs produce pairs of vortices on top of each other.

6. Conclusion. We obtained some qualitative results of flow in a simplified urban canopy for different shapes of roofs. Our next goal is using a non-stationary inlet condition by [7] to catch the turbulent properties of the atmospheric boundary layer.

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Fig. 5.2. $|\vec{u}(t)|$ at $t = 55$ for cases 1, ..., 4
REFERENCES


