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On the Hamilton-Jacobi Variational Formulation of the Vlasov Equation

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The Hamilton-Jacobi formulation of Vlasov-like systems and associated action principles, developed by the author and D. Pfirsch in a series of papers since the mid 1980s, are briefly reviewed and suggestions for their use are given.

I. INTRODUCTION

In this note we briefly review the Hamilton-Jacobi (HJ) formulation of Vlasov-like systems. This is a general formulation that applies to the Maxwell-Vlasov system and various guiding center and gyrokinetic theories with any number of species. It applies to both non-relativistic and relativistic versions of these theories and even to the Vlasov-Einstein system. Indeed, it is quite general and applies to any Vlasov-like theory, but we will review it in its simplest context of the Vlasov-Poisson system.

The formulation evolved out of early work of Pfirsch [1], but the general formulation was first given in [2]. The HJ formulation is variational – it has in fact two action principles, and so it provides a natural method via Noether's theorem for obtaining unambiguous energy-momentum tensors for general kinetic theories. These were obtained and discussed in a sequence of papers [2–4] and this work was continued in [5, 6], where errors in the literature were pointed out.

This note is organized as follows. In Sec. II we review the HJ theory in the context of classical mechanics. Then in Sec. III the action principle of [2] for the general theory is described along with a reduced version given in [7]. Finally, in Sec. IV we conclude.

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II. CLASSICAL HAMILTON-JACOBI THEORY

Hamilton-Jacobi theory arises in the context of classical mechanics where it is proposed as a means of solving Hamilton systems:

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{1}$$

where $z := (q, p)$ denotes coordinates for a $2n$ dimensional manifold $\mathcal{Z}$ and $H(q, p)$ is the Hamiltonian function that defines the system. Equations (1) can be compactly written in coordinates as follows:

$$\dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j} \tag{2}$$

where $i, j = 1 \ldots 2n$, the repeated index is to be summed, and the matrix

$$J_c := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} . \tag{3}$$

is the the cosymplectic form. Equations (1) can also be written as

$$\dot{z} = [z, H] \,, \tag{4}$$

where [,] is the Poisson bracket defined on phase space functions by

$$[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j} \,. \tag{5}$$

The basic idea underlying Hamilton-Jacobi theory is to solve Hamilton's equations by changing coordinates. Under a coordinate change $z \leftrightarrow \bar{z}$, Eqs. (2) become

$$\dot{\bar{z}}^i = \bar{J}^{ij} \frac{\partial \bar{H}}{\partial \bar{z}^j} \tag{6}$$

where the Hamiltonian transforms as a scalar, $H(z) - \bar{H}(\bar{z})$ and the cosymplectic form as a second order contravariant tensor

$$\bar{J}^{mn} = \frac{\partial \bar{z}^m}{\partial z^i} J_c^{ij} \frac{\partial \bar{z}^n}{\partial z^j} \,. \tag{7}$$

Canonical transformations or symplectomorphisms, as they are commonly referred to now when the global geometry of $\mathcal{Z}$ is under consideration, are those for which

$$\bar{J}^{mn} \equiv J_c^{mn} \,. \tag{8}$$

Two commonly used methods for generating canonical transformations are the Lie transform and the mixed variable generating function (MVGF). In recent times in plasma physics the Lie transform has been widely used, but each has their advantage. The Lie transform is a series representation that comes from exponentiating a Poisson bracket and when this series is truncated the canonical property is generally lost. However, the mixed variable generating function approach does not suffer from this defect, but it succeeds at the expense of giving an implicit form for canonical transformations, which for our purposes will be generated as follows:

$$p = \frac{\partial S}{\partial Q} \qquad \text{and} \qquad Q = \frac{\partial S}{\partial P} \qquad \text{with} \qquad \left\| \frac{\partial^2 S}{\partial q \partial P} \right\| \neq 0\,. \tag{9}$$

Here $\bar{z} = (Q, P)$ are the new canonical variables, $S(q, P, t)$ is the MVGF, $\| \, \|$ denotes determinant and the nonvanishing of $\|\partial^2 S/\partial q \partial P\|$ is a necessary condition, by the implicit function theorem, for the transformation $z \leftrightarrow \bar{z}$ to exist. If the transformation has explicit time dependence then the new Hamiltonian does not transform as a scalar, energy not being a covariant quantity, but is given by

$$\bar{H}(Q, P, t) = H(q, p, t) + \frac{\partial S}{\partial t}\,. \tag{10}$$

The strategy of HJ theory is to make $\bar{H}$ so simple that trajectories in terms of (Q, P) can be obtained, and then the complication in the orbits is embodied in the transformation back to (q, p). The transformation back is obtained by solving the HJ equation obtained by inserting (9) into (10), giving

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = \bar{H}\,. \tag{11}$$

This is the HJ equation for the *generating function* S.

Various choices for $\bar{H}$ can be considered. For example $\bar{H} \equiv 0$ would mean all of the dynamics is in the transformation back. This amounts to the use of initial conditions as coordinates which, except in the most trivial cases, are not good coordinates because of serious branching issues. Basically, these coordinates are not isolating, i.e., they do not define good surfaces in $\mathcal{Z}$. A more realistic choice is to choose $\bar{H}(P)$, where all the configuration space coordinates are ignorable. This amounts to seeking a transformation to action-angle variables. It is now known that only for integrable systems do such coordinates exist, and if the system is not nearly integrable, i.e., not near to the case where $\mathcal{Z}$ is foliated by n-tori, then such coordinates do not even approximately exist. But, near to integrability,

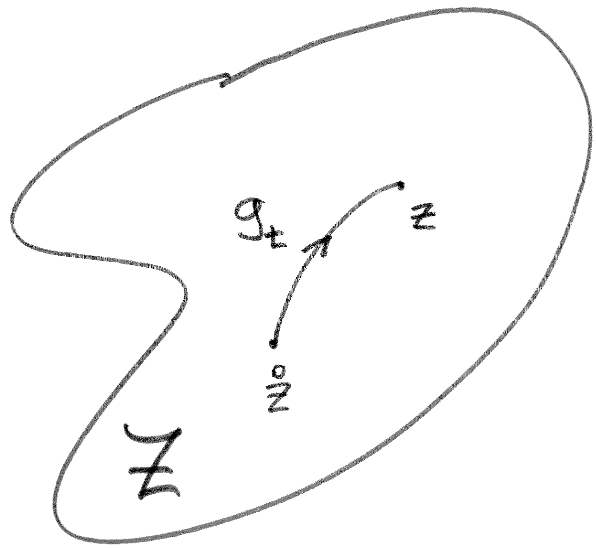

FIG. 1: Canonical transformation of phase space.

perturbation theory makes sense, as has been shown by KAM theory, which incidentally was first proven in the HJ context. Thus, there are various issues with classical existence theory of the HJ equation, tied up with small divisors, the existence of caustics, etc., which we will not pursue.

Before proceeding to how HJ theory arises in the context of Vlasov, we mention one more fact from mechanics that we will need, viz., Hamilton dynamics itself is a canonical transformation. This means that if we could integrate Hamilton's equations for all initial conditions $\mathring{z}$, then this would define a map from $\mathcal{Z}$ to itself as depicted in Fig. 1. The map $\mathring{z} \mapsto z(t)$, denoted by g_t in the figure, is a canonical transformation. The set $G = \{g_t | t \in \mathbb{R}\}$ is the Lie group of one-parameter family of canonical transformations, where $g_t : \mathcal{Z} \to \mathcal{Z}$ for all times.

III. HAMILTON-JACOBI ACTION PRINCIPLES FOR VLASOV-POISSON

Here we first consider the Vlasov-Poisson system, then proceed to construct two action principles.

A. The Vlasov-Poisson system

Global existence theories for this system were proven in the early 1990s by Pfaffelmoser, Perthame, Schaeffer and others; but, since our presentation will be formal, these theorems will not concern us. We will consider the simplest case of one spatial dimension with a single dynamical variable, the phase space density $f(q,p,t)$, that only depends on (q,p,t) and $f : U \times \mathbb{R}^2 \to \mathbb{R}^+$, where the set $U \in \mathbb{R}$ is often the circle or all of $\mathbb{R}$. Let $D := U \times \mathbb{R}$ be the phase spatial domain. The Vlasov equation of interest here is

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial q} - E\frac{\partial f}{\partial p} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial f}{\partial t} + [f, H] = 0 \tag{12}$$

which is the equation a single electron species, which we wish to solve for a given initial condition $\mathring{f}(q,p) = f(q,p,0)$. The equality of (12) follows from the definitions

$$H := \frac{p^2}{2} - \phi, \qquad E =: -\frac{\partial \phi}{\partial q}, \qquad \text{and} \qquad [f,g] := \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\,. \tag{13}$$

Here, unlike in Sec. II, (q,p) denote independent variables. From context is should be clear when we mean dependent and when we mean independent variable for these quantities.

It remains to determine the electrostatic potential $\phi(q,t)$ through Poisson's equation

$$\frac{\partial^2 \phi}{\partial q^2} = \int_{\mathbb{R}} dp\, f \qquad \longleftrightarrow \qquad \phi(q,t) = \int_D dq' dp'\, K(q|q') f(q',p',t) \tag{14}$$

where $K(q,q')$ denotes the "Green's" function. For convenience we have included charge neutrality in the definition of f.

There are two <u>essential components</u> of Vlasov theory that we will exploit:

CE: The <u>characteristic equations</u> of (12), given by

$$\dot{q} = [q, H] \qquad \text{and} \qquad \dot{p} = [p, H]\,, \tag{15}$$

which exist because of the hyperbolic nature of (12).

SR: The <u>rule</u> for constructing f from its initial condition $\mathring{f}$ given the solution to (15), i.e.,

$$f(q,p,t) = \mathring{f}\,(\mathring{q}(q,p,t), \mathring{p}(q,p,t)) = \mathring{f} \circ g_{-t}\, \mathring{z} =: z\# \mathring{f}\,. \tag{16}$$

In the first equality of (16) we have first written the solution in the usual way of plasma physics, in the second in terms of the one-parameter group discussed in Sec. II, and lastly in terms of a compact notation. Because the characteristic equations (15) are Hamiltonian we know g_t, which is the inverse of the map $\mathring{z} \mapsto z(t)$, denotes a canonical transformation. Thus, we say that the solution f is a *symplectic rearrangement* (SR) of $\mathring{f}$.

B. HJ Vlasov Formulation and the First Action Principle

The HJ formulation alters both of the essential components of Vlasov theory as follows:

CE: The characteristic equations are replaced by the generating function $S(q, P, t)$, knowledge of which is completely equivalent to the trajectories $(q(\mathring{q}, \mathring{p}, t), p(\mathring{q}, \mathring{p}))$.

SR: The rule for constructing f is replaced by a new equivalent rule given in terms of a new variable defined by

$$\Phi(q, P, t) := \left\| \frac{\partial^2 S}{\partial q \partial P} \right\| f\left(q, \frac{\partial S}{\partial q}, t\right) . \tag{17}$$

The quantity $\|\partial^2 S/\partial q \partial P\|$ was investigated in quantum mechanical contexts by Van Vleck, Pauli, and DeWitt-Morette, and is often referred to as the Van Vleck determinant.

To complete the formulation one needs equations for the pair of functions (Φ, S), such that these equations and the rule give solutions equivalent to the Vlasov equation, which can be written as

$$f\left(q, \frac{\partial S}{\partial q}, t\right) = \mathring{f}\left(\frac{\partial S}{\partial P}, P\right) . \tag{18}$$

It is evident that the HJ equation for S cannot be an ordinary HJ equation since, like the usual Vlasov equation, it must be global in nature. This arises in (12) through the electric field E that is determined by Poisson's equation. Thus, the HJ equation will be, like Vlasov, an integro-differential equation. With this in mind we rewrite the solution of Poisson's equation of (14) in a few different ways:

$$\begin{aligned} \phi(q,t) &= \int_D dq'dp'\, K(q|q')\, f(q', p', t) \\ &= \int_D dq'dP'\, K(q|q')\, f(q', p', t) \left\| \frac{\partial^2 S}{\partial q' \partial P'} \right\| \\ &= \int_D dq'dP'\, K(q|q')\, \Phi(q', P', t) , \end{aligned} \tag{19}$$

where the last expression shows a clean linear relationship between ϕ and Φ.

To obtain the equations for S and Φ we appeal to the phase space action principle, the principle of mechanics that yields Hamilton's equations upon variation. This action principle is given by

$$\mathcal{A}[q, p] = \int_{t_0}^{t_1} dt\ (p \cdot \dot{q} - H) , \tag{20}$$

which is defined on phase space paths that begin at q_0 at time t_0 and end at q_1 at time t_1. The functional derivative $\delta\mathcal{A}/\delta q = 0$ and $\delta\mathcal{A}/\delta p = 0$ imply Hamilton's equations (1); boundary conditions on p are not specified. In analogy to (20) we suppose Φ and S are like conjugate variables and write the following action for them:

$$\mathcal{A}[S,\Phi] = -\int_{t_0}^{t_1} dt \int_D dq dP\, \Phi \left(\frac{\partial S}{\partial t} + \frac{1}{2}\left|\frac{\partial S}{\partial q}\right|^2 - \frac{\phi}{2} - \bar{H} \right) \tag{21}$$

where ϕ is to be viewed as a shorthand for the expression defined by (19) and H_0 is a reference Hamiltonian analogous to the $\bar{H}$ of classical HJ theory and we are free to tailor this to the problem at hand. This amounts to a kind of gauge freedom. Variation of (21) with respect to Φ gives

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left|\frac{\partial S}{\partial q}\right|^2 - \phi - \bar{H} = 0\,, \tag{22}$$

while variation with respect to S gives

$$\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial q}\cdot\left(\Phi\,\frac{\partial H}{\partial P}\right) - \frac{\partial}{\partial P}\cdot\left(\Phi\,\frac{\partial \bar{H}}{\partial Q}\right) = 0\,. \tag{23}$$

Thus we arrive at the following

Theorem *If S satisfies (22) and Φ satisfies (23), then if f is constructed according to (17) it satisfies the Vlasov-Poisson system of (12) with (14).*

Proof The proof is mainly a chain rule exercise. For the details we refer the reader to [2, 3]. ∎

At first impression one might wonder if progress has been made since we began with one equation and now have two to solve – albeit now we have equations derivable from a variational principle. It turns out that (23) has special properties that make it easy to solve. Instead of pursing this here, in the next section we will eliminate this variable all together.

C. Reduced HJ Vlasov Formulation and the Second Action Principle

Before proceeding to our reduced action principle we describe a cartoon of the Vlasov phase space, as depicted in Fig. 2. Solutions of the Vlasov equation lie in some function space that we will denote by $\mathcal{F}$. We will not be specific about $\mathcal{F}$, but only discuss properties in a formal manner with intuition coming from finite-dimensional noncanonical Hamiltonian

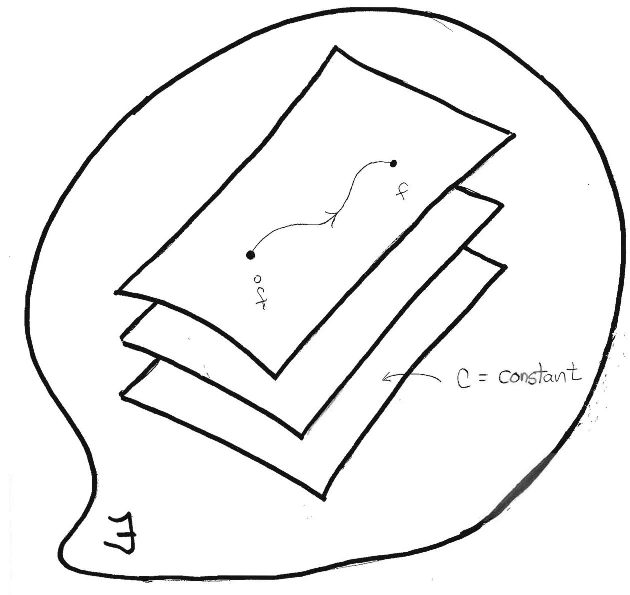

FIG. 2: Depiction of Poisson manifold foliated by constant Casimir symplectic leaves.

systems (see e.g. [8]) for which phase space is a Poisson manifold. Because $f = z\#\mathring{f}$, i.e. f is a SR of $\mathring{f}$, not all functions f are accessible from a given $\mathring{f}$. Under mild continuity conditions, the rule $f = z\#\mathring{f}$ implies certain properties of f and $\mathring{f}$ must coincide for all time, viz., the number of extrema and their values, the level set topologies, and the area between any two level sets. The SR property defines an equivalence relation $\sim$, where a phase space function $\mathring{f} \sim f$ if $\exists$ any canononical transformation z, i.e. any trajectory functions that can be generated by any Hamiltonian, such that $f = z\#\mathring{f}$. Thus, the dynamics takes place on a constraint set or, equivalently, motion lies entirely within an equivalence class. In this way the function state space $\mathcal{F}$ is foliated by leaves, each of which is labeled by an initial condition $\mathring{f}$. In [4, 9] states $f \sim \mathring{f}$ were called *dynamically accessible*.

The space $\mathcal{F}/\sim$ is formally an infinite-dimensional symplectic manifold and an explicit nondegenerate Poisson bracket on it was given in [10]. Thus we refer to the leaves as symplectic leaves. On such a leaf in the vicinity of an equilibrium $\mathring{f}$, a linear canonical form for the Poisson bracket was explicitly obtained in [11–13] and structural stability in the manner of Krein's theorem was considered in [14]. All this suggests a variational principle in terms of the single function S with a fixed symplectic leaf label. This principle is given by the following:

$$\mathcal{A}[S, \mathring{f}] = -\int_{t_0}^{t_1} dt \int_D dq dP \, \mathring{f}\left(\frac{\partial S}{\partial P}, P\right) \left\| \frac{\partial^2 S}{\partial q \partial P} \right\| \left(\frac{\partial S}{\partial t} + \frac{1}{2} \left| \frac{\partial S}{\partial q} \right|^2 - \frac{\phi}{2} \right) , \tag{24}$$

where we have explicitly displayed the dependence on $\mathring{f}$, but this quantity is not to be varied.

Since in (24) the potential ϕ is now a shorthand for

$$\phi(q,t) = \int_D dq' dP' \, K(q|q') \, \mathring{f}\left(\frac{\partial S}{\partial P'}, P'\right) \left\| \frac{\partial^2 S}{\partial q' \partial P'} \right\| , \tag{25}$$

it is clear that variation with respect to S is an onerous task with contributions from $\mathring{f}$, the Van Vleck determinant, and the other dependence on S. After some effort, one can show $\delta\mathcal{A}[S, \mathring{f}]/\delta S = 0$ implies

$$\frac{\partial S}{\partial t} + \frac{1}{2} \left| \frac{\partial S}{\partial q} \right|^2 - \phi - \bar{H}\left(\frac{\partial S}{\partial P}, P, t\right) = 0 \, , \tag{26}$$

where $\bar{H}$ is any function that satisfies

$$[\mathring{f}(Q,P), \bar{H}(Q,P,t)] = 0 \, . \tag{27}$$

We refer the reader to [7] for more details of this calculation. In priciple, if we are given $\mathring{f}$ and we solve (26) for S, then we can use (18) to construct f, which can be shown to be a solution to the Vlasov-Poisson system.

IV. CONCLUSIONS

There are many comments that can be made about the above HJ formulations and variational principles, but we will consider only two.

First, it is clear that the nonlinearity that occurs in the term $E\partial f/\partial p$ in the Vlasov equation has been redistributed in both formulations. For example, from the action principle of (24) nonlinearity is manifest in the choice of $\mathring{f}$. Thus it might be worthwhile to reinvestigate existence proofs in this setting, particularly in light of the activity on viscosity solutions for HJ and the current studies of weak KAM theory. Through the HJ equation we have a natural place where pde and ode methods meet, and it seems that techniques from Hamiltonian dynamical systems theory may prove useful here.

Second, if one obtains an approximate solution to the HJ equation (26) by any means, numerical or otherwise, then the solution constructed will be a SR. It may not be a good solution, but it will satisify $f = z\#\mathring{f}$ and all the constraints this relation implies. Thus, one might think this kind of approximation would be superior.

In closing, we reiterate that everything done here for the Vlasov-Poisson system can be done for any Vlasov-like system, including the coupling to field equations like Maxwell's or Einstein's.

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