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An Iterative Domain Decomposition Method with Mixed Formulations

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Abstract

An iterative domain decomposition method is applied to eddy current problems. In our previous methods the gauge condition is neglected, then the magnetic vector potential is only one unknown function. On the other hand, in case of magnetostatic problems, it has been well-known that some theoretical results has been introduced, where a mixed formulation with the Lagrange multiplier is introduced in order to impose the gauge condition. Therefore, in this paper, we formulate again an iterative domain decomposition method based on a mixed formulation of eddy current problem, and discuss relations with the previous one.

Keywords: *eddy current problem, mixed formulation, iterative domain decomposition method*

1 Introduction

We have introduced an iterative domain decomposition method to solve quite large scale electromagnetic field problems; see, for example, Kanayama *et al.* [10]. In our previous methods the gauge condition is neglected, then the magnetic vector potential is only one unknown function. These previous results focus themselves on the engineering points of view: the previous formulation enables us to reduce computational consts in practical large scale simulations. However this formulation yields an indeterminate linear system, it is difficult to mathematically justify numerical results, for example unique solvability of the problems and convergency of the approximate solution.

On the other hand, in case of the magnetostatic problem, some theoretical results has been introduced by, for example, Kikuchi [8], [9], where a mixed formulation with the Lagrange multiplier is introduced in order to impose the gauge condition. These results focus themselves on the mathematical point of view: owing to the introduction of the Lagrange multiplier, their mixed formulation enable us to prove unique solvability of the problems and convergency of the approximate solution. However, this formulation yields an indefinite linear system, it is difficult to find an appropriate iterative solver, which is efficient enough to reduce computational costs for practical large scale problems.

At first in this paper, we formulate again an iterative domain decomposition method based on a mixed formulation of eddy current problem. Seconed, to reduce computational costs, we simplify our iterative domain decomposition method into another one, and we discuss relations between the reduced formulation and the previous one.

2 Formulation of eddy current problems

Let Ω be a convex polyhedoral domain with its boundary Γ. Assume that the domain Ω consists of two non-overlapping subdomains, a conducting part Ω_R and a non-conducting one Ω_S, with the interface Γ_{RS} between two subdomains. Let $\boldsymbol{n}$ be the outward unit normal of Ω_S. In this paper, for simplicity, assume that the conducting part Ω_R is also a convex polyhedral domain, and that the part Ω_R is strictly included in Ω.

Let $\boldsymbol{u}$ denote the magnetic vector potential, $\boldsymbol{f}$ an excitation current density, ν the magnetic reluctivity, σ the conductivity, ω the angular frequency, and i the imaginary unit. Then, we consider the three-dimensional eddy current problem with the Coulomb gauge condition:

$$\begin{cases} \operatorname{rot}(\nu \operatorname{rot} \boldsymbol{u}) - i\omega\sigma\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \quad (1a) \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_S, \quad (1b) \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \quad (1c) \\ \displaystyle\int_{\Gamma_{RS}} \boldsymbol{u} \cdot n \, ds = 0; & \quad (1d) \end{cases}$$

for some results of the related equations, for example, see Alonso and Valli [2]. Throughout this paper, assume that ν is a piecewise positive constant, that σ is a positive constant in Ω_R, while is equal to 0 in Ω_S, and that the divergence of $\boldsymbol{f}$ vanishes in Ω:

$$\operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega. \quad (2)$$

As usual, let $L^2(\Omega)$ be the space of complex functions defined in Ω and 2nd power summable in Ω, and let $(. , .)$ be its inner product; let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with derivatives up to the 1st order and set functional spaces X, M, V, and Q by

$$X := \{\boldsymbol{v} \in (L^2(\Omega))^3;\ \operatorname{rot} \boldsymbol{v} \in (L^2(\Omega))^3\}, \quad V := \{\boldsymbol{v} \in X;\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$
$$M := H^1(\Omega), \qquad Q := \{q \in M;\ q = 0 \text{ on } \Gamma,\ \exists c \in \mathbf{C} \text{ s.t. } q = c \text{ in } \Omega_R\},$$

respectively; set bilinear forms $a(. , .)$ and $b(. , .)$ by

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\nu \operatorname{rot} \boldsymbol{u}, \operatorname{rot} \boldsymbol{v}) - i\,(\omega\sigma\,\boldsymbol{u}, \boldsymbol{v}) \qquad \forall(\boldsymbol{u}, \boldsymbol{v}) \in X \times X,$$
$$b(\boldsymbol{v}, q) := (\boldsymbol{v}, \operatorname{grad} q) \qquad \forall(\boldsymbol{v}, q) \in (L^2(\Omega))^3 \times M,$$

respectively.

Now, by introducing the Lagrange multiplier p, we obtain a mixed weak formulation of (1) as follows: given $\boldsymbol{f} \in (L^2(\Omega))^3$, find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), & \quad (3a) \\ b(\boldsymbol{u}, q) = 0, \qquad \forall(\boldsymbol{v}, q) \in V \times Q. & \quad (3b) \end{cases}$$

Remark 1 *As in case of the magnetstatic problem in Kikuchi [8], if* $\boldsymbol{f}$ *satisfies that* $\operatorname{div} \boldsymbol{f} = 0$ *in* Ω*, then* $p = 0$*. This property plays a key role in the forthcoming section.*

3 Domain decomposition method

For simplicity, the domain Ω is assumed to be decomposed into two non-overlapping subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ with their boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$, respectively:

$$\Omega^{(i)} \neq \emptyset \quad (i = 1, 2), \qquad \overline{\Omega} = \overline{\Omega}^{(1)} \cup \overline{\Omega}^{(2)}, \qquad \Omega^{(1)} \cap \Omega^{(2)} = \emptyset;$$

and let γ_{12} be the interface between $\Omega^{(1)}$ and $\Omega^{(2)}$ defined by $\gamma_{12} := \overline{\Omega}^{(1)} \cap \overline{\Omega}^{(2)}$; see Fig. 1. For $i = 1, 2$, the outward unit normal of $\Omega^{(i)}$ is denoted by $\boldsymbol{n}^{(i)}$, and set $\boldsymbol{n} = \boldsymbol{n}^{(1)} (= -\boldsymbol{n}^{(2)})$ on the interface γ_{12}. Moreover, the subdomain Ω_R is assumed to be decomposed into two non-

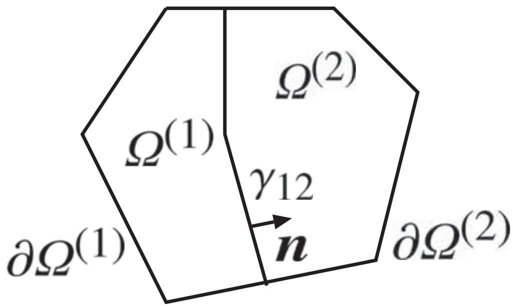

Fig. 1: Two non-overlapping subdomains of Ω.

overlapping subdomains $\Omega_R^{(1)}$ and $\Omega_R^{(2)}$ with the same assumptions as in the domain Ω.

Instead of the complex functions defined in Ω, we associate this decomposition to function spaces, bilinear forms, and inner product: let $L^2(\Omega^{(i)})$ and $H^1(\Omega^{(i)})$ be the space of real functions defined in $\Omega^{(i)}$, which are corresponding to $L^2(\Omega)$ and $H^1(\Omega)$; set function spaces $X^{(i)}$, $M^{(i)}$, $V^{(i)}_{\gamma_{12}}$, $Q^{(i)}_{\gamma_{12}}$, $V^{(i)}$, and $Q^{(i)}$ by

$$\begin{aligned}
X^{(i)} &:= \{\boldsymbol{v} \in (L^2(\Omega^{(i)}))^3;\ \operatorname{rot} \boldsymbol{v} \in (L^2(\Omega^{(i)}))^3\},\\
M^{(i)} &:= H^1(\Omega^{(i)}),\\
V^{(i)}_{\gamma_{12}} &:= \left\{\boldsymbol{v} \in X^{(i)};\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega^{(i)} \backslash \gamma_{12}\right\},\\
Q^{(i)}_{\gamma_{12}} &:= \left\{q \in M^{(i)};\ q = 0 \text{ on } \partial\Omega^{(i)} \backslash \gamma_{12}, \exists c \in \mathbf{C} \text{ s.t. } q = c \text{ in } \Omega^{(i)}_R\right\},\\
V^{(i)} &:= \left\{\boldsymbol{v} \in X^{(i)};\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega^{(i)}\right\},\\
Q^{(i)} &:= \left\{q \in M^{(i)};\ q = 0 \text{ on } \partial\Omega^{(i)}\right\},
\end{aligned}$$

respectively; and set bilinear forms and inner product $a^{(i)}(\,.\,,\,.\,)$, $b^{(i)}(\,.\,,\,.\,)$, and $(\,.\,,\,.\,)_{\Omega^{(i)}}$ by the integrals over $\Omega^{(i)}$, respectively. Moreover, set function spaces Λ and Ξ by

$$\Lambda := \left\{\boldsymbol{\lambda} : \gamma_{12} \to \mathbb{R}^3;\ \boldsymbol{\lambda} = (\boldsymbol{v} \times \boldsymbol{n})|_{\gamma_{12}},\ \boldsymbol{v} \in V\right\}, \qquad \Xi := \left\{\xi : \gamma_{12} \to \mathbb{R};\ \xi = q|_{\gamma_{12}},\ q \in Q\right\};$$

and set $\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})$ by any extension operator from Λ to $V^{(i)}_{\gamma_{12}}$ such that $\boldsymbol{\eta} = (\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})\times \boldsymbol{n})|_{\gamma_{12}}$, and $\overline{p}^{(i)}(\zeta)$ by any extension operator from Ξ to $Q^{(i)}_{\gamma_{12}}$ such that $\zeta = p(\zeta)|_{\gamma_{12}}$. A characterization of *tangential trace* spaces Λ and an *tangential extension* operator on $\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})$ has been given in Alonso–Valli [1], Buffa–Ciarlet [3], [4], Buffa, *et al.* [5], and Quarteroni–Valli [11].

Now, a two-subdomain problem is introduced by the followings: for $i = 1, 2$, find $(\boldsymbol{u}^{(i)}, p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ such that

$$\begin{cases} a^{(i)}(\boldsymbol{u}^{(i)},\ \boldsymbol{v}^{(i)}) + b^{(i)}(\boldsymbol{v}^{(i)},\ p^{(i)}) = (\boldsymbol{f}^{(i)},\ \boldsymbol{v}^{(i)})_{\Omega^{(i)}}, & & \text{(4a)}\\ b^{(i)}(\boldsymbol{u}^{(i)},\ q^{(i)}) = 0, & \forall(\boldsymbol{v}^{(i)},\ q^{(i)}) \in V^{(i)}\times Q^{(i)} & \text{(4b)}\\ \boldsymbol{u}^{(1)}\times \boldsymbol{n} = \boldsymbol{u}^{(2)}\times \boldsymbol{n} & \text{on } \gamma_{12}, & \text{(4c)}\\ p^{(1)} = p^{(2)} & \text{on } \gamma_{12}, & \text{(4d)}\\ a^{(2)}(\boldsymbol{u}^{(2)},\ \overline{\boldsymbol{u}}^{(2)}(\boldsymbol{\eta})) + b^{(2)}(\overline{\boldsymbol{u}}^{(2)}(\boldsymbol{\eta}),\ p^{(2)}) & & \\ \quad = (\boldsymbol{f}^{(1)},\ \overline{\boldsymbol{u}}^{(1)}(\boldsymbol{\eta}))_{\Omega^{(1)}} + (\boldsymbol{f}^{(2)},\ \overline{\boldsymbol{u}}^{(2)}(\boldsymbol{\eta}))_{\Omega^{(2)}} - a^{(1)}(\boldsymbol{u}^{(1)},\ \overline{\boldsymbol{u}}^{(1)}(\boldsymbol{\eta})) - b^{(1)}(\overline{\boldsymbol{u}}^{(1)}(\boldsymbol{\eta}),\ p^{(1)}), & & \text{(4e)}\\ b^{(2)}(\boldsymbol{u}^{(2)},\ \overline{p}^{(2)}(\zeta)) = b^{(1)}(\boldsymbol{u}^{(1)},\ \overline{p}^{(1)}(\zeta)), & \forall(\boldsymbol{\eta},\ \zeta) \in \Lambda \times \Xi. & \text{(4f)} \end{cases}$$

If $\{(\boldsymbol{u}^{(1)},\ p^{(1)}), (\boldsymbol{u}^{(2)},\ p^{(2)})\}$ is a pair of the solutions of two-subdomain problem (4), then the solution of the one-domain problem (3) could be constructed by

$$(\boldsymbol{u}, p) := \begin{cases} (\boldsymbol{u}^{(1)},\ p^{(1)}) & \text{in } \Omega^{(1)}, \quad \text{(5a)}\\ (\boldsymbol{u}^{(2)},\ p^{(2)}) & \text{in } \Omega^{(2)}. \quad \text{(5b)} \end{cases}$$

On the other hand, if $(\boldsymbol{u}, p)$ is a solution of the one-domain problem (3), then a pair of the solutions $\{(\boldsymbol{u}^{(1)},\ p^{(1)}), (\boldsymbol{u}^{(2)},\ p^{(2)})\}$ of the two-subdomain problem (4) could be constructed by

$$(\boldsymbol{u}^{(i)}, p^{(i)}) := (\boldsymbol{u}|_{\Omega^{(i)}}, p|_{\Omega^{(i)}}) \quad \text{in } \Omega^{(i)}. \tag{6}$$

Therefore, the equivalency between both formulations could be obtained as follows:

Theorem 1 *The one-domain problem* (3) *and the two-subdomain problem* (4) *are equivalent.*

For $i = 1, 2$, let $\mathscr{E}^{(i)}(\boldsymbol{f}, \boldsymbol{\lambda}, \xi)$ an extention operator from $(L^2(\Omega))^3 \times \Lambda \times \Xi$ to $V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ defined by $\mathscr{E}^{(i)}(\boldsymbol{f}, \boldsymbol{\lambda}, \xi) := (\boldsymbol{u}^{(i)}, p^{(i)})$, where $(\boldsymbol{u}^{(i)}, p^{(i)})$ is the solution of the following eddy current problem:

$$\begin{cases} a^{(i)}(\boldsymbol{u}^{(i)},\ \boldsymbol{v}^{(i)}) + b^{(i)}(\boldsymbol{v}^{(i)},\ p^{(i)}) = (\boldsymbol{f}^{(i)},\ \boldsymbol{v}^{(i)})_{\Omega^{(i)}}, & & \text{(7a)}\\ b^{(i)}(\boldsymbol{u}^{(i)},\ q^{(i)}) = 0, & \forall(\boldsymbol{v}^{(i)}, q^{(i)}) \in V^{(i)}\times Q^{(i)}, & \text{(7b)}\\ \boldsymbol{u}^{(i)}\times \boldsymbol{n} = \boldsymbol{\lambda} & \text{on } \gamma_{12}, & \text{(7c)}\\ p^{(i)} = \xi & \text{on } \gamma_{12}. & \text{(7d)} \end{cases}$$

Then, a *Steklov–Poincaré* operator $\mathscr{A}$ from $\Lambda \times \Xi$ to $(\Lambda \times \Xi)'$ is set by

$$\begin{aligned}&\langle \mathscr{A}(\boldsymbol{\lambda}, \xi),\ (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}}\\ &\quad := \sum_{i=1}^{2} \{a^{(i)}(\overline{\boldsymbol{u}}^{(i)},\ \overline{\boldsymbol{v}}^{(i)}) + b^{(i)}(\overline{\boldsymbol{v}}^{(i)},\ \bar{p}^{(i)}) + b^{(i)}(\overline{\boldsymbol{u}}^{(i)},\ \bar{q}^{(i)})\}, \quad \forall \boldsymbol{\lambda}, \boldsymbol{\eta} \in \Lambda,\ \forall \xi, \zeta \in \Xi \end{aligned} \tag{8}$$

where $(\overline{\boldsymbol{u}}^{(i)}, \bar{p}^{(i)}) := \mathscr{E}^{(i)}(0, \boldsymbol{\lambda}, \xi)$ and $(\overline{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)}) := \mathscr{E}^{(i)}(0, \boldsymbol{\eta}, \zeta)$; and an *interface source* $\boldsymbol{\chi} \in (\Lambda \times \Xi)'$ is set by

$$\begin{aligned}&\langle \boldsymbol{\chi},\ (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}}\\ &\quad := \sum_{i=1}^{2} \{(\boldsymbol{f}^{(i)},\ \overline{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} - a^{(i)}(\widehat{\boldsymbol{u}}^{(i)},\ \overline{\boldsymbol{v}}^{(i)}) - b^{(i)}(\overline{\boldsymbol{v}}^{(i)},\ \widehat{p}^{(i)}) - b^{(i)}(\widehat{\boldsymbol{u}}^{(i)},\ \bar{q}^{(i)})\}, \quad \forall \xi, \zeta \in \Xi \end{aligned} \tag{9}$$

where $(\widehat{\boldsymbol{u}}^{(i)}, \widehat{p}^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \boldsymbol{0}, 0)$ and $(\overline{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{0}, \boldsymbol{\eta}, \zeta)$. Now we introduce the following *interface problem* on γ_{12}:

$$\langle \mathscr{A}(\boldsymbol{\lambda}, \xi),\ (\boldsymbol{\eta},\ \zeta)\rangle_{\gamma_{12}} = \langle \boldsymbol{\chi},\ (\boldsymbol{\eta},\ \zeta)\rangle_{\gamma_{12}}, \qquad \forall (\boldsymbol{\eta},\ \zeta) \in \Lambda \times \Xi. \tag{10}$$

By using the solution $(\boldsymbol{u}^{(i)}, p^{(i)})$ of two-subdomain problem (4), let us set $(\boldsymbol{\lambda}, \xi)$ by $\boldsymbol{\lambda} := \boldsymbol{u}^{(1)} \times \boldsymbol{n}$ $(= \boldsymbol{u}^{(2)} \times \boldsymbol{n})$ and $\xi := p^{(1)} (= p^{(2)})$. Then, because of (4c)–(4f), $(\boldsymbol{\lambda}, \xi)$ satisfies the *interface problem* (10). On the other hand, once the solution $(\boldsymbol{\lambda}, \xi)$ is obtained by solving the *interface problem* (10), for $i = 1, 2$, each pair $(\boldsymbol{u}^{(i)},\ p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ could be found from the problem (4a) and (4b) in the corresponding subdomain $\Omega^{(i)}$, where the solution $(\boldsymbol{\lambda}, \xi)$ is regarded as the Dirichlet boundary on the interface: $\boldsymbol{u}^{(i)} \times \boldsymbol{n} = \boldsymbol{\lambda}$ and $p^{(i)} = \xi$ on γ_{12}. Finally, from (5), we can obtain the solution $(\boldsymbol{u}, p)$ of the one-domain problem (3).

The *interface problem* (10) complex-symmetric. Therefore, as the solver, the BiConjugate Gradient method (BiCG) is used; see Freund [6]. Then, by choosing an appropriate dual initial residual, BiCG is formally the same as the conjugate gradient method for real valued matrices; see Van der Vorst and Melissen [12]. Using these facts, we can now describe the following *biconjugate gradient* method of the linear system derived from the interface problem (10) as in Glowinski *et al* [7] (at least formally):

Choose (λ_0, ξ_0);

Compute $(\boldsymbol{g}_0, \delta_0)$ **by** (11);

$(\boldsymbol{w}_0, \omega_0) := (\boldsymbol{g}_0, \delta_0)$;

for $k = 0, 1, \ldots$;

  Compute $\mathscr{A}(\boldsymbol{w}_k, \omega_k)$ **by** (12);

  $\alpha_k := ((\boldsymbol{g}_k, \delta_k), (\boldsymbol{g}_k, \delta_k))/(\mathscr{A}(\boldsymbol{w}_k, \omega_k), (\boldsymbol{w}_k, \omega_k))$;

  $(\boldsymbol{\lambda}_{k+1}, \xi_{k+1}) := (\boldsymbol{\lambda}_k, \xi_k) - \alpha_k (\boldsymbol{w}_k, \omega_k)$;

  $(\boldsymbol{g}_{k+1}, \delta_{k+1}) := (\boldsymbol{g}_k, \delta_k) - \alpha_k \mathscr{A}(\boldsymbol{w}_k, \omega_k)$;

$\beta_k := ((\boldsymbol{g}_{k+1}, \delta_{k+1}), (\boldsymbol{g}_{k+1}, \delta_{k+1}))/((\boldsymbol{g}_k, \delta_k), (\boldsymbol{g}_k, \delta_k));$

If $((\boldsymbol{g}_{k+1}, \delta_{k+1}), (\boldsymbol{g}_{k+1}, \delta_{k+1}))/((\boldsymbol{g}_0, \delta_0), (\boldsymbol{g}_0, \delta_0)) < \varepsilon$, **break**;

$(\boldsymbol{w}_{k+1}, \omega_{k+1}) := (\boldsymbol{g}_{k+1}, \delta_{k+1}) + \beta_k (\boldsymbol{w}_k, \omega_k);$

end;

where $(\,.\,,\,.\,)$ is a just multiplication of complex numbers, and ε is a positive constant for the criterion of the convergence. In the above biconjugate gradient algorythm, $(\boldsymbol{g}_0, \delta_0)$ could be computed by the extentions $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)})$ and $(\bar{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)})$ as follow:

$$\begin{aligned}&\langle (\boldsymbol{g}_0, \delta_0),\, (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} \\ &= \sum_{i=1}^{2} \{a^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)},\ \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)},\ \widetilde{p}_0^{(i)}) - (\boldsymbol{f}^{(i)},\ \bar{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} + b^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)},\ \bar{q}^{(i)})\}, \qquad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (11)\end{aligned}$$

where $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \boldsymbol{\lambda}_0, \xi_0)$; and $\mathscr{A}(\boldsymbol{w}_k, \omega_k)$ could be computed by the extentions $(\widehat{\boldsymbol{u}}_0^{(i)}, \widehat{p}_0^{(i)})$ and $(\bar{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)})$ as follow:

$$\langle \mathscr{A}(\boldsymbol{w}_k, \omega_k),\, (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} = \sum_{i=1}^{2} \{a^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)},\ \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)}\ \widehat{p}_k^{(i)}) + b^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)},\ \bar{q}^{(i)})\}, \qquad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (12)$$

where $(\widehat{\boldsymbol{u}}_k^{(i)}, \widehat{p}_k^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{0}, \boldsymbol{w}_k, \omega_k)$. The extentions $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)})$, $(\widehat{\boldsymbol{u}}_0^{(i)}, \widehat{p}_0^{(i)})$, and $(\bar{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)})$ in (11) and (12) could be computed in $\Omega^{(1)}$ and $\Omega^{(2)}$ independently. Therefore, the above biconjugate gradient algorythm is familiar with parallel computations.

Moreover, as mentioned in Remark 1, if $\boldsymbol{f}^{(i)}$ satisfies that $\operatorname{div} \boldsymbol{f}^{(i)} = 0$ in $\Omega^{(i)}$, then $p^{(i)}$ vanishes. This implies that we can neglect the components corresponding to the Lagrange multiplier in the biconjugate gradient algorythm. Therefore we can get the reduced biconjugate gradient algorythm as follows:

Choose $\boldsymbol{\lambda}_0$;

Compute $\boldsymbol{g}_0$ **by** (13);

$\boldsymbol{w}_0 := \boldsymbol{g}_0$;

for $k = 0, 1, \ldots$;

$\quad$ **Compute** $\mathscr{A}_1(\boldsymbol{w}_k, 0)$ **by** (14);

$\quad$ $\alpha_k := (\boldsymbol{g}_k, \boldsymbol{g}_k)/(\mathscr{A}_1(\boldsymbol{w}_k, 0), \boldsymbol{w}_k,);$

$\quad$ $\boldsymbol{\lambda}_{k+1} := \boldsymbol{\lambda}_k - \alpha_k \boldsymbol{w}_k;$

$\quad$ $\boldsymbol{g}_{k+1} := \boldsymbol{g}_k - \alpha_k \mathscr{A}_1(\boldsymbol{w}_k, 0);$

$\quad$ $\beta_k := (\boldsymbol{g}_{k+1}, \boldsymbol{g}_{k+1})/(\boldsymbol{g}_k, \boldsymbol{g}_k);$

$\quad$ **If** $(\boldsymbol{g}_{k+1}, \boldsymbol{g}_{k+1})/(\boldsymbol{g}_0, \boldsymbol{g}_0) < \varepsilon$, **break**;

$w_{k+1} := g_{k+1} + \beta_k w_k;$

end;

In the reduced biconjugate gradient algorythm, $\boldsymbol{g}_0$ could be computed by the first component of the following equation:

$$\langle(\boldsymbol{g}_0, \delta_0),\ (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}}$$
$$= \sum_{i=1}^{2}\{a^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)},\ \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)},\ \widetilde{p}_0^{(i)}) - (\boldsymbol{f}^{(i)},\ \bar{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} + b^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)},\ \bar{q}^{(i)})\}, \qquad \forall(\boldsymbol{\eta},\ \zeta) \in \Lambda \times \Xi, \quad (13)$$

where $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \boldsymbol{\lambda}_0, 0)$; and $\mathscr{A}_1(\boldsymbol{w}_k, 0)$ could be computed by the first component of the following equation:

$$\langle\mathscr{A}(\boldsymbol{w}_k, 0),\ (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} = \sum_{i=1}^{2}\{a^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)},\ \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)},\ \widehat{p}_k^{(i)}) + b^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)},\ \bar{q}^{(i)})\}, \qquad \forall(\boldsymbol{\eta},\ \zeta) \in \Lambda \times \Xi, \quad (14)$$

where $(\widehat{\boldsymbol{u}}_k^{(i)}, \widehat{p}_k^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{0}, \boldsymbol{w}_k, 0)$.

References

[1] Alonso Rodríguez, A. and Valli, A., Some remarks on the characterization of the space of tangential traces of $H(\mathrm{rot}; \Omega)$ and the construction of an extension operator, *Manuscripta Math.*, 89 (1996), pp.159–178.

[2] Alonso Rodríguez, A. and Valli, A., *Eddy Current Approximation of Maxwell equations, Theory, algorithms and aplications*, MS&A Vol.4, Springer, 2010.

[3] Buffa, A. and Ciarlet, P.-G., On traces for functional spaces related to Maxwell's equations. I, An integration by parts formula in Lipschitz polyhedra, *Math. Methods Appl. Sci.*, 24 (2001), pp.9–30.

[4] Buffa, A. and Ciarlet, P.-G., On traces for functional spaces related to Maxwell's equations. II, Hodge decompositions on the boundary of Lipschitz polyhedra and applications, *Math. Methods Appl. Sci.*, 24 (2001), pp.31–48.

[5] Buffa, A., Costabel, M., and Sheen, D., On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.*, 276 (2002), pp.845–867.

[6] Freund, R. W., Conjugate gradient-type methods for linear systems with complex symmetric coefficient matrices, *SIAM J. Sci. Statist. Comput.*, 13 (1992), pp.425–448.

[7] Glowinski, R., Dinh, Q.V., and Periaux, J., Domain decomposition methods for nonlinear problems in fluid dynamics, *Compt. Meths. Appl. Mech. Engrg.*, 40 (1983), pp.27–109.

[8] Kikuchi, F., Mixed formulations for finite element analysis of magnetostatic and electrostatic problems, *Japan J. Appl. Math.*, 6 (1989), pp.209–221.

[9] Kikuchi, F., On a discrete compactness property for the Nedelec finite elements, *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.*, 36 (1989), pp.479–490.

[10] Kanayama, H., Shioya, R., Tagami, D., and Matsumoto, S., 3-D Eddy Current Computation for a Transformer Tank, *COMPEL*, 21 (2002), pp.554–562.

[11] Quarteroni, A. and Valli, A., Domain Decomposition Methods for Partial Differential Equations, Oxford, 1999.

[12] Van der Vorst, H. A., and Melissen, J. B. M., A Petrov–Galerkin type method for solving $Ax = b$, where A is symmetric complex, *IEEE Trans. Magn.*, 26 (1990), pp.706–709.