On the existence and stability of time periodic solution to the compressible Navier-Stokes equation on the whole space

Tsuda, Kazuyuki
Graduate School of Mathematics, Kyushu University

出版情報：MI Preprint Series. 2014-10, 2014-09-08. 九州大学大学院数理学研究院
バージョン：
権利関係：

MI Preprint Series

Mathematics for Industry
Kyushu University

On the existence and stability of time periodic solution to the compressible Navier-Stokes equation on the whole space

Kazuyuki Tsuda

MI 2014-10

(Received September 8, 2014)

Institute of Mathematics for Industry
Graduate School of Mathematics
Kyushu University
Fukuoka, JAPAN

On the existence and stability of time periodic solution to the compressible Navier-Stokes equation on the whole space

Kazuyuki TSUDA
Graduate School of Mathematics,
Kyushu University,
Fukuoka 819-0395, JAPAN

Abstract

The existence of a time periodic solution of the compressible Navier-Stokes equation on the whole space is proved for sufficiently small time periodic external force when the space dimension is greater than or equal to 3. The proof is based on the spectral properties of the time-T-map associated with the linearized problem around the motionless state with constant density in some weighted L^∞ and Sobolev spaces. The time periodic solution is shown to be asymptotically stable under sufficiently small initial perturbations and the L^∞ norm of the perturbation decays as time goes to infinity.

1 Introduction

We consider time periodic problem of the following compressible Navier-Stokes equation for barotropic flow in $\mathbb{R}^n$ $(n \geq 3)$:

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho v) = 0, \\ \rho(\partial_t v + (v\cdot\nabla)v) - \mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \nabla p(\rho) = \rho g. \end{cases} \tag{1.1}$$

Here $\rho = \rho(x,t)$ and $v = (v_1(x,t),\cdots,v_n(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^n$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$p'(\rho_*) > 0$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \frac{2}{n}\mu + \mu' \geq 0;$$

and $g = g(x,t)$ is a given external force periodic in t. We assume that $g = g(x,t)$ satisfies the condition

$$g(x,t+T) \;=\; g(x,t) \quad (x \in \mathbb{R}^n,\ t \in \mathbb{R}) \tag{1.2}$$

for some constant $T > 0$.

Time periodic flow is one of basic phenomena in fluid mechanics, and thus, time periodic problems for fluid dynamical equations have been extensively studied. We refer, e.g., to [8, 9, 12, 17] for the incompressible Navier-Stokes case, and to [1, 2, 3, 6, 15, 16] for the compressible case. In this paper we are interested in time periodic problem for the compressible Navier-Stokes equation on unbounded domains. Ma, Ukai, and Yang [15] proved the existence and stability of time periodic solutions on the whole space $\mathbb{R}^n$. They showed that if $n \geq 5$, there exists a time periodic solution (ρ_{per}, v_{per}) around $(\rho_*, 0)$ for a sufficiently small $g \in C^0(\mathbb{R}; H^{N-1} \cap L^1)$ with $g(x, t+T) = g(x, t)$, where $N \in \mathbb{Z}$ satisfying $N \geq n+2$. Furthermore, the time periodic solution is stable under sufficiently small perturbations and there holds the estimate

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{H^{N-1}} \leq C(1+t)^{-\frac{n}{4}} \|(\rho_0, v_0) - (\rho_{per}(t_0), v_{per}(t_0))\|_{H^{N-1} \cap L^1},$$

where t_0 is a certain initial time and $(\rho, v)|_{t=t_0} = (\rho_0, v_0)$. Here H^k denotes the L^2-Sobolev space on $\mathbb{R}^n$ of order k.

On the other hand, it was shown in [6] that, for $n \geq 3$, if the external force g satisfies the oddness condition

$$g(-x, t) = -g(x, t) \quad (x \in \mathbb{R}^n,\ t \in \mathbb{R}) \tag{1.3}$$

and if g is small enough in some weighted Sobolev space, then there exists a time periodic solution (ρ_{per}, v_{per}) for (1.1) around $(\rho_*, 0)$ and $u_{per}(t) = (\rho_{per}(t) - \rho_*, v_{per}(t))$ satisfies

$$\begin{aligned} &\sup_{t \in [0,T]} (\|u_{per}(t)\|_{L^2} + \|x \nabla u_{per}(t)\|_{L^2}) \\ &\quad \leq C\{\|(1+|x|)g\|_{C([0,T]; L^1 \cap L^2)} + \|(1+|x|)g\|_{L^2(0,T; H^{m-1})}\}. \end{aligned} \tag{1.4}$$

Furthermore, the time periodic solution (ρ_{per}, v_{per}) is asymptotically stable under sufficiently small initial perturbations, and the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^2} = O(t^{-\frac{n}{4}}) \text{ as } t \to \infty. \tag{1.5}$$

In this paper we will show the existence of a time periodic solution for (1.1) without assuming the oddness condition (1.3) for $n \geq 3$. It will be proved that if $n \geq 3$ and if g satisfies (1.2) and

$$\|g\|_{C([0,T]; L^1)} + \|(1+|x|^n)g\|_{C([0,T]; L^\infty)} + \|(1+|x|^{n-1})g\|_{L^2(0,T; H^{s-1})} \ll 1$$

with an integer $s \geq [n/2] + 1$, then there exists a time periodic solution $(\rho_{per}, v_{per}) \in C([0,T]; H^s)$ with period T for (1.1), and $u_{per}(t) = (\rho_{per}(t) - \rho_*, v_{per}(t))$ satisfies

$$\begin{aligned} &\sup_{t \in [0,T]} (\|(1+|x|^{n-1})\rho_{per}(t)\|_{L^\infty} + \sum_{j=0}^{1} \|(1+|x|^{n-2+j})\partial_x^j v_{per}(t)\|_{L^\infty}) \\ &\quad \leq C(\|g\|_{C([0,T]; L^1)} + \|(1+|x|^n)g\|_{L^\infty(0,T; L^\infty)} + \|(1+|x|^{n-1})g\|_{L^2(0,T; H^{s-1})}). \end{aligned} \tag{1.6}$$

Furthermore, if g satisfies

$$\|g\|_{C([0,T];L^1)} + \|(1+|x|^n)g\|_{C([0,T];L^\infty)} + \|(1+|x|^{n-1})g\|_{L^2(0,T;H^s)} \ll 1,$$

then the time periodic solution (ρ_{per}, v_{per}) is asymptotically stable under sufficiently small initial perturbations, and the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^\infty} \to 0$$

as $t \to \infty$. We expect that the decay estimate such as (1.5) would also hold for this case and it would be desirable to derive the optimal decay estimate of L^2 norm for the perturbations. The precise statements of our existence and stability results are given in Theorem 3.1 and Theorem 3.2 below.

We will prove the existence of a time periodic solution around $(\rho_*, 0)$ by an iteration argument by using the time-T-map associated with the linearized problem at $(\rho_*, 0)$. As in [6] we formulate the time periodic problem as a system of equations for low frequency part and high frequency part of the solution. (Cf., [7, 11].) In the proof of the existence of a time periodic solution without assuming the oddness condition (1.3), there are two key observations. One is concerned with the spectrum of the time-T-map for the low frequency part. Another one is concerned with the convection term $v \cdot \nabla v$. As for the former matter, we need to investigate $(I - S_1(T))^{-1}$, where $S_1(T) = e^{-TA}$ with A being the linearized operator around $(\rho_*, 0)$ which acts on functions whose Fourier transforms have their supports in $\{\xi \in \mathbb{R}^n; |\xi| \le r_\infty\}$ for some $r_\infty > 0$. (See (4.21) and (4.22) bellow.) We will show that the leading part of $(I - S_1(T))^{-1}$ coincides with the solution operator for the linearized stationary problem used by Shibata-Tanaka in [14]. In fact, the Fourier transform of $(I - S_1(T))^{-1}F$ takes the form $(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}$, where $\hat{F}$ is the Fourier transform of F and

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top \xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top\xi. \end{pmatrix}$$

By using the spectral resolution, we see that

$$(I - e^{-T\hat{A}_\xi})^{-1} \sim -\frac{1}{T} \begin{pmatrix} \frac{\nu+\tilde{\nu}}{\gamma^2} & -\frac{i^\top\xi}{\gamma|\xi|^2} \\ -\frac{i\xi}{\gamma|\xi|^2} & \frac{1}{\nu|\xi|^2}\left(I_n - \frac{\xi^\top\xi}{|\xi|^2}\right) \end{pmatrix} \quad \text{as} \quad \xi \to 0.$$

The right-hand side is the solution operator for the linearized stationary problem in the Fourier space. This motivates us to introduce a weighted L^∞ space for the low frequency part employed in the study of the stationary problem in [14].

As for the high frequency part, we will employ the weighted energy estimates established in [6].

Another point in our analysis is concerned with the convection term $v \cdot \nabla v$. Due to the slow decay of $v(x,t)$ as $|x| \to \infty$, there appears some difficulty in estimating $v \cdot \nabla v$. To overcome this, we will use the momentum formulation for the low frequency part, which takes a form of a conservation lows, and the velocity formulation for the high frequency part, for which the energy method works well. We also note that, in estimating the high frequency part of $v \cdot \nabla v$, we will use the fact that a Poincaré type inequality $\|f\|_{L^2} \leq C\|\nabla f\|_{L^2}$ holds for the high frequency part.

The asymptotic stability of the time periodic solution (ρ_{per}, v_{per}) can be proved as in the argument in Kagei and Kawashima [4] by using the Hardy inequality. It seems, however, that a perturbation argument for the linearized problem as in [6, 11] does not work well to derive the optimal decay estimate because of the slow decay of $v_{per}(x,t)$ as $|x| \to \infty$; and a more refined perturbation analysis would be needed.

This paper is organized as follows. In section 2, we introduce notations and auxiliary lemmas used in this paper. In section 3, we state main results of this paper. Section 4 is devoted to the reformulation of the problem. We will use the equation of the conservation of momentum for the low frequency part and the equation of motion for the high frequency part; and we will then rewrite the system for the low and high frequency parts into a system of integral equations in terms of the time-T-map. In section 5, we study the low frequency part and derive the necessary estimates for the time-T-map of the low frequency part. In section 6, we state some spectral properties of the time-T-map of the high frequency part. In section 7, we estimate nonlinear terms and then give a proof of the existence of a time periodic solution by the iteration argument.

2 Preliminaries

In this section we first introduce some notations which will be used throughout this paper. We then introduce some auxiliary lemmas which will be useful in the proof of the main results.

For a given Banach space X, the norm on X is denoted by $\|\cdot\|_X$.

Let $1 \leqq p \leqq \infty$. We denote by L^p the usual L^p space over $\mathbb{R}^n$. The inner product of L^2 is denoted by $(\cdot,\cdot)$. For a nonnegative integer k, we denote by H^k the usual L^2-Sobolev space of order k. (As usual, $H^0 = L^2$.)

We simply denote by L^p the set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in L^p$ $(j = 1, \cdots, n)$, i.e., $(L^p)^n$ and the norm $\|\cdot\|_{(L^p)^n}$ on it is denoted by $\|\cdot\|_{L^p}$ if no confusion will occur. Similarly, for a function space X, the set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in X$ $(j = 1, \cdots, n)$, i.e., X^n, is simply denoted by X; and the norm $\|\cdot\|_{X^n}$ on it is denoted by $\|\cdot\|_X$ if no confusion will occur. (For example, $(H^k)^n$ is simply denoted by H^k and the norm $\|\cdot\|_{(H^k)^n}$ is denoted by $\|\cdot\|_{H^k}$.)

Let $u = {}^\top(\phi, w)$ with $\phi \in H^k$ and $w = {}^\top(w_1, \cdots, w_n) \in H^m$. we denote the norm of u on $H^k \times H^m$ by $\|u\|_{H^k \times H^m}$:

$$\|u\|_{H^k \times H^m} = \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2\right)^{\frac{1}{2}}.$$

When $m = k$, the space $H^k \times (H^k)^n$ is simply denoted by H^k, and, also, the norm

$\|u\|_{H^k\times(H^k)^n}$ by $\|u\|_{H^k}$ if no confusion will occur :

$$H^k := H^k \times (H^k)^n, \quad \|u\|_{H^k} := \|u\|_{H^k\times(H^k)^n} \quad (u = {}^\top(\phi, w)).$$

Similarly, for $u = {}^\top(\phi, w) \in X \times Y$ with $w = {}^\top(w_1, \cdots, w_n)$, we denote its norm $\|u\|_{X\times Y}$ by $\|u\|_{X\times Y}$:

$$\|u\|_{X\times Y} = \left(\|\phi\|_X^2 + \|w\|_Y^2\right)^{\frac{1}{2}} \quad (u = {}^\top(\phi, w)).$$

If $Y = X^n$, we simply denote $X \times X^n$ by X, and, its norm $\|u\|_{X\times X^n}$ by $\|u\|_X$:

$$X := X \times X^n, \quad \|u\|_X := \|u\|_{X\times X^n} \quad (u = {}^\top(\phi, w)).$$

We will work on function spaces with spatial weight. For a nonnegative integer ℓ and $1 \le p \le \infty$, we denote by L^p_ℓ the weighted L^p space defined by

$$L^p_\ell = \{u \in L^p; \|u\|_{L^p_\ell} := \|(1 + |x|)^\ell u\|_{L^p} < \infty\}.$$

We denote the Fourier transform of f by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}^n).$$

The inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}^n).$$

Let k be a nonnegative integer and let r_1 and r_∞ be positive constants satisfying $r_1 < r_\infty$. We denote by $H^k_{(\infty)}$ the set of all $u \in H^k$ satisfying supp $\hat{u} \subset \{|\xi| \ge r_1\}$, and by $L^2_{(1)}$ the set of all $u \in L^2$ satisfying supp $\hat{f} \subset \{|\xi| \le r_\infty\}$. Note that $H^k \cap L^2_{(1)} = L^2_{(1)}$ for any nonnegative integer k. (Cf., Lemma 4.1 (ii) bellow.)

Let k and ℓ be nonnegative integers. We define the spaces H^k_ℓ and $H^k_{(\infty),\ell}$ by

$$H^k_\ell = \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$\begin{aligned} \|u\|_{H^k_\ell} &= \left(\sum_{j=0}^{\ell} |u|^2_{H^k_j}\right)^{\frac{1}{2}}, \\ |u|_{H^k_\ell} &= \left(\sum_{|\alpha|\le k} \|\, |x|^\ell \partial_x^\alpha u \,\|^2_{L^2}\right)^{\frac{1}{2}}, \end{aligned}$$

and

$$H^k_{(\infty),\ell} = \{u \in H^k_{(\infty)}; \|u\|_{H^k_\ell} < +\infty\}.$$

Let ℓ be a nonnegative integer. We denote $L^2_{(1),\ell}$ by

$$L^2_{(1),\ell} = \{f \in L^2_\ell; f \in L^2_{(1)}\}.$$

For $-\infty \le a < b \le \infty$, we denote by $C^k([a,b];X)$ the set of all C^k functions on $[a,b]$ with values in X. We denote the Bochner space on (a,b) by $L^p(a,b;X)$ and the L^2-Bochner-Sobolev space of order k by $H^k(a,b;X)$.

We define the space $\mathscr{X}_{(1)}$ by

$$\mathscr{X}_{(1)} = \{\phi \in L^\infty_{n-1}, \nabla\phi \in L^2_1; \text{supp } \hat{\phi} \subset \{|\xi| \le r_\infty\}, \|\phi\|_{\mathscr{X}_{(1)}} < +\infty\},$$

where

$$\begin{aligned}
&\|\phi\|_{\mathscr{X}_{(1)}} := \|\phi\|_{\mathscr{X}_{(1),L^\infty}} + \|\phi\|_{\mathscr{X}_{(1),L^2}},\\
&\|\phi\|_{\mathscr{X}_{(1),L^\infty}} := \|(1+|x|)^{n-1}\phi\|_{L^\infty},\ \|\phi\|_{\mathscr{X}_{(1),L^2}} := \|(1+|x|)\nabla\phi\|_{L^2}.
\end{aligned}$$

The space $\mathscr{Y}_{(1)}$ is defined by

$$\mathscr{Y}_{(1)} = \{w \in L^\infty_{n-2}, \nabla w \in H^1; \text{supp } \hat{w} \subset \{|\xi| \le r_\infty\}, \|w\|_{\mathscr{Y}_{(1)}} < +\infty\},$$

where

$$\begin{aligned}
&\|w\|_{\mathscr{Y}_{(1)}} := \|w\|_{\mathscr{Y}_{(1),L^\infty}} + \|w\|_{\mathscr{Y}_{(1),L^2}},\\
&\|w\|_{\mathscr{Y}_{(1),L^\infty}} := \sum_{j=0}^{1} \|(1+|x|)^{n-2+j}\nabla^j w\|_{L^\infty},\ \|w\|_{\mathscr{Y}_{(1),L^2}} := \sum_{j=1}^{2} \|(1+|x|)^{j-1}\nabla^j w\|_{L^2}.
\end{aligned}$$

The space $\mathscr{Z}_{(1)}(a,b)$ is defined by

$$\mathscr{Z}_{(1)}(a,b) = C^1([a,b];\mathscr{X}_{(1)}) \times \Big[C([a,b];\mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})\Big].$$

Let ℓ be a nonnegative integer and let s be a nonnegative integer satisfying $s \ge \left[\frac{n}{2}\right]+1$. For $k = s-1, s$, the space $\mathscr{Z}^k_{(\infty),\ell}(a,b)$ is defined by

$$\begin{aligned}
\mathscr{Z}^k_{(\infty),\ell}(a,b) = &\left[C([a,b];H^k_{(\infty),\ell}) \cap C^1([a,b];L^2_1)\right]\\
&\times \left[L^2(a,b;;H^{k+1}_{(\infty),\ell}) \cap C([a,b];H^k_{(\infty),\ell}) \cap H^1(a,b;H^{k-1}_{(\infty),\ell})\right].
\end{aligned}$$

Let s be a nonnegative integer satisfying $s \ge \left[\frac{n}{2}\right]+1$. and let $k = s-1, s$. The space $X^k(a,b)$ is defined by

$$\begin{aligned}
&X^k(a,b)\\
&= \Big\{\{u_1, u_\infty\}; u_1 \in \mathscr{Z}_{(1)}(a,b), u_\infty \in \mathscr{Z}^k_{(\infty),n-1}(a,b),\\
&\qquad \partial_t\phi_\infty \in C([a,b];L^2_1), u_j = {}^\top(\phi_j, w_j)\ (j=1,\infty)\Big\},
\end{aligned}$$

equipped with the norm

$$\begin{aligned}\|\{u_1, u_\infty\}\|_{X^k(a,b)} =& \|u_1\|_{\mathscr{Z}_{(1)}(a,b)} + \|u_\infty\|_{\mathscr{Z}^k_{(\infty),n-1}(a,b)} \\ &+ \|\partial_t \phi_\infty\|_{C([a,b];L^2_1)} + \|\partial_t u_1\|_{C([a,b];L^2)} + \|\partial_t \nabla u_1\|_{C([a,b];L^2_1)}.\end{aligned}$$

We also introduce function spaces of T-periodic functions in t. We denote by $C_{per}(\mathbb{R};X)$ the set of all T-periodic continuous functions with values in X equipped with the norm $\|\cdot\|_{C([0,T];X)}$; and we denote by $L^2_{per}(\mathbb{R};X)$ the set of all T-periodic locally square integrable functions with values in X equipped with the norm $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R};X)$ and $X^k_{per}(\mathbb{R})$, and so on, are defined.

For a bounded linear operator L on a Banach space X, we denote by $r_X(L)$ the spectral radius of P.

For operators L_1 and L_2, $[L_1, L_2]$ denotes the commutator of L_1 and L_2:

$$[L_1, L_2]f = L_1(L_2 f) - L_2(L_1 f).$$

We next state some lemmas which will be used in the proof of the main results.

We begin with the well-known Sobolev type inequality.

Lemma 2.1. *Let $n \geq 3$ and let $s \geq \left[\frac{n}{2}\right] + 1$. Then there holds the inequality*

$$\|f\|_{L^\infty} \leq C\|\nabla f\|_{H^{s-1}}$$

for $f \in H^s$.

We next state some inequalities concerned with composite functions.

Lemma 2.2. *Assume $n \geq 2$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$. Let s_j and μ_j ($j = 1, \cdots, \ell$) satisfy $0 \leq |\mu_j| \leq s_j \leq s + |\mu_j|$, $\mu = \mu_1 + \cdots + \mu_\ell$, $s = s_1 + \cdots + s_\ell \geq (\ell - 1)s + |\mu|$. Then there holds*

$$\| \partial_x^{\mu_1} f_1 \cdots \partial_x^{\mu_\ell} f_\ell \|_{L^2} \leq C \prod_{1\leq j\leq \ell} \| f_j \|_{H^{s_j}} .$$

See, e.g., [5], for the proof of Lemma 2.2.

Lemma 2.3. *Let $n \geq 2$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$. Suppose that F is a smooth function on I, where I is a compact interval of $\mathbb{R}$. Then for a multi-index α with $1 \leq |\alpha| \leq s$, there hold the estimates*

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{s-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^{s-1}} \|f_2\|_{H^{|\alpha|}}$$

for $f_1 \in H^s$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^n$ and $f_2 \in H^{|\alpha|}$; and

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{s-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^s} \|f_2\|_{H^{|\alpha|-1}}.$$

for $f_1 \in H^{s+1}$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^n$ and $f_2 \in H^{|\alpha|-1}$.

See, e.g., [4], for the proof of Lemma 2.3.

3 Main results

In this section, we state our results on the existence and stability of a time-periodic solution for system (1.1).

We formulate (1.1) as follows. Substituting $\phi = \frac{\rho-\rho_*}{\rho_*}$ and $w = \frac{v}{\gamma}$ with $\gamma = \sqrt{p'(\rho_*)}$ into (1.1), we see that (1.1) is rewritten as

$$\partial_t u + Au = -B[u]u + G(u,g), \tag{3.1}$$

where

$$A = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma\nabla & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div} \end{pmatrix}, \quad \nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu+\mu'}{\rho_*}, \tag{3.2}$$

$$B[\tilde{u}]u = \gamma \begin{pmatrix} \tilde{w}\cdot\nabla\phi \\ 0 \end{pmatrix} \text{ for } u = {}^\top(\phi, w),\ \tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}) \tag{3.3}$$

and

$$\begin{aligned} G(u,g) &= \begin{pmatrix} F^0(u) \\ \tilde{F}(u,g) \end{pmatrix}, && (3.4)\\ F^0(u) &= -\gamma\phi\mathrm{div}w, && (3.5)\\ \tilde{F}(u,g) &= -\gamma(1+\phi)(w\cdot\nabla w) - \phi\partial_t w - \nabla(p^{(1)}(\phi)\phi^2) + \frac{1+\phi}{\gamma}g, && (3.6)\\ p^{(1)}(\phi) &= \frac{\rho_*}{\gamma}\int_0^1 (1-\theta)p''(\rho_*(1+\theta\phi))d\theta. \end{aligned}$$

We next introduce operators which decompose a function into its low and high frequency parts. Operators P_1 and P_∞ on L^2 are defined by

$$P_j f = \mathcal{F}^{-1}\hat{\chi}_j\mathcal{F}[f] \quad (f\in L^2, j = 1,\infty),$$

where

$$\begin{aligned} &\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j=1,\infty), \quad 0 \le \hat{\chi}_j \le 1 \quad (j = 1,\infty),\\ &\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \le r_1),\\ 0 & (|\xi| \ge r_\infty), \end{cases}\\ &\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi),\\ &0 < r_1 < r_\infty. \end{aligned}$$

We fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ in such a way that the estimate (5.6) in Lemma 5.3 below holds for $|\xi| \le r_\infty$.

Our result on the existence of a time periodic solution is stated as follows.

Theorem 3.1. *Let $n \geq 3$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies (1.2) and $g(x,t) \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_n) \cap L^2_{per}(\mathbb{R}; H^{s-1}_{n-1})$. Set*

$$[g]_s = \|g\|_{C([0,T];L^1\cap L^\infty_n)} + \|g\|_{L^2(0,T;H^{s-1}_{n-1})}.$$

Then there exist constants $\delta > 0$ and $C > 0$ such that if $[g]_s \leq \delta$, then the system (3.1) has a time-periodic solution $u = u_1 + u_\infty$ satisfying $\{u_1, u_\infty\} \in X^s_{per}(\mathbb{R}^n)$ with $\|\{u_1, u_\infty\}\|_{X^s(0,T)} \leq C[g]_s$. Furthermore, the uniqueness of time periodic solutions of (3.1) holds in the class $\{u = {}^\top(\phi, w); \{P_1 u, P_\infty u\} \in X^s_{per}(\mathbb{R}), \|\{u_1, u_\infty\}\|_{X^s(0,T)} \leq C\delta\}$.

We next consider the stability of the time-periodic solution obtained in Theorem 3.1.

Let ${}^\top(\rho_{per}, v_{per})$ be the periodic solution given in Theorem 3.1. We denote the perturbation by $u = {}^\top(\phi, w)$, where $\phi = \rho - \rho_{per}, w = v - v_{per}$. Substituting $\rho = \phi + \rho_{per}$ and $v = w + v_{per}$ into (1.1), we see that the perturbation $u = {}^\top(\phi, w)$ is governed by

$$\left\{\begin{array}{l} \partial_t\phi + v_{per}\cdot\nabla\phi + \phi \mathrm{div} v_{per} + \rho_{per}\mathrm{div} w + w\cdot\nabla\rho_{per} = f^0, \\ \partial_t w + v_{per}\cdot\nabla w + w\cdot\nabla v_{per} - \frac{\mu}{\rho_{per}}\Delta w - \frac{\mu+\mu'}{\rho_{per}}\nabla\mathrm{div} w \\ \quad + \frac{\phi}{\rho^2_{per}}(\mu\Delta v_{per} + (\mu+\mu')\nabla\mathrm{div} v_{per}) + \nabla(\frac{p'(\rho_{per})}{\rho_{per}}\phi) = \tilde{f}, \end{array}\right. \tag{3.7}$$

where

$$\begin{aligned}
f^0 &= -\mathrm{div}(\phi w), \\
\tilde{f} &= -w\cdot\nabla w - \frac{\phi}{\rho_{per}(\rho_{per}+\phi)}(\mu\Delta w + (\mu+\mu')\nabla\mathrm{div} w) \\
&\quad + \frac{\phi}{\rho_{per}(\rho_{per}+\phi)}(\frac{\phi}{\rho_{per}}\mu\Delta v_{per} + \frac{\phi}{\rho_{per}}(\mu+\mu')\nabla\mathrm{div} v_{per}) \\
&\quad + \frac{\phi}{\rho^2_{per}}\nabla(p^{(2)}(\rho_{per},\phi)\phi) + \frac{\phi^2}{\rho^2_{per}(\rho_{per}+\phi)}\nabla(p(\rho_{per}+\phi)) + \frac{1}{\rho_{per}}\nabla(p^{(3)}(\rho_{per},\phi)\phi^2), \\
p^{(2)}(\rho_{per},\phi) &= \int_0^1 p'(\rho_{per}+\theta\phi)d\theta, \\
p^{(3)}(\rho_{per},\phi) &= \int_0^1 (1-\theta)p''(\rho_{per}+\theta\phi)d\theta.
\end{aligned}$$

We consider the initial value problem for (3.7) under the initial condition

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{3.8}$$

Our result on the stability of the time-periodic solution is stated as follows.

Theorem 3.2. *Let $n \geq 3$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies (1.2) and $g(x,t) \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_n) \cap L^2_{per}(\mathbb{R}; H^s_{n-1})$. Let (ρ_{per}, v_{per}) be the time-periodic solution obtained in Theorem 3.1, and let $u_0 \in H^s$. Then there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that if*

$$[g]_{s+1} \leq \epsilon_1, \quad \|u_0\|_{H^s} \leq \epsilon_2,$$

there exists a unique global solution $u = {}^\top(\phi, w)$ *of* (3.7)-(3.8) *satisfying*

$$
\begin{aligned}
&u \in C([0,\infty); H^s),\\
&\|u(t)\|_{H^s}^2 + \int_0^t \|\nabla u(\tau)\|_{H^{s-1}\times H^s}^2 d\tau \le C\|u_0\|_{H^s}^2 \quad (t \in [0,\infty)),\\
&\|u(t)\|_{L^\infty} \to 0 \quad (t \to \infty).
\end{aligned}
$$

It is not difficult to see that Theorem 3.2 can be proved by the energy method ([4], [10]), since the Hardy inequality works well to deal with the linear terms including (ρ_{per}, v_{per}) due to the estimate for (ρ_{per}, v_{per}) in Theorem 3.1; and so the proof is omitted here.

4 Reformulation of the problem

In this section, we reformulate problem (3.1). As in [6], to solve the time periodic problem for (3.1), we decompose u into a low frequency part u_1 and a high frequency part u_∞, and then, we rewrite the problem into a system of equations for u_1 and u_∞.

As in [6], we set

$$
u_1 = P_1 u, \quad u_\infty = P_\infty u.
$$

Applying the operators P_1 and P_∞ to (3.1), we obtain,

$$
\partial_t u_1 + Au_1 = F_1(u_1 + u_\infty, g), \tag{4.1}
$$

$$
\partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) = F_\infty(u_1 + u_\infty, g). \tag{4.2}
$$

Here

$$
\begin{aligned}
F_1(u_1 + u_\infty, g) &= P_1[-B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g)],\\
F_\infty(u_1 + u_\infty, g) &= P_\infty[-B[u_1 + u_\infty]u_1 + G(u_1 + u_\infty, g)].
\end{aligned}
$$

Suppose that (4.1) and (4.2) are satisfied by some functions u_1 and u_∞. Then by adding (4.1) to (4.2), we obtain

$$
\begin{aligned}
\partial_t(u_1 + u_\infty) + A(u_1 + u_\infty) &= -P_\infty(B[u_1 + u_\infty]u_\infty) + (P_1 + P_\infty)F(u_1 + u_\infty, g)\\
&= -B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g).
\end{aligned}
$$

Set $u = u_1 + u_\infty$, then we have

$$
\partial_t u + Au + B[u]u = G(u, g).
$$

Consequently, if we show the existence of a pair of functions $\{u_1, u_\infty\}$ satisfying (4.1)-(4.2), then we can obtain a solution u of (3.1).

In this paper, we consider the low frequency part u_1 in a weighted L^∞ space. To do so, the velocity formulation is not suitable, and, instead, we use the momentum formulation for the low frequency part.

Before introducing the momentum formulation, we prepare some inequalities for the low frequency part. We first derive some properties of P_1.

Lemma 4.1. (i) *Let k be a nonnegative integer. Then P_1 is a bounded linear operator from L^2 to H^k. In fact, it holds that*

$$\|\nabla^k P_1 f\|_{L^2} \le C\|f\|_{L^2} \qquad (f \in L^2).$$

As a result, for any $2 \le p \le \infty$, P_1 is bounded from L^2 to L^p.

(ii) *Let k be a nonnegative integer. Then there hold the estimates*

$$\|\nabla^k f_1\|_{L^2} + \|f_1\|_{L^p} \le C\|f_1\|_{L^2} \quad (f \in L^2_{(1)}),$$

where $2 \le p \le \infty$.

The proofs of estimates (i) and (ii) are given in [6, Lemma 4.3].

The following inequality is concerned with the estimates of the weighted L^p norm for the low frequency part.

Lemma 4.2. *Let χ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then for a nonnegative integer ℓ and $1 \le p \le \infty$, there holds*

$$\||x|^\ell(\chi * f)\|_{L^p} \le C\{\||x|^\ell \chi\|_{L^1}\|f\|_{L^p} + \|\chi\|_{L^1}\||x|^\ell f\|_{L^p}\} \qquad (f \in L^p_\ell).$$

Here C is a positive constant depending only on ℓ.

Proof. Let χ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then

$$\begin{aligned} ||x|^\ell(\chi * f)| &\le |x|^\ell \int_{\mathbb{R}^n} |\chi(x-y)f(y)|dy \\ &\le C\int_{\mathbb{R}^n} |x-y|^\ell|\chi(x-y)||f(y)|dy + C\int_{\mathbb{R}^n} |\chi(x-y)||y|^\ell|f(y)|dy. \end{aligned}$$

Therefore, the Young inequality gives

$$\||x|^\ell(\chi * f)\|_{L^p} \le C\{\||x|^\ell \chi\|_{L^1}\|f\|_{L^p} + \|\chi\|_{L^1}\||x|^\ell f\|_{L^p}\} \qquad (f \in L^p_\ell).$$

This completes the proof. □

Applying Lemma 4.2, we have the following inequality for the weighted L^p norm of the low frequency part.

Lemma 4.3. *Let k and ℓ be nonnegative integers and let $1 \le p \le \infty$. Then there holds the estimate*

$$\||x|^\ell \nabla^k f_1\|_{L^p} \le C\||x|^\ell f_1\|_{L^p} \quad (f \in L^2_{(1)} \cap L^p_\ell).$$

Proof. We define a cut-off function $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$ with $\hat{\chi}_0$ satisfying

$$\hat{\chi}_0 \in C^\infty(\mathbb{R}^n), \quad 0 \le \hat{\chi}_0 \le 1, \quad \hat{\chi}_0 = 1 \ \text{ on } \ \{|\xi| \le r_\infty\} \ \text{ and } \ \mathrm{supp}\,\hat{\chi}_0 \subset \{|\xi| \le 2r_\infty\}. \tag{4.3}$$

Since $f_1 \in L^2_{(1)}$, we see that $\nabla^k f_1 = (\nabla^k \chi_0) * f_1$ $(k \ge 0)$. Therefore, by Lemma 4.2, we obtain the desired estimate. This completes the proof. $\square$

Since $n \ge 3$, applying the Hardy inequality and Lemma 4.3, we have the following inequality for the weighted L^2 norm of the low frequency part.

Lemma 4.4. *Let $\phi \in \mathscr{X}_{(1)}$ and $w_1 \in \mathscr{Y}_{(1)}$. Then, it holds that*

$$\|P_1(\phi w_1)\|_{\mathscr{Y}_{(1),L^2}} \le C\|\phi\|_{L^\infty_{n-1}}\|\nabla w_1\|_{L^2}.$$

Here $C > 0$ is a constant depending only on n.

Proof. By Lemma 4.3, we see that

$$\|P_1(\phi w_1)\|_{\mathscr{Y}_{(1),L^2}} \le C\|\phi w_1\|_{L^2_1}. \tag{4.4}$$

Since $n \ge 3$, by the Hardy inequality, we find that

$$\|\phi w_1\|_{L^2_1} \le C\|\phi\|_{L^\infty_{n-1}}\|\nabla w_1\|_{L^2}. \tag{4.5}$$

By (4.4) and (4.5), we obtain the desired estimate. This completes the proof. $\square$

Let us now reformulate the system (4.1)-(4.2) by using the momentum. We set m_1 and $u_{1,m}$ by

$$m_1 = w_1 + P_1(\phi w), \quad u_{1,m} = {}^\top(\phi_1, m_1), \tag{4.6}$$

where $\phi = \phi_1 + \phi_\infty$, and $w = w_1 + w_\infty$. Then, we see that $\{u_{1,m}, u_\infty\}$ defined by (4.6) satisfies the following system of equations.

Lemma 4.5. *Assume that $\{u_1, u_\infty\}$ satisfies the system* (4.1)-(4.2)*. Then, $\{u_{1,m}, u_\infty\}$ satisfies the following system:*

$$\begin{aligned} \partial_t u_{1,m} + Au_{1,m} &= F_{1,m}(u_1 + u_\infty, g), \\ \partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \end{aligned} \tag{4.7}$$

Here

$$\begin{aligned} F_{1,m}(u_1 + u_\infty, g) &= {}^\top(0, \tilde{F}_{1,m}(u_1 + u_\infty, g)), \\ \tilde{F}_{1,m}(u_1 + u_\infty, g) &= -P_1\{\mu\Delta(\phi w) + \tilde{\mu}\nabla \mathrm{div}\,(\phi w) + \frac{\rho_*}{\gamma}\nabla(p^{(1)}(\phi)\phi^2) \\ &\quad + \gamma \mathrm{div}\,((1+\phi)w \otimes w) - \frac{1}{\gamma}((1+\phi)g)\}. \end{aligned} \tag{4.8}$$

Proof. If $\{u_1, u_\infty\}$ satisfies the system (4.1)-(4.2), then $u = u_1 + u_\infty$ satisfies (1.4). Hence, we see that

$$\begin{aligned}(1+\phi)w\cdot\nabla w &= \mathrm{div}\,((1+\phi)w\otimes w) - w\mathrm{div}\,((1+\phi)w)\\ &= \mathrm{div}\,((1+\phi)w\otimes w) + \frac{w}{\gamma}\partial_t\phi. \end{aligned} \tag{4.9}$$

Therefore, substituting (4.9) into (4.1), we obtain the equation (4.7). This completes the proof. $\square$

Conversely, one can see that the momentum formulation (4.2), (4.6) and (4.7) gives the solution $\{u_1, u_\infty\}$ of (4.1)-(4.2) if $\phi = \phi_1 + \phi_\infty$ is sufficiently small. In fact, we have the following Lemma.

Lemma 4.6. (i) *Let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. Then there exists a positive constant δ_0 such that if $\phi = \phi_1 + \phi_\infty$ satisfies $\sup_{t\in[a,b]} \|\phi\|_{L^\infty_{n-1}} \leq \delta_0$, then there uniquely exists $w_1 \in C([a,b]; \mathscr{Y}_{(1)}) \cap H^1(a,b; \mathscr{Y}_{(1)})$ that satisfies*

$$w_1 = m_1 - P_1(\phi(w_1 + w_\infty)) \tag{4.10}$$

where $\phi = \phi_1 + \phi_\infty$. Furthermore, there hold the estimates

$$\|w_1\|_{C([a,b];\mathscr{Y}_{(1)})} \leq C(\|m_1\|_{C([a,b];\mathscr{Y}_{(1)})} + \|w_\infty\|_{C([a,b];L^2)}), \tag{4.11}$$

$$\begin{aligned}\int_b^a \|\partial_t w_1(\tau)\|^2_{\mathscr{Y}_{(1)}} d\tau \leq\ & C((\|\partial_t\nabla\phi_1\|^2_{C([a,b];L^2_1)} + \|\partial_t\phi_\infty\|^2_{C([a,b];L^2_1)})\|w_1\|^2_{C([a,b];L^\infty_{n-2})}\\ &+\|\partial_t\phi\|^2_{C([a,b];L^2)}\|w_1\|^2_{C([a,b];\mathscr{Y}_{(1),L^\infty})})\\ &+\int_b^a C\Big(\|\partial_t m_1(\tau)\|^2_{\mathscr{Y}_{(1)}} + \|\partial_t\phi\|^2_{(C[a,b];L^2)}\|w_\infty(\tau)\|^2_{H^s_{n-1}}\\ &+\|\partial_t w_\infty(\tau)\|^2_{L^2}\Big)d\tau. \end{aligned} \tag{4.12}$$

(ii) *Let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. Assume that $\phi = \phi_1 + \phi_\infty$ satisfies $\sup_{t\in[a,b]} \|\phi\|_{L^\infty_{n-1}} \leq \delta_0$ and $\{u_{1,m}, u_\infty\}$ satisfies*

$$\begin{aligned}\partial_t u_{1,m} + Au_{1,m} &= F_{1,m}(u_1 + u_\infty, g),\\ w_1 &= m_1 - P_1(\phi w),\\ \partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g).\end{aligned}$$

Here $w = w_1 + w_\infty$ with w_1 defined by (4.10). Then $\{u_1, u_\infty\}$ with $u_1 = {}^\top(\phi_1, w_1)$ satisfies (4.1)-(4.2).

Proof. (i) Let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. For $F_1 \in \mathscr{Y}_{(1)}$, we set $\mathscr{P}[\phi]F_1 := P_1(\phi F_1)$. By Lemma 4.3 and Lemma 4.4, we see that $\mathscr{P}[\phi]F_1 \in \mathscr{Y}_{(1)}$ and

$$\|\mathscr{P}[\phi]F_1\|_{\mathscr{Y}_{(1)}} \le C\delta_0\{\|F_1\|_{L^\infty} + \|\nabla F_1\|_{L^2}\}.$$

Hence, if $\delta_0 \le \frac{C}{2}$, then $(I+\mathscr{P}[\phi])$ is boundary invertible on $\mathscr{Y}_{(1)}$ and $(I+\mathscr{P}[\phi])^{-1}$ satisfies

$$\|(I+\mathscr{P}[\phi])^{-1}F_1\|_{\mathscr{Y}_{(1)}} \le C\|F_1\|_{\mathscr{Y}_{(1)}}. \tag{4.13}$$

By Lemma 2.1 and Lemma 4.3, we see that $m_1 - P_1(\phi w_\infty) \in \mathscr{Y}_{(1)}$ and

$$\|m_1 - P_1(\phi w_\infty)\|_{\mathscr{Y}_{(1)}} \le C(\|m_1\|_{\mathscr{Y}_{(1)}} + \|w_\infty\|_{L^2}). \tag{4.14}$$

We define w_1 by

$$w_1 := (I+\mathscr{P}[\phi])^{-1}[m_1 - P_1(\phi w_\infty)].$$

Then, by (4.13) and (4.14), $w_1 \in \mathscr{Y}_{(1)}$ satisfies (4.10) and

$$\|w_1\|_{\mathscr{Y}_{(1)}} \le C(\|m_1\|_{\mathscr{Y}_{(1)}} + \|w_\infty\|_{L^2}). \tag{4.15}$$

It directly follows from (4.15) that $w_1 \in C([a,b];\mathscr{Y}_{(1)})$ and w_1 satisfies (4.11).

We next show that $\partial_t w_1 \in L^2(a,b;\mathscr{Y}_{(1)})$ and $\partial_t w_1$ satisfies (4.12). We set $K_1 := m_1 - P_1(\phi w_\infty)$. By Lemma 2.1 and Lemma 4.3, we see that $-\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1 \in \mathscr{Y}_{(1)}$ and

$$\begin{aligned} \|-\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1\|_{\mathscr{Y}_{(1)}} \le\ & C(\|\partial_t m_1\|_{\mathscr{Y}_{(1)}} + \|\partial_t\phi\|_{L^2}\|w_1\|_{\mathscr{Y}_{(1),L^\infty}} \\ & +(\|\partial_t\nabla\phi_1\|_{L^2_1} + \|\partial_t\phi_\infty\|_{L^2_1})\|w_1\|_{L^\infty_{n-2}} \\ & +\|\partial_t\phi\|_{L^2}\|w_\infty\|_{H^s_{n-1}} + \|\partial_t w_\infty\|_{L^2}). \end{aligned}$$

Therefore,

$$(I+\mathscr{P}[\phi])\partial_t w_1 = -\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1$$

and hence, $\partial_t w_1 = (I+\mathscr{P}[\phi])^{-1}[-\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1] \in L^2(a,b;\mathscr{Y}_{(1)})$ and $\partial_t w_1$ satisfies (4.12).

(ii) We see from (i) that there uniquely exists $w_1 \in C([a,b];\mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})$ satisfying (4.10). Then substituting (4.10) into (4.7), we see that

$$\partial_t\phi_1 + \gamma w_1 = -\gamma P_1(\mathrm{div}\,(\phi w)). \tag{4.16}$$

On the other hand, by $(4.2)_1$, we have

$$\partial_t\phi_\infty + \gamma w_\infty = -\gamma P_\infty(\mathrm{div}\,(\phi w)). \tag{4.17}$$

Hence, by adding (4.16) to (4.17), we see that

$$\partial_t\phi + \gamma\mathrm{div}\,((1+\phi)w) = 0 \tag{4.18}$$

where $\phi = \phi_1 + \phi_\infty$ and $w = w_1 + w_\infty$, Substituting (4.10) into (4.7), and by using a similar computation as (4.9) based on (4.18), we see that $u_1 = {}^\top(\phi_1, w_1)$ satisfies (4.1). This completes the proof. $\square$

By Lemma 4.6, if we show the existence of a pair of functions $\{u_{1,m}, u_\infty\} \in X^s(a,b)$ satisfying (4.2), (4.7) and (4.10), then we can obtain a solution $\{u_1, u_\infty\} \in X^s(a,b)$ satisfying (4.1)-(4.2). Therefore, we will consider (4.2), (4.7) and (4.10) instead of (4.1)-(4.2).

We look for a time periodic solution u for the system (4.2), (4.7) and (4.10). To solve the time periodic problem for (4.2), (4.7) and (4.10), we introduce solution operators for the following linear problems:

$$\begin{cases} \partial_t u_{1,m} + A u_{1,m} = F_{1,m}, \\ u_{1,m}|_{t=0} = u_{01,m}, \end{cases} \tag{4.19}$$

and

$$\begin{cases} \partial_t u_\infty + A u_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \\ u_\infty|_{t=0} = u_{0\infty}, \end{cases} \tag{4.20}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01,m}, u_{0\infty}, F_{1,m}$ and F_∞ are given functions.

To formulate the time periodic problem, we denote by $S_1(t)$ the solution operator for (4.19) with $F_{1,m} = 0$, and by $\mathscr{S}_1(t)$ the solution operator for (4.19) with $u_{01,m} = 0$. We also denote by $S_{\infty,\tilde{u}}(t)$ the solution operator for (4.20) with $F_\infty = 0$ and by $\mathscr{S}_{\infty,\tilde{u}}(t)$ the solution operator for (4.20) with $u_{0\infty} = 0$. (The precise definition of these operators will be given later.)

As in [6], we will look for a $\{u_{1,m}, u_\infty\}$ satisfying

$$\begin{cases} u_{1,m}(t) = S_1(t)u_{01,m} + \mathscr{S}_1(t)[F_{1,m}(u,g)], \\ u_\infty(t) = S_{\infty,u}(t)u_{0\infty} + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)], \end{cases} \tag{4.21}$$

where

$$\begin{cases} u_{01,m} = (I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)], \\ u_{0\infty} = (I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)], \end{cases} \tag{4.22}$$

$u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

Let us explain the relation between (4.21)-(4.22) and the time periodic problem (4.2), (4.7) and (4.10) for the reader's convenience.

If $\{u_{1,m}, u_\infty\}$ satisfies (4.2), (4.7) and (4.10), then $u_{1,m}(t)$ and $u_\infty(t)$ satisfy (4.21). Suppose that $\{u_{1,m}, u_\infty\}$ is a T-time periodic solution of (4.21). Then, since $u_{1,m}(T) = u_{1,m}(0)$ and $u_\infty(T) = u_\infty(0)$, we see that

$$\begin{cases} (I - S_1(T))u_{1,m}(0) = \mathscr{S}_1(T)[F_{1,m}(u,g)], \\ (I - S_{\infty,u}(T)u_\infty(0) = \mathscr{S}_{\infty,u}(T)[F_\infty(u,g)], \end{cases}$$

where $u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

Therefore if $(I - S_1(T))$ and $(I - S_{\infty,u}(T))$ are invertible in a suitable sense, then one obtains (4.21)-(4.22). So, to obtain a T-time periodic solution of (4.2), (4.7) and (4.10), we look for a pair of functions $\{u_{1,m}, u_\infty\}$ satisfying (4.21)-(4.22). We will investigate the solution operators $S_1(t)$ and $\mathscr{S}_1(t)$ in section 5; and we state some properties of $S_{\infty,u}(t)$ and $\mathscr{S}_{\infty,u}(t)$ in section 6.

In the remaining of this section we introduce some lemmas which will be used in the proof of Theorem 3.1.

For the analysis of the low frequency part, we will use the following well-known inequalities.

Lemma 4.7. *Let α and β be positive numbers satisfying $n < \alpha + \beta$. Then there holds the following estimate.*

$$\int_{\mathbb{R}^n} (1 + |x - y|)^{-\alpha} (1 + |y|^2)^{-\frac{\beta}{2}} dy \leq C \begin{cases} (1 + |x|)^{n-(\alpha+\beta)} & (\max\{\alpha, \beta\} < n), \\ (1 + |x|)^{-\min\{\alpha,\beta\}} \log |x| & (\max\{\alpha, \beta\} = n), \\ (1 + |x|)^{-\min\{\alpha,\beta\}} & (\max\{\alpha, \beta\} > n) \end{cases}$$

for $x \in \mathbb{R}^n$

The following lemma is related to the estimates for the integral kernels which will appear in the analysis of the low frequency part.

Lemma 4.8. *Let ℓ be a nonnegative integer and let $E(x) = \mathscr{F}^{-1}\hat{\Phi}_\ell$ $(x \in \mathbb{R}^n)$, where $\hat{\Phi}_\ell \in C^\infty(\mathbb{R}^n - \{0\})$ is a function satisfying*

$$\partial_\xi^\alpha \hat{\Phi}_\ell \in L^1 \quad (|\alpha| \leq n - 3 + \ell),$$
$$|\partial_\xi^\beta \hat{\Phi}_\ell| \leq C|\xi|^{-2-|\beta|+\ell} \quad (\xi \neq 0, \ |\beta| \geq 0).$$

Then the following estimate holds for $x \neq 0$.

$$|E(x)| \leq C|x|^{-(n-2+\ell)}.$$

Lemma 4.8 easily follows from a direct application of [13, Theorem 2.3]; and we omit the proof.

We will also use the following lemma for the analysis of the low frequency part.

Lemma 4.9. (i) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \leq \frac{C}{(1+|x|)^{|\alpha|+n-2}} \quad (|\alpha| = 0, 1, 2). \tag{4.23}$$

Assume that f is a scalar function satisfying $\|f\|_{L_n^\infty \cap L^1} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.

$$|[\partial_x^\alpha E * f](x)| \leq \frac{C}{(1+|x|)^{|\alpha|+n-2}} \|f\|_{L_n^\infty \cap L^1}.$$

(ii) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying* (4.23)*. Assume that f is a scalar function of the form: $f = \partial_{x_j} f_1$ for some $1 \leq j \leq n$ satisfying $\|\partial_{x_j} f_1\|_{L_n^\infty} + \|f_1\|_{L_{n-1}^\infty} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.*

$$|[\partial_x^\alpha E * f](x)| \leq \frac{C}{(1+|x|)^{|\alpha|+n-2}} (\|\partial_{x_j} f_1\|_{L_n^\infty} + \|f_1\|_{L_{n-1}^\infty}).$$

(iii) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \leq \frac{C}{(1+|x|)^{|\alpha|+n-1}} \quad (|\alpha| = 0, 1).$$

Assume that f is a scalar function satisfying $\|f\|_{L_n^\infty} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.

$$|[\partial_x^\alpha E * f](x)| \leq \frac{C \log|x|}{(1+|x|)^{|\alpha|+n-1}} \|f\|_{L_n^\infty}.$$

Lemma 4.9 (i) and (ii) is given in [14, Lemma 2.5] for $n = 3$ and the case $n \geq 4$ can be proved similarly; the assertion (iii) can be proved by a direct computation based on based on Lemma 4.7; and so the details are omitted here.

The following inequalities will be used to estimate the low frequency part of nonlinear terms.

Lemma 4.10. (i) *Let ℓ be a nonnegative integer satisfying $\ell \geq n-1$ and $E(x)$ be a scaler function satisfying that*

$$|E(x)| \leq \frac{C}{(1+|x|)^\ell} \quad \textit{for} \quad x \in \mathbb{R}^n.$$

Then for $f \in L_{n-1}^2$, it holds that

$$\|E * f\|_{L_{n-1}^\infty} \leq C\{\|(1+|y|)^{-\ell}\|_{L^2} \|f\|_{L_{n-1}^2} + \|f\|_{L_{n-1}^2}\}.$$

(ii) *Let $E(x)$ be a scaler function satisfying that*

$$|E(x)| \leq \frac{C}{(1+|x|)^{n-2}} \quad for \ \ x \in \mathbb{R}^n.$$

Then for $f \in L^1_{n-1}$*, it holds that*

$$\|E * f\|_{L^\infty_{n-1}} \leq C\|f\|_{L^1_{n-1}}.$$

Lemma 4.10 easily follows from direct computations; and we omit the proof.

The following Lemma is related to the weighted L^∞ estimate for the low frequency part.

Lemma 4.11.

$$\|F_1\|_{\mathscr{Y}_{(1),L\infty}} \leq C\|F_1\|_{L^2_{(1),n-1}}.$$

for $F_1 \in L^2_{(1),n-1}$.

Proof. We see that $\tilde{F}_1 = \chi_0 * F_1$, where $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$, $\hat{\chi}_0$ is the cut-off function defined by (4.3). Since $\hat{\chi}_0 \in \mathscr{S}$, we find that

$$|\partial_x^\alpha \chi_0(x)| \leq C(1+|x|)^{-(n+|\alpha|)} \quad \text{for} \ \ |\alpha| \geq 0. \tag{4.24}$$

Therefore, applying Lemma 4.3 and Lemma 4.10, we obtain the desired estimate. This completes the proof. $\square$

As for the high frequency part, we have the following inequalities given in [6, Lemma 4.4].

Lemma 4.12. (i) *Let* k *be a nonnegative integer. Then* P_∞ *is a bounded linear operator on* H^k*.*

(ii) *There hold the inequalities*

$$\begin{aligned} \|P_\infty f\|_{L^2} &\leq C\|\nabla f\|_{L^2} \quad (f \in H^1), \\ \|f_\infty\|_{L^2} &\leq C\|\nabla f_\infty\|_{L^2} \quad (f_\infty \in H^1_{(\infty)}). \end{aligned}$$

Lemma 4.13. *Let* $\ell \in \mathbb{N}$*. Then there exists a positive constant* C *depending only on* ℓ *such that*

$$\|P_\infty f\|_{L^2_\ell} \leq C\|\nabla f\|_{L^2_\ell}.$$

Lemma 4.13 follows from the inequalities

$$\||x|^k \nabla f_\infty\|_{L^2}^2 \geq \frac{r_1^2}{2}\||x|^k f_\infty\|_{L^2}^2 - C\||x|^{k-1} f_\infty\|_{L^2}^2 \quad (k = 1, \cdots, \ell)$$

for $f_\infty \in H^1_{(\infty),\ell}$ which are proved in [6, Lemma 4.7] by using the Plancherel theorem.

To estimate nonlinear and inhomogeneous terms, we need to estimate $w_1^{(1)} - w_1^{(2)}$ in therms of $\phi_1^{(1)} - \phi_1^{(2)}$, $\phi_\infty^{(1)} - \phi_\infty^{(2)}$, $m_1^{(1)} - m_1^{(2)}$ and $w_\infty^{(1)} - w_\infty^{(2)}$.

Let s be an integer satisfying $s \geq [\frac{n}{2}] + 1$. Let $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ and $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)})$ satisfy $\{u_{1,m}^{(k)}, u_\infty^{(k)}\} \in X^s(a,b)$. Assume that $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ satisfies $\sup_{t\in[a,b]} \|\phi^{(k)}\|_{L^\infty_{n-1}} \leq \delta_0$, which δ_0 is the one used in Lemma 4.6 for $(k = 1, 2)$. Then by Lemma 4.6 (i), There uniquely exist $w_1^{(k)} \in C([a,b]; \mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})$ satisfying

$$w_1^{(k)} = m_1^{(k)} + P_1(\phi^{(k)} w^{(k)})$$

where $w^{(k)} = w_1^{(k)} + w_\infty^{(k)}$ for $k = 1, 2$. Then $w_1^{(1)} - w_1^{(2)}$ satisfies

$$w_1^{(1)} - w_1^{(2)} = m_1^{(1)} - m_1^{(2)} - P_1(\phi^{(1)}(w^{(1)} - w^{(2)})) - P_1(w^{(2)}(\phi^{(1)} - \phi^{(2)})). \quad (4.25)$$

We obtain the following estimate for $w_1^{(1)} - w_1^{(2)}$.

Lemma 4.14. *It holds that*

$$\begin{aligned}
&\|w_1^{(1)} - w_1^{(2)}\|_{C([a,b];\mathscr{Y}_{(1)})\cap H^1(a,b;\mathscr{Y}_{(1)})} \\
&\quad \leq C\left(1 + \sum_{k=1}^{2} \|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(a,b)}\right) \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(a,b)}.
\end{aligned}$$

Lemma 4.14 directly follows from Lemma 2.1, Lemma 2.2, Lemma 4.3, Lemma 4.11 and (4.25); and we omit the proof.

5 Properties of $S_1(t)$ and $\mathscr{S}_1(t)$

In this section we investigate $S_1(t)$ and $\mathscr{S}_1(t)$ and establish estimates for a solution u_1 of

$$\partial_t u_1 + Au_1 = F_1 \quad (5.1)$$

satisfying $u_1(0) = u_1(T)$ where $F_1 = {}^\top(0, \tilde{F}_1)$.

We denote by A_1 the restriction of A on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$.

Proposition 5.1. (i) *A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and $S_1(t) = e^{-tA_1}$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. Furthermore, $S_1(t)$ satisfies*

$$S_1(t)u_1 \in C^1([0,T']; \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}), \quad \partial_t S_1(\cdot)u_1 \in C([0,T']; L^2)$$

for each $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and all $T' > 0$,

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \, (= -AS_1(t)u_1), \ S_1(0)u_1 = u_1 \ \textit{ for } \ u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty})} \le C\|u_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}},$$
$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2})} \le C\|u_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $k = 0, 1$,

$$\|\partial_t S_1(t)u_1\|_{C([0,T'];L^2)} \le C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}},$$

and

$$\|\partial_t \nabla S_1(t)u_1\|_{C([0,T'];L^2_1)} \le C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, *where* $T' > 0$ *is any given positive number and* C *is a positive constant depending on* T'.

(ii) *Let the operator* $\mathscr{S}_1(t)$ *be defined by*

$$\mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)\,d\tau$$

for $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$. *Then*

$$\mathscr{S}_1(\cdot)F_1 \in C^1([0,T];\mathscr{X}_{(1)}) \times [C([0,T];\mathscr{Y}_{(1)}) \times H^1(0,T;\mathscr{Y}_{(1)})]$$

for each $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$ *and*

$$\partial_t \mathscr{S}_1(t)F_1 + A_1\mathscr{S}_1(t)F_1 = F_1(t), \;\; \mathscr{S}_1(0)F_1 = 0,$$

$$\|\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1),L^p})\times L^2(0,T;\mathscr{Y}_{(1),L^p})},$$
$$\|\partial_t\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1),L^p})\times L^2(0,T;\mathscr{Y}_{(1),L^p})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1),L^p})\times L^2(0,T;\mathscr{Y}_{(1),L^p})},$$

for $p = 2, \infty$, *where* C *is a positive constant depending on* T. *If, in addition,* $F_1 \in C([0,T];L^2_1)$, *then* $\partial_t\mathscr{S}_1(\cdot)F_1 \in C([0,T];L^2)$, $\partial_t\nabla\mathscr{S}_1(\cdot)F_1 \in C([0,T];L^2_1)$,

$$\|\partial_t\mathscr{S}_1(\cdot)F_1\|_{C([0,T];L^2)} \le C\|F_1\|_{C([0,T];L^2)},$$

and

$$\|\partial_t\nabla\mathscr{S}_1(\cdot)F_1\|_{C([0,T];L^2_1)} \le C\|F_1\|_{C([0,T];L^2_1)},$$

where C *is a positive constant depending on* T.

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')F_1 = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ *and* $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$.

Proof of Proposition 5.1. Let

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top\xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top\xi \end{pmatrix} \quad (\xi \in \mathbb{R}^n).$$

Then, $\mathcal{F}(Au_1) = \hat{A}_\xi \hat{u}_1$. Hence, if $\mathrm{supp}\,\hat{u}_1 \subset \{\xi; |\xi| \le r_\infty\}$, then $\mathrm{supp}\,\hat{A}_\xi \hat{u}_1 \subset \{\xi; |\xi| \le r_\infty\}$. Furthermore, we see from Lemma 4.3 that

$$\|Au_1\|_{\mathscr{X}_{(1),L^p} \times \mathscr{Y}_{(1),L^p}} \le C\|u_1\|_{\mathscr{X}_{(1),L^p} \times \mathscr{Y}_{(1),L^p}}$$

for $p = 2, \infty$. Therefore, A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. It then follows that $-A_1$ generates a uniformly continuous semigroup $S_1(t) = e^{-tA_1}$ that is given by

$$S_1(t)u_1 = \mathcal{F}^{-1} e^{-t\hat{A}_\xi} \mathcal{F} u_1 \quad (u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}).$$

Furthermore, $S_1(t)$ satisfies $S_1(\cdot)u_1 \in C^1([0,T']; \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})$ for each $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, and

$$\partial_t S_1(t)u_1 = -A_1 S_1(t) u_1 \,(= -AS_1(t)u_1),\ S_1(0)u_1 = u_1 \ \text{ for } \ u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}.$$

It easily follows from the definition of $S_1(t)$ that

$$\|S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^p} \times \mathscr{Y}_{(1),L^p})} \le C\|u_1\|_{\mathscr{X}_{(1),L^p} \times \mathscr{Y}_{(1),L^p}} \ (p = 2, \infty) \ \text{ for } \ u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

and hence, by the relation that $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ and Lemma 4.3,

$$\|\partial_t S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^p} \times \mathscr{Y}_{(1),L^p})} \le C\|u_1\|_{\mathscr{X}_{(1),L^p} \times \mathscr{Y}_{(1),L^p}} \ (p = 2, \infty) \ \text{ for } \ u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

where $T' > 0$ is any given positive number and C is a positive constant depending on T'. In addition, we see from the relation $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ that $\partial_t S_1(\cdot)u_1 \in C([0,T']; L^2)$, $\partial_t \nabla S_1(\cdot)u_1 \in C([0,T']; L^2_1)$,

$$\|\partial_t S_1(\cdot)u_1\|_{C([0,T'];L^2)} \le C\|u_1\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}}$$

and

$$\|\partial_t \nabla S_1(\cdot)u_1\|_{C([0,T'];L^2_1)} \le C\|u_1\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}}.$$

The assertion (ii) follows from Lemma 4.3, the assertion (i) and the relation $\partial_t \mathscr{S}_1(t)[F_1] = -A_1 \mathscr{S}_1(t)[F_1] + F_1(t)$. The assertion (iii) easily follows from the definitions of $S_1(t)$ and $\mathscr{S}_1(t)$. This completes the proof. □

We next investigate invertibility of $I - S_1(T)$.

Proposition 5.2. *If F_1 satisfies the conditions given in* (i)-(iii)*, then, there uniquely exists $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ that satisfies $(I - S_1(T))u = F_1$ and u satisfies the estimates in each case of* (i)-(iii).

(i) $F_1 \in L^2_{(1),1} \cap L^\infty \cap L^1$;

$$\|u\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n} + \|F_1\|_{L^1}\}, \tag{5.2}$$

$$\|u\|_{\mathscr{X}_{(1),L^2} \times \mathscr{Y}_{(1),L^2}} \le C(\|F_1\|_{L^1} + \|F_1\|_{L^2_1}). \tag{5.3}$$

(ii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_n \cap L^2_{(1),1}$ *with* $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_{n-1}$ *for some* α *satisfying* $|\alpha| = 1$*;*

$$\|u\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n} + \|F_1^{(1)}\|_{L^\infty_{n-1}}\},$$
$$\|u\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C(\|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}).$$

(iii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^2_{(1)}$ *with* $F_1^{(1)} \in L^2_{(1),1} \cap L^\infty_n$ *for some* α *satisfying* $|\alpha| \ge 1$*;*

$$\|u\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\|F_1^{(1)}\|_{L^\infty_n}, \tag{5.4}$$
$$\|u\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C\|F_1^{(1)}\|_{L^2_1}. \tag{5.5}$$

To prove Proposition 5.2, we prepare some lemmas.

Lemma 5.3. ([10]) (i) *The set of all eigenvalues of* $-\hat{A}_\xi$ *consists of* $\lambda_j(\xi)\,(j = 1, \pm)$*, where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$*, then*

$$\mathrm{Re}\,\lambda_\pm = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2, \quad \mathrm{Im}\,\lambda_\pm = \pm\gamma|\xi|\sqrt{1 - \frac{(\nu+\tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *For* $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$*,* $e^{-t\hat{A}_\xi}$ *has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *is eigenprojections for* $\lambda_j(\xi)\,(j = 1, \pm)$*, and* $\Pi_j(\xi)\,(j = 1, \pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix},$$
$$\Pi_\pm(\xi) = \pm\frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} -\lambda_\mp & -i\gamma^\top\xi \\ -i\gamma\xi & \lambda_\pm\frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$*, then there exist a constant* $C > 0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \le C\,(j = 1, \pm) \tag{5.6}$$

hold for $|\xi| \le r_\infty$*.*

Hereafter we fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (5.6) in Lemma 5.3 holds for $|\xi| \le r_\infty$.

Lemma 5.4. *Let α be a multi-index. Then the following estimates hold true uniformly for ξ with $|\xi| \le r_\infty$ and $t \in [0,T]$.*

(i) $|\partial_\xi^\alpha \lambda_1| \le C|\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha \lambda_\pm| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 0)$.

(ii) $|(\partial_\xi^\alpha \Pi_1)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|$, $|(\partial_\xi^\alpha \Pi_\pm)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{F}_1|$ $(|\alpha| \ge 0)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(iii) $|\partial_\xi^\alpha (e^{\lambda_1 t})| \le C|\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1)$.

(iv) $|\partial_\xi^\alpha (e^{\lambda_\pm t})| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}_1| \le C(|\xi|^{1-|\alpha|}|\hat{F}_1^0| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|)$ $(|\alpha| \ge 1)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(vi) $|\partial_\xi^\alpha (I - e^{\lambda_1 t})^{-1}| \le C|\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0)$.

(vii) $|\partial_\xi^\alpha (I - e^{\lambda_\pm t})^{-1}| \le C|\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0)$.

Lemma 5.4 can be verified by direct computations based on Lemma 5.3.

Lemma 5.5. *Set*

$$E_{1,j}(x) := \mathcal{F}^{-1}(\hat{\chi}_0 (I - e^{\lambda_j T})^{-1}\Pi_j) \quad (j = 1, \pm), \quad (x \in \mathbb{R}^n)$$

where χ_0 is the cut-off function defined by (4.3). *Let α be a multi-index satisfying $|\alpha| \ge 0$. Then the following estimates hold true uniformly for $x \in \mathbb{R}^n$.*

(i) $|\partial_x^\alpha E_{1,1}(x)| \le C(1+|x|)^{-(n-2+|\alpha|)}$.

(ii) $|\partial_x^\alpha E_{1,\pm}(x)| \le C(1+|x|)^{-(n-1+|\alpha|)}$.

Proof. It follows from Lemma 5.4 that

$$\sum_j |\partial_x^\alpha E_{1,j}(x)| \le C \int_{|\xi| \le 2r_\infty} |\xi|^{-2} d\xi \quad (x \in \mathbb{R}^n).$$

Since $\int_{|\xi| \le r_\infty} |\xi|^{-2} d\xi < \infty$ for $n \ge 3$, we see that

$$\sum_j |\partial_x^\alpha E_{1,j}(x)| \le C \quad (x \in \mathbb{R}^n), \tag{5.7}$$

where $C > 0$ is a constant depending on α, T and n. By Lemma 5.4, we have

$$\begin{aligned}
|\partial_\xi^\beta ((i\xi)^\alpha \hat{\chi}_0 (I - e^{\lambda_1 T})^{-1}\Pi_1)| &\le C|\xi|^{-2+|\alpha|-|\beta|} \quad \text{for } |\beta| \ge 0, \\
|\partial_\xi^\beta ((i\xi)^\alpha \hat{\chi}_0 (I - e^{\lambda_\pm T})^{-1}\Pi_\pm)| &\le C|\xi|^{-1+|\alpha|-|\beta|} \quad \text{for } |\beta| \ge 0.
\end{aligned}$$

It then follows from Lemma 4.8 that

$$|\partial_x^\alpha E_{1,1}(x)| \le C|x|^{-(n-2+|\alpha|)} \ \text{ and } \ |\partial_x^\alpha E_{1,\pm}(x)| \le C|x|^{-(n-1+|\alpha|)}. \tag{5.8}$$

From (5.7) and (5.8), we obtain the desired estimates. This completes the proof. $\square$

Let us now prove Proposition 5.2.

Proof of Proposition 5.2. We define a function u by

$$u = \mathcal{F}^{-1}(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1.$$

(i) By using Lemma 5.4, one can easily obtain (5.3). As for (5.2), note that

$$u = \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1) = \sum_j E_{1,j} * F_1,$$

where $E_{1,j}$ is the ones defined in Lemma 5.5. Then by Lemma 5.5, we see that $\sum_j E_{1,j}$ satisfies

$$|\partial_x^\alpha \sum_j E_{1,j}(x)| \le C(1+|x|)^{-(n-2+|\alpha|)} \ \ (|\alpha| \ge 0).$$

Therefore, applying Lemma 4.9 (i), we obtain (5.2).

The assertion (ii) similarly follows from Lemma 4.9 (ii), Lemma 5.4 and Lemma 5.5.

(iii) By using Lemma 5.4, one can easily obtain (5.5). As for (5.4), if there exists a function $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_n$ satisfying $F_1 = \partial_x^\alpha F_1^{(1)}$ for some α satisfying $|\alpha| \ge 1$, then

$$u = \left(\sum_j \partial_x^\alpha E_{1,j}\right) * F_1^{(1)}.$$

Lemma 5.5 yields

$$|\sum_j \partial_x^{\alpha+\beta} E_{1,j}(x)| \le C(1+|x|)^{-(n-1+|\beta|)}$$

for $x \in \mathbb{R}^n$, $|\alpha| \ge 1$ and $|\beta| \ge 0$. It then follows from Lemma 4.9 (iii) that

$$\|u\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\|F_1^{(1)}\|_{L^\infty_n}.$$

This completes the proof. $\square$

In view of Proposition 5.2 (i), $I - S_1(T)$ has a bounded inverse $(I - S_1(T))^{-1}$: $L^2_{(1),1} \cap L^\infty \cap L^1 \to \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and it holds that

$$\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n} + \|F_1\|_{L^1}\},$$

$$\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}\le C(\|F_1\|_{L^1}+\|F_1\|_{L^2_1}).$$

If $F_1=\partial_x^\alpha F_1^{(1)}\in L^\infty_n\cap L^2_{(1),1}$ with $F_1^{(1)}\in L^2_{(1)}\cap L^\infty_{n-1}$ for some α satisfying $|\alpha|=1$, then $(I-S_1(T))^{-1}F_1\in\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}$ and

$$\begin{aligned}&\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}}\le C\{\|F_1\|_{L^\infty_n}+\|F_1^{(1)}\|_{L^\infty_{n-1}}\},\\&\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}\le C(\|F_1^{(1)}\|_{L^2}+\|F_1\|_{L^2_1}).\end{aligned}$$

Furthermore, if $F_1=\partial_x^\alpha F_1^{(1)}\in L^2_{(1)}$ with $F_1^{(1)}\in L^2_{(1),1}\cap L^\infty_n$ for some α satisfying $|\alpha|\ge 1$, then $(I-S_1(T))^{-1}F_1\in\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}$ and

$$\begin{aligned}&\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}}\le C\|F_1^{(1)}\|_{L^\infty_n},\\&\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}\le C\|F_1^{(1)}\|_{L^2_1}.\end{aligned}$$

We next estimate $S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1$ and $\mathscr{S}_1(t)F_1$. Let $E_1(t,\sigma)$ and $E_2(t,\tau)$ be defined by

$$\begin{aligned}&E_1(t,\sigma)=\mathcal{F}^{-1}\{\hat\chi_0e^{-t\hat A_\xi}(I-e^{-T\hat A_\xi})^{-1}e^{-(T-\sigma)\hat A_\xi}\},\\&E_2(t,\tau)=\mathcal{F}^{-1}\{\hat\chi_0e^{-(t-\tau)\hat A_\xi}\}\end{aligned}$$

for $\sigma\in[0,T]$, $0\le\tau\le t\le T$, where $\hat\chi_0$ is the cut-off function defined by (4.9). Then $\mathscr{S}_1(t)F_1$ and $S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1$ are given by

$$S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1=\int_0^T E_1(t,\sigma)*F_1(\sigma)d\sigma \tag{5.9}$$

$$\mathscr{S}_1(t)F_1=\int_0^t S_1(t-\tau)F_1(\tau)d\tau=\int_0^t E_2(t,\tau)*F_1(\tau)d\tau. \tag{5.10}$$

We have the following estimates for $E_1(t,\sigma)*F_1$ and $E_2(t,\tau)*F_1$.

Lemma 5.6. *If F_1 satisfies the conditions given in* (i)-(iii)*, then,* $E_1(t,\sigma)*F_1\in\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}$, $E_2(t,\tau)*F_1\in\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}$ $(t,\sigma,\tau\in[0,T],j=1,2)$ *and* $E_1(t,\sigma)*F_1,E_2(t,\tau)*F_1$ *satisfy the estimates in each case of* (i)-(iii)*.*

(i) $F_1\in L^2_{(1),1}\cap L^\infty\cap L^1$*;*

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}}+\|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}}\le C\{\|F_1\|_{L^\infty_n}+\|F_1\|_{L^1}\}$$

and

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}+\|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}\le C(\|F_1\|_{L^1}+\|F_1\|_{L^2_1})$$

uniformly for $\sigma\in[0,T]$ and $0\le\tau\le t\le T$.

(ii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L_n^\infty \cap L^2_{(1),1}$ *with* $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_{n-1}$ *for some* α *satisfying* $|\alpha| = 1$*;*

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L_n^\infty} + \|F_1^{(1)}\|_{L^\infty_{n-1}}\}$$

and

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C(\|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1})$$

uniformly for $\sigma \in [0,T]$ *and* $0 \le \tau \le t \le T$.

(iii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^2_{(1)}$ *with* $F_1^{(1)} \in L^2_{(1),1} \cap L_n^\infty$ *for some* α *satisfying* $|\alpha| \ge 1$*;*

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\|F_1^{(1)}\|_{L_n^\infty}$$

and

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C\|F_1^{(1)}\|_{L^2_1}$$

uniformly for $\sigma \in [0,T]$ *and* $0 \le \tau \le t \le T$.

Proof of Lemma 5.6. By Lemmas 5.3 and 5.4, we see that

$$\begin{aligned}
&|\partial_\xi^\beta(\hat{\chi}_0(i\xi)^\alpha e^{-t\hat{A}_\xi}(I - e^{-T\hat{A}_\xi})^{-1}e^{-(T-\sigma)\hat{A}_\xi})| \le C|\xi|^{-2+|\alpha|-|\beta|},\\
&|\partial_\xi^\beta(\hat{\chi}_0(i\xi)^\alpha e^{-(t-\tau)\hat{A}_\xi})| \le C|\xi|^{|\alpha|-|\beta|}
\end{aligned}$$

for $\sigma \in [0,T]$, $0 \le \tau \le t \le T$ and $|\beta| \ge 0$. It then follows from Lemma 4.8 that

$$|\partial_x^\alpha E_1(x)| \le C(1+|x|)^{-(n-2+|\alpha|)}, \quad |\partial_x^\alpha E_2(x)| \le C(1+|x|)^{-(n+|\alpha|)} \tag{5.11}$$

for $|\alpha| \ge 0$. Therefore, in a similarly manner to the proof of Proposition 5.2, we obtain the desired estimate by using Lemma 4.9 and Lemma 5.5. This completes the proof. $\square$

We see from Proposition 5.1 (i), (ii) and Lemma 5.6 that the following estimates hold for $S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}$ and $\mathscr{S}_1(t)$.

Proposition 5.7. *Let* Γ_1 *and* Γ_2 *be defined by*

$$\Gamma_1[\tilde{F}_1](t) = S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}\begin{pmatrix}0\\ \tilde{F}_1\end{pmatrix}, \ \Gamma_2[\tilde{F}_1](t) = \mathscr{S}_1(t)\begin{pmatrix}0\\ \tilde{F}_1\end{pmatrix} \tag{5.12}$$

If $\tilde{F}_1$ *satisfies the conditions given in* (i)-(iii)*, then,* $\Gamma_j[\tilde{F}_1] \in C^1([0,T];\mathscr{X}_{(1)}) \times [C([0,T];\mathscr{Y}_{(1)}) \cap H^1(0,T;\mathscr{Y}_{(1)})]$ $(j = 1,2)$ *and* $\Gamma_j[\tilde{F}_1]$ *satisfy the estimates in each case of* (i)-(iii) *for* $j = 1,2$.

(i) $\tilde{F}_1 \in L^2(0,T;L^2_{(1),1} \cap L^\infty \cap L^1 \cap \mathscr{Y}_{(1)})$*;*

$$\|\Gamma_1[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})}\le C\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)},$$
$$\|\partial_t\Gamma_1[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})}\le C\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)}$$

and

$$\|\Gamma_2[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})}\le C\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)},$$
$$\|\partial_t\Gamma_2[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})}\le C(\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)}+\|\tilde F_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).$$

(ii) $\tilde F_1=\partial_x^\alpha F_1^{(1)}\in L^2(0,T;L^\infty_n\cap L^2_{(1),1}\cap\mathscr{Y}_{(1)})$ *with* $F_1^{(1)}\in L^2(0,T;L^2_{(1)}\cap L^\infty_{n-1})$ *for some* α *satisfying* $|\alpha|=1$*;*

$$\|\Gamma_1[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})}\le C(\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)}+\|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}),$$
$$\|\partial_t\Gamma_1[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})}\le C(\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)}+\|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)})$$

and

$$\begin{aligned}\|\Gamma_2[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})}&\le C(\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)}+\|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}),\\ \|\partial_t\Gamma_2[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})}&\le C(\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)}+\|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}\\ &\quad+\|\tilde F_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).\end{aligned}$$

(iii) $\tilde F_1=\partial_x^\alpha F_1^{(1)}\in L^2(0,T;L^2_{(1)}\cap\mathscr{Y}_{(1)})$ *with* $F_1^{(1)}\in L^2(0,T;L^2_{(1),1}\cap L^\infty_n)$ *for some* α *satisfying* $|\alpha|\ge 1$*;*

$$\|\Gamma_1[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})}\le C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)},$$
$$\|\partial_t\Gamma_1[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})}\le C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)}$$

and

$$\|\Gamma_2[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})}\le C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)},$$
$$\|\partial_t\Gamma_2[\tilde F_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})}\le C(\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)}+\|\tilde F_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).$$

As for $\|\tilde F_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^p})}$ $(p=2,\infty)$, we have the following proposition.

Proposition 5.8. *If* $\tilde F_1$ *satisfies the conditions given in* (i)-(iii)*, then,* $\tilde F_1\in L^2(0,T;\mathscr{Y}_{(1)})$ *and* $\tilde F_1$ *satisfies the estimates in each case of* (i)-(iii)*.*

(i) $\tilde F_1\in L^2(0,T;L^2_{(1),1}\cap L^\infty\cap L^1)$*;*

$$\|\tilde F_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^\infty})}\le C\|\tilde F_1\|_{L^2(0,T;L^\infty_n\cap L^1)},$$

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^2})} \le C\|\tilde{F}_1\|_{L^2(0,T;L^1\cap L^2_1)}.$$

(ii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^\infty_n \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_{n-1})$ *for some* α *satisfying* $|\alpha| = 1$ *;*

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^\infty})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1})}),$$
$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^2})} \le C(\|F_1^{(1)}\|_{L^2(0,T;L^2)} + \|\tilde{F}_1\|_{L^2(0,T;L^2_1)}).$$

(iii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^2_{(1)})$ *with* $F_1^{(1)} \in L^2(0,T;L^2_{(1),1} \cap L^\infty_n)$ *for some* α *satisfying* $|\alpha| \ge 1$*;*

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^\infty})} \le C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n)},$$
$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^2})} \le C\|F_1^{(1)}\|_{L^2(0,T;L^2_1)}.$$

Proof of Proposition 5.8. We see that $\tilde{F}_1 = \chi_0 * F_1$, where $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$, $\hat{\chi}_0$ is the cut-off function defined by (4.3) satisfying (4.24). Therefore, in a similar manner to the proof of Proposition 5.2, we obtain the desired estimates. This completes the proof. □

We will also need another type of estimates for Γ_1 and Γ_2. We set

$$\Gamma_0[\tilde{F}_1] := (I - S_1(T))^{-1}\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}.$$

Proposition 5.9. (i) *Let* α *be a multi-index satisfying* $|\alpha| \ge 0$. *Suppose that* $\tilde{F}_1 \in L^1_{n-1} \cap L^2_{(1)}$. *Then* $\Gamma_0[\partial_x^\alpha \tilde{F}_1] \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ *and it holds that*

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\|\tilde{F}_1\|_{L^1_{n-1}}.$$

If $\tilde{F}_1 \in L^2(0,T;L^1_{n-1} \cap L^2_{(1)})$, *then, for* $j = 1,2$, $\Gamma_j[\partial_x^\alpha \tilde{F}_1] \in \mathscr{Z}_{(1)}(0,T)$ *and it holds that*

$$\|\Gamma_j[\partial_x^\alpha \tilde{F}_1](t)\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\tilde{F}_1\|_{L^2(0,T;L^1_{n-1})}.$$

(ii) *Let* α *be a multi-index satisfying* $|\alpha| \ge 1$. *Suppose that* $\tilde{F}_1 \in L^2_{(1),n-1}$. *Then* $\Gamma_0[\partial_x^\alpha \tilde{F}_1] \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ *and it holds that*

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\|\tilde{F}_1\|_{L^2_{n-1}}.$$

If $\tilde{F}_1 \in L^2(0,T;L^2_{(1),n-1})$, *then, for* $j = 1,2$, $\Gamma_j[\partial_x^\alpha \tilde{F}_1] \in \mathscr{Z}_{(1)}(0,T)$ *and it holds that*

$$\|\Gamma_j[\partial_x^\alpha \tilde{F}_1](t)\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\tilde{F}_1\|_{L^2(0,T;L^2_{n-1})}.$$

Proof of Proposition 5.9. (i) We have already obtained the estimate for $\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}$ in (5.3). We see from Lemma 5.5 and Lemma 4.10 (ii) that

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{L^\infty_{n-1}} \leq C\|\tilde{F}_1\|_{L^1_{n-1}}.$$

Therefore, by Lemma 4.3, we find that

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \leq C\|\tilde{F}_1\|_{L^1_{n-1}}.$$

Similarly, the estimates of Γ_j $(j=1,2)$ follow from (4.24), Lemma 4.10 (ii), Proposition 5.1, (5.9), (5.10) and (5.11).

The assertion (ii) can be proved similarly from (4.24), Lemma 4.10 (i), Proposition 5.1, (5.9), (5.10) and (5.11). This completes the proof. $\square$

We are now in a position to give estimates for a solution of (5.1) satisfying $u_1(0) = u_1(T)$.

For $F_1 = {}^\top(0, \tilde{F}_1)$ we set

$$\Gamma[\tilde{F}_1] = S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}F_1 + \mathscr{S}_1(t)F_1.$$

Then $\Gamma[\tilde{F}_1]$ is written as

$$\Gamma[\tilde{F}_1](t) = \Gamma_1[\tilde{F}_1] + \Gamma_2[\tilde{F}_1], \tag{5.13}$$

where Γ_1 and Γ_2 are the ones defined by (5.12).

Proposition 5.10. *If $\tilde{F}_1$ satisfies the conditions given in* (i)-(v)*, then, $\Gamma[\tilde{F}_1]$ is a solution of* (5.1) *with $F_1 = {}^\top(0,\tilde{F}_1)$ in $\mathscr{Z}_{(1)}(0,T)$ satisfying $\Gamma[\tilde{F}_1](0) = \Gamma[\tilde{F}_1](T)$ and $\Gamma[\tilde{F}_1]$ satisfies the estimate in each case of* (i)-(v)*.*

(i) $\tilde{F}_1 \in L^2(0,T; L^2_{(1),1} \cap L^\infty \cap L^1)$*;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n \cap L^1 \cap L^2_1)}. \tag{5.14}$$

(ii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^\infty_n \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1)} \cap L^\infty_{n-1})$ *for some α satisfying $|\alpha| = 1$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}). \tag{5.15}$$

(iii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^2_{(1)})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1),1} \cap L^\infty_n)$ *for some α satisfying $|\alpha| \geq 1$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)}. \tag{5.16}$$

(iv) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^1_{n-1} \cap L^2_{(1)})$ *for some α satisfying $|\alpha| \geq 0$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^1_{n-1})}. \tag{5.17}$$

(v) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^2_{(1),n-1})$ *for some* α *satisfying* $|\alpha| \geq 1$*;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^2_{n-1})}. \tag{5.18}$$

Proof. We find from Proposition 5.1 (iii), Proposition 5.2 and Proposition 5.9 that $\Gamma[\tilde{F}_1]$ is a solution of (5.1) with $F_1 = {}^\top(0, \tilde{F}_1)$ satisfying $\Gamma[\tilde{F}_1](0) = \Gamma[\tilde{F}_1](T)$. The estimates of $\Gamma[\tilde{F}_1]$ in (i)-(iii) follow from Proposition 5.7 and Proposition 5.8. We obtain the estimates of $\Gamma[\tilde{F}_1]$ in (iv) and (v) by Proposition 5.9. This completes the proof. □

6 Properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$

In this section we state some properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in weighted Sobolev spaces which were obtained in [6].

Let us consider the following initial value problem (4.20). Concerning the solvability of (4.20), we have the following

Proposition 6.1. ([6, Proposition 6.4]) *Let* $n \geq 3$ *and let* s *be an integer satisfying* $s \geq [\frac{n}{2}] + 1$*. Set* $k = s-1$ *or* s*. Assume that*

$$\begin{aligned}
&\nabla\tilde{w} \in C([0,T'];H^{s-1}) \cap L^2(0,T';H^s),\\
&u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},\\
&F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T';H^k_{(\infty)} \times H^{k-1}_{(\infty)}).
\end{aligned}$$

Here T' *is a given positive number. Then there exists a unique solution* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *of* (4.20) *satisfying*

$$\phi_\infty \in C([0,T'];H^k_{(\infty)}),\ w_\infty \in C([0,T'];H^k_{(\infty)}) \cap L^2(0,T';H^{k+1}_{(\infty)}) \cap H^1(0,T';H^{k-1}_{(\infty)}).$$

Remark 6.2. Concerning the condition for $\tilde{w}$, it is assumed in [6, Proposition 6.4] that $\tilde{w} \in C([0,T'];H^s) \cap L^2(0,T';H^{s+1})$. However, by taking a look at the proof of [6, Proposition 6.4], it can be replaced by the condition that $\nabla\tilde{w} \in C([0,T'];H^{s-1}) \cap L^2(0,T';H^s)$.

In view of Proposition 6.1, $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ are defined as follows.

We fix an integer s satisfying $s \geq [\frac{n}{2}] + 1$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfying

$$\tilde{\phi} \in C_{per}(\mathbb{R};H^s),\ \nabla\tilde{w} \in C_{per}(\mathbb{R};H^{s-1}) \cap L^2_{per}(\mathbb{R};H^s) \tag{6.1}$$

Let $k = s-1$ or s. The operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ is defined by

$$u_\infty(t) = S_{\infty,\tilde{u}}(t)u_{0\infty} \text{ for } u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_\infty(t)$ is the solution of (4.20) with $F_\infty = 0$; and the operator $\mathscr{S}_{\infty,\tilde{u}}(t) : L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ is defined by

$$u_\infty(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \quad \text{for} \quad F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_\infty(t)$ is the solution of (4.20) with $u_{0\infty} = 0$.

The operators $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ have the following properties.

Proposition 6.3. ([6, Proposition 6.5]) *Let $n \geq 3$ and let s be a nonnegative integer satisfying $s \geq [\frac{n}{2}] + 1$. Let $k = s-1$ or s and let ℓ be a nonnegative integer. Assume that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (6.1). Then there exists a constant $\delta > 0$ such that the following assertions hold true if $\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta$.*

(i) *It holds that $S_{\infty,\tilde{u}}(\cdot)u_{0\infty} \in C([0,\infty); H^k_{(\infty),\ell})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell}$ and there exist constants $a > 0$ and $C > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|S_{\infty,\tilde{u}}(t)u_{0\infty}\|_{H^k_{(\infty),\ell}} \leq Ce^{-at}\|u_{0\infty}\|_{H^k_{(\infty),\ell}}$$

for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),\ell}$.

(ii) *It holds that $\mathscr{S}_{\infty,\tilde{u}}(\cdot)F_\infty \in C([0,T]; H^k_{(\infty),\ell})$ for each $F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty]\|_{H^k_{(\infty),\ell}} \leq C\left\{\int_0^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell}} d\tau\right\}^{\frac{1}{2}}$$

for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ with a positive constant C depending on T.

(iii) *It holds that $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T)) < 1$.*

(iv) *$I - S_{\infty,\tilde{u}}(T)$ has a bounded inverse $(I - S_{\infty,\tilde{u}}(T))^{-1}$ on $H^k_{(\infty),\ell}$ and $(I - S_{\infty,\tilde{u}}(T))^{-1}$ satisfies*

$$\|(I - S_{\infty,\tilde{u}}(T))^{-1}u\|_{H^k_{(\infty),\ell}} \leq C\|u\|_{H^k_{(\infty),\ell}} \quad \text{for} \quad u \in H^k_{(\infty),\ell}.$$

Remark 6.4. In [6, Proposition 6.5] , it is assumed that

$$\|\tilde{w}\|_{C([0,T];H^s) \cap L^2(0,T;H^{s+1})} \leq \delta.$$

However, by taking a look at the proof of [6, Proposition 6.5, Proposition 7.1], it can be replaced by the condition that

$$\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta.$$

Applying Proposition 6.3, we easily obtain the following estimate for a solution u_∞ of (4.20) satisfying $u_\infty(0) = u_\infty(T)$.

Proposition 6.5. *Let $n \geq 3$ and let s be a nonnegative integer satisfying $s \geq [\frac{n}{2}] + 1$. Assume that*

$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),n-1} \times H^{k-1}_{(\infty),n-1})$$

with $k = s-1$ or s. Assume also that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (6.1) *. Then there exists a positive constant δ such that the following assertion holds true if*

$$\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta.$$

The function

$$u_\infty(t) := S_{\infty,\tilde{u}}(t)(I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)[F_\infty] + \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \tag{6.2}$$

is a solution of (4.20) *in $\mathscr{Z}^k_{(\infty),n-1}(0,T)$ satisfying $u_\infty(0) = u_\infty(T)$ and the estimate*

$$\|u_\infty\|_{\mathscr{Z}^k_{(\infty),n-1}(0,T)} \leq C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),n-1} \times H^{k-1}_{(\infty),n-1})}.$$

7 Proof of Theorem 3.1

In this section we give a proof of Theorem 3.1.

We first establish the estimates for the nonlinear and inhomogeneous terms $F_{1,m}(u,g)$ and $F_\infty(u,g)$:

$$F_{1,m}(u,g) = \begin{pmatrix} 0 \\ \tilde{F}_{1,m}(u,g) \end{pmatrix},$$

$$F_\infty(u,g) = P_\infty \begin{pmatrix} -\gamma w \cdot \nabla \phi_1 + F^0(u) \\ \tilde{F}(u,g) \end{pmatrix} =: \begin{pmatrix} F^0_\infty(u) \\ \tilde{F}_\infty(u,g) \end{pmatrix},$$

where $\tilde{F}_{1,m}(u,g)$, $F^0(u)$ and $\tilde{F}(u,g)$ are the same ones defined in (4.8), (3.5) and (3.6), respectively, $u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

We first state the estimates for $F_1(u,g)$ and $F_\infty(u,g)$.

For the estimates of the low frequency part, we recall that

$$\Gamma[\tilde{F}_1](t) := S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1} \begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix} + \mathscr{S}_1(t) \begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}$$

We first show the estimate of $\|\Gamma[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}(0,T)}$.

Proposition 7.1. *Let* $u_{1,m} = {}^\top(\phi_1, m_1)$ *and* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *satisfy*

$$\sup_{0\le t\le T} \|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u_\infty(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T}\|\phi(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 4.6 (i) *and* $\phi = \phi_1 + \phi_\infty$*. Then it holds that*

$$\|\Gamma[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}^{(1)}(0,T)} \le C\|\{u_{1,m}, u_\infty\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{1,m}, u_\infty\}\|_{X^s(0,T)}\Big)[g]_s$$

uniformly for $u_{1,m}$ *and* u_∞*.*

Proof. For $u^{(j)} = {}^\top(\phi^{(j)}, w^{(j)})$ $(j = 1, \infty)$, we set

$$\begin{aligned}
G_1(u^{(1)}, u^{(2)}) &= -P_1(\gamma \mathrm{div}\, w^{(1)} \otimes w^{(2)}),\\
G_2(u^{(1)}, u^{(2)}) &= -P_1(\mu\Delta(\phi^{(1)}w^{(2)}) + \tilde{\mu}\nabla \mathrm{div}\,(\phi^{(1)}w^{(2)})),\\
G_3(\phi, u^{(1)}, u^{(2)}) &= -P_1\left(\frac{\rho_*}{\gamma}\nabla(P^{(1)}(\phi)\phi^{(1)}\phi^{(2)}) + \gamma \mathrm{div}\,(\phi w^{(1)} \otimes w^{(2)})\right),\\
H_k(u^{(1)}, u^{(2)}) &= G_k(u^{(1)}, u^{(2)}) + G_k(u^{(2)}, u^{(1)}), \quad (k = 1, 2),\\
H_3(\phi, u^{(1)}, u^{(2)}) &= G_3(\phi, u^{(1)}, u^{(2)}) + G_3(\phi, u^{(2)}, u^{(1)}).
\end{aligned}$$

Then, $\Gamma[\tilde{F}_{1,m}(u,g)]$ is written as

$$\begin{aligned}
\Gamma[\tilde{F}_{1,m}(u,g)] &= \sum_{k=1}^{2}(\Gamma[G_k(u_1,u_1)] + \Gamma[H_k(u_1,u_\infty)] + \Gamma[G_k(u_\infty,u_\infty)])\\
&\quad + \Gamma[G_3(\phi,u_1,u_1)] + \Gamma[H_3(\phi,u_1,u_\infty)] + \Gamma[G_3(\phi,u_\infty,u_\infty)]\\
&\quad + \Gamma\left[\frac{1}{\gamma}(1+\phi_1)g\right] + \Gamma\left[\frac{1}{\gamma}\phi_\infty g\right].
\end{aligned}$$

Applying (5.15) to $\Gamma[G_1(u_1,u_1)]$, we have

$$\|\Gamma[G_1(u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.$$

As for $\Gamma[G_2(u_1,u_1)]$ and $\Gamma[G_3(\phi,u_1,u_1)]$, we apply (5.16) with $F_1^{(1)} = \phi_1 w_1$ $(|\alpha| = 2)$, $F_1^{(1)} = P^{(1)}(\phi)\phi_1^2$ $(|\alpha| = 1)$, and $F_1^{(1)} = \phi w_1 \otimes w_1$ $(|\alpha| = 1)$ to obtain

$$\begin{aligned}
&\|\Gamma[G_2(u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)},\\
&\|\Gamma[G_3(\phi,u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}$$

By (5.17), we have

$$\begin{aligned}
&\|\sum_{k=1}^{2}\Gamma[G_k(u_\infty,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)},\\
&\|\Gamma[G_3(\phi,u_\infty,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}$$

By (5.18), we also have

$$\begin{aligned}
&\|\sum_{k=1}^{2}\Gamma[G_k(u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)},\\
&\|\Gamma[G_3(\phi,u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}$$

Concerning $\Gamma[(1+\phi_1)g]$ and $\Gamma[\phi_\infty g]$, we see from (5.14) and (5.17) that

$$\|\Gamma[(1+\phi_1)g]\|_{\mathscr{Z}_{(1)}(0,T)} + \|\Gamma[\phi_\infty g]\|_{\mathscr{Z}_{(1)}(0,T)} \le C(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)})[g]_s.$$

Therefore, we find that

$$\|\Gamma[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}^{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)} + C\Big(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)}\Big)[g]_s.$$

Applying Lemma 4.6 (i), we obtain the desired estimate. This completes the proof. □

We next show the estimates for the nonlinear and inhomogeneous terms of the high frequency part.

Proposition 7.2. *Let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy*

$$\sup_{0\le t\le T}\|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u_\infty(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T}\|\phi(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 is the one in Lemma 4.6 (i) *and $\phi = \phi_1 + \phi_\infty$. Then it holds that*

$$\begin{aligned}
&\|F_\infty(u,g)\|_{L^2(0,T;H^s_{n-1}\times H^{s-1}_{n-1})}\\
&\quad\le\ C\|\{u_{1,m},u_\infty\}\|^2_{X^s(0,T)} + C\Big(1+\|\{u_{1,m},u_\infty\}\|_{X^s(0,T)}\Big)[g]_s
\end{aligned}$$

uniformly for $u_{1,m}$ and u_∞.

Proof. We here estimate only $P_\infty(w\cdot\nabla w)$, since the computation is not straightforward due to the slow decay of w_1 as $|x|\to\infty$. By Lemma 4.13, we see that

$$\begin{aligned}
\|P_\infty(w\cdot\nabla w)\|_{L^2_{n-1}} &\le \|\nabla(w\cdot\nabla w)\|_{L^2_{n-1}}\\
&\le C\|\nabla w\cdot\nabla w\|_{L^2_{n-1}} + \|w\cdot\nabla^2 w\|_{L^2_{n-1}}\\
&\le C(\|(1+|x|)^{n-1}\nabla w\|_{L^\infty}\|\nabla w\|_{L^2}\\
&\qquad +\|(1+|x|)^{n-2}w\|_{L^\infty}\|(1+|x|)\nabla^2 w\|_{L^2}). \qquad (7.1)
\end{aligned}$$

For $1\le|\alpha|\le s-1$, by Lemma 2.1, Lemma 2.3, Lemma 4.3 and Lemma 4.12, we see that

$$\begin{aligned}
&\|P_\infty\partial_x^\alpha(w\cdot\nabla w)\|_{L^2_{n-1}}\\
&\quad\le\ \|w\cdot\partial_x^\alpha\nabla w\|_{L^2_{n-1}} + \|[\partial_x^\alpha, w]\cdot\nabla w\|_{L^2_{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big\{\sum_{j=0}^{1}(\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty}+\|w_\infty\|_{H^s_{n-1}})\Big\}\\
&\qquad\times\Big\{\sum_{j=1}^{2}(\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}})\Big\}. \qquad (7.2)
\end{aligned}
$$

It follows from (7.1) and (7.2) that

$$
\begin{aligned}
&\|P_\infty(w\cdot\nabla w)\|_{H^{s-1}_{n-1}}\\
&\le C\Big\{\sum_{j=0}^{1}\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty}+\|w_\infty\|_{H^s_{n-1}}\Big\}\Big\{\sum_{j=1}^{2}\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}})\Big\}.
\end{aligned}
$$

Similarly to (7.2), the remaining terms can be estimated by a straightforward application of Lemma 2.1, Lemma 2.3, Lemma 4.3 and Lemma 4.12. We thus arrive at

$$
\begin{aligned}
&\|F^0_\infty(u)\|_{H^s_{n-1}}\\
&\quad\le C\{(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty}+\|\nabla\phi_1\|_{L^2}+\|\phi_\infty\|_{H^s_{n-1}})\\
&\qquad\times(\|(1+|x|)^{n-1}\nabla w_1\|_{L^\infty}+\|\nabla w_1\|_{L^2}+\|w_\infty\|_{H^{s+1}_{n-1}})\\
&\qquad+(\|(1+|x|)^{n-2}w_1\|_{L^\infty}+\|\nabla w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}})\\
&\qquad\times(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty}+\|(1+|x|)\nabla\phi_1\|_{L^2})\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\tilde F_\infty(u,g))\|_{H^{s-1}_{n-1}}\\
&\le C\Big\{\Big(\sum_{j=0}^{1}\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty}+\|w_\infty\|_{H^s_{n-1}}\Big)\Big(\sum_{j=1}^{2}\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}}\Big)\\
&\qquad+(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty}+\|\phi_\infty\|_{H^s_{n-1}})(\|\nabla\phi\|_{H^{s-1}}+\|\partial_t w\|_{H^{s-1}}+\|g\|_{H^{s-1}})\Big\}.
\end{aligned}
$$

Integrating these inequalities on $(0,T)$ and applying Lemma 4.6 (i), we obtain the desired estimate. This completes the proof. $\square$

We next estimate $F_{1,m}(u^{(1)},g)-F_{1,m}(u^{(2)},g)$.

Proposition 7.3. *Let* $u^{(k)}_{1,m}={}^\top(\phi^{(k)}_1,m^{(k)}_1)$ *and* $u^{(k)}_\infty={}^\top(\phi^{(k)}_\infty,w^{(k)}_\infty)$ *satisfy*

$$
\sup_{0\le t\le T}\|u^{(k)}_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\le t\le T}\|u^{(k)}_\infty(t)\|_{H^s_{n-1}}+\sup_{0\le t\le T}\|\phi^{(k)}(t)\|_{L^\infty}\le\min\{\delta_0,\frac{1}{2}\},
$$

where δ_0 *is the one in Lemma* 4.6 (i) *and* $\phi^{(k)}=\phi^{(k)}_1+\phi^{(k)}_\infty$ $(k=1,2)$*. Then it holds that*

$$
\|\Gamma[\tilde F_{1,m}(u^{(1)},g)-\tilde F_{1,m}(u^{(2)},g)]\|_{\mathscr{Z}^{(1)}(0,T)}
$$

$$\leq C\sum_{k=1}^{2}\|\{u_{1,m}^{(k)},u_\infty^{(k)}\}\|_{X^s(0,T)}\|\{u_{1,m}^{(1)}-u_{1,m}^{(2)},u_\infty^{(1)}-u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}$$
$$+C[g]_s\|\{u_{1,m}^{(1)}-u_{1,m}^{(2)},u_\infty^{(1)}-u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$.

Proposition 7.3 can be proved in a similar manner to the proof of Proposition 7.1; and we omit the proof.

We next estimate $F_\infty(u^{(1)},g)-F_\infty(u^{(2)},g)$.

Proposition 7.4. *Let* $u_{1,m}^{(k)}={}^\top(\phi_1^{(k)},m_1^{(k)})$ *and* $u_\infty^{(k)}={}^\top(\phi_\infty^{(k)},w_\infty^{(k)})$ *satisfy*

$$\sup_{0\leq t\leq T}\|u_{1,m}^{(k)}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\leq t\leq T}\|u_\infty^{(k)}(t)\|_{H^s_{n-1}}+\sup_{0\leq t\leq T}\|\phi^{(k)}(t)\|_{L^\infty}\leq\min\{\delta_0,\frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 4.6 (i) *and* $\phi^{(k)}=\phi_1^{(k)}+\phi_\infty^{(k)}$ $(k=1,2)$. *Then it holds that*

$$\|F_\infty(u^{(1)},g)-F_\infty(u^{(2)},g)]\|_{L^2(0,T;H^{s-1}_{n-1}\times H^{s-2}_{n-1})}$$
$$\leq C\sum_{k=1}^{2}\|\{u_{1,m}^{(k)},u_\infty^{(k)}\}\|_{X^s(0,T)}\|\{u_{1,m}^{(1)}-u_{1,m}^{(2)},u_\infty^{(1)}-u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}$$
$$+C[g]_s\|\{u_{1,m}^{(1)}-u_{1,m}^{(2)},u_\infty^{(1)}-u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$.

Proposition 7.4 directly follows from Lemmas 2.1–2.3, Lemma 4.3, Lemma 4.12 and Lemma 4.13 in a similar manner to the proof of Proposition 7.2.

We next show the following estimate which will be used in the proof of Proposition 7.6.

Proposition 7.5. (i) *Let* $u_{1,m}={}^\top(\phi_1,m_1)$ *and* $u_\infty={}^\top(\phi_\infty,w_\infty)$ *satisfy*

$$\sup_{0\leq t\leq T}\|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\leq t\leq T}\|u_\infty(t)\|_{H^s_{n-1}}+\sup_{0\leq t\leq T}\|\phi(t)\|_{L^\infty}\leq\min\{\delta_0,\frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 4.6 (i) *and* $\phi=\phi_1+\phi_\infty$. *Then it holds that*

$$\|F_{1,m}(u,g)\|_{C([0,T];L^2_1)}\leq C\|\{u_{1,m},u_\infty\}\|^2_{X^s(0,T)}+C\Big(1+\|\{u_{1,m},u_\infty\}\|_{X^s(0,T)}\Big)[g]_s$$

uniformly for $u_{1,m}$ *and* u_∞.

(ii) *Let* $u_{1,m}^{(k)} = {}^{\top}(\phi_1^{(k)}, m_1^{(k)})$ *and* $u_\infty^{(k)} = {}^{\top}(\phi_\infty^{(k)}, w_\infty^{(k)})$ *satisfy*

$$\sup_{0\le t\le T} \|u_{1,m}^{(k)}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty^{(k)}(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T} \|\phi^{(k)}(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 4.6 (i) *and* $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ $(k = 1,2)$*. Then it holds that*

$$\begin{aligned}
&\|F_{1,m}(u^{(1)}, g) - F_{1,m}(u^{(2)}, g)\|_{L^2_1} \\
&\le \quad C\sum_{k=1}^{2} \|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\
&\qquad + C[g]_s \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}
\end{aligned}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$*.*

Proof. As for (i), we here estimate ϕg. By using the Hardy inequality, since $n \ge 3$, we see that

$$\|\phi g\|_{L^2_1} \le C\|\frac{\phi}{|x|}\|_{L^2} \|(1+|x|)^{n-1} g\|_{L^\infty} \le C\|\nabla\phi\|_{L^2} \|(1+|x|)^{n-1} g\|_{L^\infty}.$$

Similarly, we can estimate the remaining terms by using Lemma 2.1, Lemma 4.3 and the Hardy inequality to obtain

$$\begin{aligned}
&\|F_{1,m}(u, g)\|_{L^2_1} \\
&\le \quad C\Big\{(\|(1+|x|)^{n-1}\phi\|_{L^\infty} + \|(1+|x|) w_1\|_{L^\infty} + \|w_\infty\|_{H^s_1})(\|\nabla w_1\|_{L^2} + \|\nabla w_\infty\|_{L^2}) \\
&\qquad + \|\nabla\phi\|_{L^2}(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty} + \|\phi_\infty\|_{H^s_{n-1}} + \|(1+|x|)^{n-1} g\|_{L^\infty}) + \|g\|_{L^2_1}\Big\}.
\end{aligned}$$

Applying Lemma 4.6 (i), we obtain the desired estimate (i).

The desired estimate in (ii) can be similarly obtained by applying Lemma 2.1, Lemma 2.2, Lemma 4.3 and the Hardy inequality. This completes the proof. □

To prove Theorem 3.1, we next show the existence of a solution $\{u_{1,m}, u_\infty\}$ of (4.2), (4.7) and (4.10) on $[0,T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$ by an iteration argument.

For $N = 0$, we define $u_{1,m}^{(0)} = {}^{\top}(\phi_1^{(0)}, m_1^{(0)})$ and $u_\infty^{(0)} = {}^{\top}(\phi_\infty^{(0)}, w_\infty^{(0)})$ by

$$\left\{\begin{array}{ll}
u_{1,m}^{(0)}(t) & = S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}\mathbb{G}_1] + \mathscr{S}_1(t)[\mathbb{G}_1], \\
w_1^{(0)} & = m_1^{(1)} - P_1(\phi^{(0)} w^{(0)}), \\
u_\infty^{(0)}(t) & = S_{\infty,0}(t)(I - S_{\infty,0}(T))^{-1}\mathscr{S}_{\infty,0}(T)[\mathbb{G}_\infty] + \mathscr{S}_{\infty,0}(t)[\mathbb{G}_\infty],
\end{array}\right. \tag{7.3}$$

where $t \in [0,T]$, $\mathbb{G} = {}^{\top}(0, \frac{1}{\gamma} g(x,t))$, $\mathbb{G}_1 = P_1\mathbb{G}$, $\mathbb{G}_\infty = P_\infty\mathbb{G}$, $\phi^{(0)} = \phi_1^{(0)} + \phi_\infty^{(0)}$ and $w^{(0)} = w_1^{(0)} + w_\infty^{(0)}$. Note that $u_{1,m}^{(0)}(0) = u_{1,m}^{(0)}(T)$ and $u_\infty^{(0)}(0) = u_\infty^{(0)}(T)$.

For $N \geq 1$, we define $u_{1,m}^{(N)} = {}^\top(\phi_1^{(N)}, m_1^{(N)})$ and $u_\infty^{(N)} = {}^\top(\phi_\infty^{(N)}, w_\infty^{(N)})$, inductively, by

$$\left\{\begin{array}{ll} u_{1,m}^{(N)}(t) & = S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}F_{1,m}(u^{(N-1)},g)] + \mathscr{S}_1(t)[F_{1,m}(u^{(N-1)},g)], \\ w_1^{(N)} & = m_1^{(N)} - P_1(\phi^{(N)}w^{(N)}), \\ u_\infty^{(N)}(t) & = S_{\infty,u^{(N-1)}}(t)(I-S_{\infty,u^{(N-1)}}(T))^{-1}\mathscr{S}_{\infty,u^{(N-1)}}(T)[F_\infty(u^{(N-1)},g)] \\ & \quad +\mathscr{S}_{\infty,u^{(N-1)}}(t)[F_\infty(u^{(N-1)},g)], \end{array}\right. \tag{7.4}$$

where $t \in [0,T]$, $u^{(N-1)} = u_1^{(N-1)} + u_\infty^{(N-1)}$, $u_1^{(N-1)} = {}^\top(\phi_1^{(N-1)}, w_1^{(N-1)})$, $\phi^{(N)} = \phi_1^{(N)} + \phi_\infty^{(N)}$ and $w^{(N)} = w_1^{(N)} + w_\infty^{(N)}$. Note that $u_{1,m}^{(N)}(0) = u_{1,m}^{(N)}(T)$ and $u_\infty^{(0)}(0) = u_\infty^{(0)}(T)$.

Proposition 7.6. *There exists a constant $\delta_1 > 0$ such that if $[g]_s \leq \delta_1$, then there holds the estimates*

(i)
$$\|\{u_{1,m}^{(N)}, u_\infty^{(N)}\}\|_{X^s(0,T)} \leq C_1[g]_s$$

for all $N \geq 0$, and

(ii)
$$\begin{aligned} &\|\{u_{1,m}^{(N+1)} - u_1^{(N)}, u_\infty^{(N+1)} - u_\infty^{(N)}\}\|_{X^{s-1}(0,T)} \\ &\leq \ C_1[g]_s\|\{u_{1,m}^{(N)} - u_{1,m}^{(N-1)}, u_\infty^{(N)} - u_\infty^{(N-1)}\}\|_{X^{s-1}(0,T)} \end{aligned}$$

for $N \geq 1$. Here C_1 is a constant independent of g and N.

Proof. If $[g]_s \leq \delta_1$ for sufficiently small δ_1, the estimate (i) easily follows from Propositions 5.1, 6.5, 7.1, 7.2, and 7.5.

Let us consider the estimate the difference $\{u_{1,m}^{(N+1)} - u_{1,m}^{(N)}, u_\infty^{(N+1)} - u_\infty^{(N)}\}$. For $N \geq 0$, we set $\bar{\phi}_j^{(N)} = \phi_j^{(N+1)} - \phi_j^{(N)}$ for $j = 1, \infty$, $\bar{m}_1^{(N)} = m_1^{(N+1)} - m_1^{(N)}$, and $\bar{w}_\infty^{(N)} = w_\infty^{(N+1)} - w_\infty^{(N)}$. Then by using (7.3) and (7.4), we see that $\bar{\phi}_j^{(N)}$, $\bar{m}_1^{(N)}$ and $\bar{w}_\infty^{(N)}$ $(N \geq 1)$ satisfy

$$\left\{\begin{array}{l} \partial_t\bar{\phi}_1^{(N)} + \gamma\mathrm{div}\bar{w}_1^{(N)} = 0, \\ \partial_t\bar{m}_1^{(N)} - \nu\Delta\bar{m}_1^{(N)} - \tilde{\nu}\nabla\mathrm{div}\bar{m}_1^{(N)} + \gamma\nabla\bar{\phi}_1^{(N)} = F_{1,m,2}(\bar{u}^{(N-1)}, g), \\ \bar{w}_1^{(N)} = \bar{m}_1^{(N)} - P_1(\phi^{(N+1)}\bar{w}_1^{(N)}) - P_1(w^{(N)}\bar{\phi}^{(N)}), \end{array}\right. \tag{7.5}$$

$$\left\{\begin{array}{l} \partial_t\bar{\phi}_\infty^{(N)} + \gamma P_\infty(w^{(N)} \cdot \nabla\bar{\phi}_\infty^{(N)}) + \gamma\mathrm{div}\bar{w}_\infty^{(N)} = F_{\infty1}(\bar{u}^{(N-1)}), \\ \partial_t\bar{w}_\infty^{(N)} - \nu\Delta\bar{w}_\infty^{(N)} - \tilde{\nu}\nabla\mathrm{div}\bar{w}_\infty^{(N)} + \gamma\nabla\bar{\phi}_\infty^{(N)} = F_{\infty2}(\bar{u}^{(N-1)}, g), \end{array}\right. \tag{7.6}$$

where

$$\begin{aligned} &F_{1,m,2}(\bar{u}^{(N-1)}, g) = \tilde{F}_{1,m}(u^{(N)}, g) - \tilde{F}_{1,m}(u^{(N-1)}, g), \\ &F_{\infty1}(\bar{u}^{(N-1)}) = F_\infty^0(u^{(N)}) - F_\infty^0(u^{(N-1)}) - \gamma P_\infty((w^{(N)} - w^{(N-1)}) \cdot \nabla\phi_\infty^{(N)}), \\ &F_{\infty2}(\bar{u}^{(N-1)}, g) = \tilde{F}_\infty(u^{(N)}, g) - \tilde{F}_\infty(u^{(N-1)}, g). \end{aligned}$$

The desired inequality (ii) can be obtained by applying Lemma 4.14, Propositions 5.1, 6.5, 7.3, 7.4, 7.5, and 7.6 (i). This completes the proof. $\square$

Before going further, we introduce a notation. We denote by $B_{X^k(a,b)}(r)$ the closed unit ball of $X^k(a,b)$ centered at 0 with radius r, i.e.,

$$B_{X^k(a,b)}(r) = \left\{ \{u_{1,m}, u_\infty\} \in X^k(a,b); \|\{u_{1,m}, u_\infty\}\|_{X^k(a,b)} \leq r \right\}.$$

Proposition 7.7. *There exists a constant $\delta_2 > 0$ such that if $[g]_s \leq \delta_2$, then the system* (4.2), (4.7) *and* (4.10) *has a unique solution $\{u_{1,m}, u_\infty\}$ on $[0,T]$ in $B_{X^s(0,T)}(C_1[g]_s)$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. The uniqueness of solutions of* (4.2), (4.7) *and* (4.10) *on $[0,T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$ holds in $B_{X^s(0,T)}(C_1\delta_2)$.*

Proof. Let $\delta_2 = \min\{\delta_1, \frac{1}{2C_1}\}$ with δ_1 given in Propositions 7.6. By Propositions 7.6, we see that if $[g]_s \leq \delta_2$, then $u_{1,m}^{(N)} = {}^\top(\phi_1^{(N)}, m_1^{(N)})$ and $u_\infty^{(N)} = {}^\top(\phi_\infty^{(N)}, w_\infty^{(N)})$ converge to $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$, respectively, in the sense

$$\{u_{1,m}^{(N)}, u_\infty^{(N)}\} \ \to \ \{u_{1,m}, u_\infty\} \text{ in } \ X^{s-1}(0,T),$$

$$u_\infty^{(N)} = {}^\top(\phi_\infty^{(N)}, w_\infty^{(N)}) \ \to \ u_\infty = {}^\top(\phi_\infty, w_\infty) \ *\text{-weakly } \text{ in } L^\infty(0,T; H^s_{(\infty),n-1}),$$

$$w_\infty^{(N)} \ \to \ w_\infty \text{ weakly } \text{ in } L^2(0,T; H^{s+1}_{(\infty),n-1}) \cap H^1(0,T; H^{s-1}_{(\infty),n-1}).$$

It is not difficult to see that $\{u_{1,m}, u_\infty\}$ is a solution of (4.2), (4.7) and (4.10) satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$.

It remains to prove $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0,T]; H^s_{n-1})$, which implies $\{u_{1,m}, u_\infty\} \in B_{X^s(0,T)}(C_1[g]_s)$ with $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. But this can be shown in the same way as in the proof of [6, Proposition. 8.4]. This completes the proof. □

By Lemma 4.6 and Proposition 7.7, we can show the existence of the solution of the system (4.1)-(4.2) satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ in therms of the velocity field w_1.

Corollary 7.8. *There exists a constant $\delta_3 > 0$ such that if $[g]_s \leq \delta_3$, then the system* (4.1)-(4.2) *has a unique solution $\{u_1, u_\infty\}$ on $[0,T]$ in $B_{X^s(0,T)}(C_2[g]_s)$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ where $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$ and C_2 is a constant independent of g. The uniqueness of solutions of* (4.1)-(4.2) *on $[0,T]$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ holds in $B_{X^s(0,T)}(C_2\delta_3)$.*

Proof. Let $[g]_s \leq \delta_2$. By Proposition 7.7, we see that the system (4.2), (4.7) and (4.10) has a unique solution $\{u_{1,m}, u_\infty\}$ on $[0,T]$ in $B_{X^s(0,T)}(C_1[g]_s)$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. The uniqueness of the solution holds in $B_{X^s(0,T)}(C_1\delta_2)$. Therefore, by Lemma 4.6, the system (4.1)-(4.2) has a solution $\{u_1, u_\infty\}$ in $X^s(0,T)$ on $[0,T]$ satisfying

$$\|\{u_1, u_\infty\}\|_{X^s(0,T)} \leq C_2[g]_s$$

and $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

We show the uniqueness of the solution. Let $\{u_1^{(k)}, u_\infty^{(k)}\}$ $(k = 1, 2)$ be solutions of the system (4.1)-(4.2) in $X^s(0, T)$ on $[0, T]$ satisfying

$$\|\{u_1^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \le C_2[g]_s$$

and $u_j^{(k)}(0) = u_j^{(k)}(T)$ $(j = 1, \infty)$. We set $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ where $m_1^{(k)} = w_1^{(k)} - P_1(\phi^{(k)} w^{(k)})$, $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ and $w^{(k)} = w_1^{(k)} + w_\infty^{(k)}$ $(k = 1, 2)$. Then by Lemmas 2.1, 4.3, 4.4 and 4.5, $\{u_{1,m}^{(k)}, u_\infty^{(k)}\}$ are solutions of the system (4.2), (4.7) and (4.10) on $[0, T]$ in $B_{X^s(0,T)}(CC_2[g]_s)$ satisfying $u_{1,m}^{(k)}(0) = u_{1,m}^{(k)}(T)$ and $u_\infty^{(k)}(0) = u_\infty^{(k)}(T)$ $(k = 1, 2)$. If $\delta_3 = \min\{\frac{C_1}{CC_2}\delta_2, \delta_2\}$ and $[g]_s \le \delta_3$, then $\{u_{1,m}^{(k)}, u_\infty^{(k)}\} \in B_{X^s(0,T)}(C_1\delta_2)$ $(k = 1, 2)$. Therefore, by the uniqueness of the solution of (4.2), (4.7) and (4.10), we see that $u_{1,m}^{(1)} = u_{1,m}^{(2)}$ and $u_\infty^{(1)} = u_\infty^{(2)}$. It follows from Lemma 2.1 and Lemma 4.3 that $m_1^{(k)} - P_1(\phi^{(k)} w_\infty^{(k)}) \in \mathscr{Y}^{(1)}$ $(k = 1, 2)$, hence,

$$\begin{aligned} w_1^{(1)} &= (I - \mathscr{P}[\phi^{(1)}])^{-1}[m^{(1)} - P_1(\phi^{(1)} w_\infty^{(1)})] \\ &= (I - \mathscr{P}[\phi^{(2)}])^{-1}[m^{(2)} - P_1(\phi^{(2)} w_\infty^{(2)})] \\ &= w_1^{(2)} \end{aligned}$$

where $\mathscr{P}$ is the one in the proof of Lemma 4.6 (i). Therefore, we see that $u_1^{(1)} = u_1^{(2)}$ and $u_\infty^{(1)} = u_\infty^{(2)}$. This completes the proof. $\square$

We can now construct a time periodic solution of (4.1)-(4.2) in the same argument as that in [6]. As in [6], based on the estimates in sections 6-8, one can show the following proposition on the unique existence of solutions of the initial value problem.

Proposition 7.9. *Let $h \in \mathbb{R}$ and let $U_0 = U_{01} + U_{0\infty}$ with $U_{01} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and $U_{0\infty} \in H^s_{(\infty),n-1}$. Then there exist constants $\delta_4 > 0$ and $C_3 > 0$ such that if*

$$M(U_{01}, U_{0\infty}, g) := \|U_{01}\|_{\mathscr{X}^{(1)} \times \mathscr{Y}^{(1)}} + \|U_{0\infty}\|_{H^s_{(\infty),n-1}} + [g]_s \le \delta_4,$$

there exists a solution $\{U_1, U_\infty\}$ of the initial value problem for (4.1)-(4.2) on $[h, h+T]$ in $B_{X^s(h,h+T)}(C_3 M(U_{01}, U_{0\infty}, g))$ satisfying the initial condition $U_j|_{t=h} = U_{0j}$ $(j = 0, \infty)$. The uniqueness for this initial value problem holds in $B_{X^s(h,h+T)}(C_3\delta_4)$.

By using Corollary 7.8 and Proposition 7.9, one can extend $\{u_1, u_\infty\}$ periodically on $\mathbb{R}$ as a time periodic solution of (4.1)-(4.2). Since the argument for extension is the same as that given in [6], we here omit the details. Consequently, we obtain Theorem 3.1. This completes the proof.

Acknowledgements. The author is very grateful to Yoshiyuki Kagei for valuable suggestions and comments. This work was partly supported by the JSPS Japanese-German Graduate Externship.

References

[1] J. Brezina, Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow, SIAM J. Math. Anal., **45** (2013), pp. 3514–3574.

[2] E. Feireisl, Š. Matušu-Necasová, H. Petzeltová and Straškrava, On the motion of a viscous compressible fluid driven by a time-periodic external force, Arch. Rational Mech. Anal., **149** (1999), pp. 69–96.

[3] E. Feireisl, P. B. Mucha, A. Novotný and M. Pokorný, Time-periodic solutions to the full Navier-Stokes-Fourier system, Arch. Rational Mech. Anal., **204** (2012), pp. 745–786.

[4] Y. Kagei and S. Kawashima, Stability of planar stationary solutions to the compressible Navier-Stokes equation on the half space, Commun. Math. Phys., **266** (2006), pp. 401–430.

[5] Y. Kagei and T. Kobayashi, Asymptotic Behavior of Solutions of the Compressible Navier-Stokes Equation on the Half Space, Arch. Rational Mech. Anal., **177** (2005), pp. 231–330.

[6] Y. Kagei and K. Tsuda, Existence and stability of time periodic solution to the compressible Navier-Stokes equation for time periodic external force with symmetry, preprint, 2014, MI Preprint Series, MI 2014-5, Kyushu University.

[7] Y. Kagei, Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow, Arch. Ration. Mech. Anal., **205** (2012), pp. 585–650.

[8] S. Kaniel and M. Shinbrot, A reproductive property of the Navier-Stokes equations, Arch. Rational Mech. Anal., **24** (1967), pp. 363–369.

[9] H. Kozono and M. Nakao, Periodic solutions of the Navier-Stokes equations in unbounded domains, Tohoku Math. J., **48** (1996), pp. 33–50.

[10] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A, **55** (1979), pp. 337–342.

[11] M. Okita, On the convergence rates for the compressible Navier- Stokes equations with potential force, preprint, MI Preprint Series 2012-13, Kyushu University, 2012.

[12] J. Serrin, A note on the existence of periodic solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal., **3** (1959), pp. 120–122.

[13] Y. Shibata and S. Shimizu, A decay property of the Fourier transform and its application to the Stokes problem, J. Math. Fluid Mech, **3** (2001), pp. 213 – 230.

[14] Y. Shibata and K. Tanaka, On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance, J. Math. Soc. Japan, **55** (2003), pp. 797–826.

[15] H. Ma, S. Ukai and T. Yang, Time periodic solutions of compressible Navier-Stokes equations, Journal of Differential Equations, **248** (2010), pp. 2275–2293.

[16] A. Valli, Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (1983), pp. 607–647.

[17] M. Yamazaki, The Navier-Stokes equations in the weak-L^n space with timedependent external force, Math. Ann., **317** (2000), pp. 635–675.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization

MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5 Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings

MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2-projection

MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)

MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type$(A_2 + A_1)^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO
Abstract collision systems on groups

MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres

MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space

MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field

MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling

MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE
Lagrangian approach to weakly nonlinear stability of an elliptical flow

MI2010-21 Hiroki MASUDA
Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test

MI2010-22 Toshimitsu TAKAESU
A Generalized Scaling Limit and its Application to the Semi-Relativistic Particles System Coupled to a Bose Field with Removing Ultraviolet Cutoffs

MI2010-23 Takahiro ITO, Mitsuhiko FUJIO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Composition, union and division of cellular automata on groups

MI2010-24 Toshimitsu TAKAESU
A Hardy's Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra

MI2010-25 Toshimitsu TAKAESU
On the Essential Self-Adjointness of Anti-Commutative Operators

MI2010-26 Reiichiro KAWAI & Hiroki MASUDA
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling

MI2010-27 Chikashi ARITA & Daichi YANAGISAWA
Exclusive Queueing Process with Discrete Time

MI2010-28 Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Motion and Bäcklund transformations of discrete plane curves

MI2010-29 Takanori YASUDA, Masaya YASUDA, Takeshi SHIMOYAMA & Jun KOGURE
On the Number of the Pairing-friendly Curves

MI2010-30 Chikashi ARITA & Kohei MOTEGI
Spin-spin correlation functions of the q-VBS state of an integer spin model

MI2010-31 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and spike detection via Gaussian basis expansions

MI2010-32 Nobutaka NAKAZONO
Hypergeometric τ functions of the q-Painlevé systems of type $(A_2 + A_1)^{(1)}$

MI2010-33 Yoshiyuki KAGEI
Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

MI2010-34 Nobushige KUROKAWA, Masato WAKAYAMA & Yoshinori YAMASAKI
Milnor-Selberg zeta functions and zeta regularizations

MI2010-35 Kissani PERERA & Yoshihiro MIZOGUCHI
Laplacian energy of directed graphs and minimizing maximum outdegree algorithms

MI2010-36 Takanori YASUDA
CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup

MI2010-37 Chikashi ARITA & Andreas SCHADSCHNEIDER
Dynamical analysis of the exclusive queueing process

MI2011-1 Yasuhide FUKUMOTO& Alexander B. SAMOKHIN
Singular electromagnetic modes in an anisotropic medium

MI2011-2 Hiroki KONDO, Shingo SAITO & Setsuo TANIGUCHI
Asymptotic tail dependence of the normal copula

MI2011-3 Takehiro HIROTSU, Hiroki KONDO, Shingo SAITO, Takuya SATO, Tatsushi TANAKA & Setsuo TANIGUCHI
Anderson-Darling test and the Malliavin calculus

MI2011-4 Hiroshi INOUE, Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling via Compressed Sensing

MI2011-5 Hiroshi INOUE
Implications in Compressed Sensing and the Restricted Isometry Property

MI2011-6 Daeju KIM & Sadanori KONISHI
Predictive information criterion for nonlinear regression model based on basis expansion methods

MI2011-7 Shohei TATEISHI, Chiaki KINJYO & Sadanori KONISHI
Group variable selection via relevance vector machine

MI2011-8 Jan BREZINA & Yoshiyuki KAGEI
Decay properties of solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow
Group variable selection via relevance vector machine

MI2011-9 Chikashi ARITA, Arvind AYYER, Kirone MALLICK & Sylvain PROLHAC
Recursive structures in the multispecies TASEP

MI2011-10 Kazunori YASUTAKE
On projective space bundle with nef normalized tautological line bundle

MI2011-11 Hisashi ANDO, Mike HAY, Kenji KAJIWARA & Tetsu MASUDA
An explicit formula for the discrete power function associated with circle patterns of Schramm type

MI2011-12 Yoshiyuki KAGEI
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow

MI2011-13 Vladimír CHALUPECKÝ & Adrian MUNTEAN
Semi-discrete finite difference multiscale scheme for a concrete corrosion model: approximation estimates and convergence

MI2011-14 Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Explicit solutions to the semi-discrete modified KdV equation and motion of discrete plane curves

MI2011-15 Hiroshi INOUE
A generalization of restricted isometry property and applications to compressed sensing

MI2011-16 Yu KAWAKAMI
A ramification theorem for the ratio of canonical forms of flat surfaces in hyperbolic three-space

MI2011-17 Naoyuki KAMIYAMA
Matroid intersection with priority constraints

MI2012-1 Kazufumi KIMOTO & Masato WAKAYAMA
Spectrum of non-commutative harmonic oscillators and residual modular forms

MI2012-2 Hiroki MASUDA
Mighty convergence of the Gaussian quasi-likelihood random fields for ergodic Levy driven SDE observed at high frequency

MI2012-3 Hiroshi INOUE
A Weak RIP of theory of compressed sensing and LASSO

MI2012-4 Yasuhide FUKUMOTO & Youich MIE
Hamiltonian bifurcation theory for a rotating flow subject to elliptic straining field

MI2012-5 Yu KAWAKAMI
On the maximal number of exceptional values of Gauss maps for various classes of surfaces

MI2012-6 Marcio GAMEIRO, Yasuaki HIRAOKA, Shunsuke IZUMI, Miroslav KRAMAR, Konstantin MISCHAIKOW & Vidit NANDA
Topological Measurement of Protein Compressibility via Persistence Diagrams

MI2012-7 Nobutaka NAKAZONO & Seiji NISHIOKA
Solutions to a q-analog of Painlevé III equation of type $D_7^{(1)}$

MI2012-8 Naoyuki KAMIYAMA
A new approach to the Pareto stable matching problem

MI2012-9 Jan BREZINA & Yoshiyuki KAGEI
Spectral properties of the linearized compressible Navier-Stokes equation around time-periodic parallel flow

MI2012-10 Jan BREZINA
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow

MI2012-11 Daeju KIM, Shuichi KAWANO & Yoshiyuki NINOMIYA
Adaptive basis expansion via the extended fused lasso

MI2012-12 Masato WAKAYAMA
On simplicity of the lowest eigenvalue of non-commutative harmonic oscillators

MI2012-13 Masatoshi OKITA
On the convergence rates for the compressible
Navier- Stokes equations with potential force

MI2013-1 Abuduwaili PAERHATI & Yasuhide FUKUMOTO
A Counter-example to Thomson-Tait-Chetayev's Theorem

MI2013-2 Yasuhide FUKUMOTO & Hirofumi SAKUMA
A unified view of topological invariants of barotropic and baroclinic fluids and their application to formal stability analysis of three-dimensional ideal gas flows

MI2013-3 Hiroki MASUDA
Asymptotics for functionals of self-normalized residuals of discretely observed stochastic processes

MI2013-4 Naoyuki KAMIYAMA
On Counting Output Patterns of Logic Circuits

MI2013-5 Hiroshi INOUE
RIPless Theory for Compressed Sensing

MI2013-6 Hiroshi INOUE
Improved bounds on Restricted isometry for compressed sensing

MI2013-7 Hidetoshi MATSUI
Variable and boundary selection for functional data via multiclass logistic regression modeling

MI2013-8 Hidetoshi MATSUI
Variable selection for varying coefficient models with the sparse regularization

MI2013-9 Naoyuki KAMIYAMA
Packing Arborescences in Acyclic Temporal Networks

MI2013-10 Masato WAKAYAMA
Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun's differential equations, eigenstates degeneration, and Rabi's model

MI2013-11 Masatoshi OKITA
Optimal decay rate for strong solutions in critical spaces to the compressible Navier-Stokes equations

MI2013-12 Shuichi KAWANO, Ibuki HOSHINA, Kazuki MATSUDA & Sadanori KONISHI
Predictive model selection criteria for Bayesian lasso

MI2013-13 Hayato CHIBA
The First Painleve Equation on the Weighted Projective Space

MI2013-14 Hidetoshi MATSUI
Variable selection for functional linear models with functional predictors and a functional response

MI2013-15 Naoyuki KAMIYAMA
The Fault-Tolerant Facility Location Problem with Submodular Penalties

MI2013-16 Hidetoshi MATSUI
Selection of classification boundaries using the logistic regression

MI2014-1 Naoyuki KAMIYAMA
Popular Matchings under Matroid Constraints

MI2014-2 Yasuhide FUKUMOTO & Youichi MIE
Lagrangian approach to weakly nonlinear interaction of Kelvin waves and a symmetry-breaking bifurcation of a rotating flow

MI2014-3 Reika AOYAMA
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Parallel flow in a cylindrical domain

MI2014-4 Naoyuki KAMIYAMA
The Popular Condensation Problem under Matroid Constraints