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### Abstract

An iterative domain decomposition method is applied to magnetostatic problems. In our previous methods the gauge condition is neglected, then the magnetic vector potential is only one unknown function. On the other hand, it has been well-known that some theoretical results has been introduced, where a mixed formulation with the Lagrange multiplier is introduced in order to impose the gauge condition. Therefore, in this paper, we formulate again an iterative domain decomposition method based on a mixed formulation of magnetostatic problem, and discuss relations with the previous one.

**Keywords:** *magnetostatic problem, mixed formulation, iterative domain decomposition method*

## 1   Introduction

We have introduced an iterative domain decomposition method to solve quite large scale electromagnetic field problems; see, for example, Kanayama *et al*. [8]. In our previous methods the gauge condition is neglected, then the magnetic vector potential is only one unknown function. These previous results focus themselves on the engineering points of view: the previous formulation enables us to reduce computational consts in practical large scale simulations. However this formulation yields an indeterminate linear system, it is difficult to mathematically justify numerical results, for example unique solvability of the problems and convergency of the approximate solution.

On the other hand, some theoretical results has been introduced by, for example, Kikuchi [6], [7], where a mixed formulation with the Lagrange multiplier is introduced in order to impose the gauge condition. These results focus themselves on the mathematical point of view: owing to the introduction of the Lagrange multiplier, their mixed formulation enable us to prove unique solvability of the problems and convergency of the approximate solution. However this formulation yields an indefinite linear system, it is difficult to find an appropriate iterative solver, which is efficient enough to reduce computational costs for practical large scale problems.

At first in this paper, we formulate again an iterative domain decomposition method based on a mixed formulation of magnetostatic problem introduced in Kikuchi [6], [7], which enable us to prove unique solvability of the problems and convergency of the approximate solution. Seconed, to reduce computational costs, we simplify our iterative domain decomposition method into another one, and we discuss relations between the reduced formulation and the previous one.

## 2   Formulation of magnetostatic problems

Let $\Omega$ be a polyhedoral domain with its boundary $\Gamma$, and $\boldsymbol{n}$ the outward unit normal of $\Omega$. Let $\boldsymbol{u}$ denote the magnetic vector potential, $\boldsymbol{f}$ an excitation current density, and $\nu$ the magnetic reluctivity. Then, we

consider the magnetostatic equation with the Coulomb gauge condition:

$$\begin{cases} \operatorname{rot}(\nu \operatorname{rot} \boldsymbol{u}) = \boldsymbol{f} & \text{in } \Omega, & \text{(1a)} \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, & \text{(1b)} \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma; & \text{(1c)} \end{cases}$$

for example, see Kikuchi [6].

As usual, let $L^2(\Omega)$ be the space of real functions defined in $\Omega$ and 2nd power summable in $\Omega$, let $\|\,.\,\|$ be its norm, and let $(\,.\,,\,.\,)$ be its inner product; let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with derivatives up to the 1st order, and let $\|\,.\,\|_1$ and $|\,.\,|_1$ be its norm and seminorm, respectively; and set functional spaces $X$, $M$, $V$, and $Q$ by

$$\begin{aligned} X &:= \{\boldsymbol{v} \in (L^2(\Omega))^3; \ \operatorname{rot} \boldsymbol{v} \in (L^2(\Omega))^3\}, & M &:= H^1(\Omega), \\ V &:= \{\boldsymbol{v} \in X; \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}, & Q &:= \{q \in M; \ q = 0 \text{ on } \Gamma\}, \end{aligned}$$

respectively; set bilinear forms $a(\,.\,,\,.\,)$ and $b(\,.\,,\,.\,)$ by

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &:= \int_\Omega \nu \operatorname{rot} \boldsymbol{u} \operatorname{rot} \boldsymbol{v}\, dx, & \forall (\boldsymbol{u},\boldsymbol{v}) \in X \times X, \\ b(\boldsymbol{v},q) &:= \int_\Omega \boldsymbol{v} \operatorname{grad} q\, dx, & \forall (\boldsymbol{v},q) \in (L^2(\Omega))^3 \times M, \end{aligned}$$

respectively.

Now, by introducing the Lagrange multiplier $p$, we obtain a mixed weak formulation of (1) as follows: given $\boldsymbol{f} \in (L^2(\Omega))^3$, find $(\boldsymbol{u},p) \in V \times Q$ such that

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}), & & \text{(2a)} \\ b(\boldsymbol{u},q) = 0, & \forall (\boldsymbol{v},q) \in V \times Q. & \text{(2b)} \end{cases}$$

Some theoretical results of (2) such as the unique solvability have been proved in Kikuchi [6].

**Remark 1** *As in mentioned in Kikuchi [6], if $\boldsymbol{f}$ satisfies that* $\operatorname{div} \boldsymbol{f} = 0$ *in $\Omega$, then $p = 0$. This property plays a key role in the forthcoming section.*

## 3  Domain decomposition method

For simplicity, the domain $\Omega$ is assumed to be decomposed into two non-overlapping subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ with their boundaries $\partial \Omega^{(1)}$ and $\partial \Omega^{(2)}$, respectively:

$$\Omega^{(i)} \neq \emptyset \quad (i = 1, 2), \qquad \overline{\Omega} = \overline{\Omega}^{(1)} \cup \overline{\Omega}^{(2)}, \qquad \Omega^{(1)} \cap \Omega^{(2)} = \emptyset;$$

and let $\gamma_{12}$ be the interface between $\Omega^{(1)}$ and $\Omega^{(2)}$ defined by $\gamma_{12} := \overline{\Omega}^{(1)} \cap \overline{\Omega}^{(2)}$; see Fig. 1. For $i = 1, 2$, the outward unit normal of $\Omega^{(i)}$ is denoted by $\boldsymbol{n}^{(i)}$, and set $\boldsymbol{n} = \boldsymbol{n}^{(1)}(= -\boldsymbol{n}^{(2)})$ on the interface $\gamma_{12}$.

Instead of the real functions defined in $\Omega$, we associate this decomposition to function spaces, bilinear forms, and inner product: let $L^2(\Omega^{(i)})$ and $H^1(\Omega^{(i)})$ be the space of real functions defined in $\Omega^{(i)}$, which are corresponding to $L^2(\Omega)$ and $H^1(\Omega)$; set function spaces $X^{(i)}$, $M^{(i)}$, $V_{\gamma_{12}}^{(i)}$, $Q_{\gamma_{12}}^{(i)}$, $V^{(i)}$, and $Q^{(i)}$ by

$$\begin{aligned} X^{(i)} &:= \{\boldsymbol{v} \in (L^2(\Omega^{(i)}))^3; \ \operatorname{rot} \boldsymbol{v} \in (L^2(\Omega^{(i)}))^3\}, & M^{(i)} &:= H^1(\Omega^{(i)}), \\ V_{\gamma_{12}}^{(i)} &:= \left\{\boldsymbol{v} \in X^{(i)}; \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega^{(i)} \backslash \gamma_{12}\right\}, & Q_{\gamma_{12}}^{(i)} &:= \left\{q \in M^{(i)}; \ q = 0 \text{ on } \partial\Omega^{(i)} \backslash \gamma_{12}\right\}, \\ V^{(i)} &:= \left\{\boldsymbol{v} \in X^{(i)}; \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega^{(i)}\right\}, & Q^{(i)} &:= \left\{q \in M^{(i)}; \ q = 0 \text{ on } \partial\Omega^{(i)}\right\}, \end{aligned}$$
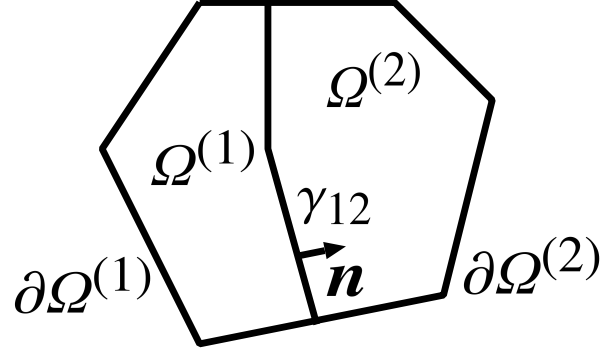
Fig. 1: Two non-overlapping subdomains of $\Omega$.

respectively; and set bilinear forms and inner product $a^{(i)}(.\,,.)$, $b^{(i)}(.\,,.)$, and $(.\,,.)_{\Omega^{(i)}}$ by

$$a^{(i)}(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) := \int_{\Omega^{(i)}} \nu \operatorname{rot} \boldsymbol{u}^{(i)} \operatorname{rot} \boldsymbol{v}^{(i)} \, dx, \qquad \forall (\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) \in X^{(i)} \times X^{(i)},$$

$$b^{(i)}(\boldsymbol{v}^{(i)}, p^{(i)}) := \int_{\Omega^{(i)}} \boldsymbol{v}^{(i)} \operatorname{grad} p^{(i)} \, dx, \qquad \forall (\boldsymbol{v}^{(i)}, p^{(i)}) \in (L^2(\Omega^{(i)}))^3 \times M^{(i)},$$

$$(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)})_{\Omega^{(i)}} := \int_{\Omega^{(i)}} \boldsymbol{u}^{(i)} \boldsymbol{v}^{(i)} \, dx, \qquad \forall (\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) \in (L^2(\Omega^{(i)}))^3 \times (L^2(\Omega^{(i)}))^3,$$

respectively. Moreover, set function spaces $\Lambda$ and $\Xi$ by

$$\Lambda := \left\{ \boldsymbol{\lambda} : \gamma_{12} \to \mathbb{R}^3; \; \boldsymbol{\lambda} = (\boldsymbol{v} \times \boldsymbol{n})|_{\gamma_{12}}, \; \boldsymbol{v} \in V \right\}, \qquad \Xi := \left\{ \xi : \gamma_{12} \to \mathbb{R}; \; \xi = q|_{\gamma_{12}}, \; q \in Q \right\};$$

and set $\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})$ by any extension operator from $\Lambda$ to $V^{(i)}_{\gamma_{12}}$ such that $\boldsymbol{\eta} = (\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta}) \times \boldsymbol{n})|_{\gamma_{12}}$, and $\overline{p}^{(i)}(\zeta)$ by any extension operator from $\Xi$ to $Q^{(i)}_{\gamma_{12}}$ such that $\zeta = p(\zeta)|_{\gamma_{12}}$. A characterization of *tangential trace* spaces $\Lambda$ and an *tangential extension* operator on $\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})$ has been given in Alonso–Valli [1], Buffa–Ciarlet [2], [3], Buffa, *et al.* [4], and Quarteroni–Valli [9].

Now, a two-subdomain problem is introduced by the followings: for $i = 1, 2$, find $(\boldsymbol{u}^{(i)}, p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ such that

$$\left\{ \begin{array}{lll} a^{(i)}(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) + b^{(i)}(\boldsymbol{v}^{(i)}, p^{(i)}) = (\boldsymbol{f}^{(i)}, \boldsymbol{v}^{(i)})_{\Omega^{(i)}}, & & \text{(3a)} \\[4pt] b^{(i)}(\boldsymbol{u}^{(i)}, q^{(i)}) = 0, & \forall (\boldsymbol{v}^{(i)}, q^{(i)}) \in V^{(i)} \times Q^{(i)} & \text{(3b)} \\[4pt] \boldsymbol{u}^{(1)} \times \boldsymbol{n} = \boldsymbol{u}^{(2)} \times \boldsymbol{n} & \text{on } \gamma_{12}, & \text{(3c)} \\[4pt] p^{(1)} = p^{(2)} & \text{on } \gamma_{12}, & \text{(3d)} \\[4pt] a^{(2)}(\boldsymbol{u}^{(2)}, \overline{\boldsymbol{u}}^{(2)}(\boldsymbol{\eta})) + b^{(2)}(\overline{\boldsymbol{u}}^{(2)}(\boldsymbol{\eta}), p^{(2)}) & & \\[4pt] \quad = (\boldsymbol{f}^{(1)}, \overline{\boldsymbol{u}}^{(1)}(\boldsymbol{\eta}))_{\Omega^{(1)}} + (\boldsymbol{f}^{(2)}, \overline{\boldsymbol{u}}^{(2)}(\boldsymbol{\eta}))_{\Omega^{(2)}} - a^{(1)}(\boldsymbol{u}^{(1)}, \overline{\boldsymbol{u}}^{(1)}(\boldsymbol{\eta})) - b^{(1)}(\overline{\boldsymbol{u}}^{(1)}(\boldsymbol{\eta}), p^{(1)}), & & \text{(3e)} \\[4pt] b^{(2)}(\boldsymbol{u}^{(2)}, \overline{p}^{(2)}(\zeta)) = b^{(1)}(\boldsymbol{u}^{(1)}, \overline{p}^{(1)}(\zeta)), & \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi. & \text{(3f)} \end{array} \right.$$

If $\{(\boldsymbol{u}^{(1)}, p^{(1)}), (\boldsymbol{u}^{(2)}, p^{(2)})\}$ is a pair of the solutions of two-subdomain problem (3), then the solution of the one-domain problem (2) could be constructed by

$$(\boldsymbol{u}, p) := \left\{ \begin{array}{ll} (\boldsymbol{u}^{(1)}, p^{(1)}) & \text{in } \Omega^{(1)}, \qquad\qquad\qquad\qquad\quad \text{(4a)} \\[4pt] (\boldsymbol{u}^{(2)}, p^{(2)}) & \text{in } \Omega^{(2)}. \qquad\qquad\qquad\qquad\quad \text{(4b)} \end{array} \right.$$

On the other hand, if $(\boldsymbol{u}, p)$ is a solution of the one-domain problem (2), then a pair of the solutions $\{(\boldsymbol{u}^{(1)}, p^{(1)}), (\boldsymbol{u}^{(2)}, p^{(2)})\}$ of the two-subdomain problem (3) could be constructed by

$$(\boldsymbol{u}^{(i)}, p^{(i)}) := (\boldsymbol{u}|_{\Omega^{(i)}}, p|_{\Omega^{(i)}}) \quad \text{in } \Omega^{(i)}. \tag{5}$$

Moreover, as mentioned in Sec 2, the one-domain problem (2) is uniquely solvable. Then two-subdomain problem (3) is also uniquely solvable. Therefore, the equivalency between both formulations and unique solvability could be obtained as follows:

**Theorem 1** *There exists a unique solution* $(\boldsymbol{u}, p) \in V \times Q$ *of the two-subdomain problem* (3)*. Moreover, the one-domain problem* (2) *and the two-subdomain problem* (3) *are equivalent.*

For $i = 1, 2$, let $\mathscr{E}^{(i)}(\boldsymbol{f}, \lambda, \xi)$ an extention operator from $(L^2(\Omega))^3 \times \Lambda \times \Xi$ to $V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ defined by $\mathscr{E}^{(i)}(\boldsymbol{f}, \lambda, \xi) := (\boldsymbol{u}^{(i)}, p^{(i)})$, where $(\boldsymbol{u}^{(i)}, p^{(i)})$ is the solution of the following magnetostatic problem:

$$
\begin{cases}
a^{(i)}(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) + b^{(i)}(\boldsymbol{v}^{(i)}, p^{(i)}) = (\boldsymbol{f}^{(i)}, \boldsymbol{v}^{(i)})_{\Omega^{(i)}}, & & \text{(6a)} \\
b^{(i)}(\boldsymbol{u}^{(i)}, q^{(i)}) = 0, & \forall(\boldsymbol{v}^{(i)}, q^{(i)}) \in V^{(i)} \times Q^{(i)}, & \text{(6b)} \\
\boldsymbol{u}^{(i)} \times \boldsymbol{n} = \lambda & \text{on } \gamma_{12}, & \text{(6c)} \\
p^{(i)} = \xi & \text{on } \gamma_{12}. & \text{(6d)}
\end{cases}
$$

Then, a *Steklov–Poincaré* operator $\mathscr{A}$ from $\Lambda \times \Xi$ to $(\Lambda \times \Xi)'$ is set by

$$
\langle \mathscr{A}(\lambda, \xi), (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}} := \sum_{i=1}^{2} \{ a^{(i)}(\overline{\boldsymbol{u}}^{(i)}, \overline{\boldsymbol{v}}^{(i)}) + b^{(i)}(\overline{\boldsymbol{v}}^{(i)}, \overline{p}^{(i)}) + b^{(i)}(\overline{\boldsymbol{u}}^{(i)}, \overline{q}^{(i)}) \}, \quad \forall \lambda, \boldsymbol{\eta} \in \Lambda, \forall \xi, \zeta \in \Xi \quad (7)
$$

where $(\overline{\boldsymbol{u}}^{(i)}, \overline{p}^{(i)}) := \mathscr{E}^{(i)}(0, \lambda, \xi)$ and $(\overline{\boldsymbol{v}}^{(i)}, \overline{q}^{(i)}) := \mathscr{E}^{(i)}(0, \boldsymbol{\eta}, \zeta)$; and an *interface source* $\chi \in (\Lambda \times \Xi)'$ is set by

$$
\langle \chi, (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}} := \sum_{i=1}^{2} \{ (\boldsymbol{f}^{(i)}, \overline{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} - a^{(i)}(\widehat{\boldsymbol{u}}^{(i)}, \overline{\boldsymbol{v}}^{(i)}) - b^{(i)}(\overline{\boldsymbol{v}}^{(i)}, \widehat{p}^{(i)}) - b^{(i)}(\widehat{\boldsymbol{u}}^{(i)}, \overline{q}^{(i)}) \}, \quad \forall \xi, \zeta \in \Xi \quad (8)
$$

where $(\widehat{\boldsymbol{u}}^{(i)}, \widehat{p}^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, 0, 0)$ and $(\overline{\boldsymbol{v}}^{(i)}, \overline{q}^{(i)}) := \mathscr{E}^{(i)}(0, \boldsymbol{\eta}, \zeta)$. Now we introduce the following *interface problem* on $\gamma_{12}$:

$$
\langle \mathscr{A}(\lambda, \xi), (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}} = \langle \chi, (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}}, \qquad \forall(\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi. \quad (9)
$$

By using the solution $(\boldsymbol{u}^{(i)}, p^{(i)})$ of two-subdomain problem (3), let us set $(\lambda, \xi)$ by $\lambda := \boldsymbol{u}^{(1)} \times \boldsymbol{n} \, (= \boldsymbol{u}^{(2)} \times \boldsymbol{n})$ and $\xi := p^{(1)}(= p^{(2)})$. Then, because of (3c)–(3f), $(\lambda, \xi)$ satisfies the *interface problem* (9). On the other hand, once the solution $(\lambda, \xi)$ is obtained by solving the *interface problem* (9), for $i = 1, 2$, each pair $(\boldsymbol{u}^{(i)}, p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ could be found from the problem (3a) and (3b) in the corresponding subdomain $\Omega^{(i)}$, where the solution $(\lambda, \xi)$ is regarded as the Dirichlet boundary on the interface: $\boldsymbol{u}^{(i)} \times \boldsymbol{n} = \lambda$ and $p^{(i)} = \xi$ on $\gamma_{12}$. Finally, from (4), we can obtain the solution $(\boldsymbol{u}, p)$ of the one-domain problem (2).

Therefore, error analysis of the approximate solution of the problem (3) could be reduced into error analysis of the one of each subdomain problem; For example, when magnetic vector potential $\boldsymbol{u}$ is approximated by the Nedelec element of the first order and the Lagrange multiplier $p$ is approximated by the conventional $P1$-element; see Kikuchi [7].

The *interface problem* (9) is symmetric, and not positive definite. Then, by following Glowinski *et al.* [5], the following conjugate gradient algorithm could be obtained (at least formally):

> **Choose** $(\lambda_0, \xi_0)$;
>
> **Compute** $(\boldsymbol{g}_0, \delta_0)$ **by** (10);
>
> $(\boldsymbol{w}_0, \omega_0) := (\boldsymbol{g}_0, \delta_0)$;
>
> **for** $k = 0, 1, \dots$;
>
>     **Compute** $\mathscr{A}(\boldsymbol{w}_k, \omega_k)$ **by** (11);
>
>     $\alpha_k := ((\boldsymbol{g}_k, \delta_k), (\boldsymbol{g}_k, \delta_k)) / (\mathscr{A}(\boldsymbol{w}_k, \omega_k), (\boldsymbol{w}_k, \omega_k))$;
>
>     $(\lambda_{k+1}, \xi_{k+1}) := (\lambda_k, \xi_k) - \alpha_k (\boldsymbol{w}_k, \omega_k)$;
>
>     $(\boldsymbol{g}_{k+1}, \delta_{k+1}) := (\boldsymbol{g}_k, \delta_k) - \alpha_k \mathscr{A}(\boldsymbol{w}_k, \omega_k)$;

$$\beta_k := ((\boldsymbol{g}_{k+1}, \delta_{k+1}), (\boldsymbol{g}_{k+1}, \delta_{k+1})) / ((\boldsymbol{g}_k, \delta_k), (\boldsymbol{g}_k, \delta_k));$$

**If** $((\boldsymbol{g}_{k+1}, \delta_{k+1}), (\boldsymbol{g}_{k+1}, \delta_{k+1})) / ((\boldsymbol{g}_0, \delta_0), (\boldsymbol{g}_0, \delta_0)) < \varepsilon,$ **break**;

$$(\boldsymbol{w}_{k+1}, \omega_{k+1}) := (\boldsymbol{g}_{k+1}, \delta_{k+1}) + \beta_k (\boldsymbol{w}_k, \omega_k);$$

**end**;

where $\varepsilon$ is a positive constant for the criterion of the convergence. In the above conjugate gradient algorythm, $(\boldsymbol{g}_0, \delta_0)$ could be computed by the extentions $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)})$ and $(\bar{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)})$ as follow:

$$\langle (\boldsymbol{g}_0, \delta_0), (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}}$$
$$= \sum_{i=1}^{2} \{ a^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)}, \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)}, \widetilde{p}_0^{(i)}) - (\boldsymbol{f}^{(i)}, \bar{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} + b^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)}, \bar{q}^{(i)}) \}, \qquad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (10)$$

where $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \lambda_0, \xi_0)$; and $\mathscr{A}(\boldsymbol{w}_k, \omega_k)$ could be computed by the extentions $(\widehat{\boldsymbol{u}}_0^{(i)}, \widehat{p}_0^{(i)})$ and $(\bar{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)})$ as follow:

$$\langle \mathscr{A}(\boldsymbol{w}_k, \omega_k), (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}} = \sum_{i=1}^{2} \{ a^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)}, \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)}, \widehat{p}_k^{(i)}) + b^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)}, \bar{q}^{(i)}) \}, \qquad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (11)$$

where $(\widehat{\boldsymbol{u}}_k^{(i)}, \widehat{p}_k^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{0}, \boldsymbol{w}_k, \omega_k)$. The extentions $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)})$, $(\widehat{\boldsymbol{u}}_0^{(i)}, \widehat{p}_0^{(i)})$, and $(\bar{\boldsymbol{v}}^{(i)}, \bar{q}^{(i)})$ in (10) and (11) could be computed in $\Omega^{(1)}$ and $\Omega^{(2)}$ independently. Therefore, the above conjugate gradient algorythm is familiar with parallel computations.

Moreover, as mentioned in Remark 1, if $\boldsymbol{f}^{(i)}$ satisfies that $\operatorname{div} \boldsymbol{f}^{(i)} = 0$ in $\Omega^{(i)}$, then $p^{(i)}$ vanishes. This implies that we can neglect the components corresponding to the Lagrange multiplier in the conjugate gradient algorythm. Therefore we can get the reduced conjugate gradient algorythm as follows:

**Choose** $\lambda_0$;

**Compute** $\boldsymbol{g}_0$ **by** (12);

$\boldsymbol{w}_0 := \boldsymbol{g}_0$;

**for** $k = 0, 1, \ldots$;

    **Compute** $\mathscr{A}_1(\boldsymbol{w}_k, 0)$ **by** (13);

    $\alpha_k := (\boldsymbol{g}_k, \boldsymbol{g}_k) / (\mathscr{A}_1(\boldsymbol{w}_k, 0), \boldsymbol{w}_k,)$;

    $\lambda_{k+1} := \lambda_k - \alpha_k \boldsymbol{w}_k$;

    $\boldsymbol{g}_{k+1} := \boldsymbol{g}_k - \alpha_k \mathscr{A}_1(\boldsymbol{w}_k, 0)$;

    $\beta_k := (\boldsymbol{g}_{k+1}, \boldsymbol{g}_{k+1}) / (\boldsymbol{g}_k, \boldsymbol{g}_k)$;

    **If** $(\boldsymbol{g}_{k+1}, \boldsymbol{g}_{k+1}) / (\boldsymbol{g}_0, \boldsymbol{g}_0) < \varepsilon,$ **break**;

    $\boldsymbol{w}_{k+1} := \boldsymbol{g}_{k+1} + \beta_k \boldsymbol{w}_k$;

**end**;

In the reduced conjugate gradient algorythm, $\boldsymbol{g}_0$ could be computed by the first component of the following equation:

$$\langle (\boldsymbol{g}_0, \delta_0), (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}}$$
$$= \sum_{i=1}^{2} \{ a^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)}, \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)}, \widetilde{p}_0^{(i)}) - (\boldsymbol{f}^{(i)}, \bar{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} + b^{(i)}(\widetilde{\boldsymbol{u}}_0^{(i)}, \bar{q}^{(i)}) \}, \qquad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (12)$$

where $(\widetilde{\boldsymbol{u}}_0^{(i)}, \widetilde{p}_0^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \lambda_0, 0)$; and $\mathscr{A}_1(\boldsymbol{w}_k, 0)$ could be computed by the first component of the following equation:

$$\langle \mathscr{A}(\boldsymbol{w}_k, 0), (\boldsymbol{\eta}, \zeta) \rangle_{\gamma_{12}} = \sum_{i=1}^{2} \{ a^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)}, \bar{\boldsymbol{v}}^{(i)}) + b^{(i)}(\bar{\boldsymbol{v}}^{(i)}, \widehat{p}_k^{(i)}) + b^{(i)}(\widehat{\boldsymbol{u}}_k^{(i)}, \bar{q}^{(i)}) \}, \qquad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (13)$$

where $(\widehat{\boldsymbol{u}}_k^{(i)}, \widehat{p}_k^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{0}, \boldsymbol{w}_k, 0)$.

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