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出版情報：COE Lecture Note. 39, pp.141-150, 2012-03-13. 九州大学マス・フォア・インダストリ研究所

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In this paper, I consider matching models with divisible fiat money. It is shown that there always exists a conservation law in the stationarity condition for money holdings distribution in such models and thus real indeterminacy of stationary equilibria arises. Surprisingly it has nothing to do with the other specifications, e.g., the bargaining procedures, of the models. I also introduce a policy which breaks the conservation law.

1 Introduction

Recently, real indeterminacy of stationary equilibria has been found in matching models with divisible fiat money. Kamiya and Shimizu (2006) show that there always exists a conservation law in such models and thus real indeterminacy generically arises. Surprisingly it has nothing to do with the other specifications, e.g., the bargaining procedures, of the models. Kamiya and Shimizu (2007) also present a way to break the conservation law, and to induce an efficient equilibrium. The purpose of this paper is to give a unified approach to the above two results.

A sketch of the idea is as follows. Suppose the nominal stock of money is given. When the price level is lower, there is more liquidity in the economy, the trade is more frequent, and therefore the welfare level is higher. When the price level is higher, there is less liquidity in the economy, the trade is less frequent, and therefore the welfare level is lower. If the corresponding equilibrium values of the other variables can be found, such as the money holdings distribution and the value function, as the price level continuously varies, then the real indeterminacy follows. More precisely, if the number of variables is larger than that of equations, then by applying the implicit function theorem this property holds. I can show that the stationary condition of money holdings has a conservation law and thus there is at least one more variable than the number of equations. Thus the stationary equilibria in such models are indeterminate.

In this paper, I consider the case of one fiat money. Suppose it is divisible and there is an upper bound of its holdings. I focus on the stationary equilibria in which, for some positive number p, all trades occur with its integer multiple amounts of money. I focus on stationary distributions on $\{0, \dots, N\}$ expressed by $h = (h(0), \dots, h(N))$, where $h(n)$ is the measure of the set of agents with np amount of money, and $N < \infty$ is the upper bound. The condition for stationarity of money holdings is $O_n = I_n$, $n = 0, 1, \dots, N$, and $\sum_{n=0}^{N} h(n) = 1$, where O_n (I_n) is the outflow (inflow resp.) at n. Since $\sum_{n=0}^{N} O_n = \sum_{n=0}^{N} I_n$ always holds, then, at first glance, there seem to be $(N+1)$ independent equations. Thus it seems that the numbers of independent equations and variables, $h(n), n = 0, \dots, N$, are the same. However, it can be shown that one more equation is always redundant and that the system of equations has always at least one degree of freedom; namely, $\sum_{n=0}^{N} nO_n = \sum_{n=0}^{N} nI_n$, a conservation law, always holds. This fact is the key to the real indeterminacy of stationary equilibria.

I also show that a tax-subsidy policy can break the identity, and induce an efficient equilibrium. The identity means that the total money holding are the same before and after trades. Thus the government can break the identity by absorbing and discarding money by using tax and subsidy.

The plan of this paper is as follows. In the next section, I present a general model and discuss the case without tax-subsidy policies. Section 3 is devoted the case with the policy.

2 The Model

I first present a general model, where the private sector is a special case of Kamiya and Shimizu (2006). Hereafter, I call it KS1 simply.

There is a continuum of private agents with a mass of measure one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. A type i good is produced by a type $i-1$ agent. A type i agent obtains some positive utility only when she consumes type i good. I make no assumption on the divisibility of goods. I assume that fiat money is durable and perfectly divisible. Time is continuous, and pairwise random matchings take place according to Poisson process with parameter $\mu > 0$.

I focus on the case that, for some positive number p, all trades occur with its integer multiple amounts of money. In what follows, I focus on a stationary distribution of economy-wide money holdings on $\{0, \dots, N\}$ expressed by $h = (h_0, \dots, h_N)$, where h_n is the measure of agents with np amount of money, and $N < \infty$ is the upper bound of the distribution. For simplicity, I assume that N is exogenously given. I also focus on the case of $h_n \geq 0$ and $\sum_{n=0}^{N} h_n = 1$. Let $M > 0$ be a given nominal stock of money circulating in the private sector. Since p is uniquely determined by $\sum_{n=0}^{N} pnh_n = M$ for a given h for $h_0 \neq 1$, then, deleting p from $\{0, p, \dots, Np\}$, the set $\{0, \dots, N\}$ can be considered as the state space.

Since I adopt a general framework, various types of bargaining procedures are allowed.[1] An agent with n, or an agent with np amount of money, chooses an action in $A_n = \{a_{n1}, \dots, a_{ns_n}\}$. Let $A = \Pi_{n=0}^{N} A_n$. For simplicity, I focus on the stationary equilibrium in which all agents choose pure strategies. Let $S = \sum_{n=0}^{N} s_n$. Given an equilibrium action profile $a = (a_0, \dots, a_N)$, where a_n is the action taken at np in the equilibrium, define $\alpha(a) = \{(n, j) \mid a_n = a_{nj}\}$.

The monetary transition resulted from transaction among a matched pair is described by a function f. When an agent with money holdings np and action a_{nj} meets an agent with $n'p$ and $a_{n'j'}$, the former's and the latter's states, i.e., money holdings, will be $n + f(n, j; n', j')$ and $n' - f(n, j; n', j')$, respectively. That is, f maps an ordered pair $(n, j; n', j')$ to a non-negative integer $f(n, j; n', j')$. Here "ordered" means, for example, that the former is a seller and the latter is a buyer. When N is exogenously determined, I assume

$$N \geq n + f(n, j; n', j') \quad \text{and} \quad n' - f(n, j; n', j') \geq 0.$$

Next, I introduce government agents following Aiyagari and Wallace (1997). They follow a rule which prescribes them how to collect tax from or give subsidy to the agents they are matched with. I assume that government agents can observe current money holdings of agents they are matched with. Let $G > 0$ be the measure of the government agents. Thus the total measure of agents is $1 + G$. Note that in the following arguments G can be any small positive number.

Then I describe government's policy by $(t_0, t_1, \dots, t_N)$, where $t_n \in [-1, 1]$, $t_0 \geq 0$, and $t_N \leq 0$. Each government agent gives subsidy p to the matched agent with n with probability $|t_n|$ when $t_n > 0$, while she collects tax p with probability $|t_n|$ when $t_n < 0$. As seen in the previous section,

[1] See Remark 1 for the details.

the budget of the government may not be balanced out of equilibria.

Let $\theta \in \mathbb{R}^L$ be the parameters of the model besides t. Note that θ includes k, μ, and G.

Below, I adopt dynamic programming approach. Let V_n be the value of state n, $n = 0, \ldots, N$. The variables in the model are denoted by $x = (h, V, a)$. Let $W_{nj}(x; \theta, t)$ be the value of action j at state n. Thus, in equilibria, $W_{nj}(x; \theta, t) = V_n$ holds for $(n, j) \in \alpha(a)$. Note that $W_{nj}(x; \theta, t)$ includes the utility and/or the production cost of perishable goods.

2.1 Stationary Equilibria without Tax-Subsidy

First, I consider the case that $t_n = 0$ for all n.

Define

$$h_{nj} = \begin{cases} h_n & \text{if } a_{nj} = a_n, \\ 0 & \text{if } a_{nj} \neq a_n. \end{cases}$$

Then by the random matching assumption and the definition of f, the inflow I_n into state n and the outflow O_n from state n are defined as follows:

$$I_n(h, a; \theta) = \frac{\mu\kappa}{1+G}\left[\sum_{(i,j,i',j')\in X_n} h_{ij}h_{i'j'} + \sum_{(i,j,i',j')\in X'_n} h_{ij}h_{i'j'}\right],$$

$$O_n(h, a; \theta) = \frac{\mu\kappa}{1+G}\left[\sum_{(j,i',j')\in Y_n} h_{nj}h_{i'j'} + \sum_{(j,i',j')\in Y'_n} h_{nj}h_{i'j'}\right],$$

where

$$X_n = \{(i, j, i', j') \mid f(i, j; i', j') > 0,\ i + f(i, j; i', j') = n\},$$

$$X'_n = \{(i, j, i', j') \mid f(i, j, i', j') > 0,\ i' - f(i, j; i', j') = n\},$$

$$Y_n = \{(j, i', j') \mid f(n, j; i', j') > 0\},$$

$$Y'_n = \{(j, i', j') \mid f(i', j'; n, j) > 0\}.$$

I denote $I_n - O_n$ by D_n. Then the condition for stationarity is $D_n = 0$ for $n = 0, \ldots, N$ and $\sum_{n=0}^{N} h_n = 1$. Clearly, $\sum_{n=0}^{N} D_n = 0$ holds as an identity, and thus at least one equation is redundant. Moreover, the following theorem prunes the above conditions of another redundant equation other than this.

Theorem 1 (Kamiya and Shimizu (2006)) For any a,

$$\sum_{n=0}^{N} nD_n(h,a;\theta) = 0, \tag{1}$$

is an identity.

The identity can be considered as a conservation law. The economic interpretation of the law is as follows. Suppose that two agents, say a buyer and a seller, meet and a monetary trade occurs. Then the amount of money the buyer pays is equal to that of the seller obtains; in other words, the sum of their money holdings before trade is equal to that after trade. Since this holds in each trade, the total amount of money before trades, expressed by $\sum_{n=0}^{N} pnO_n(h,a;\theta)$, is equal to the total amount of money after trades, expressed by $\sum_{n=0}^{N} pnI_n(h,a;\theta)$, and thus $\sum_{n=0}^{N} nD_n(h,a;\theta) = 0$ always holds.

Together with the other identity $\sum_{n=0}^{N} D_n(h,a;\theta) = 0$, the above theorem implies that h is a stationary distribution if and only if $D_n(h,a;\theta) = 0, n = 2,\ldots,N$, and $\sum_{n=0}^{N} h_n = 1$ hold. Namely, the condition for stationarity has at least one-degree of freedom. This is the main cause of the indeterminacy.

Now the equilibrium condition is expressed as follows:

Definition 1 Given θ, $x = (h,V,a) \in \mathbb{R}^{N+1} \times \mathbb{R}_+^{N+1} \times A$ is a (pure strategy) *stationary equilibrium without tax-subsidy* if it satisfies the following:

$$\begin{aligned} D_n(h,a;\theta) &= 0, && n = 2,\ldots,N \\ \sum_{n=0}^{N} h_n - 1 &= 0, && \\ V_n - W_{nj}(x;\theta,0) &= 0, && (n,j) \in \alpha(a) \\ V_n - W_{nj}(x;\theta,0) &\geq 0, && (n,j) \notin \alpha(a). \end{aligned} \tag{2}$$

(h,V) is called a stationary equilibrium for a and θ if (h,V,a) is a stationary equilibrium for θ. Let E_θ^a be the set of such (h,V)s, and $g^a : \mathbb{R}_+^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{R}^L (\ni (h,V,\theta)) \to \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{S-N-1}$ be the LHS of the above condition.

Remark 1 In addition to the above equilibrium conditions, the following conditions are typically required: (i) the existence of $p > 0$ satisfying $\sum_{n=0}^{N} pnh_n = M$, (ii) the incentive not to choose an action out of the action space, and (iii) the incentive to take the equilibrium strategy at state $\eta \notin \{0, p, \ldots, Np\}$. However, they are not very restrictive. As for (i), it immediately follows from $h_0 \neq 1$. As for (ii) and (iii), KS1 presents a sufficient condition to assure that (ii) and (iii) hold, and it is satisfied in all of the matching models with divisible money known so far, such as Zhou (1999)'s model, a divisible money version of Camera and Corbae (1999)'s model, and a divisible money version of Trejos and Wright (1995)'s model.

Let

$$C^a = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{S-N-1},$$

and, for $(n, j) \notin \alpha(a)$,

$$C^{a(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{S-N-1},$$

where the last $\{0\}$ corresponds to $V_n - W_{nj}(x; \theta, 0)$. Moreover, for $(n, j), (n', j') \notin \alpha(a)$,

$$C^{a(n,j)(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+1} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{S-N-1},$$

where the last two $\{0\}$s correspond to $V_n - W_{nj}(x; \theta, 0)$ and $V_n - W_{n'j'}(x; \theta, 0)$, respectively. Below, it is verified that there is the indeterminacy of the stationary equilibrium under some regularity conditions.

Assumption 1 Given a, g^a is of class C^2 and is transversal to C^a, $C^{a(n,j)}$, and $C^{a(n,j)(n',j')}$ for all $(n, j) \notin \alpha(a)$ and $(n', j') \notin \alpha(a)$.[2]

Assumption 2 Given a, there exists a C^2-manifold without boundary, $\Theta \subset \mathbb{R}^L$, such that $E_\theta^a \neq \emptyset$ holds for all $\theta \in \Theta$.

Theorem 2 (Kamiya and Shimizu (2006)) For a given a, suppose Assumptions 1 and 2 are satisfied for some Θ. Then, for almost every $\theta \in \Theta$, E_θ^a is a one-dimensional manifold with

[2]This assumption implies that that $D_n = 0, n = 2, \ldots, N$, are linearly independent in stationary equilibria. See KS1 for indeterminacy results of the other cases.

boundary. Moreover, at any endpoint of the manifold, only one $V_n - W_{nj}(x;\theta,0) \geq 0$, $(n,j) \notin \alpha(a)$, is binding, and at points in the relative interior of the manifold, no inequality is binding.

KS1 also shows that this indeterminacy is real. That is, the welfare are typically not the same in a connected component of the equilibrium manifold.

2.2 Stationary Equilibria with Tax-Subsidy

In this section, I investigate the case of $t \neq (0,\ldots,0)$. In what follows, variables and functions with "tilde" denote the ones with nonzero t. The inflow at n, $\tilde{I}_n$, and the outflow at n, $\tilde{O}_n$, are defined as follows:

$$\begin{aligned}\tilde{I}_n(\tilde{h},a;\theta,t) &= I_n(\tilde{h},a;\theta) + \frac{\mu G}{1+G}\left(t^+_{n-1}\tilde{h}_{n-1} + t^-_{n+1}\tilde{h}_{n+1}\right),\\ \tilde{O}_n(\tilde{h},a;\theta,t) &= O_n(\tilde{h},a;\theta) + \frac{\mu G}{1+G}|t_n|\tilde{h}_n,\end{aligned}$$

where $t^+_n = \max\{0,t_n\}$, $t^-_n = -\min\{0,t_n\}$, and $t_{-1} = t_{N+1} = 0$. Let $\tilde{D}_n(\tilde{h},a;\theta,t) = \tilde{I}_n(\tilde{h},a;\theta,t) - \tilde{O}_n(\tilde{h},a;\theta,t)$.

Since $\sum_{n=0}^N n\tilde{D}_n$ is not identically zero, then I define a stationary equilibrium with tax-subsidy as follows. In other words, the tax-subsidy breaks the conservation law.

Definition 2 Given θ, $\tilde{x} = (\tilde{h},\tilde{V},a) \in \mathbb{R}^{N+1}\times\mathbb{R}^{N+1}_+\times A$ is a (pure strategy) *stationary equilibrium with tax-subsidy scheme* t if it satisfies the following:

$$\begin{aligned}\tilde{D}_n(\tilde{h},a;\theta,t) &= 0, && n=1,\ldots,N\\ \sum_{n=0}^N \tilde{h}_n - 1 &= 0, &&\\ \tilde{V}_n - W_{nj}(\tilde{x};\theta,t) &= 0, && (n,j)\in\alpha(a)\\ \tilde{V}_n - W_{nj}(\tilde{x};\theta,t) &\geq 0, && (n,j)\notin\alpha(a).\end{aligned} \tag{3}$$

Theorem 3 (Kamiya and Shimizu (2007)) Given a, consider the following system of the stationary condition:

$$(\tilde{D}_1,\ldots,\tilde{D}_N,\sum_{n=0}^N \tilde{h}_n - 1)^T = (0,\ldots,0)^T,$$

where T denotes transpose. If the Jacobian matrix with respect to $\tilde{h}$ of the LHS of the above system is of full rank at a stationary distribution, then the stationary distribution is locally determinate. Moreover, the budget is balanced on this stationary distribution.

Next, I discuss the existence of a locally determinate stationary equilibrium which has the following property; it is induced by a certain tax-subsidy scheme, and it exists in any given neighborhood of the stationary equilibrium which is not induced by tax-subsidy. I choose an arbitrary stationary equilibrium without tax-subsidy, denoted by $x^* = (h^*, V^*, a^*)$, which is in the relative interior of the equilibrium manifold. Thus, by Theorem 2, (2) is satisfied with strict inequalities.

First, the following vector can be found:

Lemma 1 There exists an $(N+1)$-dimensional vector τ satisfying

(a) $\tau \neq (0, \ldots, 0)$,

(b) $\left(\frac{\partial D_n(h^*, a^*; \theta)}{\partial h_i}\right)_{i=0,\ldots,N} \cdot \tau = 0$ for $n = 2, \ldots, N$,

(c) $h^* \cdot \tau = 0$.

The above lemma clearly holds, since (b) and (c) have at least one-degree of freedom.

Using this vector, a tax-subsidy scheme $t = \epsilon\tau$ is constructed. Here $\epsilon > 0$ is the size of the policy. For such a t to be a tax-subsidy scheme, I need the following assumption:

Assumption 3 It is also satisfied for τ in Lemma 1 that

(d) $\tau_N \leq 0$, and

(e) $\tau_0 \geq 0$.

Next, I make the following assumption.

Assumption 4 W_{nj} is C^2 with respect to ϵ for any (n, j).

Under the above conditions and assumptions, the government can approximately induce any stationary equilibrium.

Theorem 4 Suppose Assumptions 1, 2, 3, and 4 hold. Then, for almost every $\theta \in \Theta$, almost every $(h^*, V^*) \in E_\theta^{a^*}$, and any δ-neighborhood of (h^*, V^*), there exists a tax-subsidy scheme such that a stationary equilibrium with tax-subsidy is locally determinate and lies in the neighborhood.

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