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Abstract: This paper is concerned with the convergence rates of the global strong solutions to the motionless state with constant density of the compressible Navier-Stokes equations in the whole space $\mathbb{R}^n$ for $n \geq 2$. The optimal decay estimates in critical spaces are established if the initial perturbations of density and velocity are small in $\dot{B}^{\frac{n}{2}}_{2,1}(\mathbb{R}^n) \cap \dot{B}^0_{p,\infty}(\mathbb{R}^n)$ and $\dot{B}^{\frac{n}{2}-1}_{2,1}(\mathbb{R}^n) \cap \dot{B}^0_{p,\infty}(\mathbb{R}^n)$ with $1 \leq p < \frac{2n}{n+1}$, respectively, for $n \geq 2$.

Key Words: compressible Navier-Stokes equations; critical space; convergence rate.

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1 Introduction

This paper studies the initial value problem for the compressible Navier-Stokes equation in $\mathbb{R}^n$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + (u \cdot \nabla) u + \frac{\nabla P(\rho)}{\rho} = \frac{\mu}{\rho} \triangle u + \frac{\mu + \mu'}{\rho} \nabla(\nabla \cdot u), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x). \end{cases} \tag{1}$$

Here $t > 0$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$; the unknown functions $\rho = \rho(t, x) > 0$ and $u = u(t, x) = (u_1(t, x), u_2(t, x), \cdots, u_n(t, x))$ denote the density and velocity, respectively; $P = P(\rho)$ is the pressure that is assumed to be a function of the density ρ; μ and μ' are the viscosity coefficients satisfying the conditions $\mu > 0$ and $\mu' + \frac{2}{n}\mu \geq 0$; and $\nabla\cdot$, ∇ and $\triangle$ denote the usual divergence, gradient and Laplacian with respect to x, respectively.

We assume that $P(\rho)$ is smooth in a neighborhood of $\bar{\rho}$ with $P'(\bar{\rho}) > 0$, where $\bar{\rho}$ is a given positive constant.

In this paper we derive the convergence rate of solutions of problem (1) to the constant stationary solution $(\bar\rho, 0)$ as $t \to \infty$ when the initial perturbation $(\rho_0-\bar\rho, u_0)$ is sufficiently small in critical spaces and $\dot B^0_{p,\infty}$ with $1 \le p < \frac{2n}{n+1}$.

Matsumura-Nishida [9] showed the global in time existence of the solution of (1) for $n = 3$, provided that the initial perturbation $(\rho_0 - \bar\rho, u_0)$ is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Furthermore, the following decay estimates were obtained in [9]

$$\|\nabla^k(\rho - \bar\rho, u)(t)\|_{L^2} \le C(1 + t)^{-\frac{3}{4}-\frac{k}{2}} \quad k = 0, 1. \tag{2}$$

These results were proved by combining the energy method and the decay estimates of the semigroup $E(t)$ generated by the linearized operator A at the constant state $(\bar\rho, 0)$.

On the other hand, Kawashita [7] showed the global existence of solutions for initial perturbations sufficiently small in $H^{s_0}(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$, $n \ge 2$. (Note that $s_0 = 2$ for $n = 3$). Wang-Tan [13] then considered the case $n = 3$ when the initial perturbation $(\rho_0 - \bar\rho, u_0)$ is sufficiently small in $H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and proved the decay estimates (2). Okita [11] showed that if $n \ge 2$ then the following estimates hold for the solution (ρ, u) of (1) :

$$\|\nabla^k(\rho - \bar\rho, u)(t)\|_{L^2} \le C(1 + t)^{-\frac{n}{4}-\frac{k}{2}} \quad k = 0, \cdots, s_0,$$

provided that $(\rho_0 - \bar\rho, u_0)$ is sufficiently small in $H^{s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$. Li-Zhang [8] showed that the density and momentum converge at the rates $(1 + t)^{-\frac{3}{4}-\frac{s}{2}}$ in the L^2-norm, when initial perturbation sufficiently small in $H^l(\mathbb{R}^3) \cap \dot B^{-s}_{1,\infty}(\mathbb{R}^3)$ with $l \ge 4$ and $s \in [0, 1]$. Note that L^1 is included in $\dot B^0_{1,\infty}$. The optimal $L^p - L^q$ convergence rate with $1 \le p < \frac{6}{5}$ and $2 \le q \le 6$ in $\mathbb{R}^3$ was established by Duan-Liu-Ukai-Yang [4] such as

$$\|(\rho - \rho_*, u)(t)\|_{L^q} \le C(1 + t)^{-\frac{3}{2}(1-\frac{1}{q})}, \quad 2 \le q \le 6,$$

where $(\rho_*, 0)$ is the stationary solution of the compressible Navier-Stokes equation with external potential force, under the assumptions that initial perturbation and external potential force are sufficiently small in in some function spaces respectively.

Danchin [2] proved the global existence in a critical homogeneous Besov space, i.e., a scaling invariant Besov space. The system $(1)_1 - (1)_2$ is invariant under the following transformation

$$\rho_\lambda(t, x) := \rho(\lambda^2 t, \lambda x), \quad u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x).$$

More precisely, if (ρ, u) solves (1), so dose $(\rho_\lambda, u_\lambda)$ provided that the pressure law P has been changed into $\lambda^2 P$. Usually, we call that a functional space is a critical space for (1) if the associated norm is invariant under the transformation $(\rho, u) \to (\rho_\lambda, u_\lambda)$ (up to a constant independent of λ). Homogeneous Besov space $C\big([0,\infty); \dot B^{\frac{n}{p}}_{p,1} \times \dot B^{\frac{n}{p}-1}_{p,1}\big)$ is a critical space for (1); and Danchin [2] proved the global existence in $C\big([0,\infty); \dot B^{\frac{n}{p}}_{p,1}\big) \times \big(C\big([0,\infty); \dot B^{\frac{n}{p}-1}_{p,1}\big) \cap L^1\big(0,\infty; \dot B^{\frac{n}{p}+1}_{p,1}\big)\big)$ and the estimate

$$\begin{aligned} &\sup_{t\ge0}\{\|\rho(t) - \bar\rho\|_{\dot B^{\frac{n}{2}-1}_{2,1}} + \|u(t)\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\} + \int_0^\infty \|u\|_{\dot B^{\frac{n}{2}+1}_{2,1}} dt \\ \le\ & M\big(\|\rho_0 - \bar\rho\|_{\dot B^{\frac{n}{2}}_{2,1}\cap \dot B^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\big), \end{aligned} \tag{3}$$

if the initial perturbation is sufficiently small in $\left(\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}\right) \times \dot{B}^{\frac{n}{2}-1}_{2,1}$ for $n \geq 2$.

On the other hand, nonhomogeneous Besov space $B^{\frac{n}{p}}_{p,1} \times B^{\frac{n}{p}-1}_{p,1}$ is called a critical regularity space for (1). Haspot [5] proved the local solvability in a critical regularity space.

In [12] the decay estimate of L^2-norm was studied for initial perturbations sufficiently small in critical spaces. It was proved in [12] that the perturbation satisfies

$$\|(\rho - \bar{\rho}, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}}$$

if the initial perturbation is sufficiently small in $(\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^0_{1,\infty}) \times (\dot{B}^{\frac{n}{2}-1}_{2,1} \cap \dot{B}^0_{1,\infty})$ for $n \geq 3$.

In this paper we improve the results in [12] and establish the decay estimate of the $\dot{B}^{\frac{n}{2}}_{2,1} \times \dot{B}^{\frac{n}{2}-1}_{2,1}$-norm of the perturbation for $n \geq 2$. Our main result of this paper gives the optimal decay rate for strong solutions in critical Besov spaces, which is stated as follows.

Theorem 1.1. *Assume that $n \geq 2$ and $1 \leq p < \frac{2n}{n+1}$. Then there exists $\epsilon > 0$ such that if*

$$u_0 \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^0_{p,\infty}, \ \ (\rho_0 - \bar{\rho}) \in \dot{B}^{\frac{n}{2}-1}_{2,1} \cap \dot{B}^0_{p,\infty}$$

and

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^0_{p,\infty}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1} \cap \dot{B}^0_{p,\infty}} \leq \epsilon,$$

then problem (1) has a unique global solution (ρ, u) satisfying

$$(\rho - \bar{\rho}, u) \in C([0,\infty); B^{\frac{n}{2}}_{2,1}) \times \left(C([0,\infty); B^{\frac{n}{2}-1}_{2,1}) \cap L^1(0,\infty; \dot{B}^{\frac{n}{2}+1}_{2,1})\right).$$

Furthermore, there exists a constant $C_0 > 0$ such that the estimates

$$\|(\rho - \bar{\rho}, u)(t)\|_{B^{\frac{n}{2}-1}_{2,1}} \leq C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}, \tag{4}$$

$$\|(\rho - \bar{\rho}, u)(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq C_0(1+t)^{-\frac{n}{2p}+\frac{1}{2}}, \tag{5}$$

$$\|(\rho - \bar{\rho})(t)\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \leq C_0(1+t)^{-\frac{n}{2p}}, \tag{6}$$

hold for $t \geq 0$. Moreover, if $2 \leq q \leq n$, then

$$\|(\rho - \bar{\rho}, u)(t)\|_{\dot{B}^0_{q,1}} \leq C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \tag{7}$$

for $t \geq 0$.

To prove Theorem 1.1, as in [6, 11, 12], we introduce a decomposition of the perturbation $U(t) = (\rho - \bar{\rho}, u)(t)$ associated with the spectral properties of the linearized operator A. In the case of our problem, we simply decompose the perturbation $U(t)$ into low and high frequency parts. As for the low frequency part, we apply the decay estimates for the low frequency part of the semigroup $E(t)$ generated by the linearized operator A; while the high frequency part is estimated by using the energy method. One of the points of our approach is that by restricting the use of the decay estimates for $E(t)$ to its low frequency part, one can avoid the derivative loss due to the convective term of the transport equation $(1)_1$. We note that in estimating the low frequency part, we also make use of the fact that any order of differentiation acts as a bounded operator on the low frequency part, so that we can establish the decay estimate for the norm of the velocity with critical regularity. (See Remark 4.4.) On the other hand, the convective term of $(1)_1$ can be controlled by the energy method and commutator estimate which we apply to the high frequency part. In the estimates of nonlinearities we carefully compute nonlinear interactions between low-low, low-high and high-high frequency parts. We also use the estimate $\int_0^\infty \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} dt < M\epsilon$, that follows from (3) established by Danchin [2].

The paper is organized as follows. In Section 2 we introduce the notation and some properties of Besov spaces. In Section 3 we rewrite the system into the one for the perturbation and introduce auxiliary lemmas used in this paper. In Section 4 we give a proof of Theorem 1.1.

2 Preliminaries

In this section we first introduce the notation which will be used throughout this paper. We then introduce Besov spaces and some properties of Besov spaces.

2.1 Notation

Let $L^p (1 \le p \le \infty)$ denote the usual L^p-Lebesgue space on $\mathbb{R}^n$. For a nonnegative integer m, we denote by H^m the usual L^2-Sobolev space of order m. $\mathcal{S}'$ denotes dual space of the Schwartz space. The inner-product of L^2 is denoted by $(\cdot,\cdot)$. If S is any nonempty subset of $\mathbb{Z}$, sequence space $l^p(S)$ denote the usual l^p sequence space on S.

For any integer $l \ge 0, \nabla^l f$ denotes all of l-th derivatives of f.

For a function f, we denote its Fourier transform by $\mathfrak{F}[f] = \hat{f}$:

$$\mathfrak{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx \quad (\xi \in \mathbb{R}).$$

The inverse Fourier transform is denoted by $\mathfrak{F}^{-1}[f] = \check{f}$,

$$\mathfrak{F}^{-1}[f](x) = \check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi) e^{i\xi\cdot x} d\xi \quad (x \in \mathbb{R}).$$

2.2 Besov spaces

Let us now define the homogeneous and nonhomogeneous Besov spaces. First we introduce the dyadic partition of unity. We can use for instance any $\{\phi, \chi\} \in C^\infty$, such that

$$\operatorname{Supp} \phi \subset \{\xi \in \mathbb{R}^n | \frac{3}{4} \le |\xi| \le \frac{8}{3}\},$$

$$\operatorname{Supp}\chi \subset \{\xi \in \mathbb{R}^n \,|\, |\xi| \le \frac{4}{3}\},$$

$$\chi(\xi) + \sum_{j\ge 0}\phi(2^{-j}\xi) = 1 \ \ for\ \xi \in \mathbb{R}^n,$$
$$\sum_{j\in\mathbb{Z}}\phi(2^{-j}\xi) = 1 \quad for\ \xi \in \mathbb{R}^n\backslash\{0\},$$

$$\operatorname{Supp}\phi(2^{-j}\cdot) \cap \operatorname{Supp}\phi(2^{-j'}\cdot) = \emptyset \quad for\ |j-j'| \ge 2,$$
$$\operatorname{Supp}\chi \cap \operatorname{Supp}\phi(2^{-j}\cdot) = \emptyset \quad for\ j \ge 1.$$

Denoting $h = \mathfrak{F}^{-1}\phi$ and $\tilde{h} = \mathfrak{F}^{-1}\chi$, we then define the dyadic blocks by

$$\begin{aligned}
\triangle_{-1}u &= \tilde{h} * u,\\
\triangle_j u &= 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy \quad \text{if } j \ge 0,\\
\dot{\triangle}_j u &= 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy \quad \text{if } j \in \mathbb{Z}.
\end{aligned}$$

The low-frequency cut-off operators are defined by

$$S_j u = \sum_{-1\le k\le j-1}\triangle_k u, \quad \dot{S}_j u = \sum_{k\le j-1}\dot{\triangle}_k u.$$

Obviously we can write that: $Id = \sum_j \triangle_j$. The high-frequency cut-off operators $\tilde{S}_j$ are defined by

$$\tilde{S}_j u = \sum_{k\ge j}\dot{\triangle}_k u.$$

We define ϕ_j by $\phi_j(\xi) = \phi(2^{-j}\xi)$.

To begin with, we define Besov spaces.

Definition 1. For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, and $u \in \mathcal{S}'$ we set

$$\|u\|_{B^s_{p,r}} := \left\|2^{js}\|\triangle_j u\|_{L^p}\right\|_{l^r(\{j\ge -1\})},$$

$$\|u\|_{\dot{B}^s_{p,r}} := \left\|2^{js}\|\dot{\triangle}_j u\|_{L^p}\right\|_{l^r(\mathbb{Z})}.$$

The nonhomogeneous Besov space $B^s_{p,r}$ and the homogeneous Besov space $\dot{B}^s_{p,r}$ are the sets of functions $u \in \mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}$ and $\|u\|_{\dot{B}^s_{p,r}} < \infty$ respectively.

Let us state some basic lemmas for Besov spaces.

Lemma 2.1. *The following inequalities hold:*

(i) $\|\nabla\triangle_{-1}u\|_{L^2} \le C\|\triangle_{-1}u\|_{L^2}$.

(ii) $C^{-1}2^j\|\dot{\triangle}_j u\|_{L^2} \le \|\nabla\dot{\triangle}_j u\|_{L^2} \le C2^j\|\dot{\triangle}_j u\|_{L^2} \quad (j \in \mathbb{Z})$.

(iii) $\|\nabla S_j u\|_{L^2} \le C 2^j \|S_j u\|_{L^2} \quad (j \ge 0)$.

(iv) $\|\tilde{S}_j u\|_{L^2} \le C 2^{-j} \|\nabla \tilde{S}_j u\|_{L^2} \quad (j \ge 0)$.

Lemma 2.1 easily follows from the Plancherel theorem.

Remark 2.2. *For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, we have*

(i) $C^{-1}\big(\sum_{k\le j-1} 2^{srk}\|\dot{\triangle} u\|_{L^p}^r\big)^{\frac{1}{r}} \le \|\dot{S}_j u\|_{\dot{B}^s_{p,r}} \le C\big(\sum_{k\le j-1} 2^{srk}\|\dot{\triangle} u\|_{L^p}^r\big)^{\frac{1}{r}}$

(ii) $C^{-1}\big(\sum_{k\ge j} 2^{srk}\|\dot{\triangle} u\|_{L^p}^r\big)^{\frac{1}{r}} \le \|\tilde{S}_j u\|_{\dot{B}^s_{p,r}} \le C\big(\sum_{k\ge j} 2^{srk}\|\dot{\triangle} u\|_{L^p}^r\big)^{\frac{1}{r}}$

One can easily prove Remark 2.2.

Lemma 2.3. *The following properties hold:*

(i) $C^{-1}\|u\|_{\dot{B}^s_{p,r}} \le \|\nabla u\|_{\dot{B}^{s-1}_{p,r}} \le C\|u\|_{\dot{B}^s_{p,r}}$.

(ii) $\|\nabla u\|_{B^{s-1}_{p,r}} \le C\|u\|_{B^s_{p,r}}$.

(iii) *If $s' > s$ or if $s' = s$ and $r_1 \le r$ then $B^{s'}_{p,r_1} \subset B^s_{p,r}$.*

(iv) *If $r_1 \le r$ then $\dot{B}^s_{p,r_1} \subset \dot{B}^s_{p,r}$.*

(v) *Let $\Lambda := \sqrt{-\triangle}$ and $t \in \mathbb{R}$. Then the operator Λ^t is an isomorphism from $\dot{B}^s_{2,1}$ to $\dot{B}^{s-t}_{2,1}$.*

See, e.g., [2], [3] and [5] for a proof of Lemma 2.3.

Lemma 2.4. *The following properties hold:*

(i) $\|u\|_{L^\infty} \le C\|u\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \qquad \big(\dot{B}^{\frac{n}{2}}_{2,1} \subset L^\infty\big)$.

(ii) $\dot{B}^0_{1,1} \subset L^1 \subset \dot{B}^0_{1,\infty}$.

(iii) $B^s_{2,2} = H^s$.

(iv) $B^s_{p,r} \subset \dot{B}^s_{p,r}$ $(s > 0)$.

See, e.g., [2], [3] and [5] for a proof of Lemma 2.4.

Lemma 2.5. *Let $1 \le p \le q \le \infty$. Assume that $f \in L^p(\mathbb{R}^n)$. Then for any $\alpha \in (\mathbb{N}\cup\{0\})^n$, there exist constants C_1, C_2 independent of f, j such that*

$$\operatorname{Supp} \hat{f} \subseteq \{|\xi| \le A_0 2^j\} \Longrightarrow \|\partial_x^\alpha f\|_{L^q} \le C_1 2^{j|\alpha| + jn(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$

$$\operatorname{Supp} \hat{f} \subseteq \{A_1 2^j \le |\xi| \le A_2 2^j\} \Longrightarrow \|f\|_{L^p} \le C_2 2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial_x^\beta f\|_{L^p}.$$

See, e.g., [1] for a proof of Lemma 2.5.

By Lemma 2.5, we see that

$$\sum_{j\in\mathbb{Z}} \|\dot{\triangle} f\|_{L^n} \leq C \sum_{j\in\mathbb{Z}} 2^{j(\frac{n}{2}-1)} \|\dot{\triangle} f\|_{L^2}, \tag{8}$$

hence, we obtain $\dot{B}^{\frac{n}{2}-1}_{2,1} \subset \dot{B}^0_{n,1}$.

Remark 2.6. *Let $s \geq 0$ and $1 \leq p < 2$. Then*

$$\dot{B}^s_{2,1} \cap \dot{B}^0_{p,\infty} \subset B^s_{2,1} \subset L^2.$$

Proof. By using Lemma 2.5, we have

$$\begin{aligned}
\|u\|_{L^2} &= \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j u\|^2_{L^2}\big)^{\frac{1}{2}} \leq \big(\sum_{j<0} \|\dot{\triangle}_j u\|^2_{L^2}\big)^{\frac{1}{2}} + \big(\sum_{j\geq 0} \|\dot{\triangle}_j u\|^2_{L^2}\big)^{\frac{1}{2}} \\
&\leq C\big(\sum_{j<0} 2^{2jn(\frac{1}{p}-\frac{1}{2})} \|\dot{\triangle}_j u\|^2_{L^p}\big)^{\frac{1}{2}} + \sum_{j\geq 0} 2^{js} \|\dot{\triangle}_j u\|_{L^2} \\
&\leq C \sup_{j<0} \|\dot{\triangle}_j u\|_{L^p} \big(\sum_{j<0} 2^{2jn(\frac{1}{p}-\frac{1}{2})}\big)^{\frac{1}{2}} + \sum_{j\geq 0} 2^{js} \|\dot{\triangle}_j u\|_{L^2}.
\end{aligned}$$

This completes the proof. □

3 Reformulation of the problem

In this section we first rewrite system (1) into the one for the perturbation. We then introduce some auxiliary lemmas which will be useful in the proof of the main result.

Let us rewrite the problem (1). We define μ_1, μ_2 and γ by

$$\mu_1 = \frac{\mu}{\bar{\rho}},\ \mu_2 = \frac{\mu + \mu'}{\bar{\rho}},\ \gamma = \sqrt{P'(\bar{\rho})}.$$

By using the new unknown function

$$\sigma(t,x) = \frac{\rho(t,x) - \bar{\rho}}{\bar{\rho}}, \quad w(t,x) = \frac{1}{\gamma} u(t,x),$$

the initial value problem (1) is reformulated as

$$\begin{cases} \partial_t \sigma + \gamma \nabla \cdot w = F_1(U), \\ \partial_t w - \mu_1 \triangle w - \mu_2 \nabla(\nabla \cdot w) + \gamma \nabla \sigma = F_2(U), \\ (\sigma, w)(0,x) = (\sigma_0, w_0)(x), \end{cases} \tag{9}$$

where, $U = \begin{pmatrix} \sigma \\ w \end{pmatrix}$,

$$F_1(U) = -\gamma(w \cdot \nabla \sigma + \sigma \nabla \cdot w),$$

$$
\begin{aligned}
F_2(U) &= -\gamma(w\cdot\nabla)w - \mu_1\frac{\sigma}{\sigma+1}\triangle w - \mu_2\frac{\sigma}{\sigma+1}\nabla(\nabla\cdot w)\\
&\quad + \Big(\frac{\bar\rho\gamma}{\sigma+1} - \frac{\bar\rho}{\gamma}\frac{\int_0^1 P''(s\bar\rho\sigma+\bar\rho)ds}{\sigma+1}\Big)\sigma\nabla\sigma.
\end{aligned}
$$

We set

$$
A = \begin{pmatrix} 0 & -\gamma\nabla\cdot \\ -\gamma\nabla & \mu_1\triangle + \mu_2\nabla\nabla\cdot \end{pmatrix}.
$$

By using operator A, problem (9) is written as

$$
\partial_t U - AU = F(U), \quad U|_{t=0} = U_0, \tag{10}
$$

where

$$
F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \end{pmatrix}, \quad U_0 = \begin{pmatrix} \sigma_0 \\ w_0 \end{pmatrix}.
$$

We introduce a semigroup generated by A. We set

$$
E(t)u := \mathfrak{F}^{-1}[e^{\hat A(\xi)t}\hat u] \quad \text{for } u \in L^2,
$$

where

$$
\hat A(\xi) = \begin{pmatrix} 0 & -i\gamma\xi^{\mathrm{t}} \\ -i\gamma\xi & -\mu_1|\xi|^2 I_n - \mu_2\xi\xi^{\mathrm{t}} \end{pmatrix}.
$$

Here and in what follows the superscript $\cdot^{\mathrm{t}}$ means the transposition.

We next state some basic lemmas.

Lemma 3.1. *Let $s_1, s_2 \le \frac{n}{2}$ such that $s_1 + s_2 > 0$; and let $u \in \dot B^{s_1}_{2,1}$ and $v \in \dot B^{s_2}_{2,1}$. Then $uv \in \dot B^{s_1+s_2-\frac{n}{2}}_{2,1}$ and*

$$
\|uv\|_{\dot B^{s_1+s_2-\frac{n}{2}}_{2,1}} \le C\|u\|_{\dot B^{s_1}_{2,1}}\|v\|_{\dot B^{s_2}_{2,1}}.
$$

See, e.g., [1], for a proof of Lemma 3.1.

Lemma 3.2. *Let $s > 0$ and let $u \in \dot B^s_{2,1} \cap L^\infty$. Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^n)$ such that $F(0) = 0$. Then $F(u) \in \dot B^s_{2,1}$. Moreover, there exists a function C_1 of one variable depending only on s, n and F such that*

$$
\|F(u)\|_{\dot B^s_{2,1}} \le C_1(\|u\|_{L^\infty})\|u\|_{\dot B^s_{2,1}}.
$$

See, e.g., [2], for a proof of Lemma 3.2.

Lemma 3.3. (*i*) *Let $a, b > 0$ satisfying* $\max\{a,b\} > 1$. *Then*

$$
\int_0^t (1+s)^{-a}(1+t-s)^{-b}ds \le C(1+t)^{-\min\{a,b\}}, \quad t \ge 0.
$$

(*ii*) *Let* $f \in L^p(0,\infty)$ *and* $a, b > 0$ *satisfying* $\max\{a,b\} > \frac{1}{p'}$ *for* $1 \le p \le \infty$ *and* p' *is the conjugate exponent to* p. *Then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b} f ds \le C(1+t)^{-\min\{a,b\}}\Big(\int_0^t |f|^p ds\Big)^{\frac{1}{p}}, \quad t \ge 0.$$

For a proof of (i), see [10]. Proof of (ii) is given by using Hölder inequality; we omit it.

Let us now introduce a few bilinear estimates in Besov spaces. We will use the Bony decomposition

$$uv = T_u v + T_v u + R(u,v), \tag{11}$$

with

$$T_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\triangle}_j v, \quad R(u,v) = \sum_{j\in\mathbb{Z}} \dot{\triangle}_j u \tilde{\triangle}_j v, \quad \tilde{\triangle}_j v = \dot{\triangle}_{j-1} v + \dot{\triangle}_j v + \dot{\triangle}_{j+1} v.$$

Lemma 3.4. *It holds that*

(i)

$$\sup_{j<0} \|\dot{\triangle}_j T_g f\|_{L^1} \le C\|\dot{S}_4 f\|_{L^2}\|\dot{S}_4 g\|_{L^2},$$

$$\sup_{j<0} \|\dot{\triangle}_j R(f,g)\|_{L^1} \le C\big(\|\dot{S}_3 f\|_{L^2}\|\dot{S}_3 g\|_{L^2} + \|\tilde{S}_0 f\|_{L^2}\|\tilde{S}_0 g\|_{L^2}\big).$$

(ii) *If* $0 \le s_1, s_2, s_3, s_4 \le \frac{n}{2}$, *then*

$$\sum_{j\ge 0} 2^{s_1 j}\|\triangle_j T_g f\|_{L^2} \le C\big(\|\dot{S}_{-5} g\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5} f\|_{\dot{B}^{s_1+s_2}_{2,1}} + \|\tilde{S}_{-5} g\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5} f\|_{\dot{B}^{s_1+s_3}_{2,1}}\big),$$

$$\sum_{j\ge 0} 2^{s_1 j}\|\triangle_j R(f,g)\|_{L^2} \le C\|\tilde{S}_{-4} f\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-4} g\|_{\dot{B}^{s_1+s_4}_{2,1}}.$$

Remark 3.5. *By Lemma* 3.4, *we have*

(i)

$$\sup_{j<0} \|\dot{\triangle} uv\|_{L^1} \le C(\|\dot{S}_4 u\|_{L^2}\|\dot{S}_4 v\|_{L^2} + \|\tilde{S}_0 u\|_{L^2}\|\tilde{S}_0 v\|_{L^2}).$$

(ii) *If* $0 \le s_1, s_2, s_3, s_4 \le \frac{n}{2}$, *then*

$$\begin{aligned}\sum_{j\ge 0} 2^{s_1 j}\|\triangle_j uv\|_{L^2} \quad \le \quad & C\big(\|\dot{S}_{-5} u\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5} v\|_{\dot{B}^{s_1+s_2}_{2,1}} + \|\dot{S}_{-5} v\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5} v\|_{\dot{B}^{s_1+s_3}_{2,1}} \\ & + \|\tilde{S}_{-5} u\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-5} v\|_{\dot{B}^{s_1+s_4}_{2,1}}\big).\end{aligned}$$

Proof of Lemma 3.4**.** We have

$$\dot{\triangle}_j T_g f = \sum_{|j'-j|\le 4} \dot{\triangle}_j(\dot{S}_{j'-1} g \dot{\triangle}_{j'} f), \quad \dot{\triangle}_j R(f,g) = \sum_{j'\ge j-3} \dot{\triangle}_j(\dot{\triangle}_{j'} f \tilde{\triangle}_{j'} g).$$

For any $j < 0$, by the Hölder inequality, we have

$$\begin{aligned}\|\dot{\triangle}_j T_g f\|_{L^1} &\le C\sum_{|j'-j|\le 4}\|\dot{S}_{j'-1}g\dot{\triangle}_{j'}f\|_{L^1}\\ &\le C\|\dot{S}_4 g\|_{L^2}\|\dot{S}_4 f\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\|\dot{\triangle}_j R(f,g)\|_{L^1} &\le C\|\sum_{j'\ge j-3}\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^1}\\ &\le C\sum_{j'\le 0}\|\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g\|_{L^1}+\sum_{j'\ge 1}\|\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g\|_{L^1}\\ &\le C(\|\dot{S}_3 f\|_{L^2}\|\dot{S}_3 g\|_{L^2}+\|\tilde{S}_0 f\|_{L^2}\|\tilde{S}_0 g\|_{L^2}).\end{aligned}$$

Taking the supremum in $j < 0$, we obtain the desired estimates of (i).

We next prove (ii). Choose $s_1 \in [0, \frac{n}{2}]$. We then obtain by Hölder inequality and Lemma 2.5 that

$$\begin{aligned}\sum_{j\ge 0}2^{s_1 j}\|\dot{\triangle}_j T_g f\|_{L^2} &\le C\sum_{j\ge 0}\sum_{|j'-j|\le 4}2^{s_1 j}\|\dot{\triangle}_j(\dot{S}_{j'-1}g\dot{\triangle}_{j'}f)\|_{L^2}\\ &\le C\sum_{j'\ge -4}2^{s_1 j'}\|\dot{S}_{j'-1}g\dot{\triangle}_{j'}f\|_{L^2}\\ &\le C\sum_{j'\ge -4}2^{s_1 j'}\|\{\dot{S}_{-5}g+(\dot{S}_{j'-1}-\dot{S}_{-5})g\}\dot{\triangle}_{j'}f\|_{L^2}\\ &\le C\sum_{j'\ge -4}2^{s_1 j'}\{\|\dot{S}_{-5}g\|_{L^{\frac{n}{s_2}}}\|\dot{\triangle}_{j'}f\|_{L^{\frac{2n}{n-2s_2}}}\\ &\quad+\|(\dot{S}_{j'-1}-\dot{S}_{-5})g\|_{L^{\frac{n}{s_3}}}\|\dot{\triangle}_{j'}f\|_{L^{\frac{2n}{n-2s_3}}}\}\\ &\le C(\|\dot{S}_{-5}g\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5}g\|_{\dot{B}^{s_1+s_2}_{2,1}}+\|\tilde{S}_{-5}g\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5}g\|_{\dot{B}^{s_1+s_3}_{2,1}}),\end{aligned}$$

$$\begin{aligned}\sum_{j\ge 0}2^{s_1 j}\|\triangle_j R(f,g)\|_{L^2} &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{s_1 j}\|\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^2}\\ &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{(s_1+\frac{n}{2})j}\|\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^1}\\ &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{(s_1+\frac{n}{2})(j-j')}2^{(\frac{n}{2}-s_4)j'}\|\dot{\triangle}_{j'}f\|_{L^2}2^{(s_1+s_4)j'}\|\tilde{\triangle}_{j'}g\|_{L^2}\\ &\le C\|\tilde{S}_{-4}f\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-4}g\|_{\dot{B}^{s_1+s_4}_{2,1}}.\end{aligned}$$

This completes the proof. □

We now introduce commutator estimates.

Lemma 3.6. *Let $s \in \big(-\frac{n}{2}, \frac{n}{2}+1\big]$. There exists a sequence $c_j \in l^1(\mathbb{Z})$ such that $\|c_j\|_{l^1} = 1$ and a constant C depending only on n and s such that*

$$\forall j\in\mathbb{Z},\quad \|[f\cdot\nabla,\dot{\triangle}_j]g\|_{L^2}\le Cc_j 2^{-sj}\|\nabla f\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|g\|_{\dot{B}^{s}_{2,1}}.$$

See, e.g., [1] for a proof of Lemma 3.6.

Lemma 3.7. *Let* $0 < p < q < r \leq \infty$ *and set* $\theta = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}} \in (0,1)$. *Then it holds that*

(*i*) $L^p \cap L^r \subset L^q$ *and* $\|f\|_{L^q} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^r}^{1-\theta}$,

(*ii*) $\dot{B}^0_{p,l} \cap \dot{B}^0_{r,l} \subset \dot{B}^0_{q,l}$ *and* $\|f\|_{\dot{B}^0_{q,l}} \leq \|f\|^{\theta}_{\dot{B}^0_{p,l}} \|f\|^{1-\theta}_{\dot{B}^0_{r,l}}$, *for* $1 \leq l \leq \infty$.

Proof. (*i*) is a well-known inequality. Let us prove (*ii*). Let $1 \leq l < \infty$. Then by using (*i*) and Hölder inequality of sequence space, we have

$$\begin{aligned}
\|f\|_{\dot{B}^0_{q,l}} &= \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^l_{L^q}\big)^{\frac{1}{l}} \\
&\leq \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^{l\theta}_{L^p} \|\dot{\triangle}_j f\|^{l(1-\theta)}_{L^r}\big)^{\frac{1}{l}} \\
&\leq \big\{\big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^l_{L^p}\big)^{\theta} \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^l_{L^r}\big)^{(1-\theta)}\big\}^{\frac{1}{l}} \\
&\leq \|f\|^{\theta}_{\dot{B}^0_{p,l}} \|f\|^{1-\theta}_{\dot{B}^0_{r,l}}.
\end{aligned}$$

When $l = \infty$, we have

$$\|f\|_{\dot{B}^0_{q,\infty}} \leq \sup_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^{l\theta}_{L^p} \|\dot{\triangle}_j f\|^{l(1-\theta)}_{L^r} \leq \|f\|^{\theta}_{\dot{B}^0_{p,\infty}} \|f\|^{1-\theta}_{\dot{B}^0_{r,\infty}}.$$

This completes the proof. □

4 Proof of main result

In this section we prove Theorem 1.1. In subsections 4.1 and 4.2 we establish the necessary estimates for $\triangle_{-1}U(t)$ and $\triangle_j U(t)$ for $j \geq 0$, respectively. In subsection 4.3 we derive the a priori estimate to complete the proof of Theorem 1.1.

We first explain known results which are used to prove Theorem 1.1.

Danchin [2] proved the following global existence result in nonhomogeneous Besov space.

Proposition 4.1 (Danchin [2]). *Let* $n \geq 2$. *There are two positive constants* ϵ_1 *and* M *such that for all* (ρ_0, u_0) *with* $(\rho_0 - \bar{\rho}) \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}$, $u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$ *and*

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq \epsilon_1, \tag{12}$$

problem (1) *has a unique global solution* $(\rho, u) \in C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}) \times \big(L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}+1}_{2,1}) \cap C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1})\big)$ *that satisfies the estimate*

$$\sup_{t\geq 0}\{\|\rho(t)-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\} + \int_0^{\infty} \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} dt \leq M\big(\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big).$$

Haspot [5] proved the following local existence result in nonhomogeneous Besov space.

Proposition 4.2 (Haspot [5])**.** *Let $n \geq 2$ and $1 \leq p < 2n$. Let $u_0 \in B_{p,1}^{\frac{n}{p}-1}$ and $(\rho_0 - \bar{\rho}) \in B_{p,1}^{\frac{n}{p}}$ with $\frac{1}{\rho_0}$ bounded away from zero. Then there exist a constant $T > 0$ such that the problem (1) has a local solution (ρ, u) on $[0, T]$ with $\frac{1}{\rho} > 0$ bounded away from zero and:*

$$\rho - \bar{\rho} \in C([0,T]; B_{p,1}^{\frac{n}{p}}), \quad u \in \big(C([0,T]; B_{p,1}^{\frac{n}{p}-1}) \cap L^1(0,T; B_{p,1}^{\frac{n}{p}+1})\big).$$

Moreover, this solution is unique if

$$p \leq n.$$

Proposition 4.3. *Let $T > 0$ and let (σ, w) be a solution of problem (10) on $[0,T]$ such that*

$$\sigma \in C([0,T]; B_{2,1}^{\frac{n}{2}}), w \in C([0,T]; B_{2,1}^{\frac{n}{2}}) \cap L^1(0,T; B_{2,1}^{\frac{n}{2}+1}), \tag{13}$$

Then, $\triangle_j U(t) = (\triangle_j \sigma, \triangle_j w)^{\mathrm{t}}$ for $j \geq -1$ satisfy

$$\partial_t \triangle_j U - A \triangle_j U = \triangle_j F(U), \tag{14}$$

$$\triangle_j U|_{t=0} = \triangle_j U_0. \tag{15}$$

Moreover, $\triangle_{-1} U(t)$ satisfy

$$\triangle_{-1} U(t) \in C\big([0,T]; \dot{B}_{2,1}^{k}\big), \quad \forall k \in [0, \infty) \tag{16}$$

and

$$\triangle_{-1} U(t) = E(t) \triangle_{-1} U_0 + \int_0^t E(t-s) \triangle_{-1} F(U)(s) ds. \tag{17}$$

Proof. Let $U(t) = (\sigma, w)^{\mathrm{t}}$ be a solution of (10) satisfying (13). Since $\triangle_j AU = A \triangle_j U$, applying $\triangle_j$ to (10), we obtain (14) and (15). It then follows that

$$\triangle_j U(t) = E(t) \triangle_j U_0 + \int_0^t E(t-s) \triangle_j F(U)(s) ds.$$

We also have (16) from Lemma 2.1. This completes the proof. □

Set

$$\begin{aligned} M_1(t) \;:=\; & \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|\triangle_{-1} U(\tau)\|_{L^2} \\ & + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}} \sum_{j<0} 2^j \|\dot{\triangle}_j U(\tau)\|_{L^2} \\ & + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2p}-\frac{1}{2}} \sum_{j<0} 2^{(\frac{n}{2}-1)j} \|\dot{\triangle}_j U(\tau)\|_{L^2} \\ & + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2p}} \sum_{j<0} 2^{\frac{n}{2}j} \|\dot{\triangle}_j U(\tau)\|_{L^2}, \end{aligned}$$

$$M_\infty(t) := \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2p}}\sum_{j=0}^{\infty}2^{(\frac{n}{2}-1)j}\{\|\triangle_j U(\tau)\|_{L^2}+2^j\|\triangle_j\sigma\|_{L^2}\},$$

$$M(t) := M_1(t)+M_\infty(t).$$

If we could obtain uniform estimates of $M_1(t)$ and $M_\infty(t)$, then Theorem 1.1 would be proved.

Remark 4.4. *$M_1(t)$ includes the $B^{\frac{n}{2}}_{2,1}$-norm of the low frequency part of perturbation with time weigh. Since any order of differentiation acts as a bounded operator on the low frequency part, we can treat $\dot{B}^{\frac{n}{2}}_{2,1}$-norm of the low frequency part of velocity, although the velocity itself belongs to $C([0,\infty);\dot{B}^{n/2-1}_{2,1})$. $M_\infty(t)$ is $\dot{B}^{\frac{n}{2}}_{2,1}\times\dot{B}^{\frac{n}{2}-1}_{2,1}$ -norm of the high frequency part of perturbation with time weigh. We note that the decay order of high frequency part is faster than the low frequency part. These facts are used to obtain decay estimates of nonlinear term.*

4.1 Estimate of low frequency parts

In this subsection we derive the estimate of $\triangle_{-1}U(t)$, in other words, we estimate $M_1(t)$.

Lemma 4.5 (Matsumura-Nishida [10]). *(i) The set of all eigenvalues of $\hat{A}(\xi)$ consists of $\lambda_i(\xi)$ $(i=1,2,3)$, where*

$$\begin{cases} \lambda_1(\xi) = \frac{-(\mu_1+\mu_2)|\xi|^2+i|\xi|\sqrt{4\gamma^2-(\mu_1+\mu_2)|\xi|^2}}{2}, \\ \lambda_2(\xi) = \frac{-(\mu_1+\mu_2)|\xi|^2-i|\xi|\sqrt{4\gamma^2-(\mu_1+\mu_2)|\xi|^2}}{2}, \\ \lambda_3(\xi) = -\mu_1|\xi|^2, \end{cases}$$

for all $\xi\in\mathbb{R}^n$.

(ii) $e^{t\hat{A}(\xi)}$ has the spectral resolution

$$e^{t\hat{A}(\xi)} = \sum_{j=1}^{3}e^{t\lambda_j(\xi)}P_j(\xi),$$

for all $|\xi|$ except at most points of $|\xi|>0$, where $P_j(\xi)$ is the eigenprojection for $\lambda_j(\xi)$.

Remark 4.6. *For each $M>0$ there exist $C_2=C_2(M)>0$ and $\beta_2=\beta_2(M)>0$ such that the estimate*

$$\|e^{t\hat{A}(\xi)}\|\le C_2e^{-\beta_2|\xi|^2t}$$

holds for $|\xi|\le M$ and $t>0$.

Lemma 4.7. *Let $s \geq 0$ and let $1 \leq p \leq 2$. Then $E(t)$ satisfies the estimates*

$$\|E(t)\triangle_{-1}U_0\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\sup_{j<0}\|\dot{\triangle}_j U_0\|_{L^p},$$

$$\sum_{j<0} 2^{sj}\|E(t)\dot{\triangle}_j U_0\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_j U_0\|_{L^p}$$

for $t \geq 0$.

To prove Lemma 4.7, we will use the following inequalities.

Lemma 4.8. *Let $\alpha > 0$, $p_0 > 0$ and $s > -\frac{n}{p_0}$. Then there holds the estimate*

$$\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}}|\xi|^{p_0 s}e^{-p_0\alpha|\xi|^2 t}d\xi\Big)^{\frac{1}{p_0}} \leq C(1+t)^{\frac{n}{2p_0}-\frac{s}{2}}$$

for all $t > 0$.

We will prove Lemma 4.8 later. Now we prove Lemma 4.7.

Proof of Lemma 4.7. Let $1 \leq p < 2$ and p' be the Hölder conjugate exponent to p. By Plancherel's theorem and Lemma 4.5 (ii), we have that there exists a constant $\beta' > 0$ such that

$$\begin{aligned}
\|E(t)\triangle_{-1}U_0(t)\|_{L^2} &\leq C\Big(\int_{|\xi|\leq 2}\big|e^{\hat{A}(\xi)t}\chi(\xi)\hat{U_0}(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}}\\
&\leq C\sup_{j<0}\|\phi_j(\xi)\hat{U_0}\|_{L^{p'}}\Big(\sum_{j<0}\int_{2^{j-1}|\xi|<2^{j+2}} e^{-\frac{2p}{2-p}\beta'|\xi|^2 t}d\xi\Big)^{\frac{1}{p}-\frac{1}{2}}\\
&\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\sup_{j<0}\|\dot{\triangle}_j U_0\|_{L^p}, \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j<0}2^{sj}\|E(t)\dot{\triangle}_j U_0(t)\|_{L^2} &\leq C\sum_{j<0}2^{sj}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}}\big|e^{\hat{A}(\xi)t}\phi_j(\xi)\hat{U_0}(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}}\\
&\leq C\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|\leq 2^{j+2}}|\xi|^{2s}e^{-2\beta'|\xi|^2 t}|\phi_j(\xi)\hat{U_0}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}\\
&\leq C\sum_{j<0}\|\dot{\triangle}_j U_0\|_{L^p}\Big(\int_{2^{j-1}<|\xi|\leq 2^{j+2}}|\xi|^{\frac{2p}{2-p}s}e^{-\frac{2p}{2-p}\beta'|\xi|^2 t}d\xi\Big)^{\frac{1}{p}-\frac{1}{2}}\\
&\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_j U_0\|_{L^p}. \qquad (19)
\end{aligned}$$

Here we used Lemma 4.8.

The desired estimates of Lemma 4.7 for $1 \le p < 2$ follow from (18) and (19). We can easily prove for $p = 2$. $\square$

It remains to prove Lemma 4.8.

Proof of Lemma 4.8. Let $\alpha > 0$, $p_0 > 0$ and $s > -\frac{n}{p_0}$. We have

$$\begin{aligned} & \sum_{j<0} \Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{p_0 s} e^{-p_0 \alpha |\xi|^2 t} d\xi \Big)^{\frac{1}{p_0}} \\ \le\ & C \sum_{j<0} 2^{js} \Big(\int_{|\xi|<2^{j+2}} d\xi \Big)^{\frac{1}{p_0}} \\ \le\ & C \sum_{j<0} 2^{j(s+\frac{n}{p_0})} \le C. \end{aligned} \tag{20}$$

We will next show the the inequality

$$\sum_{j<0} \Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{p_0 s} e^{-p_0 \alpha |\xi|^2 t} d\xi \Big)^{\frac{1}{p_0}} \le C t^{-\frac{n}{2p_0} - \frac{s}{2}}. \tag{21}$$

By the substitution $\eta = t^{\frac{1}{2}} \xi$, we obtain

$$\begin{aligned} & \sum_{j<0} \Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{p_0 s} e^{-p_0 \alpha |\xi|^2 t} d\xi \Big)^{\frac{1}{p_0}} \\ =\ & t^{-\frac{n}{2p_0} - \frac{s}{2}} \sum_{j<0} \Big(\int_{2^{j-1}\sqrt{t}<|\xi|<2^{j+2}\sqrt{t}} |\eta|^{p_0 s} e^{-p_0 \alpha |\eta|^2} d\xi \Big)^{\frac{1}{p_0}}. \end{aligned}$$

If $t \le 1$, we can easily prove (21).

We suppose $t > 1$. There exist an integer $J < 0$ such that $2^{-2J} < t < 2^{-2(J-1)}$. We have

$$\begin{aligned} & \sum_{j<0} \Big(\int_{2^{j-1}\sqrt{t}<|\xi|<2^{j+2}\sqrt{t}} |\eta|^{p_0 s} e^{-p_0 \alpha |\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\ \le\ & \sum_{j\le J} \Big(\int_{2^{j-J-1}<|\xi|<2^{j-J+3}} |\eta|^{p_0 s} e^{-p_0 \alpha |\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\ & + \sum_{J<j<0} \Big(\int_{2^{j-J-1}<|\xi|<2^{j-J+3}} |\eta|^{p_0 s} e^{-p_0 \alpha |\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\ =:\ & I_1 + I_2. \end{aligned}$$

By the substitution $k = j - J$, we have

$$I_1 = \sum_{k\le 0} \Big(\int_{2^{k-1}<|\xi|<2^{k+3}} |\eta|^{p_0 s} e^{-p_0 \alpha |\eta|^2} d\xi \Big)^{\frac{1}{p_0}} < C,$$

and

$$\begin{aligned} I_2 &\leq \sum_{k>0} \Big(\int_{2^{k-1}<|\xi|<2^{k+3}} |\eta|^{p_0 s} e^{-p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\ &\leq C\sum_{k>0} e^{-\frac{1}{2}2^k} \Big(\int_{2^{k-1}<|\xi|<2^{k+3}} |\eta|^{p_0 s} e^{-\frac{1}{2}p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\ &\leq C\sum_{k>0} e^{-\frac{1}{2}2^k} \leq C. \end{aligned}$$

Hence we obtain (21). By (20) and (21) we have the desired inequality . □

As for $M_1(t)$, we show the following estimate.

Proposition 4.9. *Let $1 \leq p < \frac{2n}{n+1}$. Then there exists a constant $C > 0$ independent of T such that*

$$M_1(t) \leq C\|U_0\|_{\dot{B}^0_{p,\infty}} + CM(t)\int_0^t \|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} d\tau + CM^2(t)$$

for $t \in [0, T]$.

To prove Proposition 4.9, we will use the following estimate on $F(U)$.

Lemma 4.10. *Suppose that $1 \leq p < \frac{2n}{n+1}$. Then there exists a constant $C > 0$ independent of T such that*

$$\sum_{j<0} \|\dot{\triangle}_j F(U)\|_{L^1} \leq C(1+t)^{-\frac{n}{2p}} M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} M^2(t)$$

for $t \in [0, T]$.

We will prove Lemma 4.10 later. Now we prove Proposition 4.9.

Proof of Proposition 4.9. By Lemma 4.7 and (17), we see that

$$\begin{aligned} \|\triangle_{-1}U(\tau)\|_{L^2} &\leq \|E(\tau)\triangle_{-1}U_0\|_{L^2} + \int_0^\tau \|E(\tau-\tau')\triangle_{-1}F(U(\tau'))\|_{L^2} d\tau' \\ &\leq C(1+\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \sup_{j<0} \|\dot{\triangle}_j U_0\|_{L^p} \\ &\quad + \int_0^t (1+\tau-\tau')^{-\frac{n}{4}} \sup_{j<0} \|\dot{\triangle}_j F(U(\tau'))\|_{L^1} ds, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \sum_{j<0} 2^{sj}\|\dot{\triangle}_j U(\tau)\|_{L^2} &\leq \sum_{j<0} \|E(\tau)\dot{\triangle}_j U_0\|_{L^2} + \int_0^\tau \sum_{j<0} \|E(\tau-\tau')\dot{\triangle}_j F(U(\tau'))\|_{L^2} d\tau' \\ &\leq C(1+\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s}{2}} \sup_{j<0} \|\dot{\triangle}_j U_0\|_{L^p} \\ &\quad + \int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}} \sup_{j<0} \|\dot{\triangle}_j F(U(\tau'))\|_{L^1} d\tau' \end{aligned} \tag{23}$$

for $s>0$.

Using Lemma 4.10, for $0\le s\le \frac{n}{2}$, we have

$$
\begin{aligned}
&\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\sup_{j<0}\|\dot\triangle_j F(U(\tau'))\|_{L^1}d\tau'\\
\le\ & C\int_0^t (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\{(1+\tau')^{-\frac{n}{2p}}M(\tau')\|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}+(1+\tau')^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(\tau')\}d\tau'\\
\le\ & CM(t)\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}(1+\tau')^{-n(\frac{1}{p}-\frac{1}{2})}\|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'\\
&+CM^2(t)\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}(1+\tau')^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}d\tau'\\
\le\ & C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}M(t)\int_0^\tau \|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'+C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}M^2(t). \qquad (24)
\end{aligned}
$$

Here we used Lemma 3.3 and the facts that $n(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}>1$ for $n\ge 2$ and $1\le p<\frac{2n}{n+1}$. By (22) and (24), we obtain

$$
\begin{aligned}
\|\triangle_{-1}U(\tau)\|_{L^2}\ \le\ & C(1+\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|U_0\|_{\dot B^0_{p,\infty}}\\
&+C(1+\tau)^{-\frac{n}{4}}M(t)\int_0^t \|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'+C(1+\tau)^{-\frac{n}{4}}M^2(t),
\end{aligned}
$$

and hence,

$$
(1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|\triangle_{-1}U(\tau)\|_2\le C\|U_0\|_{\dot B^0_{p,\infty}}+CM(t)\int_0^t \|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t).
$$

Similarly, we get estimates

$$
(1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}\sum_{j<0}2^j\|\dot\triangle_jU(\tau)\|_2\le C\|U_0\|_{\dot B^0_{p,\infty}}+CM(t)\int_0^t \|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t),
$$

$$
(1+\tau)^{\frac{n}{2p}-\frac{1}{2}}\sum_{j<0}2^{(\frac{n}{2}-1)j}\|\dot\triangle_jU(\tau)\|_2\le C\|U_0\|_{\dot B^0_{p,\infty}}+CM(t)\int_0^t \|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t),
$$

$$
(1+\tau)^{\frac{n}{2p}}\sum_{j<0}2^{\frac{n}{2}j}\|\dot\triangle_jU(\tau)\|_2\le C\|U_0\|_{\dot B^0_{p,\infty}}+CM(t)\int_0^t \|w(\tau')\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t).
$$

Taking the supremum in $\tau\in[0,t]$, we obtain the desired estimate. □

It remains to prove Lemma 4.10.

Proof of Lemma 4.10. We consider each term of $F(U)$. By Lemma 3.4, we have

$$
\begin{aligned}
\sup_{j<0}\|\dot\triangle_j(w\cdot\nabla\sigma)\|_{L^1}\ \le\ & C\{\|\dot S_4w\|_{L^2}\|\dot S_4\nabla\sigma\|_{L^2}+\|\tilde S_0w\|_{L^2}\|\tilde S_0\nabla\sigma\|_{L^2}\}\\
\le\ & C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t),
\end{aligned}
$$

$$
\begin{aligned}
\sup_{j<0}\|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^1} &\leq C\{\|\dot{S}_4\sigma\|_{L^2}\|\dot{S}_4\nabla w\|_{L^2}+\|\tilde{S}_0\sigma\|_{L^2}\|\tilde{S}_0\nabla w\|_{L^2}\}\\
&\leq C\{\|\dot{S}_4\sigma\|_{L^2}(\|\dot{S}_0\nabla w\|_{L^2}+\|\dot{\triangle}_0 w\|_{L^2}+\|\dot{\triangle}_1 w\|_{L^2}\\
&\quad+\|\dot{\triangle}_2 w\|_{L^2}+\|\dot{\triangle}_3 w\|_{L^2})+\|\tilde{S}_0\sigma\|_{L^2}\|\tilde{S}_0\nabla w\|_{L^2}\}\\
&\leq C\{(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.
\end{aligned}
$$

Similarly, we have

$$
\sup_{j<0}\|\dot{\triangle}_j(w\cdot\nabla w)\|_{L^1} \leq C\{(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.
$$

We obtain by Lemma 2.1, 3.2 and 3.4

$$
\begin{aligned}
\sup_{j<0}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\triangle w)\|_{L^1} &\leq C\{\|\dot{S}_4(\frac{\sigma}{\sigma+1})\|_{L^2}\|\dot{S}_4\triangle w\|_{L^2}+\|\tilde{S}_0(\frac{\sigma}{\sigma+1})\|_{L^2}\|\tilde{S}_0\triangle w\|_{L^2}\}\\
&\leq C\{\|\sigma\|_{L^2}\|\dot{S}_4 w\|_{\dot{B}^1_{2,1}}++\|\tilde{S}_0(\frac{\sigma}{\sigma+1})\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\tilde{S}_0\triangle w\|_{L^2}\}\\
&\leq C\{(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.
\end{aligned}
$$

The other terms are estimated similarly, and we arrive at

$$
\sup_{j<0}\|\dot{\triangle}F(U)\|_{L^1}\leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+C(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.
$$

This completes the proof. □

4.2 Estimate of high frequency parts

We next derive estimates for $M_\infty(t)$.

$$
\begin{cases}
\partial_t\triangle_j\sigma+\gamma\nabla\cdot\triangle_j w=\triangle_j F_1(U),\\
\partial_t\triangle_j w-\mu_1\triangle\triangle_j w-\mu_2\nabla\cdot(\nabla\triangle_j w)+\gamma\nabla\triangle_j\sigma=\triangle_j F_2(U).
\end{cases}
\tag{25}
$$

Proposition 4.11. *Let $j\geq 0$. There holds*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\triangle_j U(t)\|^2_{L^2}+\mu_1\|\nabla\triangle_j w(t)\|^2_{L^2}+\mu_2\|\nabla\cdot\triangle_j w(t)\|^2_{L^2}\\
=\ &(\triangle_j F_1(U),\triangle_j\sigma)+(\triangle_j F_2(U),\triangle_j w)
\end{aligned}
\tag{26}
$$

for a.e. $t\in[0,T]$.

See, e.g., [12], for the proof of Lemma 4.11.

We recall that for $s\in\mathbb{R}$, Λ^s is defined by $\Lambda^s z:=\mathfrak{F}^{-1}[|\xi|^s\hat{z}]$. Let $d=\Lambda^{-1}\nabla\cdot w$ be the "compressible part" of the velocity. Applying $\Lambda^{-1}\nabla\cdot$ to $(25)_2$, system (25) writes

$$
\begin{cases}
\partial_t\triangle_j\sigma+\gamma\Lambda\triangle_j d=\triangle_j F_1(U),\\
\partial_t\triangle_j d-\nu\triangle\triangle_j d-\gamma\Lambda\triangle_j\sigma=\Lambda^{-1}\nabla\cdot\triangle_j F_2(U),
\end{cases}
\tag{27}
$$

where we denote $\nu=\mu_1+\mu_2$.

Proposition 4.12. *Let $j \geq 0$. There holds*

$$\frac{1}{2}\frac{\nu}{\gamma}\frac{d}{dt}\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \frac{d}{dt}(\Lambda\triangle_j\sigma, \triangle_j d) + \|\Lambda\triangle_j\sigma\|_{L^2}^2 = \gamma\|\Lambda\triangle_j d\|_{L^2}^2$$
$$-(\Lambda\triangle_j F_1(U), \triangle_j d) - (\Lambda^{-1}\nabla\cdot\triangle_j F_2(U), \Lambda\triangle_j\sigma) + \frac{\nu}{\gamma}(\Lambda\triangle_j F_1(U), \Lambda\triangle_j\sigma) \quad (28)$$

for a.e. $t \in [0, T]$.

See, e.g., [12], for the proof of Lemma 4.12.
We introduce a lemma for estimates of the right-hand side of (28).

Lemma 4.13. *The following inequalities hold*

(i)

$$\big|\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \Lambda\triangle_j\sigma\big)\big| \leq C\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\Lambda\triangle_j\sigma\|_{L^2},$$

(ii)

$$\begin{aligned}
&\big|\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \triangle_j d\big)\big| \\
\leq\ & C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} \\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} + 2^{2j}\|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\},
\end{aligned}$$

where C is independ of $j \in \mathbb{Z}$ and $\{\alpha_j\}$ with $\|\{\alpha_j\}\|_{l^1} \leq 1$.

Proof. As for (i), see, e.g., [2].
Let us prove (ii). By using Lemma 3.6, we obtain

$$\begin{aligned}
&\big|\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \triangle_j d\big)\big| \\
\leq\ & \big|\big([w\cdot\nabla, \triangle_j]\sigma, \Lambda\triangle_j d\big)\big| + \big|\big(w\cdot\nabla\triangle_j\sigma, \Lambda\triangle_j d\big)\big| \\
\leq\ & C\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|\nabla w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} \\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(\|\dot{S}_0 w\|_{L^\infty}\|\Lambda\triangle_j d\|_{L^2} + \|\tilde{S}_0 w\|_{L^n}\|\Lambda\triangle_j d\|_{L^{\frac{2n}{n-2}}}\big)\} \\
\leq\ & C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|\nabla w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} \\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} + 2^{2j}\|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\}.
\end{aligned}$$

This completes the proof. □

Proposition 4.14. *There holds*

$$\begin{aligned}
&\frac{d}{dt}E_j(t) + c_0 E_j(t) \\
\leq\ & C\{\alpha_j(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + (1+t)^{-\frac{n}{2p}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t) \\
&+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2} + 2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2} \\
&+2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\},
\end{aligned} \quad (29)$$

for $t \in [0,T]$ and $j \geq 1$, where $\sum_{j\in\mathbb{Z}} \alpha_j \leq 1$, and c_0 is a positive constant independent of j. Here, $E_j(t)$ is equivalent to $2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2} + 2^{\frac{n}{2}j}\|\triangle_j \sigma(t)\|_{L^2}$. That is, there exists a positive constant D_1 such that

$$\begin{aligned}&\frac{1}{D_1}\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2} + 2^{\frac{n}{2}j}\|\triangle_j \sigma(t)\|_{L^2}\big)\\ \leq\ & E_j(t)\\ \leq\ & D_1\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2} + 2^{\frac{n}{2}j}\|\triangle_j \sigma(t)\|_{L^2}\big).\end{aligned}$$

Proof. We add (26) to $\kappa \times$ (28) with a constant $\kappa > 0$ to be determined later. Then, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\{\frac{1}{2}\|\triangle_j U\|_{L^2}^2 + \frac{\kappa}{2}\frac{\nu}{\gamma}\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \kappa(\Lambda\triangle_j\sigma, \triangle_j d)\Big\}\\ &+\mu_1\|\nabla\triangle_j w\|_{L^2}^2 + \mu_2\|\nabla\cdot\triangle_j w\|_{L^2}^2 + \kappa\|\Lambda\triangle_j\sigma\|_{L^2}^2\\ =\ & \gamma\kappa\|\Lambda\triangle_j w\|_{L^2}^2 + \big(\triangle_j F_1(U), \triangle_j\sigma\big) + \big(\triangle_j F_1(U), \triangle_j w\big) + \kappa\frac{\nu}{\gamma}\big(\Lambda\triangle_j F_1(U), \Lambda\triangle_j\sigma\big)\\ &-\kappa\big(\Lambda\triangle_j F_1(U), \triangle_j d\big) - \kappa\big(\Lambda^{-1}\nabla\cdot\triangle_j F_2(U), \Lambda\triangle_j\sigma\big).\end{aligned} \tag{30}$$

We set

$$E_j^2(t) = 2^{2(\frac{n}{2}-1)j}\Big\{\frac{1}{2}\|\triangle_j U\|_{L^2}^2 + \frac{\kappa}{2}\frac{\nu}{\gamma}\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \kappa(\Lambda\triangle_j\sigma, \triangle_j d)\Big\}.$$

It is not difficult to see that there exists $D_1 > 0$ such that, if $\kappa = \min\{D_1\frac{\mu}{\gamma}, 1\}$, then $E_j(t)$ is equivalent to $2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2} + 2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}$ and that there exists a $c_0 > 0$ such that

$$2c_0 E_j^2 \leq 2^{2(\frac{n}{2}-1)j}\big\{\mu_1\|\nabla\triangle_j w\|_{L^2}^2 + \mu_1\|\nabla\cdot\triangle_j w\|_{L^2}^2 + \kappa\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \gamma\kappa\|\Lambda\triangle_j w\|_{L^2}^2\big\}.$$

Let us next estimate the right-hand side of $2^{2(\frac{n}{2}-1)j}\times$(30). By Hölder's inequality, we obtain

$$\begin{aligned}2^{2(\frac{n}{2}-1)j}\big(\triangle_j F_1(U), \triangle_j\sigma\big) &\leq 2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\triangle_j\sigma\|_{L^2},\\ 2^{2(\frac{n}{2}-1)j}\big(\triangle_j F_2(U), \triangle_j\sigma\big) &\leq 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\triangle_j\sigma\|_{L^2},\\ 2^{2(\frac{n}{2}-1)j}\big(\Lambda^{-1}\nabla\cdot\triangle_j F_2(U), \triangle_j\sigma\big) &\leq 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\triangle_j\sigma\|_{L^2}.\end{aligned}$$

By Lemma 4.13 we have

$$\begin{aligned}&2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j F_1(U), \Lambda\triangle_j\sigma\big)\\ =\ & 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \Lambda\triangle_j\sigma\big) + 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(\sigma\nabla\cdot w), \Lambda\triangle_j\sigma\big)\\ \leq\ & C\alpha_j\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j\sigma\|_{L^2} + 2^{2(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\|\Lambda\triangle_j\sigma\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}&2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j F_1(U), \triangle_j d\big)\\ =\ & 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \triangle_j d\big) + 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(\sigma\nabla\cdot w), \triangle_j d\big)\\ \leq\ & C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}\\ &+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot S_0 w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} + 2^{2j}\|\tilde S_0 w\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\}\\ &+2^{2(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\|\triangle_j d\|_{L^2},\end{aligned}$$

where $\sum_{j\in\mathbb{Z}}\alpha_j \le 1$. Hence we obtain

$$\begin{aligned}
&\frac{d}{dt}E_j^2+2c_0E_j^2\le CE_j\big\{\alpha_j(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot B_{2,1}^{\frac{n}{2}+1}}\\
&+(1+t)^{-\frac{n}{2p}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t)+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\\
&+2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2}+2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\}. \qquad (31)
\end{aligned}$$

From (31) and dividing by E_j, we get the desired result. □

4.3 Proof of Theorem 1.1.

Proposition 4.15. *Let $1\le p<\frac{2n}{n+1}$. There exists a constant $\epsilon_2>0$ such that if*

$$\|U_0\|_{\dot B_{2,1}^{\frac{n}{2}-1}\cap\dot B_{p,\infty}^0}+\|\sigma_0\|_{\dot B_{2,1}^{\frac{n}{2}}}\le\epsilon_2,$$

then there holds

$$M(t)\le C\big\{\|U_0\|_{\dot B_{2,1}^{\frac{n}{2}-1}\cap\dot B_{p,\infty}^0}+\|\sigma_0\|_{\dot B_{2,1}^{\frac{n}{2}}}\big\}$$

for $0\le t\le T$, where the constant C does not depend on T.

Proof. By (29) we have

$$\begin{aligned}
E_j(t)\ \le\ & e^{-c_0t}E_j(0)\\
&+C\int_0^t e^{-c_0(t-\tau)}\{\alpha_j(1+\tau)^{-\frac{n}{2p}}M(t)\|w\|_{\dot B_{2,1}^{\frac{n}{2}+1}}\\
&+(1+\tau)^{-\frac{n}{2p}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t)\\
&+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\\
&+2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2}+2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\}d\tau, \qquad (32)
\end{aligned}$$

where $\sum_{j=0}^{\infty}\alpha_j\le 1$. Hence summing up on $j\ge 0$, by the monotone convergence theorem, we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}E_j(t)\le e^{-c_0t}\sum_{j=0}^{\infty}E_j(0)\\
&+C\int_0^t e^{-c_0(t-\tau)}\{(1+\tau)^{-\frac{n}{2p}}M(t)\|w\|_{\dot B_{2,1}^{\frac{n}{2}+1}}+\sum_{j=0}^{\infty}2^{j\frac{n}{2}}\|\dot\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\\
&+\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot\triangle_j F_1(U)\|_{L^2}+\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot\triangle_j F_2(U)\|_{L^2}\}d\tau. \qquad (33)
\end{aligned}$$

We next estimate the right-hand side of (33). From Lemma 3.1, we have

$$\sum_{j=0}^{\infty}2^{j\frac{n}{2}}\|\dot\triangle_j\sigma\nabla\cdot w\|_{L^2}\le\|\sigma\nabla\cdot w\|_{\dot B_{2,1}^{\frac{n}{2}}}\le C\|\sigma\|_{\dot B_{2,1}^{\frac{n}{2}}}\|\nabla\cdot w\|_{\dot B_{2,1}^{\frac{n}{2}}}\le C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot B_{2,1}^{\frac{n}{2}+1}}.$$

Let us next consider the quantities $\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2}$:

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(w\cdot\nabla\sigma)\|_{L^2} &\leq \|w\cdot\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\big(\|\dot{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big)\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{p}}M^2(\tau) + C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}},
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^2} &\leq \|\sigma\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{p}}M^2(\tau) + C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.
\end{aligned}$$

Hence, we obtain the estimate of $\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2}$. By using Lemma 3.1, Lemma 3.2 and Lemma 3.4, $\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_2(U)\|_{L^2}$ is estimated as

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(w\cdot\nabla)w\| &\leq C\{\|\dot{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\tilde{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\quad +\|\dot{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\tilde{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\} \\
&\leq C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.
\end{aligned}$$

Here we used

$$\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq C\{\big(\sum_{j=-5}^{-1} 2^{j\frac{n}{2}}\|\dot{\triangle}_j w\|_{L^2}\big) + \|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\} \leq C(1+\tau)^{-\frac{n}{2p}}M(\tau),$$

$$\|\tilde{S}_{-4}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \leq C\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.$$

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\triangle w)\|_{L^2} &\leq \|\frac{\sigma}{\sigma+1}\triangle w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\frac{\sigma}{\sigma+1}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}},
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\nabla\sigma)\|_{L^2} &\leq \|\frac{\sigma}{\sigma+1}\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\frac{\sigma}{\sigma+1}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{p}}M^2(\tau).
\end{aligned}$$

In the same way as above, we can obtain estimates of other terms on $\|F_2(U)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}$. Hence, by using Lemma 3.3, the integral of the right-hand side of (33) is estimated as

$$
\begin{aligned}
&\int_0^t e^{-c_0(t-\tau)}\big\{(1+\tau)^{-\frac{n}{p}}M^2(\tau)+(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\big\}d\tau\\
\le\ & M(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{n}{2p}}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau+M^2(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{n}{p}}d\tau\\
\le\ & C(1+t)^{-\frac{n}{2p}}\epsilon_2 M(t)+C(1+t)^{-\frac{n}{p}}M^2(t).
\end{aligned}
$$

Hence, we obtain

$$
M_\infty(t)\le C\big(\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big)+C\epsilon_2 M(t)+CM^2(t). \tag{34}
$$

By Proposition 4.9 and (34), we have

$$
M(t)\le C\big(\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^0_{p,\infty}}+\|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big)+C\epsilon_2 M(t)+CM^2(t).
$$

By taking $\epsilon_2>0$ suitably small, we obtain

$$
M(t)\le C\big(\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^0_{p,\infty}}+\|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big)
$$

for all $0\le t\le T$ with C independent of T. This completes the proof. □

It follows from Proposition 4.2 and Proposition 4.15 that

$$
M(t)\le C_3 \qquad \text{for all } t,
$$

if the initial perturbation is sufficiently small. Hence we obtain the desired decay estimate (4), (5) and (6) of Theorem 1.1.

Finally, by (8), we have

$$
\|U(t)\|_{\dot{B}^0_{n,1}}\le C\|U(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{n})}.
$$

By Lemma 3.7, for any $2<q<n$, we obtain

$$
\|U(t)\|_{\dot{B}^0_{q,1}}\le\|U(t)\|^{\theta}_{\dot{B}^0_{2,1}}\|U(t)\|^{1-\theta}_{\dot{B}^0_{n,1}}\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})},
$$

where $\theta=\frac{2n}{q(n-2)}-\frac{2}{n-2}$. This gives (7). The proof of Theorem 1.1 is thus complete.

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