九州大学学術情報リポジトリ
Kyushu University Institutional Repository

Self-organization in a foliated phase space

Yoshida, Z
Graduate School of Frontier Science, The University of Tokyo

Mahajan, S. M.
Institute for Fusion Studies, The University of Texas

https://hdl.handle.net/2324/1462241

出版情報：COE Lecture Note. 39, pp.170-180, 2012-03-13. 九州大学マス・フォア・インダストリ研究所
バージョン：
権利関係：

KYUSHU UNIVERSITY

Self-organization in a foliated phase space

Z. Yoshida[1] and S. M. Mahajan[2]

[1] *Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, Chiba 277-8561, Japan*
[2] *Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, U.S.A.*

Abstract. Self-organization of a structure is, at its surface, an antithesis of the entropy *ansatz*. However, disorder can still develop at micro scale while a structure emerges on some macro scale; it seems more common in various nonlinear systems that order and disorder are simultaneous, and such coexistence may be possible if the self-organization and the entropy principle work on different scales. The self-organization of a magnetospheric plasma vortex is an example. A systematic formulation of a scale hierarchy is given by a "foliation" of the kinetic phase space in terms of actions (adiabatic invariants). Interpreting an action μ (the magnetic moment, for instance) as the number of quantized "quasi-particles," we can construct a grand-canonical distribution function $f_\alpha \propto e^{-\beta H - \alpha\mu}$ (H: Hamiltonian, β: inverse temperature, α: chemical potential of the quasi-particle) that maximizes entropy on a macroscopic leaf where the microscopic action-angle variables are abstracted as quasi-particle numbers. While embedding the macroscopic leaf in the (laboratory-frame) total phase space, the transforming Jacobian weight forces an inhomogeneous density profile in the laboratory flat space. An inhomogeneous magnetic field (typically a dipole) yields a strongly distorted invariant measure of the magnetized particles, creating a self- organized vortex with steep density gradient; this theory explains "inward diffusion" in magnetospheres, as well as their laboratory simulations [Z. Yoshida *et al.*, Phys. Rev. Lett. **104** (2010) 235004].

1 Introduction

Superficially, the process of self-organization of a structure may appear to be an antithesis of the maximum entropy *ansatz*. And yet various nonlinear systems display what may be viewed as the simultaneous existence of order and disorder [1]. This co-existence will begin to make sense if the self-organization processes and the entropy principle were to manifest on different scales; disorder can still develop at a microscopic scale while an ordered structure emerges on some appropriate macroscopic scale. Writing a theory of self-organization, then, will be an exercise in delineating and understanding the characteristic *scale hierarchy* of the physical system.

Scale hierarchy is a popular keyword in various arguments on "structures"; a biological body is a typical example in which an evident hierarchical structure is preprogrammed enabling effective consumption of energy and materials as well as emission of entropy and waste. A physical macrosystem —a collective system of "simple" elements (a gravitational system, a plasma, etc.)— is anchored on a different framework. An automatic emergence of scale hierarchy is not "programmed" and the structures are not assigned specific functions *a priori*; yet the controlling nonlinear dynamics can mimic a fundamental and elementary process of *creation*.

The ordering principal is generally epitomized in a "constraint" —a possible conservation law, a constant of motion— that, by restricting the class of motions available to the system, limits its ability to degenerate into general disorder. In complicated dynamics, the emergence of "complete" order would surely seem like a tall order; what one may, realistically, expect is the creation of order at some scale coming at the cost of enhanced disorder at another.

Although this scenario seems philosophically sensible, one must trace the theoretical path through which such a self-organization could occur. In this Letter we analyze the *creation* process embodied in the formation of a self-organized magnetospheric plasma. Through this example, we delineate a clear and distinct scale hierarchy in terms of Hamiltonian mechanics and foliation (provided by adiabatic invariants), and show how an ordered structure (heterogeneity) can emerge while maximizing the entropy.

Magnetospheres are self-organized structures found, commonly, in the Universe. The naturally occurring ones (the planetary magnetospheres [2, 3, 4], for instance) as well as their laboratory simulations [5, 6, 7, 8, 9] are built around the dipole magnetic fields. In a dipole magnetic field, the charged particles can cause a variety of interesting phenomena. Of particular interest is the often observed *inward diffusion* (or up-hill diffusion) of magnetized particles. This process is driven by some spontaneous fluctuations (symmetry breaking) that violate the constancy of angular momentum. In a strong enough magnetic field, the canonical angular momentum P_θ is dominated by the magnetic part $q\psi$: the charge multiplied by the flux function (Gauss'potential). The conservation of $P_\theta \approx q\psi$, therefore, restricts the charged particle motion to the magnetic surface (level-set of ψ). It is only via randomly-phased fluctuations that the particles can diffuse across magnetic surfaces. Although the “diffusion” is normally a process of diminishing gradients, numerical experiments do exhibit preferential inward shifts through random motions of test particles [10, 11]. Detailed specification of the fluctuations or the microscopic motion of particles is not the subject of present effort. We plan to construct, instead, a clear-cut description of the self-organized state, which, through with entropy principle, allows steep gradients to form at the goal of diffusion. [1]

Here we describe the scale hierarchy by a phase-space foliation. Heterogeneity is, then, created by the distortion of the metric (invariant measure) dictating the *equipartition* on the leaf of the macroscopic hierarchy; in a strongly inhomogeneous magnetic field (typically a dipole magnetic field), the phase-space metric of *magnetized particles* is distorted, thus the projection of the equipartition distribution onto the flat space of the laboratory frame yields peaked profile by the Jacobian weight (this idea was proposed by Hasegawa [12, 13]; in later discussion we will put the present theory in perspective). The theoretical path for the self-organization scenario will be charted invoking a Hamiltonian formalism of the magnetized charged particles and the Boltzmann distribution on a foliated phase space endowed with a *non-canonical* Poisson bracket [14].

2 Separation of action-angle variables and scale hierarchy

To analyze the effect of magnetization in an inhomogeneous magnetic field (specifically, a dipole magnetic field), we invoke the hierarchical Hamiltonian structure built by adiabatic invariants (actions).

2.1 Hamiltonian of charged particle

The Hamiltonian of a charged particle is a sum of the kinetic energy and the potential energy:

$$H = \frac{m}{2}v^2 + q\phi, \tag{1}$$

[1] A density f (on an n-dimensional space) is an extensive quantity (n-form), which differs from an intensive quantity (scalar, or 0-form). A diffusion equation for f, with respect to an invariant measure $dx^1 \wedge \cdots \wedge dx^n$, is written as $\partial_t f = d(\mathcal{D}\delta f)$, where $\mathcal{D}$ is a diffusion coefficient, d is the exterior derivative (gradient) and $\delta = (-1)^{n+1}\, {}^*d^*$ is the codifferential (* is the Hodge star); a diffusion equation for a scalar ϕ is such that $\partial_t \phi = \delta(\mathcal{D}d\phi)$. Thus, the diffusion of f is a process of flattening ${}^*f$, while that of ϕ simply flattens ϕ. The Hodge star converts $f = f(y^1, \cdots, y^n)$ to ${}^*f = f(y^1, \cdots, y^n)D(y^1, \cdots, y^n)/D(x^1, \cdots, x^n)$.

where $\boldsymbol{v} := (\boldsymbol{P}-q\boldsymbol{A})/m$ is the velocity, $\boldsymbol{P}$ is the canonical momentum, $(\phi, \boldsymbol{A})$ is the electromagnetic 4-potential, m (q) is the particle mass (charge). In the present work, we may treat electrons and ions equally (in later discussion, we will neglect ϕ assuming charge neutrality, but generalization to a non-neutral plasma will be interesting [7, 15]). Denoting by $\boldsymbol{v}_{\|}$ and $\boldsymbol{v}_{\perp}$ the parallel and perpendicular (with respect to the local magnetic field) components of the velocity, we may write

$$H = \frac{m}{2}v_{\perp}^2 + \frac{m}{2}v_{\|}^2 + q\phi. \tag{2}$$

The velocities are related to the mechanical momentum as $\boldsymbol{p} := m\boldsymbol{v}$, $\boldsymbol{p}_{\|} := m\boldsymbol{v}_{\|}$, and $\boldsymbol{p}_{\perp} := m\boldsymbol{v}_{\perp}$.

2.2 Creation of an action-angle pair by magnetization

In a strong magnetic field, $\boldsymbol{v}_{\perp}$ can be decomposed into a small-scale cyclotron motion $\boldsymbol{v}_c$ and a macroscopic guiding-center drift motion $\boldsymbol{v}_d$. The periodic cyclotron motion $\boldsymbol{v}_c$ can be *quantized* to write $(m/2)v_c^2 = \mu\omega_c(\boldsymbol{x})$ in terms of the magnetic moment μ and the cyclotron frequency $\omega_c(\boldsymbol{x})$; the adiabatic invariant μ and the gyration phase $\vartheta_c := \omega_c t$ constitute an action-angle pair. In the standard interpretation, in analogy with the Landau levels in quantum theory, ω_c is the *energy level* and μ is the *number* of quasi-particles (quantized periodic motions) at the corresponding energy level.

For an axisymmetric system with a poloidal (but no toroidal) magnetic field, let (ψ, ζ, θ) be a magnetic coordinate system such that $\boldsymbol{B} = \nabla\psi\times\nabla\theta$ (θ is the toroidal angle). An approximately-vacuum magnetic field may also be written as $\boldsymbol{B} = \nabla\xi = B\nabla\zeta$. [2] The macroscopic part of the perpendicular kinetic energy is expressed as $mv_d^2/2 = (P_\theta - q\psi)^2/(2mr^2)$, where P_θ is the angular momentum in the θ direction and r is the radius from the geometric axis. In terms of the canonical-variable set $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_{\|}, \theta, P_\theta)$ the Hamiltonian of the guiding center (or, the quasi-particle) becomes

$$H_c = \mu\omega_c + \frac{1}{2m}\frac{(P_\theta - q\psi)^2}{r^2} + \frac{1}{2m}p_{\|}^2 + q\phi. \tag{3}$$

Note that the energy of the cyclotron motion has been quantized in term of the frequency $\omega_c(\boldsymbol{x})$ and the action μ; the gyro-phase ϑ_c has been coarse grained (integrated to yield 2π).

[2] The flux function ψ is equal to rA_θ (A_θ is the θ component of $\boldsymbol{A}$). For a point dipole of magnetic moment M, $\psi(r,z) = Mr^2(r^2+z^2)^{-3/2}$. On the other hand, the scalar potential is $\xi(r,z) = Mz(r^2+z^2)^{-3/2}$. The magnetic field strength is $B = M\sqrt{(r^4+5r^2z^2+4z^4)/(r^2+z^2)^5}$.

2.3 Boltzmann distribution

The standard Boltzmann distribution function is derived when we assume that the Lebesgue measure d^3vd^3x is an invariant measure and the Hamiltonian H is the determinant of the ensemble. Maximizing the entropy $S = -\int f \log f d^3vd^3x$ keeping the total energy $E = \int Hfd^3vd^3x$ and the total particle number $N = \int fd^3vd^3x$ constant, we obtain

$$f(\boldsymbol{x}, \boldsymbol{v}) = Z^{-1}e^{-\beta H}, \tag{4}$$

where Z is the normalization factor ($\log Z - 1$ is the Lagrange multiplier on N) and β is the inverse temperature (the Lagrange multiplier on E). The corresponding configuration-space density,

$$\rho(\boldsymbol{x}) = \int fd^3v \propto e^{-\beta q\phi}, \tag{5}$$

becomes constant for a charge neutral system ($\phi = 0$).

Needless to say that the Boltzmann distribution or the corresponding configuration-space density, with an appropriate Jacobian multiplication, is independent of the choice of phase-space coordinates. Moreover, the density is invariant no matter whether we quantize the cyclotron morion or not. Let us confirm this fact by a direct calculation. For the Boltzmann distribution of the "guiding-center plasma"

$$\begin{aligned} f(\mu, v_d, v_\parallel; \boldsymbol{x}) &= Z^{-1}e^{-\beta H_c} \\ &= Z^{-1}e^{-\beta\left(\mu\omega_c(\boldsymbol{x}) + mv_d^2/2 + mv_\parallel^2/2 + q\phi(\boldsymbol{x})\right)}, \end{aligned} \tag{6}$$

the density is given by

$$\rho(\boldsymbol{x}) = \int fd^3v = \int f\,\frac{2\pi\omega_c}{m}d\mu dv_d dv_\parallel \propto e^{-\beta q\phi}, \tag{7}$$

exactly reproducing (5).

2.4 Equilibrium on macroscopic hierarchy

The adiabatic invariance of the magnetic moment μ (the *number* of the quantized quasi-particles) imposes a *topological constraint* on the motion of particles; it is this constraint that is the root-cause of a macroscopic hierarchy and of structure formation. Mathematically, the *scale hierarchy* is equivalent to a foliation of the phase space. To explain how the scale hierarchy is formulated, we start by the general (micro-macro total) formulation, and then separate the microscopic action-angle pair μ-ϑ_c; the *macroscopic phase space* is the remaining sub-manifold immersed in the general phase space, which we delineate as a leaf of the foliation in terms of a *Casimir element* [14].

The Poisson bracket on the total phase space, spanned by the canonical variables $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_{\|}, \theta, P_\theta)$, is

$$\{F, G\} := \langle \mathcal{J} \partial_{\boldsymbol{z}} F, \partial_{\boldsymbol{z}} G \rangle,$$

where $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \int u_j v^j d^6 z$ is the inner-product and $\mathcal{J}$ is the canonical symplectic matrix (Poisson tensor):

$$\mathcal{J} := \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

The equation of motion for the Hamiltonian H_c is written as $dz^j/dt = \{H_c, z^j\}$. Notice that the quantization of the cyclotron motion (i.e., the replacement of the microscopic variables ϑ_c-v_c by ϑ_c-μ) suppresses change in μ. Liouville's theorem determines the invariant measure $d^6 z$, by which we obtain the Boltzmann distribution (6).

To extract the macroscopic hierarchy, we "separate" the microscopic variables (ϑ_c, μ) by modifying the symplectic matrix as

$$\mathcal{J}_{nc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}. \tag{9}$$

The Poisson bracket, determining the kinematics on the macroscopic hierarchy, is defined as $\{F, G\}_{nc} := \langle \mathcal{J}_{nc} \partial_{\boldsymbol{z}} F, \partial_{\boldsymbol{z}} G \rangle$. The macroscopic kinetic equation, $\partial_t f + \{H_c, f\}_{nc} = 0$, reproduces the familiar drift-kinetic equation (see for example [16]).

$$\begin{aligned} &\partial_t f + \{H_c, f\}_{nc} \\ &\quad = \partial_t f + v_{\|} \frac{\partial f}{\partial \zeta} + F_{\|} \frac{\partial f}{\partial v_{\|}} + \frac{v_d}{r} \frac{\partial f}{\partial \theta} + F_\theta \frac{\partial f}{\partial P_\theta} = 0, \end{aligned}$$

where $F_{\|} := -\partial_\zeta H_c$ and $F_\theta := -\partial_\theta H_c$ (the last two terms on the right-hand side vanishes because of the symmetry $\partial_\theta = 0$).

The nullity of $\mathcal{J}_{nc}$ makes the Poisson bracket $\{\,,\,\}_{nc}$ *non-canonical* [14], i.e., there is a nontrivial $C(\boldsymbol{z})$ such that $\{G, C\}_{nc} = 0$ for every G. We call

$C(\boldsymbol{z})$ a Casimir element (since $\{H_c, C\} \equiv 0$, C is an invariant). Evidently, μ is a Casimir element (more generally $C = g(\mu)$ with g being any smooth function). The level-set of $C(\boldsymbol{z}) = \mu$, a leaf of the Casimir foliation, identifies what we may call the *macroscopic hierarchy*.

By applying Liouville's theorem to the Poisson bracket $\{\ ,\ \}_{nc}$, the invariant measure on the macroscopic hierarchy is $d^4z = d^6z/(2\pi d\mu)$, the the total phase-space measure modulo the microscopic measure. The most probable state (statistical equilibrium) on the macroscopic ensemble must maximize the entropy with respect to this invariant measure. The variational principle is set up following the standard procedure —immersing the macroscopic hierarchy into the general phase space, and incorporating the constraints through the Lagrange multipliers: We maximize entropy $S = -\int f \log f \, d^6z$ for a given particle number $N = \int f d^6z$, a quasi-particle number $M = \int \mu f d^6z$, and an energy $E = \int H_c f d^6z$, to obtain the distribution function

$$f = f_\alpha := Z^{-1} e^{-(\beta H_c + \alpha\mu)}, \tag{10}$$

where α, β, and $\log Z - 1$ are, respectively the Lagrange multipliers on M, E, and N. In this "grand-canonical" distribution function, α is the chemical potential associated with the quasi-particles. [3]

The factor $e^{-\alpha\mu}$ in f_α yields a direct ω_c dependence of the configuration-space density:

$$\rho = \int f_\alpha \frac{2\pi\omega_c}{m} d\mu dv_d dv_{||} \propto \frac{\omega_c(\boldsymbol{x})}{\beta\omega_c(\boldsymbol{x}) + \alpha}, \tag{11}$$

which may be compared with the density (7) evaluated for the Boltzmann distribution ($\phi = 0$ assuming charge neutrality). Notice that the Jacobian $(2\pi\omega_c/m)d\mu$ multiplying the macroscopic measure d^4z reflects the distortion of the macroscopic phase space (Casimir leaf) caused by the magnetic field; see Fig. 1.

3 Macro-scale action-angle pairs in an axisymmetric system

In an axisymmetric system, the quasi-particle motion, periodic in both the parallel and θ directions, may be described in terms of the macroscopic

[3] We can also derive (10) by an *energy-Casimir function*. With a Casimir element μ, we can transform the Hamiltonian as $H_c \mapsto H_\alpha := H_c + \alpha\mu$ (α is an arbitrary constant) without changing the macroscopic dynamics; H_α is called an energy-Casimir function [14]. The Boltzmann distribution with respect to H_α is equivalent to (10).

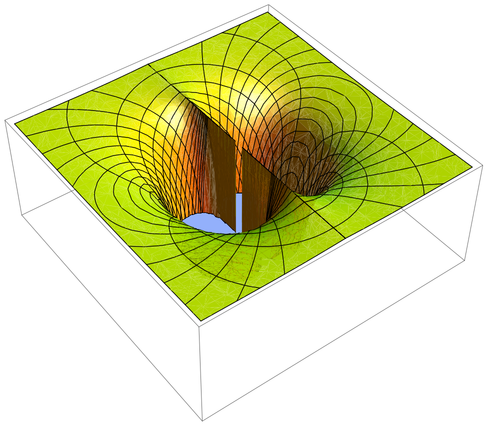

Figure 1: Jacobian weight on the foliated phase space of quasi-particles —the macroscopic hierarchy of magnetized charged particles.

action-angle pairs: $J_{\|}$-$\vartheta_{\|}$ ($:= \sin^{-1}(\zeta/\ell_{\|})$; $\ell_{\|}$ is the bounce orbit length) and P_θ ($\approx q\psi$)-θ (for a hierarchy of adiabatic invariants, see [17] and papers cited there). To find explicit expressions for the parallel action-angle variables, we invoke the Hamiltonian H_c of (3) in which the gyration action-angle pair μ-ϑ_c is "quantized" in a sense that $\omega_c = \dot{\vartheta}_c$ is given as a function of $\boldsymbol{x}$ (configuration space coordinate). Neglecting the curvature effect [4] and putting $\phi = 0$, the equation of the parallel motion reads as

$$\left\{ \begin{aligned} \frac{d\zeta}{dt} &= \frac{\partial H_c}{\partial p_{\|}} = \frac{p_{\|}}{m}, \\ \frac{dp_{\|}}{dt} &= -\frac{\partial H_c}{\partial \zeta} = -\mu \nabla_{\|} \omega_c, \end{aligned} \right. \tag{12}$$

which combine to yield

$$m\frac{d^2}{dt}\zeta = -\mu\nabla_{\|}\omega_c. \tag{13}$$

In the vicinity of $\zeta = 0$, where ω_c has a minimum on each magnetic surface, we may expand

$$\omega_c = \Omega_c(\psi) + \Omega_c''(\psi)\zeta^2/2,$$

[4] With $\boldsymbol{b} = \boldsymbol{B}/B$, we define $\boldsymbol{v}_{\|} = \boldsymbol{b}\cdot\boldsymbol{v}$, and hence, $\partial/\partial\zeta$ operates on $\boldsymbol{b}$. If $v_{\|}/R$ (R: curvature) is comparable to the bounce-frequency (14), we have to take into account the centrifugal force; see for example [16].

where $\Omega_c(\psi)$ is the minimum of ω_c and $\Omega_c''(\psi) := d^2\omega_c/d\zeta^2|_\psi$. In terms of the length

$$L_{||}(\psi) := \left(\frac{2\Omega_c(\psi)}{\Omega_c''(\psi)}\right)^{1/2},$$

which scales the variation of ω_c along ζ, (13) is integrated to identify the corresponding action-angle variables: $\zeta = \ell_{||}\sin\vartheta_{||}$, $\vartheta_{||} = \omega_b t$ with the bounce frequency

$$\omega_b = \sqrt{\frac{\Omega_c''(\psi)\mu}{m}} = \frac{v_\perp}{L_{||}(\psi)}. \tag{14}$$

The bounce amplitude $\ell_{||} = [2E_{||}/(m\omega_b^2)]^{1/2}$ is evaluated in terms of the parallel energy $E_{||} := (mv_{||}^2)/2|_{\zeta=0}$. Assuming $E_{||} \approx E_\perp := \mu\Omega_c$, we estimate $\ell_{||} \approx L_{||}$. The action

$$J_{||} := \frac{1}{2\pi}\oint mv_{||}d\zeta$$

is related to $E_{||} = J_{||}\omega_b$, and $dv_{||} = \left[\omega_b/(mv_{||})\right] dJ_{||}$. Using the relation $\omega_b/(mv_{||}) = v_\perp/(L_{||}mv_{||}) \approx 1/(mL_{||})$, we may write

$$dv_{||} \approx \frac{1}{mL_{||}}dJ_{||}.$$

The quantization of the parallel action-angle pair $J_{||}$-$\vartheta_{||}$, adds an additional constraint leading to a new equilibrium distribution function:

$$f_{\alpha,\gamma} = Z^{-1}e^{-(\beta H_c + \alpha\mu + \gamma J_{||})}, \tag{15}$$

and the corresponding density

$$\begin{aligned} \rho &= \int f_{\alpha,\gamma}\frac{2\pi\omega_c d\mu}{m}\frac{dJ_{||}}{mL_{||}(\psi)}dv_d \\ &\propto \frac{\omega_c(\boldsymbol{x})}{m^2}\int_0^\infty \frac{e^{-(\beta\omega_c+\alpha)\mu}d\mu}{\beta\sqrt{2\omega_c\mu/m} + \gamma L_{||}(\psi)}. \end{aligned} \tag{16}$$

Through $L_{||}(\psi)$, the density ρ acquires a dependence on ψ. We may estimate $L_{||}(\psi) \sim \psi^{-1}$.

The second part of the macroscopic dynamics is contained in the equation of motion in the θ direction. Solving the θ motion reveals that the self-organized clump of density turns out to be a *vortex*. Averaging the cyclotron and bounce motions, the Hamiltonian becomes (neglecting ϕ and approximating $P_\theta \approx q\psi$)

$$H_{cb} = \mu\omega_c(\psi,\zeta) + J_{||}\omega_b(\psi,\mu). \tag{17}$$

The governing equations for the canonical pair $q\psi$-θ

$$\begin{cases} \dfrac{d}{dt}\theta = \dfrac{\partial H_{cb}}{\partial(q\psi)} = \dfrac{\mu}{q}\dfrac{\partial \omega_c}{\partial \psi} + \dfrac{J_{\|}}{q}\dfrac{\partial \omega_b}{\partial \psi} =: \omega_d, \\ \\ \dfrac{d}{dt}(q\psi) = -\dfrac{\partial H_{cb}}{\partial \theta} = 0 \end{cases} \tag{18}$$

tell us that a particle involved in the density clump rotates with a frequency ω_d.

4 Concluding remarks

We have constructed a grand-canonical distribution function f_α (or $f_{\alpha,\gamma}$) that maximizes entropy on a macroscopic leaf where the microscopic action-angle variables are abstracted as quasi-particle numbers. While embedding the macroscopic leaf in the (laboratory-frame) total phase space, the transforming Jacobian weight forces an inhomogeneous density profile in the laboratory flat space. Evidently, f_α (or $f_{\alpha,\gamma}$) is a "particular" solution of the stationary kinetic equation $\{H_c, f\}_{nc} = 0$; the general solution, freed from the maximum-entropy condition, is any arbitrary function of the constants of motion, $f = F(H_c, \mu, J_{\|}, \psi)$. The solution $f = F(\mu, J_{\|})$, for instance, yields a density function $\rho \propto \omega_c/L_{\|}$. In a dipole magnetic field, this function scales as $\propto r^{-4}$, precisely the density profile given by Hasegawa [12, 13], and is the asymptotic form of (16) in the limit $r \to \infty$ ($\omega_c \to 0$ so that $\beta H_c \ll \alpha\mu + \gamma J_{\|}$).

The chosen example of self-organization, simultaneous with maximum entropy, is distinguished from the standard narrative based on the integral fluid invariants (like a helicity) with motion taking place in the 3-dimensional space [1]. Here the domain is the 6-dimensional phase space, and the constants of motion are the adiabatic invariants (actions) associated with particle motion. The conserved actions force the trajectories in the action-angle space to be straight lines. It is the translation of these straight action-angle trajectories (microscopic) into configuration space of observation (macroscopic) that connect the micro with the macro with the emergence of structure in the latter while the former may be totally structureless.

Acknowledgment

The authors acknowledge the stimulating discussions and suggestions of Professor A. Hasegawa and Professor Philip. J Morrison. ZY's research was supported by Grant-in-Aid for Scientific Research (23224014) from MEXT-Japan. SMM's research was supported by the U.S. DOE Grant DE-FG02-04ER54742.

References

[1] A. Hasegawa, Adv. Phys. **34**, 1 (1985).

[2] M. Schulz and L. J. Lanzerotti, *Particle Diffusion in the Radiation Belts* (Springer, New York, 1974).

[3] D. H. Brautigam and J. M. Albert, J. Geophys. Res. **105**, 291 (2000).

[4] Y. Chen, G. D. Reeves, and R. H. W. Friedel, Nature Phys. **3**, 614 (2007).

[5] D. T. Garnier *et al.*, Phys. Plasmas **13**, 056111 (2006).

[6] A. C. Boxer *et al.*, Nature Phys. **6**, 207 (2010).

[7] Z. Yoshida, H. Saitoh, J. Morikawa, Y. Yano, S. Watanabe and Y. Ogawa, Phys. Rev. Lett. **104**, 235004 (2010).

[8] H. Saitoh *et al.*, Phys. Plasmas **17**, 112111 (2010).

[9] H. Saitoh *et al.*, Nucl. Fusion **51**, 063034 (2011).

[10] T.J. Birmingham, T.G. Northrop, and C.-G. Falthammar, Phys. Fluids **10**, 2389 (1967).

[11] M. Walt, Space Sci. Rev. **12**, 446 (1971).

[12] A. Hasegawa, Comments Plasma Phys. Contr. Fusion **1**, 147 (1987).

[13] A. Hasegawa, Phys. Scr. **T116**, 72 (2005).

[14] P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).

[15] D. H. E. Dubin and T. M. O'Neil, Rev. Mod. Phys. **71**, 87 (1999).

[16] K. Nishikawa and M. Wakatani, *Plasma Physics*, 3rd ed. (Springer-Verlag, Berlin-Heidelberg, 2000), Sec. 4.3.

[17] A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*, 2nd ed. (Springer-Verlag, New York, 1992), Sec. 2.3b.