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Abstract

Time periodic problem for the compressible Navier-Stokes equation on the whole space is studied. The existence of a time periodic solution is proved for sufficiently small time periodic external force with some symmetry when the space dimension is greater than or equal to 3. The proof is based on the spectral properties of the time-T map associated with the linearized problem around the motionless state with constant density in some weighted Sobolev space. The stability of the time periodic solution is also proved and the decay estimate of the perturbation is established.

1 Introduction

This paper studies time periodic problem of the following compressible Navier-Stokes equation in $\mathbb{R}^n$ $(n \geq 3)$:

$$\left\{ \begin{array}{l} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \rho(\partial_t v + (v \cdot \nabla) v) - \mu \triangle v - (\mu + \mu')\nabla(\nabla \cdot v) + \nabla P(\rho) = \rho g. \end{array} \right. \tag{1.1}$$

Here $\rho = \rho(x,t)$ and $v = (v_1(x,t), \cdots, v_n(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^n$; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$P'(\rho_*) > 0$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \frac{2}{n}\mu + \mu' \geq 0;$$

and $g = g(x,t)$ is a given external force periodic in t. We assume that $g = g(x,t)$ satisfies the conditions

$$\begin{array}{rcl} g(x, t+T) &=& g(x,t) \quad (x \in \mathbb{R}^n,\, t \in \mathbb{R}), \\ g(-x,t) &=& -g(x,t) \quad (x \in \mathbb{R}^n,\, t \in \mathbb{R}) \end{array} \tag{1.2}$$

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for some constant $T > 0$.

The purpose of this paper is to investigate the existence and stability of a time periodic solution of system (1.1) around the constant state $(\rho_*, 0)$.

Concerning the time periodic problem for the compressible Navier-Stokes equations, Valli ([11]) proved the existence and (exponential) stability of time periodic solutions on a bounded domain of $\mathbb{R}^3$ for sufficiently small time periodic external forces. On the other hand, for large time periodic external forces, the existence of time periodic solutions on a bounded domain of $\mathbb{R}^3$ was proved by Feireisl, Matušu-Necasová, Petzeltová and Straškrava ([2]) and Feireisl, Mucha, Novotný and Pokorný ([3]) in the framework of weak solutions. As for the time periodic problem on unbounded domains, Ma, Ukai, and Yang ([8]) studied the existence and stability of time periodic solutions on the whole space $\mathbb{R}^n$. It was shown in [8] that if $n \geq 5$, there exists a time periodic solution (ρ_{per}, v_{per}) for a sufficiently small $g \in C^0(\mathbb{R}; H^{N-1} \cap L^1)$ with $g(x, t+T) = g(x,t)$, where $N \in \mathbb{Z}$ satisfying $N \geq n+2$. Furthermore, the time periodic solution is stable under sufficiently small perturbations and there holds the estimate

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{H^{N-1}} \leq C(1+t)^{-\frac{n}{4}} \|(\rho_0, v_0) - (\rho_{per}(t_0), v_{per}(t_0))\|_{H^{N-1}\cap L^1},$$

where t_0 is a certain initial time and $(\rho, v)|_{t=t_0} = (\rho_0, v_0)$. Here H^k denotes the L^2-Sobolev space on $\mathbb{R}^n$ of order k.

In this paper, we will show that for $n \geq 3$, if the external force g satisfies the oddness condition (1.2) and is small enough in some weighted Sobolev space, then (1.1) has a time periodic solution (ρ_{per}, v_{per}) and $u_{per}(t) = (\rho_{per}(t) - \rho_*, v_{per}(t))$ satisfies

$$\begin{aligned}&\sup_{t\in[0,T]} (\|u_{per}(t)\|_{L^2} + \|x\nabla u_{per}(t)\|_{L^2}) \\ &\quad \leq \|g\|_{C([0,T];L^1\cap L^2)} + \|xg\|_{C([0,T];L^1\cap L^2)} + \|g\|_{L^2(0,T;H^{m-1})} + \|xg\|_{L^2(0,T;H^{m-1})}. \qquad (1.3)\end{aligned}$$

In addition, we will prove that the time periodic solution is stable under sufficiently small initial perturbation, and that the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^2} = O(t^{-\frac{n}{4}}) \text{ as } t \to \infty. \qquad (1.4)$$

The precise statements are given in Theorem 3.1 and Theorem 3.2 below.

The proof of the existence of a time periodic solution is given by an iteration argument by using the time-T-map associated with the linearized problem around $(\rho_*, 0)$. Substituting $\phi = \frac{\rho-\rho_*}{\rho_*}$ and $w = \frac{v}{\gamma}$ with $\gamma = \sqrt{P'(\rho_*)}$ into (1.1), we see that (1.1) is rewritten as

$$\partial_t u + Au = -B[u]u + G(u,g), \qquad (1.5)$$

where

$$A = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma\nabla & -\nu\triangle - \tilde{\nu}\nabla\mathrm{div} \end{pmatrix}, \quad \nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu+\mu'}{\rho_*}, \qquad (1.6)$$

$$B[\tilde{u}]u = \gamma \begin{pmatrix} \tilde{w}\cdot\nabla\phi \\ 0 \end{pmatrix} \text{ for } u = {}^\top(\phi, w),\ \tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}) \qquad (1.7)$$

and

$$\begin{aligned}
G(u,g) &= \begin{pmatrix} f^0(u) \\ \tilde{f}(u,g) \end{pmatrix}, && (1.8)\\
f^0(u) &= -\gamma\phi \mathrm{div} w, && (1.9)\\
\tilde{f}(u,g) &= -\gamma(1+\phi)(w\cdot\nabla w) - \phi\partial_t w - \nabla(P^{(1)}(\phi)\phi^2) + \frac{1+\phi}{\gamma}g, && (1.10)\\
P^{(1)}(\phi) &= \frac{\rho_*}{\gamma}\int_0^1 (1-\theta)P''(\rho_*(1+\theta\phi))d\theta.
\end{aligned}$$

To solve the time periodic problem for (1.5), we decompose u into a low frequency part u_1 and a high frequency part u_∞. Then u_1 and u_∞ satisfy

$$\partial_t u_1 + Au_1 = F_1(u,g), \qquad (1.11)$$

$$\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty(u,g), \qquad (1.12)$$

where

$$F_1(u,g) = P_1[-B[\tilde{u}]u + G(u,g)],$$
$$F_\infty(u,g) = P_\infty[-B[\tilde{u}]u_1 + G(u,g)]$$

and

$$\tilde{u} = u = u_1 + u_\infty, \quad u_j = P_j u \quad (j = 1, \infty).$$

Here P_1 and P_∞ are bounded linear operators from L^2 into a low frequency part and a high frequency part, respectively, satisfying $P_1 + P_\infty = I$. (See sections 3 and 4 for the definitions and properties of P_1 and P_∞.)

We rewrite (1.11)-(1.12) as

$$u_1(t) = S_1(t)u_{01} + \mathscr{S}_1(t)F_1(u,g), \qquad (1.13)$$

$$u_\infty(t) = S_{\infty,\tilde{u}}(t)u_{0\infty} + \mathscr{S}_{\infty,\tilde{u}}(t)F_\infty(u,g), \qquad (1.14)$$

where

$$u_{01} = (I - S_1(T))^{-1}\mathscr{S}_1(T)F_1(u,g),$$
$$u_{0\infty} = (I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)F_\infty(u,g)$$

with

$$\tilde{u} = u = u_1 + u_\infty.$$

Here $S_1(t)$ is the solution operator for the linear initial value problem for (1.11) with the inhomogeneous term $F_1(u,g) \equiv 0$ under the initial condition $u_1|_{t=0} = u_{01}$; $\mathscr{S}_1(t)$ is the one for (1.11) with a given inhomogeneous term $F_1(u,g)$ under the initial condition $u_1|_{t=0} = 0$; and $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ are similarly defined by the solution operators for the linear initial value problem for (1.12). We will investigate properties of $S_1(t)$, $\mathscr{S}_1(t)$, $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in weighted Sobolev spaces. The necessary estimates for $S_1(t)$ and $\mathscr{S}_1(t)$ will be obtained by the explicit formulas for these operators through the Fourier transform, while those for $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ will be established by a weighted energy method.

The stability of the time periodic solution will be also shown by a decomposition method associated with the spectral properties of the linearized operator which, in this case, is a decomposition into low and high frequency parts (cf., [4, 10]). Based on the estimate (1.3) for $u_{per}(t) = {}^\top(\rho_{per}(t) - \rho_*, v_{per}(t))$, we can apply the Hardy inequality to show the stability of the time periodic solution ${}^\top(\rho_{per}(t), v_{per}(t))$ under sufficiently small initial perturbations and the decay estimate (1.4) in a similar manner to [10]. In contrast to the problem in [10], the terms $v_{per} \cdot \nabla\phi + \phi \mathrm{div} v_{per}$ appear in the transport equation for the perturbation. These terms can be handled by using the energy method and the boundedness properties of the projection onto the low frequency part as in [4], together with the Hardy inequality. (See also [1]).

This paper is organized as follows. In section 2, we introduce notations and auxiliary lemmas used in this paper. In section 3, we state main results of this paper. In section 4, we reformulate the problem. Section 5 is devoted to studying $S_1(t)$ and $\mathscr{S}_1(t)$ and we derive the estimates for low frequency part. In section 6, we state some spectral properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ and derive the estimates for high frequency part. In section 7, we establish the weighted energy estimate for high frequency part which gives spectral information on $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$. In section 8, we estimate nonlinear terms and then give a proof of the existence of a time periodic solution.

2 Preliminaries

In this section we first introduce some notations which will be used throughout this paper. We then introduce some auxiliary lemmas which will be useful in the proof of the main results.

For a given Banach space X, the norm on X is denoted by $\| \cdot \|_X$.

Let $1 \leqq p \leqq \infty$. L^p stands for the usual L^p space over $\mathbb{R}^n$. The inner product of L^2 is denoted by $(\cdot, \cdot)$. For a nonnegative integer k, H^k stands for the usual L^2-Sobolev space of order k. (As usual, $H^0 = L^2$.)

The set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in L^p$ $(j = 1, \cdots, n)$, i.e., $(L^p)^n$, is simply denoted by L^p; and the norm $\| \cdot \|_{(L^p)^n}$ on it is denoted by $\| \cdot \|_{L^p}$ if no confusion will occur. Similarly, for a function space X, the set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in X$ $(j = 1, \cdots, n)$, i.e., X^n, is simply denoted by X; and the norm $\| \cdot \|_{X^n}$ on it is denoted by $\| \cdot \|_X$ if no confusion will occur. (For example, $(H^k)^n$ is simply denoted by H^k and the norm $\| \cdot \|_{(H^k)^n}$ is denoted by $\| \cdot \|_{H^k}$.)

For $u = {}^\top(\phi, w)$ with $\phi \in H^k$ and $w = {}^\top(w_1, \cdots, w_n) \in H^m$, we define the norm $\|u\|_{H^k \times H^m}$ of u on $H^k \times H^m$ by

$$\|u\|_{H^k \times H^m} = \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2 \right)^{\frac{1}{2}}.$$

When $m = k$, we simply write $H^k \times (H^k)^n$ as H^k, and, also, $\|u\|_{H^k \times (H^k)^n}$ as $\|u\|_{H^k}$ if no confusion will occur :

$$H^k := H^k \times (H^k)^n, \quad \|u\|_{H^k} := \|u\|_{H^k\times(H^k)^n} \quad (u = {}^\top(\phi, w)).$$

Similarly, when $u = {}^\top(\phi, w) \in X \times Y$ with $w = {}^\top(w_1, \cdots, w_n)$ for function spaces X and Y, we denote its norm $\|u\|_{X\times Y}$ by

$$\|u\|_{X\times Y} = \left(\|\phi\|_X^2 + \|w\|_Y^2\right)^{\frac{1}{2}} \quad (u = {}^\top(\phi, w)).$$

When $Y = X^n$, we simply write $X \times X^n$ as X, and also its norm $\|u\|_{X\times X^n}$ as $\|u\|_X$:

$$X := X \times X^n, \quad \|u\|_X := \|u\|_{X\times X^n} \quad (u = {}^\top(\phi, w)).$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index. We use the following notation

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \qquad |\alpha| = \sum_{j=1}^n \alpha_j.$$

For any integer $l \geq 0$, $\nabla^l f$ denotes x-derivatives of order l of a function f.

For $1 \leq p < \infty$, $L^p((1 + |x|^p)dx)$ stands for the weighted L^p space over $\mathbb{R}^n$ defined by

$$L^p((1 + |x|^p)dx) = \left\{f; \int_{\mathbb{R}^n} |f(x)|^p(1 + |x|^p)dx < +\infty\right\}.$$

In particular, we denote $L^1((1 + |x|)dx)$ by L_1^1.

We denote by $\hat{f}$ or $\mathcal{F}[f]$ the Fourier transform of f :

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}^n).$$

The inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}^n).$$

For a nonnegative integer k and positive constants r_1 and r_∞ with $r_1 < r_\infty$, $H^k_{(\infty)}$ denotes the set of all $f \in H^k$ satisfying supp $\hat{f} \subset \{|\xi| \geq r_1\}$, and $H^k_{(1)}$ denotes the set of all $f \in H^k$ satisfying supp $\hat{f} \subset \{|\xi| \leq r_\infty\}$.

For a nonnegative integers k and ℓ, the spaces H^k_ℓ and $H^k_{(\infty),\ell}$ are defined by

$$H^k_\ell = \{f \in H^k; \|f\|_{H^k_\ell} < +\infty\},$$

where

$$\|f\|_{H^k_\ell} = \left(\sum_{j=0}^{\ell} |f|^2_{H^k_j}\right)^{\frac{1}{2}},$$

$$|f|_{H^k_\ell} = \left(\sum_{|\alpha|\le k} \| \, |x|^\ell \partial_x^\alpha f \, \|_{L^2}^2 \right)^{\frac{1}{2}},$$

and

$$H^k_{(\infty),\ell} = \{ f \in H^k_{(\infty)}; \|f\|_{H^k_\ell} < +\infty \}.$$

The space $L^2_{(1),1}$ is defined by

$$L^2_{(1),1} = \{ f \in L^2_1; f \in L^2_{(1)} \};$$

and the space $H^1_{(1),1}$ is defined by

$$H^1_{(1),1} = \{ f \in H^1_{(1)}; \|f\|_{H^1_{(1),1}} < +\infty \},$$

where

$$\|f\|_{H^1_{(1),1}} = \left(\|f\|_{L^2}^2 + \|x\nabla f\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

We note that $H^k_{(\infty),\ell}$ and $H^1_{(1),1}$ are closed subspaces of H^k_ℓ and H^1_1, respectively.

We will consider the time periodic problem in function spaces with some symmetry. We define Γ by

$$(\Gamma u)(x) = {}^\top(\phi(-x), -w(-x)) \quad (u(x) = {}^\top(\phi(x), w(x)), \quad x \in \mathbb{R}^n).$$

We indicate function spaces satisfying the symmetric condition $\Gamma u = u$ by the subscript $\cdot_{sym}$. More precisely, We denote by X_{sym} the set of all $u = {}^\top(\phi, w) \in X$ satisfying the symmetric conditions $\Gamma u = u$, i.e., $\phi(-x) = \phi(x)$ and $w(-x) = -w(x)$ $(x \in \mathbb{R}^n)$:

$$X_{sym} = \{ u = {}^\top(\phi, w) \in X; \Gamma u = u \}.$$

Let $-\infty \le a < b \le \infty$. We denote by $C^k([a,b]; X)$ the set of all C^k functions on $[a,b]$ with values in X. The Bochner space on (a,b) is denoted by $L^p(a,b;X)$ and the L^2-Bochner-Sobolev space of order k is denoted by $H^k(a,b;X)$.

Let k be a nonnegative integer satisfying $k \ge 1$. The space $\mathscr{X}^k(a,b)$ is defined by

$$\mathscr{X}^k(a,b) = \mathscr{Y}_1(a,b) \times \mathscr{Y}^k_\infty(a,b)$$

equipped with the norm

$$\|\{u_1, u_\infty\}\|_{\mathscr{X}^k(a,b)} = \left(\|u_1\|^2_{\mathscr{Y}_1(a,b)} + \|u_\infty\|^2_{\mathscr{Y}^k_\infty(a,b)} \right)^{\frac{1}{2}},$$

where

$$\mathscr{Y}_1(a,b) = \left\{ u_1 = {}^\top(\phi_1, w_1) \in C^1([a,b]; (H^1_{(1),1})_{sym}); \partial_t w_1 \in C([a,b]; L^2_1) \right\},$$

$$\mathscr{Y}^k_\infty(a,b) = \{u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([a,b];(H^k_{(\infty),1})_{sym}); w_\infty \in L^2(a,b;H^{k+1}_{(\infty),1}) \cap H^1(a,b;H^{k-1}_{(\infty),1})\},$$

$$\begin{aligned} \|u_1\|_{\mathscr{Y}_1(a,b)} &= \left(\|u_1\|^2_{C^1([a,b];H^1_{(1),1})} + \|\partial_t w_1\|^2_{C([a,b];L^2_1)}\right)^{\frac{1}{2}}, \\ \|u_\infty\|_{\mathscr{Y}^k_\infty(a,b)} &= \left(\|u_\infty\|^2_{C([a,b];H^k_{(\infty),1})} + \|w_\infty\|^2_{L^2(a,b;H^{k+1}_{(\infty),1}) \cap H^1(a,b;H^{k-1}_{(\infty),1})}\right)^{\frac{1}{2}}. \end{aligned}$$

We also introduce function spaces of T-periodic functions in t. $C_{per}(\mathbb{R};X)$ denotes the set of all T-periodic continuous functions with values in X equipped with the norm $\|\cdot\|_{C([0,T];X)}$; and $L^2_{per}(\mathbb{R};X)$ denotes the set of all T-periodic locally square integrable functions with values in X equipped with the norm $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R};X)$ and $\mathscr{X}^k_{per}(\mathbb{R})$, and so on, are defined.

Let X be a Banach space and let P be a bounded linear operator on X. We denote by $r_X(P)$ the spectral radius of P.

For operators A and B, $[A,B]$ denotes the commutator of A and B :

$$[A,B]f = A(Bf) - B(Af).$$

We next state some lemmas which will be used in the proof of the main results.

Lemma 2.1. *Let $n \geq 3$ and let $m \geq \left[\frac{n}{2}\right] + 1$. Then there holds the inequality*

$$\|f\|_{L^\infty} \leq C\|\nabla f\|_{H^{m-1}}$$

for $f \in H^m$.

Lemma 2.1 is proved as follows. Let $n \geq 3$ and set $2^* := \frac{2n}{n-2}$. Since $m \geq \left[\frac{n}{2}\right] + 1$, we see that $m - 1 \geq \frac{n}{2^*}$. It then follows from the Sobolev inequalities that

$$\|f\|_{L^\infty} \leq C\|f\|_{W^{m,2^*}} \leq C\|\nabla f\|_{H^{m-1}},$$

which shows Lemma 2.1.

Lemma 2.2. *Assume $n \geq 3$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Let m_j and μ_j $(j = 1, \cdots, \ell)$ satisfy $0 \leq |\mu_j| \leq m_j \leq m + |\mu_j|$, $\mu = \mu_1 + \cdots + \mu_\ell$, $m = m_1 + \cdots + m_\ell \geq (\ell - 1)m + |\mu|$. Then there holds*

$$\| \partial_x^{\mu_1} f_1 \cdots \partial_x^{\mu_\ell} f_\ell \|_{L^2} \leq C \prod_{1 \leq j \leq \ell} \| f_j \|_{H^{m_j}} .$$

See, e.g., [7], for the proof of Lemma 2.2.

Lemma 2.3. *Let $n \geq 3$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Suppose that F is a smooth function on I, where I is a compact interval of $\mathbb{R}$. Then for a multi-index α with $1 \leq |\alpha| \leq m$, there hold the estimates*

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{m-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^{m-1}} \|f_2\|_{H^{|\alpha|}}$$

for $f_1 \in H^m$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^n$ and $f_2 \in H^{|\alpha|}$; and

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{m-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^m} \|f_2\|_{H^{|\alpha|-1}}.$$

for $f_1 \in H^{m+1}$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^n$ and $f_2 \in H^{|\alpha|-1}$.

See, e.g., [5], for the proof of Lemma 2.3.

3 Main results

In this section we state our results on the existence and stability of a time periodic solution for system (1.1).

We begin with the existence of a time periodic solution. To state the existence result, we introduce operators which decompose a function into its low and high frequency parts. We define operators P_1 and P_∞ on L^2 by

$$P_j f = \mathcal{F}^{-1}\hat{\chi}_j \mathcal{F}[f] \quad (f \in L^2, j = 1, \infty),$$

where

$$\begin{aligned}
&\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j = 1, \infty), \quad 0 \leq \hat{\chi}_j \leq 1 \quad (j = 1, \infty),\\
&\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \leq r_1), \\ 0 & (|\xi| \geq r_\infty), \end{cases}\\
&\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi),\\
&0 < r_1 < r_\infty.
\end{aligned}$$

We fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (5.4) in Lemma 5.5 below holds for $|\xi| \leq r_\infty$.

Theorem 3.1. *Let $n \geq 3$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies* (1.2) *and $g \in C_{per}(\mathbb{R}; L_1^1 \cap L_1^2) \cap L_{per}^2(\mathbb{R}; H_1^{m-1})$. Set*

$$[g]_m = \|g\|_{C([0,T]; L_1^1 \cap L_1^2)} + \|g\|_{L^2(0,T; H_1^{m-1})}.$$

Then there exist constants $\delta > 0$ and $C > 0$ such that if $[g]_m \leq \delta$, then the system (1.5) *has a time periodic solution $u = u_1 + u_\infty$ satisfying $\{u_1, u_\infty\} \in \mathscr{X}_{per}^m(\mathbb{R})$ with $\|\{u_1, u_\infty\}\|_{\mathscr{X}^m(0,T)} \leq C[g]_m$.*

Furthermore, the uniqueness of time periodic solutions of (1.5) *holds in the class $\{u = {}^\top(\phi, w); \{P_1 u, P_\infty u\} \in \mathscr{X}_{per}^m(\mathbb{R}),\ \|\{P_1 u, P_\infty u\}\|_{\mathscr{X}^m(0,T)} \leq C\delta\}$.*

Our next issue is to study the stability of the time periodic solution obtained in Theorem 3.1.

Let ${}^\top(\rho_{per}, v_{per})$ be the time periodic solution given in Theorem 3.1. We denote the perturbation by $u = {}^\top(\phi, w)$, where $\phi = \rho - \rho_{per}, w = v - v_{per}$. Substituting $\rho = \phi + \rho_{per}$ and $v = w + v_{per}$ into (1.1), we see that the perturbation $u = {}^\top(\phi, w)$ is governed by

$$\begin{cases} \partial_t \phi + v_{per} \cdot \nabla \phi + \phi \mathrm{div} v_{per} + \rho_{per} \mathrm{div} w + w \cdot \nabla \rho_{per} = F^0, \\ \partial_t w + v_{per} \cdot \nabla w + w \cdot \nabla v_{per} - \frac{\mu}{\rho_{per}} \triangle w - \frac{\mu+\mu'}{\rho_{per}} \nabla \mathrm{div} w \\ \quad + \frac{\phi}{\rho_{per}^2} (\mu \triangle v_{per} + (\mu + \mu') \nabla \mathrm{div} v_{per}) + \nabla (\frac{P'(\rho_{per})}{\rho_{per}} \phi) = \tilde{F}, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} F^0 &= -\mathrm{div}(\phi w), \\ \tilde{F} &= -w \cdot \nabla w - \frac{\phi}{\rho_{per}(\rho_{per} + \phi)} (\mu \triangle w + (\mu + \mu') \nabla \mathrm{div} w) \\ &\quad + \frac{\phi}{\rho_{per}(\rho_{per} + \phi)} (\frac{\phi}{\rho_{per}} \mu \triangle v_{per} + \frac{\phi}{\rho_{per}} (\mu + \mu') \nabla \mathrm{div} v_{per}) \\ &\quad + \frac{\phi}{\rho_{per}^2} \nabla (P^{(2)}(\rho_{per}, \phi)\phi) + \frac{\phi^2}{\rho_{per}^2 (\rho_{per} + \phi)} \nabla (P(\rho_{per} + \phi)) + \frac{1}{\rho_{per}} \nabla (P_{(3)}(\rho_{per}, \phi)\phi^2), \\ P^{(2)}(\rho_{per}, \phi) &= \int_0^1 P'(\rho_{per} + \theta\phi) d\theta, \\ P^{(3)}(\rho_{per}, \phi) &= \int_0^1 (1 - \theta) P''(\rho_{per} + \theta\phi) d\theta. \end{aligned}$$

We consider the initial value problem for (3.1) under the initial condition

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{3.2}$$

Our result on the stability of the time periodic solution is stated as follows.

Theorem 3.2. *Let $n \geq 3$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies (1.2) and $g \in C_{per}(\mathbb{R}; L_1^1 \cap L_1^2) \cap L_{per}^2(\mathbb{R}; H_1^m)$. Let ${}^\top(\rho_{per}, v_{per})$ be the time periodic solution obtained in Theorem 3.1 and let $u_0 = {}^\top(\phi_0, w_0) \in H^m \cap L^1$. Then there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that if*

$$[g]_{m+1} < \epsilon_1, \quad \|u_0\|_{H^m \cap L^1} < \epsilon_2,$$

there exists a unique global solution $u = {}^\top(\phi, w) \in C([0, \infty); H^m)$ of (3.1)-(3.2) and u satisfies

$$\|\nabla^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4} - \frac{k}{2}} \qquad (t \in [0, +\infty), \quad k = 0, 1).$$

Theorem 3.2 follows from the same argument as that in [10]; and we omit the details. In contrast to the problem in [10], several linear terms with coefficients including v_{per} appear in the equations for the perturbation. In the transport equation for the perturbation, there appear the terms $v_{per} \cdot \nabla\phi + \phi \mathrm{div} v_{per}$ and these terms can be handled by using the energy method and the boundedness properties of the projection onto the low frequency part as in [4, 10], together with the Hardy inequality; the linear terms including v_{per} in the equation of motion for the perturbation can be handled by using the Hardy inequality. (See also [1]).

4 Reformulation of the problem

In this section we reformulate the time periodic problem for (1.5).

To solve the time periodic problem for (1.5), we decompose u into a low frequency part u_1 and a high frequency part u_∞; and we rewrite the problem into a system of equations for u_1 and u_∞.

Set

$$u_1 = P_1 u, \quad u_\infty = P_\infty u.$$

Applying the operators P_1 and P_∞ to (1.5), we obtain,

$$\partial_t u_1 + A u_1 = F_1(u_1 + u_\infty, g), \tag{4.1}$$

$$\partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) = F_\infty(u_1 + u_\infty, g). \tag{4.2}$$

Here

$$\begin{aligned} F_1(u_1 + u_\infty, g) &= P_1[-B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g)], \\ F_\infty(u_1 + u_\infty, g) &= P_\infty[-B[u_1 + u_\infty]u_1 + G(u_1 + u_\infty, g)]. \end{aligned}$$

Suppose that (4.1) and (4.2) are satisfied by some functions u_1 and u_∞. Then, since $P_1 + P_\infty = I$, by adding (4.1) to (4.2), we obtain

$$\begin{aligned} \partial_t(u_1 + u_\infty) + A(u_1 + u_\infty) &= -P_\infty(B[u_1 + u_\infty]u_\infty) + (P_1 + P_\infty)F(u_1 + u_\infty, g) \\ &= -B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g). \end{aligned}$$

Set $u = u_1 + u_\infty$, then we have

$$\partial_t u + Au + B[u]u = G(u, g).$$

Consequently, if we show the existence of a pair of functions $\{u_1, u_\infty\}$ satisfying (4.1)-(4.2), then we can obtain a solution u of (1.5). Therefore, we will consider (4.1)-(4.2) to solve the time periodic problem for (1.5).

The following two lemmas are concerned with symmetry of (1.5) and (4.1)-(4.2). We recall that Γ is defined by

$$(\Gamma u)(x) = {}^\top(\phi(-x), -w(-x)) \quad (u(x) = {}^\top(\phi(x), w(x)), \quad x \in \mathbb{R}^n).$$

Lemma 4.1. *Set* $\boldsymbol{g}(x,t) = {}^\top(0, g(x,t))$ *and assume that* $(\Gamma\boldsymbol{g})(x,t) = \boldsymbol{g}(x,t)$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$.

(i) *If* $u = {}^\top(\phi, w)$ *is a solution of* (1.5)*, then* Γu *is also a solution of* (1.5)*.*

(ii) *If* $\{u_1, u_\infty\}$ *is a solution of* (4.1)*-*(4.2)*, then* $\{\Gamma u_1, \Gamma u_\infty\}$ *is also a solution of* (4.1)*-*(4.2)*.*

Lemma 4.2. *Assume that* $(\Gamma\boldsymbol{g})(x,t) = \boldsymbol{g}(x,t)$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$.

(i) *If* $(\Gamma u)(x,t) = u(x,t)$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$*, then*

$$[\Gamma(\partial_t u + Au + B[u]u - G(u,g))](x,t) = [\partial_t u + Au + B[u]u - G(u,g)](x,t)$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$.

(ii) *If* $\{\Gamma u_1(x,t), \Gamma u_\infty(x,t)\} = \{u_1(x,t), u_\infty(x,t)\}$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$*, then*

$$[\Gamma(\partial_t u_1 + Au_1 - F_1(u_1 + u_\infty, g))](x,t) = [\partial_t u_1 + Au_1 - F_1(u_1 + u_\infty, g)](x,t)$$

and

$$\begin{aligned}
&[\Gamma(\partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) - F_\infty(u_1 + u_\infty, g))](x,t) \\
&\quad = \ [\partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) - F_\infty(u_1 + u_\infty, g)](x,t)
\end{aligned}$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$.

Lemma 4.1 (i) and Lemma 4.2 (i) can be verified by direct computations. As for Lemma 4.1 (ii) and Lemma 4.2 (ii), by using the facts $\hat{f}(-\xi) = \widehat{f(-\cdot)}(\xi)$ and $\chi_j(-\xi) = \chi_j(\xi)$ $(j = 1, \infty)$, we see that $\Gamma P_j = P_j \Gamma$ $(j = 1, \infty)$. Based on these relations, Lemma 4.1 (ii) and Lemma 4.2 (ii) can be proved by a straightforward computation.

By Lemma 4.1 and Lemma 4.2, one can consider (4.1)-(4.2) in space of functions satisfying $\{\Gamma u_1, \Gamma u_\infty\} = \{u_1, u_\infty\}$, i.e., $u_j = {}^\top(\phi_j(x,t), w_j(x,t)) = {}^\top(\phi_j(-x,t), -w_j(-x,t))$ $(j = 1, \infty)$.

We look for a time periodic solution $\{u_1, u_\infty\}$ for the system (4.1)-(4.2). To solve the time periodic problem for (4.1)-(4.2), we introduce solution operators for the following linear problems:

$$\left\{ \begin{array}{l} \partial_t u_1 + Au_1 = F_1, \\ u|_{t=0} = u_{01}, \end{array} \right. \tag{4.3}$$

and

$$\left\{ \begin{array}{l} \partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \\ u|_{t=0} = u_{0\infty}, \end{array} \right. \tag{4.4}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01}, u_{0\infty}, F_1$ and F_∞ are given functions.

To formulate the time periodic problem, we denote by $S_1(t)$ the solution operator for (4.3) with $F_1 = 0$, and by $\mathscr{S}_1(t)$ the solution operator for (4.3) with $u_{01} = 0$. We also denote by $S_{\infty,\tilde{u}}(t)$ the solution operator for (4.4) with $F_\infty = 0$ and by $\mathscr{S}_{\infty,\tilde{u}}(t)$ the solution operator for (4.4) with $u_{0\infty} = 0$. (The precise definition of these operators will be given later.)

If $\{u_1, u_\infty\}$ satisfies (4.1)-(4.2), then $u_1(t)$ and $u_\infty(t)$ are written as

$$\begin{aligned} u_1(t) &= S_1(t)u_1(0) + \mathscr{S}_1(t)[F_1(u,g)], && (4.5)\\ u_\infty(t) &= S_{\infty,u}u_\infty(0) + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)] && (4.6)\end{aligned}$$

with $u = u_1 + u_\infty$.

Suppose that $\{u_1, u_\infty\}$ is a T-time periodic solution of (4.5)-(4.6). Then, since $u_1(T) = u_1(0)$ and $u_\infty(T) = u_\infty(0)$, we see that

$$\begin{cases} (I - S_1(T))u_1(0) = \mathscr{S}_1(T)[F_1(u,g)],\\ (I - S_{\infty,u}(T))u_\infty(0) = \mathscr{S}_{\infty,u}(T)[F_\infty(u,g)],\\ u = u_1 + u_\infty. \end{cases}$$

Therefore if $(I - S_1(T))$ and $(I - S_{\infty,u}(T))$ are invertible in a suitable sense, then one obtains

$$\begin{cases} u_1(t) = S_1(t)u_{01} + \mathscr{S}_1(t)[F_1(u,g)],\\ u_\infty(t) = S_{\infty,u}(t)u_{0\infty} + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)] \end{cases} \tag{4.7}$$

with

$$\begin{cases} u = u_1 + u_\infty,\\ u_{01} = (I - S_1(T))^{-1}\mathscr{S}_1(T)[F_1(u,g)],\\ u_{0\infty} = (I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)]. \end{cases} \tag{4.8}$$

Therefore, to obtain a T-time periodic solution of (4.1)-(4.2), we look for a pair of functions $\{u_1, u_\infty\}$ satisfying (4.7)-(4.8). We will investigate the solution operators $S_1(t)$, $\mathscr{S}_1(t)$, $S_{\infty,u}(t)$ and $\mathscr{S}_{\infty,u}(t)$ in sections 5 and 6.

Next, we introduce some lemmas which will be used in the proof of Theorem 3.1. We first derive some inequalities for the low frequency part.

Lemma 4.3. (i) *Let k be a nonnegative integer. Then P_1 is a bounded linear operator from L^2 to H^k. In fact, it holds that*

$$\|\nabla^k P_1 f\|_{L^2} \le C_k \|f\|_{L^2} \qquad (f \in L^2).$$

As a result, for any $2 \le p \le \infty$, P_1 is bounded from L^2 to L^p.

(ii) *Let k be a nonnegative integer. Then there hold the estimates*

$$\|\nabla^k f_1\|_{L^2} + \|f_1\|_{L^p} \le C_{k,p}\|f_1\|_{L^2} \quad (f \in L^2_{(1)}),$$

where $2 \le p \le \infty$,

$$\begin{aligned} \|f_1\|_{H_1^k} &\le C_k \|f_1\|_{L_1^2} \quad (f \in L^2_{(1),1}), \\ \|\nabla f_1\|_{H_1^k} &\le C_k \|f_1\|_{H^1_{(1),1}} \quad (f \in H^1_{(1),1}), \\ \|f_1\|_{L_1^2} + \|f_1\|_{H^1_{(1),1}} &\le C_k \|f_1\|_{L_1^1} \quad (f \in L^2_{(1)} \cap L_1^1). \end{aligned}$$

Proof. The boundedness of P_1 from L^2 to H^k can be easily verified by using the Plancherel theorem, since $\operatorname{supp} \widehat{P_1 f} \subset \{\xi; |\xi| \le r_\infty\}$; and, then, the boundedness of P_1 from L^2 to L^p with $2 \le p \le \infty$ follows from the Sobolev inequality.

As for (ii), the first inequality can be obtained as in the same reason for (i). The second inequality is obtained by (i) and the following computation. For $0 \le |\alpha| \le k$ and $f_1 \in L^2_{(1),1}$, we see that

$$\begin{aligned} \|x \partial_x^\alpha f_1\|_{L^2} &= (2\pi)^{-\frac{n}{2}} \|\partial_\xi (\xi^\alpha \hat{f}_1)\|_{L^2(|\xi| \le r_\infty)} \\ &\le C\big\{ \||\xi|^{(|\alpha|-1)_+} \hat{f}_1\|_{L^2(|\xi|\le r_\infty)} + \||\xi|^{|\alpha|} \partial_\xi \hat{f}_1\|_{L^2(|\xi|\le r_\infty)} \big\} \\ &\le C\big\{ \|\hat{f}_1\|_{L^2(|\xi|\le r_\infty)} + \|\partial_\xi \hat{f}_1\|_{L^2(|\xi|\le r_\infty)} \big\} \\ &\le C \|f_1\|_{L_1^2}. \end{aligned}$$

The third inequality follows from the second inequality with f_1 replaced by ∇f_1, since, by the first inequality, we have $\|\nabla f_1\|_{L_1^2} \le C \|f_1\|_{H^1_{(1),1}}$. As for the last inequality, we have

$$\begin{aligned} \|f_1\|^2_{L_1^2} &= (2\pi)^{-n} \big\{ \|\hat{f}_1\|^2_{L^2(|\xi|\le r_\infty)} + \|\partial_\xi \hat{f}_1\|^2_{L^2(|\xi|\le r_\infty)} \big\} \\ &\le C \big\{ \sup_{|\xi| \le r_\infty} (|\hat{f}_1(\xi)| + |\partial_\xi \hat{f}_1(\xi)|) \big\}^2 \\ &\le C \|f_1\|^2_{L_1^1}, \end{aligned}$$

and, likewise, we can obtain $\|f_1\|_{H^1_{(1),1}} \le C \|f_1\|_{L_1^1}$. This completes the proof. □

As for the high frequency part, we have the following inequalities.

Lemma 4.4. (i) *Let* k *be a nonnegative integer. Then* P_∞ *is a bounded linear operator on* H^k.

(ii) *There hold the inequalities*

$$\begin{aligned} \|P_\infty f\|_{L^2} &\le C \|\nabla f\|_{L^2} \quad (f \in H^1), \\ \|f_\infty\|_{L^2} &\le C \|\nabla f_\infty\|_{L^2} \quad (f_\infty \in H^1_{(\infty)}). \end{aligned}$$

Lemma 4.4 (i) immediately follows from the definition of P_∞ by using the Plancherel theorem; and, similarly, inequalities in (ii) can be easily seen since supp $\widehat{P_\infty f} \subset \{\xi; |\xi| \geq r_1\}$ and supp $\hat{f}_\infty \subset \{\xi; |\xi| \geq r_1\}$ for $f_\infty \in H^1_{(\infty)}$. We omit the proof.

We next introduce a cut-off function ζ_R. Let $\tilde{\zeta} \in C^\infty([0,\infty))$ be a nonincreasing function satisfying

$$0 \leq \tilde{\zeta} \leq 1, \quad \tilde{\zeta}(r) = \begin{cases} 1 & (|r| \leq 1), \\ 0 & (|r| > 2). \end{cases}$$

Set

$$\zeta_R(x) = \tilde{\zeta}\left(\frac{|x|}{R}\right). \tag{4.9}$$

Then, $\zeta_R \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \zeta_R \leq 1$, and

$$\zeta_R(x) = \begin{cases} 1 & (|x| \leq R), \\ 0 & (|x| > 2R). \end{cases}$$

An elementary computation gives the following lemma.

Lemma 4.5. *Let ζ_R be defined in* (4.9). *For a nonnegative integer ℓ and a multi-index α, there holds*

$$||x|^\ell \partial_x^\alpha (\zeta_R(x))| \leq C_{\alpha,\ell} R^{\ell - |\alpha|} \qquad (x \in \mathbb{R}^n).$$

If $|\alpha| \geq 1$, then it holds that

$$\mathrm{supp}(\partial_x^\alpha \zeta_R) \subset \{R \leq |x| \leq 2R\}.$$

Lemma 4.6. *Let χ_1 be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then for a nonnegative integer ℓ, there holds*

$$\||x|^\ell \chi_1 * f\|_{L^2} \leq C\{\||x|^\ell \chi_1\|_{L^1} \|f\|_{L^2} + \|\chi_1\|_{L^1} \||x|^\ell f\|_{L^2}\} \qquad (f \in L^2_\ell).$$

Here C is a positive constant depending only on ℓ.

Proof. Let χ_1 be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then

$$\begin{aligned} ||x|^\ell \chi_1 * f| &\leq |x|^\ell \int_{\mathbb{R}^n} |\chi_1(x-y) f(y)| dy \\ &\leq C \int_{\mathbb{R}^n} |x-y|^\ell |\chi_1(x-y)||f(y)| dy + C \int_{\mathbb{R}^n} |\chi_1(x-y)||y|^\ell |f(y)| dy. \end{aligned}$$

Therefore, the Young inequality gives

$$\||x|^\ell \chi_1 * f\|_{L^2} \leq C\{\||x|^\ell \chi_1\|_{L^1} \|f\|_{L^2} + \|\chi_1\|_{L^1} \||x|^\ell f\|_{L^2}\} \qquad (f \in L^2_\ell).$$

This completes the proof. $\square$

Lemma 4.7. *Let $\ell \in \mathbb{N}$. Then there exists a positive constant C depending only on ℓ such that*

$$\||x|^\ell \nabla f_\infty\|_{L^2}^2 \geq \frac{r_1^2}{2}\||x|^\ell f_\infty\|_{L^2}^2 - C\||x|^{\ell-1} f_\infty\|_{L^2}^2 \quad (f_\infty \in H^1_{(\infty),\ell}).$$

Proof. By the multinomial theorem, we have

$$\begin{aligned}
\||x|^\ell \nabla f_\infty\|_{L^2}^2 &= \int_{\mathbb{R}^n} |x_1^2 + \cdots + x_n^2|^\ell |\nabla f_\infty|^2 dx \\
&= \int_{\mathbb{R}^n} \sum_{|\alpha|=\ell} \binom{|\alpha|}{\alpha} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} |\nabla f_\infty|^2 dx \\
&= \sum_{|\alpha|=\ell} \binom{|\alpha|}{\alpha} \||x_1^{\alpha_1} \cdots x_n^{\alpha_n}|\nabla f_\infty\|_{L^2}^2,
\end{aligned}$$

where

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}, \quad \alpha! = \alpha_1! \cdots \alpha_n!.$$

Therefore, it follows from the Plancherel theorem that

$$\begin{aligned}
\||x|^\ell \nabla f_\infty\|_{L^2}^2 &\geq \frac{1}{2} \sum_{|\alpha|=\ell} \binom{|\alpha|}{\alpha} \|\nabla((x_1^{\alpha_1} \cdots x_n^{\alpha_n}) f_\infty)\|_{L^2}^2 - C\||x|^{\ell-1} f_\infty\|_{L^2}^2 \\
&= \frac{1}{2}(2\pi)^{-n} \sum_{|\alpha|=\ell} \binom{|\alpha|}{\alpha} \|\xi(\partial_\xi^{\alpha_1} \cdots \partial_\xi^{\alpha_n}) \hat{f}_\infty\|_{L^2}^2 - C\||x|^{\ell-1} f_\infty\|_{L^2}^2.
\end{aligned}$$

Since $\operatorname{supp} \hat{f}_\infty \subset \{|\xi| \geq r_1\}$, we see that

$$\begin{aligned}
\||x|^\ell \nabla f_\infty\|_{L^2}^2 &\geq \frac{r_1^2}{2}(2\pi)^{-n} \sum_{|\alpha|=\ell} \binom{|\alpha|}{\alpha} \|(\partial_\xi^{\alpha_1} \cdots \partial_\xi^{\alpha_n}) \hat{f}_\infty\|_{L^2}^2 - C\||x|^{\ell-1} f_\infty\|_{L^2}^2 \\
&\geq \frac{r_1^2}{2}\||x|^\ell f_\infty\|_{L^2}^2 - C\||x|^{\ell-1} f_\infty\|_{L^2}^2.
\end{aligned}$$

This completes the proof. □

5 Properties of $S_1(t)$ and $\mathscr{S}_1(t)$

In this section we investigate $S_1(t)$ and $\mathscr{S}_1(t)$ and establish an estimate for a solution u_1 of

$$\partial_t u_1 + A u_1 = F_1 \tag{5.1}$$

satisfying $u_1(0) = u_1(T)$.

We consider the restriction of A on $L^2_{(1)}$. By Lemma 4.3 (ii), we see that $\|Au_1\|_{L^2} \leq C\|u_1\|_{L^2}$ for $u_1 \in L^2_{(1)}$.

Let

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top \xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top \xi \end{pmatrix} \quad (\xi \in \mathbb{R}^n).$$

Then, since $Au_1 = \mathcal{F}^{-1}\hat{A}_\xi \hat{u}_1$, we see that $\operatorname{supp} \hat{A}_\xi \hat{u}_1 \subset \{\xi; |\xi| \leq r_\infty\}$ for $u_1 \in L^2_{(1)}$. Therefore, the restriction of A on $L^2_{(1)}$ is a bounded linear operator on $L^2_{(1)}$.

We denote by A_1 the restriction of A on $L^2_{(1)}$. Then A_1 is a bounded linear operator on $L^2_{(1)}$ and it satisfies $\|A_1 u_1\|_{L^2} \leq C\|u_1\|_{L^2}$ for $u_1 \in L^2_{(1)}$ and

$$A_1 u_1 = \mathcal{F}^{-1}\hat{A}_\xi \mathcal{F} u_1 \quad (u_1 \in L^2_{(1)}).$$

Furthermore, $-A_1$ generates a uniformly continuous semigroup $S_1(t) = e^{-tA_1}$ that is given by

$$S_1(t)u_1 = \mathcal{F}^{-1} e^{-t\hat{A}_\xi}\mathcal{F}u_1 \quad (u_1 \in L^2_{(1)});$$

and it holds that $S_1(t)$ satisfies $S_1(\cdot)u_1 \in C^1([0,\infty); L^2_{(1)})$ for each $u_1 \in L^2$ and

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \,(= -AS_1(t)u_1), \;\; S_1(0)u_1 = u_1 \;\text{ for }\; u_1 \in L^2_{(1)},$$

$$\|\partial_t^k S_1(t)u_1\|_{L^2} \leq \|A_1\|^k \|u_1\|_{L^2} \;\text{ for }\; u_1 \in L^2_{(1)}, \;\; t \geq 0, \;\; k = 0, 1,$$

where $\|A_1\|$ denotes the operator norm of A_1. The estimates can be obtained by the energy method based on the relation

$$(Au, u) = \nu\|\nabla u\|^2_{L^2} + \tilde{\nu}\|\nabla \cdot u\|^2_{L^2}.$$

We also define the operator $\mathscr{S}_1(t)$ by

$$\mathscr{S}_1(t)[F_1] = \int_0^t S_1(t-\tau)F_1(\tau)\, d\tau$$

for $F_1 \in C([0,T]; L^2_{(1)})$. It follows that

$$\mathscr{S}_1(t)[F_1] = \mathcal{F}^{-1}\Big[\int_0^t e^{-(t-\tau)\hat{A}_\xi}\hat{F}_1(\tau)\, d\tau\Big],$$

$\mathscr{S}_1(\cdot)[F_1] \in C^1([0,T]; L^2_{(1)})$ for each $F_1 \in C([0,T]; L^2_{(1)})$ and

$$\partial_t \mathscr{S}_1(t)[F_1] + A_1\mathscr{S}_1(t)[F_1] = F_1(t), \;\; \mathscr{S}_1(0)[F_1] = 0,$$

$$\|\partial_t^k \mathscr{S}_1(t)[F_1]\|_{L^2} \leq C\|F_1\|_{C([0,T];L^2)} \;\text{ for }\; t \in [0,T], \;\; k = 0, 1,$$

where $C = C(T) > 0$.

We next show that A_1 has similar properties on $H^1_{(1),1}$.

Proposition 5.1. (i) *A_1 is a bounded linear operator on $H^1_{(1),1}$ and $S_1(t) = e^{-tA_1}$ is a uniformly continuous semigroup on $H^1_{(1),1}$. Furthermore, it holds that $S_1(\cdot)u_1 \in C^1([0,T']; H^1_{(1),1})$, $\partial_t S_1(\cdot)u_1 \in C([0,T']; L^2_1)$ for each $u_1 \in H^1_{(1),1}$ and all $T' > 0$,*

$$\|\partial_t^k S_1(t)u_1\|_{H^1_{(1),1}} \le C\|u_1\|_{H^1_{(1),1}} \quad for \;\; u_1 \in H^1_{(1),1}, \;\; t \in [0,T'], \;\; k = 0,1,$$

and

$$\|\partial_t S_1(t)u_1\|_{L^2_1} \le C\|u_1\|_{H^1_{(1),1}} \quad for \;\; u_1 \in H^1_{(1),1}, \;\; t \in [0,T'],$$

where $T' > 0$ is any given positive number and C is a positive constant depending on T'.

(ii) *$\mathscr{S}_1(\cdot)$ satisfies that $\mathscr{S}_1(\cdot)[F_1] \in C^1([0,T]; H^1_{(1),1})$ for each $F_1 \in C([0,T]; H^1_{(1),1})$ and*

$$\|\partial_t^k \mathscr{S}_1(t)[F_1]\|_{H^1_{(1),1}} \le C\|F_1\|_{C([0,T];H^1_{(1),1})} \; for \;\; F_1 \in C([0,T]; H^1_{(1),1}),\, t \in [0,T],\, k = 0,1,$$

where C is a positive constant depending on T. If, in addition, $F_1 \in C([0,T]; L^2_1)$, then $\partial_t \mathscr{S}_1(\cdot)[F_1] \in C([0,T]; L^2_1)$ and

$$\|\partial_t \mathscr{S}_1(t)[F_1]\|_{L^2_1} \le C\|F_1\|_{C([0,T];L^2_1)} \quad for \; F_1 \in C([0,T]; H^1_{(1),1} \cap L^2_1), \;\; t \in [0,T],$$

where C is a positive constant depending on T.

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')[F_1] = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ and $F_1 \in C([0,T]; X)$, where $X = L^2_{(1)}$, $H^1_{(1),1}$.

(iv) *It holds that $\Gamma S_1(t) = S_1(t)\Gamma$ and $\Gamma \mathscr{S}_1(t) = \mathscr{S}_1(t)\Gamma$. Consequently, the assertions* (i)–(iii) *above hold with function spaces $L^2_{(1)}$, $H^1_{(1),1}$ and L^2_1 replaced by $(L^2_{(1)})_{sym}$, $(H^1_{(1),1})_{sym}$ and $(L^2_1)_{sym}$, respectively.*

The proof of Proposition 5.1 will be given later.

We next investigate invertibility of $I - S_1(T)$.

Proposition 5.2. *Let $F_1 = {}^\top(F_1^0(x), \tilde{F}_1(x)) \in L^2_{(1)} \cap L^1_1$ and suppose that $\tilde{F}_1(-x) = -\tilde{F}_1(x)$ for $x \in \mathbb{R}^n$. Then there uniquely exists $u \in H^1_{(1),1}$ that satisfies*

$$(I - S_1(T))u = F_1 \quad and \quad \|u\|_{H^1_{(1),1}} \le C\|F_1\|_{L^1_1}. \tag{5.2}$$

Furthermore, if $\Gamma F_1 = F_1$, then $\Gamma u = u$.

The proof of Proposition 5.2 will be given later.

In view of Proposition 5.2, $I - S_1(T)$ has a bounded inverse $(I - S_1(T))^{-1}$: $(L^2_{(1)} \cap L^1_1)_{sym} \to (H^1_{(1),1})_{sym}$ and it holds that

$$\|(I - S_1(T))^{-1}F_1\|_{H^1_{(1),1}} \le C\|F_1\|_{L^1_1}.$$

Using Proposition 5.1 (ii) and Proposition 5.2, we can obtain the following estimate for $\mathscr{S}_1(T)(I - S_1(T))^{-1}$.

Proposition 5.3. *For $F_1 \in C([0,T]; (L^2_{(1)} \cap L^1_1)_{sym})$, it holds that $\mathscr{S}_1(T)[(I - S_1(T))^{-1}F_1] \in (H^1_{(1),1})_{sym}$ and*

$$\|\mathscr{S}_1(T)[(I - S_1(T))^{-1}F_1]\|_{H^1_{(1),1}} \leq C\|F_1\|_{C([0,T];L^1_1)}.$$

We are now in a position to give an estimate for a solution of (5.1) satisfying $u_1(0) = u_1(T)$.

Proposition 5.4. *Set*

$$u_1(t) = S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}F_1] + \mathscr{S}_1(t)[F_1] \tag{5.3}$$

for $F_1 = {}^\top(F_1^0(x,t), \tilde{F}_1(x,t)) \in C([0,T]; (L^2_{(1)} \cap L^1_1)_{sym})$. Then u_1 is a solution of (5.1) in $\mathscr{Y}_1(0,T)$ satisfying $u_1(0) = u_1(T)$ and

$$\|u_1\|_{\mathscr{Y}_1(0,T)} \leq C\|F_1\|_{C([0,T];L^1_1)}.$$

Proof. We find from Proposition 5.1 (iii) and Proposition 5.2 that $u_1(0) = u_1(T)$. As for the estimate for u_1, the first term on the right-hand side of (5.3) is estimated by using Proposition 5.1 (i) and Proposition 5.3. The second term on the right-hand side of (5.3) is estimated by using Proposition 5.1 (ii) and Lemma 4.3 (ii). Hence, we obtain the desired estimate. This completes the proof. □

In the rest of this section we will give proofs of Proposition 5.1 and Proposition 5.2.

Lemma 5.5. ([9]) (i) *The set of all eigenvalues of $-\hat{A}_\xi$ consists of $\lambda_j(\xi)$ $(j = 1, \pm)$, where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\frac{1}{2}(\nu + \tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < \frac{2\gamma}{\nu + \tilde{\nu}}$, then

$$\mathrm{Re}\,\lambda_\pm = -\frac{1}{2}(\nu + \tilde{\nu})|\xi|^2, \quad \mathrm{Im}\,\lambda_\pm = \pm\gamma|\xi|\sqrt{1 - \frac{(\nu + \tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *$e^{-t\hat{A}_\xi}$ has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *is eigenprojections for* $\lambda_j(\xi)\,(j=1,\pm)$*, and* $\Pi_j(\xi)\,(j=1,\pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \frac{\xi^\top \xi}{|\xi|^2} \end{pmatrix},$$
$$\Pi_+(\xi) = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_- & i\gamma^\top \xi \\ i\gamma\xi & \lambda_+ \frac{\xi^\top \xi}{|\xi|^2} \end{pmatrix},$$
$$\Pi_-(\xi) = -\frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_+ & i\gamma^\top \xi \\ i\gamma\xi & \lambda_- \frac{\xi^\top \xi}{|\xi|^2} \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$*, then there exist a constant* $C > 0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \le C\,(j=1,\pm) \tag{5.4}$$

hold for $|\xi| \le r_\infty$*.*

Hereafter we fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (5.4) in Lemma 5.5 holds for $|\xi| \le r_\infty$.

Lemma 5.6. *Let* α *be a multi-index. Then the following estimates hold true uniformly for* ξ *with* $|\xi| \le r_\infty$ *and* $t \in [0,T]$*.*

(i) $|\partial_\xi^\alpha \lambda_1| \le C_\alpha |\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha \lambda_\pm| \le C_\alpha |\xi|^{1-|\alpha|}$ $(|\alpha| \ge 0)$.

(ii) $|(\partial_\xi^\alpha \Pi_1)\hat{F}| \le C_\alpha |\xi|^{-|\alpha|} |\hat{\tilde{F}}_1|$, $|(\partial_\xi^\alpha \Pi_\pm)\hat{F}| \le C_\alpha |\xi|^{-|\alpha|} |\hat{F}_1|$ $(|\alpha| \ge 0)$*, where* $F = {}^\top(F_1^0, \tilde{F}_1)$*.*

(iii) $|\partial_\xi^\alpha (e^{\lambda_1 t})| \le C_{\alpha,T} |\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1)$.

(iv) $|\partial_\xi^\alpha (e^{\lambda_\pm t})| \le C_{\alpha,T} |\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}| \le C_{\alpha,T}(|\xi|^{1-|\alpha|}|\hat{F}_1^0| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|)$ $(|\alpha| \ge 1)$*, where* $F = {}^\top(F_1^0, \tilde{F}_1)$*.*

(vi) $|\partial_\xi^\alpha (I - e^{\lambda_1 t})^{-1}| \le C_{\alpha,T} |\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0)$.

(vii) $|\partial_\xi^\alpha (I - e^{\lambda_\pm t})^{-1}| \le C_{\alpha,T} |\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0)$.

Lemma 5.6 can be verified by direct computations based on Lemma 5.5.

Let us prove Proposition 5.1.

Proof of Proposition 5.1. We see from Lemma 4.3 (ii) that

$$\|A_1 u_1\|_{H^1_{(1),1}} \le C\|\nabla u_1\|_{H^1_1} \le C\|u_1\|_{H^1_{(1),1}} \quad (u_1 \in H^1_{(1),1}),$$

and so, A_1 is bounded on $H^1_{(1),1}$. It then follows that $S_1(\cdot)u_1 \in C^1([0,T']; H^1_{(1),1})$ for each $u_1 \in H^1_{(1),1}$ and

$$\|\partial_t^k S_1(t) u_1\|_{H^1_{(1),1}} \le C\|u_1\|_{H^1_{(1),1}} \quad \text{for} \;\; u_1 \in H^1_{(1),1}, \;\; t \in [0,T'], \;\; k = 0,1,$$

where $T' > 0$ is any given positive number and C is a positive constant depending on T'. Since $\|A_1 u_1\|_{L^2_1} \le C\|\nabla u_1\|_{H^1_1} \le C\|u_1\|_{H^1_{(1),1}}$ for $u_1 \in H^1_{(1),1}$ by Lemma 4.3 (ii), we see from the relation $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ that $\partial_t S_1(\cdot)u_1 \in C([0,T']; L^2_1)$ and

$$\|\partial_t S_1(t)u_1\|_{L^2_1} \le \|S_1(t)u_1\|_{H^1_{(1),1}} \le C\|u_1\|_{H^1_{(1),1}}.$$

The assertion (ii) follows from (i) and the relation $\partial_t \mathscr{S}_1(t)[F_1] = -A_1\mathscr{S}_1(t)[F_1] + F_1(t)$. The assertion (iii) easily follows from the definitions of $S_1(t)$ and $\mathscr{S}_1(t)$. As for (iv), we observe that $\Gamma A_1 = A_1\Gamma$, from which we find that $\Gamma S_1(t) = S_1(t)\Gamma$, and hence, $\Gamma\mathscr{S}_1(t) = \mathscr{S}_1(t)\Gamma$. This completes the proof. $\square$

Let us finally prove Proposition 5.2.

Proof of Proposition 5.2. We define a function u

$$u = \mathcal{F}^{-1}(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1$$

for $F_1 = {}^\top(F_1^0, \tilde{F}_1)$. It suffices to show that $\|u\|_{H^1_{(1),1}} \le C\|F_1\|_{L^1_1}$. By the Plancherel theorem, we see that

$$\begin{aligned}\|u\|_{L^2_{(1)}} &= (2\pi)^{-\frac{n}{2}}\|(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} \\ &\le (2\pi)^{-\frac{n}{2}}\{\|(I - e^{T\lambda_1})^{-1}\Pi_1\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} + \|(I - e^{T\lambda_+})^{-1}\Pi_+\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} \\ &\quad + \|(I - e^{T\lambda_-})^{-1}\Pi_-\hat{F}_1\|_{L^2(|\xi|\le r_\infty)}\} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Observe that $\Pi_1\hat{F}_1$ depends only on $\hat{\tilde{F}}_1$ but not on $\hat{F}_1^0$.

By using Lemma 5.5, Lemma 5.6 and the fact $\hat{\tilde{F}}_1(0) = 0$, we see that

$$\begin{aligned}I_1 &\le C\left\|\frac{1}{|\xi|^2}\hat{\tilde{F}}_1\right\|_{L^2(|\xi|\le r_\infty)} \\ &\le C\left\|\frac{1}{|\xi|}\right\|_{L^2(|\xi|\le r_\infty)}\|x\tilde{F}_1\|_{L^1}.\end{aligned}$$

Since

$$\left\|\frac{1}{|\xi|}\right\|_{L^2(|\xi|\le r_\infty)} < +\infty$$

for $n \ge 3$, we find that

$$I_1 \le C\|x\tilde{F}_1\|_{L^1}.$$

Similarly, we can obtain $I_2 + I_3 \le C\|F_1\|_{L^1}$, and hence, we see that

$$\|u\|_{L^2_{(1)}} \le C\{\|F_1\|_{L^1} + \|x\tilde{F}_1\|_{L^1}\}. \tag{5.5}$$

Next, by the Plancherel theorem, it follows that

$$
\begin{aligned}
\|x\nabla u\|_{L^2_{(1)}} &= (2\pi)^{-\frac{n}{2}} \left\|(i\partial_\xi)\left(i\xi(I-e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\right)\right\|_{L^2(|\xi|\le r_\infty)} \\
&\le C\Big\{\|(I-e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} + \left\|i\xi\partial_\xi\left((I-e^{-T\hat{A}_\xi})^{-1}\right)\hat{F}_1\right\|_{L^2(|\xi|\le r_\infty)} \\
&\quad +\|i\xi(I-e^{-T\hat{A}_\xi})^{-1}\partial_\xi\hat{F}_1\|_{L^2(|\xi|\le r_\infty)}\Big\}.
\end{aligned}
$$

The first term on right-hand side has already been estimated and it is bounded by the right-hand side of (5.5). As for the second and third terms on the right-hand side, similarly to above, one can find from Lemma 5.6 that

$$
\left\|i\xi\partial_\xi\left((I-e^{-T\hat{A}_\xi})^{-1}\right)\hat{F}_1\right\|_{L^2(|\xi|\le r_\infty)} + \|i\xi(I-e^{-T\hat{A}_\xi})^{-1}\partial_\xi\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} \le C\|F_1\|_{L^1_1}.
$$

We thus obtain

$$
\|x\nabla u\|_{L^2_{(1)}} \le C\|F_1\|_{L^1_1}.
$$

Finally, we see from Proposition 5.1 (iv) that if $\Gamma F_1 = F_1$, then $\Gamma u = u$. This completes the proof. □

6 Properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$

In this section we investigate $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$.

Let us consider the following initial value problem

$$
\begin{cases} \partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \\ u|_{t=0} = u_{0\infty}, \end{cases} \tag{6.1}
$$

where

$$
F_\infty = \begin{pmatrix} F^0_\infty \\ \tilde{F}_\infty \end{pmatrix},\ P_\infty(B[\tilde{u}]u) = \begin{pmatrix} \gamma P_\infty(\tilde{w}\cdot\nabla\phi) \\ 0 \end{pmatrix},\ u = \begin{pmatrix} \phi \\ w \end{pmatrix},\ \tilde{u} = \begin{pmatrix} \tilde{\phi} \\ \tilde{w} \end{pmatrix}.
$$

We begin with the solvability of (6.1). Let us first consider the following system:

$$
\begin{cases} \partial_t\phi + \gamma(\tilde{w}\cdot\nabla\phi) = f^0, \\ \phi|_{t=0} = \phi_0. \end{cases} \tag{6.2}
$$

Lemma 6.1. ([6, Theorem 4.1].) Let $n \ge 3$ and let m be an integer satisfying $m \ge [\frac{n}{2}]+1$. Set $k = m-1$ or m. Assume that $\tilde{w} \in C([0,T']; H^m)\cap L^2(0,T'; H^{m+1})$, $f^0 \in L^2(0,T'; H^k)$ and $\phi_0 \in H^k$. Here T' is a given positive number. Then (6.2) has a unique solution $\phi \in C([0,T']; H^k)$ and ϕ satisfies

$$
\|\phi(t)\|^2_{H^k} \le C\left\{\|\phi_0\|^2_{H^k} + \int_0^t \|\tilde{w}\|_{H^{m+1}}\|\phi\|^2_{H^k}ds + \int_0^t \|f^0\|_{H^k}\|\phi\|_{H^k}ds\right\}
$$

and

$$\|\phi(t)\|_{H^k}^2 \le Ce^{C\int_0^t(1+\|\tilde{w}\|_{H^{m+1}})ds}\left\{\|\phi_0\|_{H^k}^2 + \int_0^t \|f^0\|_{H^k}^2 ds\right\}$$

for $t \in [0, T']$. Moreover, the solution is unique in $C([0, T']; H^1)$.

We next consider the following system:

$$\begin{cases} \partial_t \phi_\infty + \gamma P_\infty(\tilde{w} \cdot \nabla \phi_\infty) = F_\infty^0, \\ \phi_\infty|_{t=0} = \phi_{0\infty}. \end{cases} \tag{6.3}$$

Note that (6.3) is rewritten as

$$\partial_t \phi_\infty + \gamma(\tilde{w} \cdot \nabla \phi_\infty) = F_\infty^0 + \gamma P_1(\tilde{w} \cdot \nabla \phi_\infty). \tag{6.4}$$

As for the solvability of (6.3), we have the following lemma.

Lemma 6.2. *Let $n \ge 3$ and let m be an integer satisfying $m \ge [\frac{n}{2}]+1$. Set $k = m-1$ or m. Assume that $\tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1})$, $F_\infty^0 \in L^2(0, T'; H^k_{(\infty)})$ and $\phi_{0\infty} \in H^k_{(\infty)}$. Here T' is a given positive number. Then (6.3) has a unique solution $\phi_\infty \in C([0, T']; H^k_{(\infty)})$ and ϕ_∞ satisfies*

$$\begin{aligned} \|\phi_\infty(t)\|_{H^k}^2 \;\le\; & C\Big\{\|\phi_{0\infty}\|_{H^k}^2 + \int_0^t (\|\tilde{w}\|_{H^{m+1}} + \|\tilde{w}\|_{H^m}^2)\|\phi_\infty\|_{H^k}^2 ds \\ & + \int_0^t \|F_\infty^0\|_{H^k}\|\phi\|_{H^k} ds\Big\} \end{aligned}$$

and

$$\|\phi_\infty(t)\|_{H^k}^2 \le Ce^{C\int_0^t(1+\|\tilde{w}\|_{H^{m+1}}+\|\tilde{w}\|_{H^m}^2)ds}\left\{\|\phi_{0\infty}\|_{H^k}^2 + \int_0^t \|F_\infty^0\|_{H^k}^2 ds\right\}$$

for $t \in [0, T']$.

Proof. We define $\{\phi_\infty^{(p)}\}_{p=0}^\infty$ as follows. For $p = 0$, $\phi_\infty^{(0)}$ is the solution of

$$\begin{cases} \partial_t \phi_\infty^{(0)} + \gamma(\tilde{w} \cdot \nabla \phi_\infty^{(0)}) = F_\infty^0, \\ \phi_\infty^{(0)}|_{t=0} = \phi_{0\infty}. \end{cases} \tag{6.5}$$

For $p \ge 1$, $\phi_\infty^{(p)}$ is the solution of

$$\begin{cases} \partial_t \phi_\infty^{(p)} + \gamma(\tilde{w} \cdot \nabla \phi_\infty^{(p)}) = F_\infty^0 + \gamma P_1(\tilde{w} \cdot \nabla \phi_\infty^{(p-1)}), \\ \phi_\infty^{(p)}|_{t=0} = \phi_{0\infty}. \end{cases} \tag{6.6}$$

By Lemma 4.3 (i), we have

$$\|P_1(\tilde{w} \cdot \nabla \phi_\infty)\|_{H^m} \le C\|\tilde{w}\|_{L^\infty}\|\nabla \phi_\infty\|_{L^2} \le C\|\tilde{w}\|_{H^m}\|\phi_\infty\|_{H^k} \tag{6.7}$$

since $m \geq [\frac{n}{2}]+1 \geq 2$. In view of Lemma 6.1 and (6.7), we find by a standard argument that

$$\|\phi_\infty^{(p+1)}(t) - \phi_\infty^{(p)}(t)\|_{H^k}^2 \leq A_0 \frac{(A_1 t)^{p+1}}{(p+1)!} \quad (p \geq 0),$$

where

$$A_0 = Ce^{C\int_0^{T'}(1+\|\tilde{w}\|_{H^{m+1}}+\|\tilde{w}\|_{H^m}^2)\,d\tau}\Big\{\|\phi_0\|_{H^k}^2 + \int_0^{T'} \|F_\infty^0\|_{H^k}^2\,d\tau\Big\},$$

$$A_1 = C\|\tilde{w}\|_{C([0,T'];H^m)}^2 e^{C\int_0^{T'}(1+\|\tilde{w}\|_{H^{m+1}})\,d\tau}.$$

Therefore, one can see that $\phi_\infty^{(p)}$ converges in $C([0,T'];H^k)$ to a function $\phi_\infty \in C([0,T'];H^k)$ that satisfies

$$\begin{cases} \partial_t\phi_\infty + \gamma(\tilde{w}\cdot\nabla\phi_\infty) = F_\infty^0 + \gamma P_1(\tilde{w}\cdot\nabla\phi_\infty), \\ \phi_\infty|_{t=0} = \phi_{0\infty}. \end{cases} \tag{6.8}$$

In view of (6.4), we see that ϕ_∞ is a solution of (6.3). The estimates for ϕ_∞ follows from Lemma 6.1 and (6.7).

It remains to prove supp $\hat{\phi}_\infty(t) \subset \{|\xi| \geq r_1\}$ for $t \in [0,T']$. Let $\tilde{\chi}_\infty \in C_c^\infty(\mathbb{R}^n)$ with supp $\tilde{\chi}_\infty \subset \{|\xi| < r_1\}$. Let us consider the Fourier transform of (6.3):

$$\partial_t\hat{\phi}_\infty + \gamma\hat{\chi}_\infty\widehat{(\tilde{w}\cdot\nabla\phi_\infty)} = \hat{F}_\infty^0, \quad \hat{\phi}_\infty|_{t=0} = \hat{\phi}_{0\infty}.$$

Taking the inner product of this equation with $\tilde{\chi}_\infty^2\hat{\phi}_\infty$, we have $\frac{d}{dt}\|\tilde{\chi}_\infty\hat{\phi}_\infty\|_{L^2}^2 = 0$. We thus deduce that $\|\tilde{\chi}_\infty\hat{\phi}_\infty(t)\|_{L^2}^2 = \|\tilde{\chi}_\infty\hat{\phi}_{0\infty}\|_{L^2}^2 = 0$ for $t \in [0,T']$. It then follows that supp $\hat{\phi}_\infty(t) \subset \{|\xi| \geq r_1\}$ for $t \in [0,T']$. This completes the proof. □

We next consider the following system:

$$\begin{cases} \partial_t w_\infty - \nu\triangle w_\infty - \tilde{\nu}\nabla\mathrm{div} w_\infty = \tilde{F}_\infty, \\ w_\infty|_{t=0} = w_{0\infty}. \end{cases} \tag{6.9}$$

Lemma 6.3. (i) *Let $n \geq 3$ and let m be an integer satisfying $m \geq [\frac{n}{2}]+1$. Set $k = m-1$ or m. Assume that $\tilde{F}_\infty \in L^2(0,T';H^{k-1})$ and $w_{0\infty} \in H^k$. Here T' is a given positive number. Then (6.9) has a unique solution $w_\infty \in C([0,T'];H^k) \cap L^2(0,T';H^{k+1}) \cap H^1(0,T';H^{k-1})$ and*

$$\|w_\infty(t)\|_{H^k}^2 + \int_0^t \|w_\infty\|_{H^{k+1}}^2 + \|\partial_\tau w_\infty\|_{H^{k-1}}^2\,d\tau \leq C\left\{\|w_{0\infty}\|_{H^k}^2 + \int_0^t \|\tilde{F}_\infty\|_{H^{k-1}}^2 ds\right\}$$

for $t \in [0,T']$ with $C = C(T') > 0$.

(ii) *Assume, further, that $\tilde{F}_\infty \in L^2(0,T';H_{(\infty)}^{k-1})$ and $w_{0\infty} \in H_{(\infty)}^k$. Then the solution w_∞ satisfies*

$$w_\infty \in C([0,T'];H_{(\infty)}^k) \cap L^2(0,T';H_{(\infty)}^{k+1}) \cap H^1(0,T';H_{(\infty)}^{k-1}).$$

Lemma 6.3 (i) follows from standard theory of parabolic equation. The assertion (ii) can be proved in a similar manner to the proof of Lemma 6.2. We omit the details.

By using Lemma 6.2 and Lemma 6.3, we show the solvability of (6.1).

Proposition 6.4. *Let $n \geq 3$ and let m be an integer satisfying $m \geq [\frac{n}{2}]+1$. Set $k = m-1$ or m. Assume that*

$$\begin{aligned}
&\tilde{w} \in C([0,T']; H^m) \cap L^2(0,T'; H^{m+1}),\\
&u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},\\
&F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T'; H^k_{(\infty)} \times H^{k-1}_{(\infty)}).
\end{aligned}$$

Here T' is a given positive number. Then there exists a unique solution $u_\infty = {}^\top(\phi_\infty, w_\infty)$ of (6.1) satisfying

$$\phi_\infty \in C([0,T']; H^k_{(\infty)}),\ w_\infty \in C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)}) \cap H^1(0,T'; H^{k-1}_{(\infty)}).$$

Proof. We write (6.1) as

$$\left\{\begin{array}{l}
\partial_t\phi_\infty + \gamma P_\infty(\tilde{w}\cdot\nabla\phi_\infty) + \gamma \mathrm{div} w_\infty = F^0_\infty,\\
\partial_t w_\infty - \nu\triangle w_\infty - \tilde{\nu}\nabla\mathrm{div} w_\infty + \gamma\nabla\phi_\infty = \tilde{F}_\infty,\\
\phi_\infty|_{t=0} = \phi_{0\infty},\ w_\infty|_{t=0} = w_{0\infty}.
\end{array}\right. \tag{6.10}$$

We define $u^{(p)}_\infty = {}^\top(\phi^{(p)}_\infty, w^{(p)}_\infty)$ $(p = 0, 1, \cdots)$ as follows. For $p = 0$, $w^{(0)}_\infty = 0$ and $\phi^{(0)}_\infty$ is the solution of

$$\left\{\begin{array}{l}
\partial_t\phi^{(0)}_\infty + \gamma P_\infty(\tilde{w}\cdot\nabla\phi^{(0)}_\infty) = F^0_\infty,\\
\phi^{(0)}_\infty|_{t=0} = \phi_{0\infty}.
\end{array}\right. \tag{6.11}$$

For $p \geq 1$, $w^{(p)}_\infty$ is the solution of

$$\left\{\begin{array}{l}
\partial_t w^{(p)}_\infty - \nu\triangle w^{(p)}_\infty - \tilde{\nu}\nabla\mathrm{div} w^{(p)}_\infty = -\gamma\nabla\phi^{(p-1)}_\infty + \tilde{F}_\infty,\\
w^{(p)}_\infty|_{t=0} = w_{0\infty},
\end{array}\right. \tag{6.12}$$

and $\phi^{(p)}_\infty$ is the solution of

$$\left\{\begin{array}{l}
\partial_t\phi^{(p)}_\infty + \gamma P_\infty(\tilde{w}\cdot\nabla\phi^{(p)}_\infty) = -\gamma\mathrm{div} w^{(p)}_\infty + F^0_\infty,\\
\phi^{(p)}_\infty|_{t=0} = \phi_{0\infty}.
\end{array}\right. \tag{6.13}$$

As in the proof of Lemma 6.2, by using Lemma 6.2 and Lemma 6.3, one can show that $u^{(p)}_\infty = {}^\top(\phi^{(p)}_\infty, w^{(p)}_\infty)$ converges to a pair of function $u_\infty = {}^\top(\phi_\infty, w_\infty)$ in $C([0,T']; H^k_{(\infty)}) \times \Big[C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)})\Big]$. It is not difficult to see that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ is a unique solution of (6.1). This completes the proof. □

We now define $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ formally introduced in section 4.

In the remaining of this section we fix an integer m satisfying $m \geq [\frac{n}{2}]+1$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfying

$$\tilde{\phi} \in C_{per}(\mathbb{R}; H^m), \ \ \tilde{w} \in C_{per}(\mathbb{R}; H^m) \cap L^2_{per}(\mathbb{R}; H^{m+1}) \tag{6.14}$$

In view of Proposition 6.4, we define $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ as follows.

Let $k = m-1$ or m. The operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ is defined by

$$u_\infty(t) = S_{\infty,\tilde{u}}(t) u_{0\infty} \ \text{ for } \ u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_\infty(t)$ is the solution of (6.1) with $F_\infty = 0$; and the operator $\mathscr{S}_{\infty,\tilde{u}}(t) : L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ is defined by

$$u_\infty(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \ \text{ for } \ F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_\infty(t)$ is the solution of (6.1) with $u_{0\infty} = 0$.

The operators $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ have the following properties in weighted Sobolev spaces.

Proposition 6.5. *Let $n \geq 3$ and let m be a nonnegative integer satisfying $m \geq [\frac{n}{2}]+1$. Let $k = m-1$ or m and let ℓ be a nonnegative integer. Assume that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (6.14). Then there exists a constant $\delta > 0$ such that if $\|\tilde{w}\|_{C([0,T];H^m) \cap L^2(0,T;H^{m+1})} \leq \delta$, the following assertions hold true.*

(i) *It holds that $S_{\infty,\tilde{u}}(\cdot) u_{0\infty} \in C([0,\infty); H^k_{(\infty),\ell})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell}$ and there exists a constant $a > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|S_{\infty,\tilde{u}}(t) u_{0\infty}\|_{H^k_{(\infty),\ell}} \leq C e^{-at} \|u_{0\infty}\|_{H^k_{(\infty),\ell}}$$

for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),\ell}$ with a constant $C = C(T) > 0$.

(ii) *It holds that $\mathscr{S}_{\infty,\tilde{u}}(\cdot) F_\infty \in C([0,T]; H^k_{(\infty),\ell})$ for each $F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty]\|_{H^k_{(\infty),\ell}} \leq C \left\{ \int_0^t e^{-a(t-\tau)} \|F_\infty\|^2_{H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell}} d\tau \right\}^{\frac{1}{2}}$$

for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ with $C = C(T) > 0$.

(iii) *It holds that $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T)) < 1$.*

(iv) $I - S_{\infty,\tilde{u}}(T)$ *has a bounded inverse* $(I - S_{\infty,\tilde{u}}(T))^{-1}$ *on* $H^k_{(\infty),\ell}$ *and* $(I - S_{\infty,\tilde{u}}(T))^{-1}$ *satisfies*

$$\|(I - S_{\infty,\tilde{u}}(T))^{-1}u\|_{H^k_{(\infty),\ell}} \le C\|u\|_{H^k_{(\infty),\ell}} \quad for \quad u \in H^k_{(\infty),\ell}.$$

(v) *If* $\Gamma\tilde{u} = \tilde{u}$, *then* $\Gamma S_{\infty,\tilde{u}}(t) = S_{\infty,\tilde{u}}(t)\Gamma$ *and* $\Gamma\mathscr{S}_{\infty,\tilde{u}}(t) = \mathscr{S}_{\infty,\tilde{u}}(t)\Gamma$. *Consequently, if* $\Gamma\tilde{u} = \tilde{u}$, *then the assertions* (i)–(iv) *above hold with function spaces* $H^k_{\infty,\ell}$ *and* $H^k_{\infty,\ell} \times H^{k-1}_{\infty,\ell}$ *replaced by* $(H^k_{\infty,\ell})_{sym}$ *and* $(H^k_{\infty,\ell} \times H^{k-1}_{\infty,\ell})_{sym}$, *respectively.*

Remark 6.6. In this paper we will apply Proposition 6.5 with $\ell = 1$ to prove Theorem 3.1. For the purpose of future use, we formulate and prove it for a general nonnegative integer ℓ.

Proposition 6.5 will be proved by the weighted energy method. In fact, Proposition 6.5 follows from the weighted energy estimate in the following proposition.

Proposition 6.7. *Let* $n \ge 3$ *and let* m *be a nonnegative integer satisfying* $m \ge [\frac{n}{2}] + 1$. *Let* $k = m - 1$ *or* m *and let* ℓ *be a nonnegative integer. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell},$$
$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0, T'; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$$

for all $T' > 0$ *and that* $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ *satisfies* (6.14). *Assume also that* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *is the solution of* (6.1) *satisfying*

$$\phi_\infty \in C([0, T']; H^k_{(\infty)}), \; w_\infty \in C([0, T']; H^k_{(\infty)}) \cap L^2(0, T'; H^{k+1}_{(\infty)})$$

for all $T' > 0$.

Then there exist a positive constant δ *and an energy functional* $\mathcal{E}^k_\ell[u_\infty]$ *such that if*

$$\|\tilde{w}\|_{C([0,T];H^m)\cap L^2(0,T;H^{m+1})} \le \delta,$$

there holds the estimate

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}^k_\ell[u_\infty](t) + d(\|\phi_\infty(t)\|^2_{H^k_\ell} + \|w_\infty(t)\|^2_{H^{k+1}_\ell}) \\ &\le \; C\{\|F_\infty(t)\|^2_{H^k_\ell \times H^{k-1}_\ell} + (\|\nabla\tilde{w}(t)\|_{H^m} + \|\nabla\tilde{w}(t)\|^2_{H^m})\|\phi_\infty(t)\|^2_{H^k_\ell}\} \end{aligned} \tag{6.15}$$

on $(0, T')$ *for all* $T' > 0$. *Here* d *is a positive constant depending on* ℓ; C *is a positive constant depending on* T *but not on* T'; $\mathcal{E}^k_\ell[u_\infty]$ *is equivalent to* $\|u_\infty\|^2_{H^k_\ell}$, *i.e,*

$$C^{-1}\|u_\infty\|^2_{H^k_\ell} \le \mathcal{E}^k_\ell[u_\infty] \le C\|u_\infty\|^2_{H^k_\ell};$$

and $\mathcal{E}^k_\ell[u_\infty](t)$ *is absolutely continuous in* $t \in [0, T']$ *for all* $T' > 0$.

The proof of Proposition 6.7 will be given in section 7.

By using Proposition 6.7, we prove Proposition 6.5.

Proof of Proposition 6.5. Set

$$\begin{aligned} \omega &= \frac{1}{T}\int_0^T (\|\nabla \tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|_{H^m}^2)\, dt, \\ z(t) &= (\|\nabla \tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|_{H^m}^2) - \omega, \\ Z(t) &= \int_0^t z(\tau) d\tau. \end{aligned}$$

Observe that $Z(t)$ satisfies $Z(t+T) = Z(t)$ for any $t \in \mathbb{R}$, and so it holds that

$$\sup_{t\in\mathbb{R}} |Z(t)| \le \sup_{\tau\in[0,T]} |Z(\tau)| \le C(1 + \|\tilde{w}\|_{L^2(0,T;H^{m+1})}^2),$$

where $C = C(T) > 0$.

By Proposition 6.7 with $F_\infty = 0$, we see that there exists a positive constant d_1 such that

$$\frac{d}{dt}\mathcal{E}_\ell^k[u_\infty](t) + d_1\mathcal{E}_\ell^k[u_\infty](t) \le C\omega\mathcal{E}_\ell^k[u_\infty](t) + Cz(t)\mathcal{E}_\ell^k[u_\infty](t) \quad (t \ge 0). \tag{6.16}$$

If $\omega \le \frac{d_1}{2C}$, then we find from (6.16) that

$$\frac{d}{dt}\mathcal{E}_\ell^k[u_\infty](t) + \frac{d_1}{2}\mathcal{E}_\ell^k[u_\infty](t) \le Cz(t)\mathcal{E}_\ell^k[u_\infty](t) \quad (t \ge 0).$$

We thus obtain

$$\frac{d}{dt}\left(e^{\frac{d_1}{2}t}e^{-CZ(t)}\mathcal{E}_\ell^k[u_\infty](t)\right) \le 0 \quad (t \ge 0),$$

and hence,

$$\mathcal{E}_\ell^k[u_\infty](t) \le \mathcal{E}_\ell^k[u_\infty](0)e^{-\frac{d_1}{2}t}e^{CZ(t)} \le e^{C(1+\|\tilde{w}\|_{L^2(0,T;H^{m+1})}^2)}\mathcal{E}_\ell^k[u_\infty](0)e^{-\frac{d_1}{2}t} \quad (t \ge 0).$$

Consequently, we have

$$\|S_{\infty,\tilde{u}}(t)u_{0\infty}\|_{H^k_{(\infty),\ell}} \le Ce^{-\frac{d_1}{4}t}\|u_{0\infty}\|_{H^k_{(\infty),\ell}} \quad (t \ge 0).$$

This proves (i). The assertion (ii) is proved similarly; and we omit the proof.

As for (iii), since $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}) \in C_{per}(\mathbb{R}; H^m)$, it follows from (i) that, for each $j \in \mathbb{N}$,

$$\|(S_{\infty,\tilde{u}}(T))^j u\|_{H^k_{(\infty),\ell}} = \|S_{\infty,\tilde{u}}(jT)u\|_{H^k_{(\infty),\ell}} \le Ce^{-d_2 jT}\|u\|_{H^k_{(\infty),\ell}},$$

where $d_2 = \frac{d_1}{4} > 0$. Hence, we have

$$\|(S_{\infty,\tilde{u}}(T))^j\| \le Ce^{-d_2 jT}.$$

We thus obtain

$$\lim_{j\to\infty} \|(S_{\infty,\tilde{u}}(T))^j\|^{\frac{1}{j}} \quad\le\quad \lim_{j\to\infty} C^{\frac{1}{j}} e^{-d_2 T} = e^{-d_2 T} < 1.$$

This shows (iii). The assertion (iv) is an immediate consequence of (iii).

As for (v), we see that if $\Gamma\tilde{u} = \tilde{u}$, then $\Gamma P_\infty(B[\tilde{u}]u_\infty) = P_\infty(B[\tilde{u}]\Gamma u_\infty)$, and so,

$$\Gamma(\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty)) = \partial_t \Gamma u_\infty + A\Gamma u_\infty + P_\infty(B[\tilde{u}]\Gamma u_\infty).$$

It then follows from the uniqueness of solutions of (6.1) that $\Gamma S_{\infty,\tilde{u}}(t) = S_{\infty,\tilde{u}}(t)\Gamma$ and $\Gamma \mathscr{S}_{\infty,\tilde{u}}(t) = \mathscr{S}_{\infty,\tilde{u}}(t)\Gamma$. This completes the proof. □

We conclude this section with the estimate for a solution u_∞ of (6.1) satisfying $u_\infty(0) = u_\infty(T)$.

Proposition 6.8. *Let $n \ge 3$ and let m be a nonnegative integer satisfying $m \ge [\frac{n}{2}] + 1$. Assume that*

$$F_\infty = {}^\top(F_\infty^0, \tilde{F}_\infty) \in L^2(0,T; (H^k_{(\infty),1} \times H^{k-1}_{(\infty),1})_{sym})$$

with $k = m-1$ or m. Assume also that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (6.14) and $\Gamma\tilde{u} = \tilde{u}$. Then there exists a positive constant δ such that if

$$\|\tilde{w}\|_{C([0,T];H^m)\cap L^2(0,T;H^{m+1})} \le \delta,$$

the following assertion holds true.

The function

$$u_\infty(t) := S_{\infty,\tilde{u}}(t)(I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)[F_\infty] + \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \tag{6.17}$$

is a solution of (6.1) in $\mathscr{Y}^k_\infty(0,T)$ satisfying $u_\infty(0) = u_\infty(T)$ and the estimate

$$\|u_\infty\|_{\mathscr{Y}^k_\infty(0,T)} \le C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}.$$

Proof. By Proposition 6.7 and Proposition 6.5, we see that

$$\begin{aligned}
&\|u_\infty(t)\|^2_{H^k_1} + \|w_\infty\|^2_{L^2(0,t;H^{k+1}_1)} \\
&\le C\Big\{\|(I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)[F_\infty]\|^2_{H^k_1} + \|F_\infty\|^2_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})} \\
&\quad + \int_0^T (\|\nabla\tilde{w}\|_{H^m} + \|\tilde{w}\|^2_{H^m})\|\phi_\infty\|^2_{H^k_1}\,ds\Big\} \\
&\le C\left\{\|F_\infty\|^2_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})} + \delta\|\phi_\infty\|^2_{C([0,T];H^k_1)}\right\}
\end{aligned}$$

for $t \in [0,T]$. Therefore, if δ is so small that $C\delta \le \frac{1}{2}$, then we obtain

$$\|u_\infty\|^2_{C([0,T];H^k_1)} + \|w_\infty\|^2_{L^2(0,T;H^{k+1}_1)} \le C\|F_\infty\|^2_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}. \tag{6.18}$$

Next, since $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfies the system of equations

$$\begin{cases} \partial_t\phi_\infty + \gamma P_\infty(\tilde{w}\cdot\nabla\phi_\infty) + \gamma \mathrm{div} w_\infty = F^0_\infty, \\ \partial_t w_\infty - \nu\triangle w_\infty - \tilde{\nu}\nabla\mathrm{div} w_\infty + \gamma\nabla\phi_\infty = \tilde{F}_\infty, \end{cases}$$

we obtain

$$\|\partial_t w_\infty\|_{H^{k-1}_{(\infty),1}} \le C\{\|w_\infty\|_{H^{k+1}_{(\infty),1}} + \|\phi_\infty\|_{H^{k}_{(\infty),1}} + \|\tilde{F}_\infty\|_{H^{k-1}_{(\infty),1}}\}.$$

Hence, it follows from (6.18) that

$$\|\partial_t w_\infty\|_{L^2(0,T;H^{k-1}_{(\infty),1})} \le C\|F_\infty\|_{L^2(0,T;H^{k}_{(\infty),1}\times H^{k-1}_{(\infty),1})}. \tag{6.19}$$

Consequently, we see from (6.18) and (6.19) that

$$\|u_\infty\|_{\mathscr{Y}^k_\infty(0,T)} \le C\|F_\infty\|_{L^2(0,T;H^{k}_{(\infty),1}\times H^{k-1}_{(\infty),1})}.$$

This completes the proof. □

7 Weighted energy estimates for P_∞ part

In this section we prove Proposition 6.7 by a weighted energy method.

We first consider the following equation.

$$\begin{cases} \partial_t u_\infty + Au_\infty + B[\tilde{u}]u_\infty = F_\infty, \\ u|_{t=0} = u_{0\infty}, \end{cases} \tag{7.1}$$

where

$$F_\infty = \begin{pmatrix} F^0_\infty \\ \tilde{F}_\infty \end{pmatrix}, B[\tilde{u}]u = \begin{pmatrix} \gamma\tilde{w}\cdot\nabla\phi \\ 0 \end{pmatrix}, u = \begin{pmatrix} \phi \\ w \end{pmatrix}, \tilde{u} = \begin{pmatrix} \tilde{\phi} \\ \tilde{w} \end{pmatrix}.$$

We introduce some notations. For nonnegative integers k and ℓ, we define $E^k_\ell[u_\infty]$ by

$$E^k_\ell[u_\infty] = \kappa(|\phi_\infty|^2_{H^k_\ell} + |w_\infty|^2_{H^k_\ell}) + \sum_{|\alpha|\le k-1}(\partial^\alpha_x w_\infty, |x|^{2\ell}\nabla\partial^\alpha_x\phi_\infty).$$

Here κ is a positive constant to be determined later.

Note that there exists a constant $\kappa_0 > 0$ such that if $\kappa \ge \kappa_0$, then $E^k_\ell[u_\infty]$ is equivalent to $|u_\infty|^2_{H^k_\ell}$, i.e.,

$$C^{-1}|u_\infty|^2_{H^k_\ell} \le E^k_\ell[u_\infty] \le C|u_\infty|^2_{H^k_\ell}$$

for some constant $C > 0$.

We also define $D^k_\ell[u_\infty]$ for integers $k \ge 1$ and $\ell \ge 0$ by

$$D^k_\ell[u_\infty] = |\nabla\phi_\infty|^2_{H^{k-1}_\ell} + |\nabla w_\infty|^2_{H^k_\ell}.$$

Proposition 7.1. *Let m be a nonnegative integer satisfying $m \geq [\frac{n}{2}]+1$ and let ℓ be a nonnegative integer. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k,$$
$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T'; H^k \times H^{k-1})$$

for $k = m-1$ or $k = m$. Here T' is a given positive number. Assume also that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ is the solution of (7.1) with $\tilde{w} \in C([0,T']; H^m) \cap L^2(0,T'; H^{m+1})$ and that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfies

$$\phi_\infty \in C([0,T']; H^k),\ w_\infty \in C([0,T']; H^k) \cap L^2(0,T'; H^{k+1}).$$

Then there exist positive constants $\kappa \geq \kappa_0$ and $d > 0$ such that the estimate

$$\begin{aligned}
&\frac{d}{dt} E^k_\ell[\zeta_R u_\infty] + d D^k_\ell[\zeta_R u_\infty] \\
&\leq\ C\Big\{\epsilon |\zeta_R u_\infty|^2_{L^2_\ell} + \Big(\big(1+\frac{\ell^2}{\epsilon}\big)\|\tilde{w}\|^2_{H^m} + \|\nabla \tilde{w}\|_{H^m}\Big)\|\zeta_R \phi_\infty\|^2_{H^k_\ell} \\
&\quad + \Big(1+\frac{1}{\epsilon}\Big)|\zeta_R F_\infty|^2_{H^k_\ell \times H^{k-1}_\ell} \\
&\quad + \ell^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m}\right)|\zeta_R u_\infty|^2_{H^k_{\ell-1}} \\
&\quad + \Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m(N_R)}\right)|u_\infty|^2_{H^k_{\ell-1}(N_R)\times H^{k+1}_{\ell-1}(N_R)}\Big\}
\end{aligned} \tag{7.2}$$

holds on $(0,T')$, where ϵ is any positive number; C is a positive constant independent of T', ϵ and $R \geq 1$; and N_R denotes the set $N_R = \{x \in \mathbb{R}^n; R \leq |x| \leq 2R\}$.

Proof. By multiplying ζ_R to (7.1), we obtain

$$\left\{\begin{array}{l}
\partial_t(\zeta_R\phi_\infty) + \gamma\tilde{w}\cdot\nabla(\zeta_R\phi_\infty) + \gamma\mathrm{div}(\zeta_R w_\infty) = \zeta_R F^0_\infty + K_1(\nabla\zeta_R), \\[2ex]
\partial_t(\zeta_R w_\infty) - \nu\triangle(\zeta_R w_\infty) - \tilde{\nu}\nabla\mathrm{div}(\zeta_R w_\infty) + \gamma\nabla(\zeta_R\phi_\infty) = \zeta_R\tilde{F}_\infty + K_2(\nabla\zeta_R),
\end{array}\right. \tag{7.3}$$

where

$$\begin{aligned}
K_1(\nabla\zeta_R) &= \gamma(w_\infty\cdot\nabla\zeta_R + \tilde{w}\cdot\nabla\zeta_R\phi_\infty), \\
K_2(\nabla\zeta_R) &= -\nu([\zeta_R, \triangle]w_\infty) - \tilde{\nu}([\zeta_R, \nabla\mathrm{div}]w_\infty) + \gamma\nabla\zeta_R\phi_\infty.
\end{aligned}$$

For a multi-index α satisfying $|\alpha| \leq k$, we take the inner product of $\partial^\alpha_x(7.3)_1$ with $|x|^{2\ell}\partial^\alpha_x(\zeta_R\phi_\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\||x|^\ell\partial^\alpha_x(\zeta_R\phi_\infty)\|^2_{L^2} + \gamma(\partial^\alpha_x\mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial^\alpha_x(\zeta_R\phi_\infty)) \\
&= \sum_{j=1}^{2} I^{(j)}_{\alpha,\ell,R} + \mathscr{P}^{(1)}_{\alpha,\ell}[\zeta_R u_\infty] + Q_{1,\alpha,\ell}(\nabla\zeta_R),
\end{aligned} \tag{7.4}$$

where

$$I^{(1)}_{\alpha,\ell,R} \;=\; -\gamma\left\{\frac{1}{2}(\mathrm{div}\tilde{w},|x|^{2\ell}|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)+([\partial_x^\alpha,\tilde{w}]\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))\right\},$$

$$I^{(2)}_{\alpha,\ell,R} \;=\; (\partial_x^\alpha(\zeta_R F^0_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)),$$

$$\mathscr{P}^{(1)}_{\alpha,\ell}[\zeta_R u_\infty]=\frac{\gamma}{2}(\tilde{w}\cdot\nabla(|x|^{2\ell}),|\partial_x^\alpha(\zeta_R\phi_\infty)|^2),$$

$$Q_{1,\alpha,\ell}(\nabla\zeta_R)=(\partial_x^\alpha K_1(\nabla\zeta_R),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)).$$

Here we used

$$\begin{aligned}
&(\partial_x^\alpha(\gamma\tilde{w}\cdot\nabla(\zeta_R\phi_\infty)),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))\\
&=\;\gamma(\tilde{w}\cdot\nabla\partial_x^\alpha(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))+\gamma([\partial_x^\alpha,\tilde{w}]\cdot\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))\\
&=\;\frac{1}{2}\gamma(|x|^{2\ell}\tilde{w},\nabla|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)+\gamma([\partial_x^\alpha,\tilde{w}]\cdot\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))\\
&=\;-\frac{1}{2}\gamma(|x|^{2\ell}\mathrm{div}\tilde{w},|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)-\frac{1}{2}\gamma(\tilde{w}\cdot\nabla(|x|^{2\ell}),|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)\\
&\qquad+\gamma([\partial_x^\alpha,\tilde{w}]\cdot\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))\\
&=\;I^{(1)}_{\alpha,\ell,R}+\mathscr{P}^{(1)}_{\alpha,R}(\nabla(|x|^{2\ell})).
\end{aligned}$$

This calculation can be justified by using the standard Friedrichs commutator argument.

We take the inner product of $\partial_x^\alpha(7.3)_2$ with $|x|^{2\ell}\partial_x^\alpha(\zeta_R w_\infty)$ and integrate by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\||x|^\ell\partial_x^\alpha(\zeta_R w_\infty)\|^2_{L^2}+\nu\||x|^\ell\nabla\partial_x^\alpha(\zeta_R w_\infty)\|^2_{L^2}+\tilde{\nu}\||x|^\ell\mathrm{div}\partial_x^\alpha(\zeta_R w_\infty)\|^2_{L^2}\\
&\quad-\gamma(\partial_x^\alpha(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty))\\
&\quad=I^{(3)}_{\alpha,\ell,R}+\mathscr{P}^{(2)}_{\alpha,\ell}[\zeta_R u_\infty]+Q_{2,\alpha,\ell}(\nabla\zeta_R),
\end{aligned}\tag{7.5}$$

where

$$I^{(3)}_{\alpha,\ell,R}=\begin{cases}((\zeta_R\tilde{F}_\infty),|x|^{2\ell}(\zeta_R w_\infty)) & (\alpha=0),\\ -(\partial_x^{\alpha-1}(\zeta_R\tilde{F}_\infty),|x|^{2\ell}\partial_x^{\alpha+1}(\zeta_R w_\infty)) & (|\alpha|\geq 1),\end{cases}$$

$$\begin{aligned}
\mathscr{P}^{(2)}_{\alpha,\ell}[\zeta_R u_\infty] \;=\;& (\nu\partial_x^\alpha\nabla(\zeta_R w_\infty)+\tilde{\nu}\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty)+\gamma\partial_x^\alpha(\zeta_R\phi_\infty),\nabla(|x|^{2\ell})\partial_x^\alpha(\zeta_R w_\infty))\\
&-(\partial_x^{\alpha-1}(\zeta_R\tilde{F}_\infty),\partial_x(|x|^{2\ell})\partial_x^\alpha(\zeta_R w_\infty)),
\end{aligned}$$

$$Q_{2,\alpha,\ell}(\nabla\zeta_R)=(\partial_x^\alpha(K_2(\nabla\zeta_R)),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)).$$

By adding (7.4) to (7.5), we see that

$$\frac{1}{2}\frac{d}{dt}\left\{\||x|^\ell\partial_x^\alpha(\zeta_R\phi_\infty)\|^2_{L^2}+\||x|^\ell\partial_x^\alpha(\zeta_R w_\infty)\|^2_{L^2}\right\}$$

$$
\begin{aligned}
&+\nu\||x|^\ell\nabla\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2+\tilde{\nu}\||x|^\ell\mathrm{div}\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2\\
&= \sum_{j=1}^{3} I_{\alpha,\ell,R}^{(j)}+\mathscr{P}_{\alpha,\ell}^{(1)}[\zeta_R u_\infty]+\mathscr{P}_{\alpha,\ell}^{(2)}[\zeta_R u_\infty]+Q_{1,\alpha,\ell}(\nabla\zeta_R)+Q_{2,\alpha,\ell}(\nabla\zeta_R).
\end{aligned}\tag{7.6}
$$

By using Lemma 2.2 and Lemma 2.3, we obtain

$$
\begin{aligned}
|\sum_{|\alpha|\le k}\sum_{j=1}^{3} I_{\alpha,\ell,R}^{(j)}| \;\le\;& \epsilon|\zeta_R u_\infty|_{L_\ell^2}^2+\epsilon_1|\nabla(\zeta_R\phi_\infty)|_{H_\ell^{k-1}}^2+\epsilon_2|\nabla(\zeta_R w_\infty)|_{H_\ell^k}^2\\
&+C\|\nabla\tilde{w}\|_{H^m}\|\zeta_R\phi_\infty\|_{H_\ell^k}^2\\
&+C\Big(\frac{1}{\epsilon}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\Big)|\zeta_R F_\infty|_{H_\ell^k\times H_\ell^{k-1}}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{|\alpha|\le k}\sum_{j=1}^{2}\mathscr{P}_{\alpha,\ell}^{(j)}[\zeta_R u_\infty]| \;\le\;& \epsilon|\zeta_R u_\infty|_{L_\ell^2}^2+\epsilon_1|\nabla(\zeta_R\phi_\infty)|_{H_\ell^{k-1}}^2+\epsilon_2|\nabla(\zeta_R w_\infty)|_{H_\ell^k}^2\\
&+C\ell^2\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\Big)|\zeta_R w_\infty|_{H_{\ell-1}^k}^2\\
&+C\ell^2\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}\Big)\|\tilde{w}\|_{H^m}^2|\zeta_R\phi_\infty|_{H_{\ell-1}^k}^2\\
&+C\ell^2|\zeta_R F_\infty|_{H_\ell^k\times H_\ell^{k-1}}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{|\alpha|\le k}\sum_{j=1}^{2} Q_{j,\alpha,\ell}(\nabla\zeta_R)| \;\le\;& \epsilon|\zeta_R u_\infty|_{L_\ell^2(N_R)}^2+\epsilon_1|\nabla(\zeta_R\phi_\infty)|_{H_\ell^{k-1}(N_R)}^2\\
&+\epsilon_2|\nabla(\zeta_R w_\infty)|_{H_\ell^k(N_R)}^2\\
&+C\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\Big)\Big\{|w_\infty|_{H_{\ell-1}^{k+1}(N_R)}^2\\
&+(1+\|\tilde{w}\|_{H^m(N_R)}^2)|\phi_\infty|_{H_{\ell-1}^k(N_R)}^2\Big\}^2.
\end{aligned}
$$

Taking $\epsilon_2>0$ suitably small, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\zeta_R u_\infty|_{H_\ell^k}^2 \quad&+\frac{\nu}{2}|\nabla(\zeta_R w_\infty)|_{H_\ell^k}^2+\frac{\tilde{\nu}}{2}|\mathrm{div}(\zeta_R w_\infty)|_{H_\ell^k}^2\\
\le\;& \epsilon|\zeta_R u_\infty|_{L_\ell^2}^2+\epsilon_1|\nabla(\zeta_R\phi_\infty)|_{H_\ell^{k-1}}^2+C\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}\Big)|\zeta_R F|_{H_\ell^k\times H_\ell^{k-1}}^2\\
&+C\|\nabla\tilde{w}\|_{H^m}\|\zeta_R\phi_\infty\|_{H_\ell^k}^2\\
&+C\ell^2\Big(1+(\frac{1}{\epsilon}+\frac{1}{\epsilon_1})(1+\|\tilde{w}\|_{H^m}^2)\Big)|\zeta_R u_\infty|_{H_{\ell-1}^k}^2\\
&+C\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}\Big)(1+\|\tilde{w}\|_{H^m}^2)|u_\infty|_{H_{\ell-1}^k(N_R)\times H_{\ell-1}^{k+1}(N_R)}^2.
\end{aligned}\tag{7.7}
$$

We next estimate $\||x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty\|_{L^2}^2$ for α with $|\alpha|\le k-1$. For a multi-index α satisfying $|\alpha|\le k-1$, we take the inner product of $\partial_x^\alpha(7.3)_2$ with $|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)$ to obtain

$$\begin{aligned}
&(\partial_t\partial_x^\alpha(\zeta_R w_\infty),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty))+\gamma|\nabla\partial_x^\alpha(\zeta_R\phi_\infty)|_{L^2_\ell}^2\\
&\quad=\sum_{i=1}^{3}J^{(i)}_{\alpha,\ell,R}+\left(\partial_x^\alpha K_2(\nabla\zeta_R),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)\right),
\end{aligned}\tag{7.8}$$

where

$$\begin{aligned}
J^{(1)}_{\alpha,\ell,R}&=(\nu(\partial_x^\alpha\triangle(\zeta_R w_\infty),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)),\\
J^{(2)}_{\alpha,\ell,R}&=(\tilde\nu(\partial_x^\alpha(\nabla\mathrm{div}(\zeta_R w_\infty)),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)),\\
J^{(3)}_{\alpha,\ell,R}&=(\partial_x^\alpha(\zeta_R\tilde F_\infty),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)).
\end{aligned}$$

As for the first term on the left-hand side, we have

$$\begin{aligned}
&(\partial_t\partial_x^\alpha(\zeta_R w_\infty),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)\\
&\quad=\frac{d}{dt}(\partial_x^\alpha(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty))+(\partial_x^\alpha(\zeta_R w_\infty),\nabla(|x|^{2\ell})\partial_x^\alpha\partial_t(\zeta_R\phi_\infty))\\
&\qquad+(\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha\partial_t(\zeta_R\phi_\infty)).
\end{aligned}\tag{7.9}$$

By (7.3), we have

$$\partial_t(\zeta_R\phi_\infty)=-\gamma\tilde w\cdot\nabla(\zeta_R\phi_\infty)-\gamma\mathrm{div}(\zeta_R w_\infty)+\zeta_R F^0_\infty+K_1(\nabla\zeta_R).$$

Substituting this into (7.9), we obtain

$$\begin{aligned}
&(\partial_t\partial_x^\alpha(\zeta_R\phi_\infty),|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R w_\infty))\\
&\quad=\frac{d}{dt}(\partial_x^\alpha(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty))-\sum_{i=4}^{6}J^{(i)}_{\alpha,\ell,R}-\mathscr{P}^{(3)}_{\alpha,\ell}[\zeta_R u_\infty]\\
&\qquad+(\partial_x^\alpha(\zeta_R w_\infty),\nabla(|x|^{2\ell})\partial_x^\alpha K_1(\nabla\zeta_R))+(\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha K_1(\nabla\zeta_R)),
\end{aligned}$$

where

$$\begin{aligned}
J^{(4)}_{\alpha,\ell,R}&=\gamma(\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha(\tilde w\cdot\nabla(\zeta_R\phi_\infty))),\\
J^{(5)}_{\alpha,\ell,R}&=\gamma(\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha(\mathrm{div}(\zeta_R w_\infty))),\\
J^{(6)}_{\alpha,\ell,R}&=-(\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R F^0_\infty)),
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{P}^{(3)}_{\alpha,\ell}[\zeta_R u_\infty]&=\gamma(\partial_x^\alpha(\zeta_R w_\infty),\nabla(|x|^{2\ell})\partial_x^\alpha(\tilde w\cdot\nabla(\zeta_R\phi_\infty)))\\
&\quad+\gamma(\partial_x^\alpha(\zeta_R w_\infty),\nabla(|x|^{2\ell})\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty))\\
&\quad-(\partial_x^\alpha(\zeta_R w_\infty),\nabla(|x|^{2\ell})\partial_x^\alpha(\zeta_R F^0_\infty)).
\end{aligned}$$

This, together with (7.8), gives

$$\begin{aligned}
&\frac{d}{dt}(\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty)) + \gamma|\nabla\partial_x^\alpha(\zeta_R\phi_\infty)|_{L^2_\ell}^2 \\
&\quad = \sum_{i=4}^{6} J_{\alpha,\ell,R}^{(i)} + \mathscr{P}_{\alpha,\ell}^{(3)}[\zeta_R u_\infty] + Q_{3,\alpha,\ell}(\nabla\zeta_R),
\end{aligned} \tag{7.10}$$

where

$$\begin{aligned}
Q_{3,\alpha,\ell} &= (\partial_x^\alpha K_2(\nabla\zeta_R), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)) \\
&\quad -(\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha K_1(\nabla\zeta_R)) \\
&\quad -(\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha K_1(\nabla\zeta_R)).
\end{aligned}$$

By Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}
|\sum_{|\alpha|\le k-1}\sum_{i=1}^{6} J_{\alpha,\ell,R}^{(i)}| &\le \frac{\gamma}{4}|\nabla\partial_x^\alpha(\zeta_R\phi_\infty)|_{L^2_\ell}^2 + C\Big(\gamma+\frac{1}{\gamma}\Big)|\nabla(\zeta_R w_\infty)|_{H^k_\ell}^2 \\
&\quad +\gamma\|\tilde{w}\|_{H^m}^2\|\nabla(\zeta_R\phi_\infty)\|_{H^{k-1}_\ell}^2 + \frac{C}{\gamma}|\zeta_R F_\infty|_{H^{k-1}_\ell}^2,
\end{aligned}$$

$$\begin{aligned}
|\sum_{|\alpha|\le k-1}\mathscr{P}_{\alpha,\ell}^{(3)}[\zeta_R u_\infty]| &\le \tilde{\epsilon}\ell|\zeta_R w_\infty|_{L^2_\ell}^2 + \tilde{\epsilon}\ell|\nabla(\zeta_R w_\infty)|_{H^{k-1}_\ell}^2 \\
&\quad +\frac{C\ell}{\tilde{\epsilon}}\|\tilde{w}\|_{H^m}^2\|\nabla(\zeta_R\phi_\infty)\|_{H^{k-1}_{\ell-1}}^2 \\
&\quad +C\ell\Big(1+\frac{1}{\tilde{\epsilon}}\Big)|\zeta_R w_\infty|_{H^{k-1}_{\ell-1}}^2 + C\ell|\zeta_R F_\infty^0|_{H^{k-1}_{\ell-1}}^2,
\end{aligned}$$

$$\begin{aligned}
|\sum_{|\alpha|\le k-1} Q_{3,\alpha,\ell}(\nabla\zeta_R)| &\le \frac{\gamma}{4}|\nabla(\zeta_R\phi_\infty)|_{H^{k-1}_\ell(N_R)}^2 + \tilde{\epsilon}\ell|\zeta_R w_\infty|_{L^2_\ell(N_R)}^2 \\
&\quad +C|\nabla(\zeta_R w_\infty)|_{H^{k-1}_\ell(N_R)}^2 \\
&\quad +C\Big(\frac{1}{\gamma}+\frac{1}{\tilde{\epsilon}}\Big)(1+\|\tilde{w}\|_{H^m(N_R)}^2)|u_\infty|_{H^k_{\ell-1}(N_R)}^2
\end{aligned}$$

for any $\tilde{\epsilon} > 0$ with $C > 0$ independent of $\tilde{\epsilon}$.

Combining these estimates with (7.8) and (7.10), we see that

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le k-1}(\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty)) + \frac{\gamma}{2}|\nabla(\zeta_R\phi_\infty)|_{H^{k-1}_\ell}^2 \\
&\le \tilde{\epsilon}\ell|\zeta_R w_\infty|_{L^2_\ell}^2 \\
&\quad +C\Big\{|\nabla(\zeta_R w_\infty)|_{H^k_\ell}^2 + \Big(1+\frac{1}{\tilde{\epsilon}}\Big)\|\tilde{w}\|_{H^m}^2\|\nabla(\zeta_R\phi_\infty)\|_{H^{k-1}_\ell}^2 + |\zeta_R F_\infty|_{H^{k-1}_\ell}^2\Big\}
\end{aligned}$$

$$+C\Big(1+\frac{1}{\tilde{\epsilon}}\Big)\Big\{\ell|\zeta_R w_\infty|^2_{H^{k-1}_{\ell-1}} + (1+\|\tilde{w}\|^2_{H^m(N_R)})|u_\infty|^2_{H^k_{\ell-1}(N_R)}\Big\} \tag{7.11}$$

for any $\tilde{\epsilon} > 0$ with $C > 0$ independent of $\tilde{\epsilon}$.

Consider now $\kappa \times (7.7) + (7.11)$ with a constant $\kappa > 0$. Taking $\kappa > 0$ so large that $|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell}$ on the right-hand side is absorbed into the left-hand side and setting $\epsilon_1 = \frac{\gamma}{4\kappa}$ and $\tilde{\epsilon} = \ell^{-1}\epsilon$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}E^k_\ell[\zeta_R u_\infty](t) + dD^k_\ell[\zeta_R u_\infty] \\
&\le \ \epsilon|\zeta_R u_\infty|^2_{L^2_\ell} + C\Big(\big(1+\frac{\ell^2}{\epsilon}\big)\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}\Big)\|\zeta_R\phi_\infty\|^2_{H^k_\ell} \\
&\quad +C\Big(1+\frac{1}{\epsilon}\Big)|\zeta_R F_\infty|^2_{H^k_\ell\times H^{k-1}_\ell} \\
&\quad +C\ell^2\Big(1+\frac{1}{\epsilon}\Big)(1+\|\tilde{w}\|^2_{H^m})|\zeta_R u_\infty|^2_{H^k_{\ell-1}} \\
&\quad +C\Big(1+\frac{1}{\epsilon}\Big)(1+\|\tilde{w}\|^2_{H^m(N_R)})|u_\infty|^2_{H^k_{\ell-1}(N_R)\times H^{k+1}_{\ell-1}(N_R)}
\end{aligned}$$

for any$\epsilon > 0$ with $C > 0$ independent of ϵ. This completes the proof. □

Remark 7.2. Similarly to the proof of Proposition 6.4, one can prove that if

$$\begin{aligned}
&\tilde{w} \in C([0,T'];H^m) \cap L^2(0,T';H^{m+1}), \\
&u_{0\infty} \in H^k, \\
&F_\infty \in L^2(0,T';H^k \times H^{k-1}),
\end{aligned}$$

then there exists a unique solution $u_\infty = {}^\top(\phi_\infty, w_\infty)$ of (7.1) in $C([0,T'];H^k)\cap L^2(0,T';H^k\times H^{k+1})$. Furthermore, by setting $\ell = 0$ and $\zeta_R \equiv 1$ in the proof of Proposition 7.1, one can see that $E^k_0[u_\infty](t)$ is absolutely continuous in $t \in [0,T']$ and there holds the estimate

$$\begin{aligned}
\frac{d}{dt}E^k_0[u_\infty] + dD^k_0[u_\infty] \ \le\ & C\Big\{\epsilon\|u_\infty\|^2_2 + (\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m})\|\nabla\phi_\infty\|^2_{H^{k-1}} \\
&+\Big(1+\frac{1}{\epsilon}\Big)\|F_\infty\|^2_{H^k\times H^{k-1}}\Big\}
\end{aligned} \tag{7.12}$$

on $(0,T')$, where ϵ is any positive number; and C is a positive constant independent of T' and ϵ.

Remark 7.3. One can easily see that (7.2) holds with ζ_R and N_R replaced by $\zeta_R - \zeta_{R'}$ and $N_{R,R'}$ for $R' > R \ge 1$, where $N_{R,R'}$ denotes the set $N_{R,R'} = \{x \in \mathbb{R}^n; R \le |x| \le 2R'\}$.

Proposition 7.4. *Let m be a nonnegative integer satisfying $m \ge [\frac{n}{2}]+1$ and let ℓ be an integer satisfying $\ell \ge 1$. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_\ell,$$

$$F_\infty = {}^\top(F^0_\infty, \tilde F_\infty) \in L^2(0,T'; H^k_\ell \times H^{k-1}_\ell)$$

for $k = m-1$ or $k = m$. Here T' is a given positive number. Assume also that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ is the solution of (7.1) with $\tilde w \in C([0,T']; H^m) \cap L^2(0,T'; H^{m+1})$ and that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfies

$$\phi_\infty \in C([0,T']; H^k),\ w_\infty \in C([0,T']; H^k) \cap L^2(0,T'; H^{k+1}).$$

Then it holds that

$$\phi_\infty \in C([0,T']; H^k_\ell),\ \ w_\infty \in C([0,T']; H^k_\ell) \cap L^2(0,T'; H^{k+1}_\ell).$$

Furthermore, there exist positive constants $\kappa \geq \kappa_0$ and $d > 0$ such that $E^k_\ell[u_\infty](t)$ is absolutely continuous in $t \in [0,T']$ and there holds the estimate

$$\begin{aligned} &\frac{d}{dt}E^k_\ell[u_\infty] + dD^k_\ell[u_\infty] \\ &\leq\ C\Big\{\epsilon|u_\infty|^2_{L^2_\ell} + \Big(\big(1+\frac{\ell^2}{\epsilon}\big)\|\tilde w\|^2_{H^m} + \|\nabla \tilde w\|_{H^m}\Big)\|\phi_\infty\|^2_{H^k_\ell} \\ &\qquad + \Big(1+\frac{1}{\epsilon}\Big)|F_\infty|^2_{H^k_\ell \times H^{k-1}_\ell} \\ &\qquad + \ell^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde w\|^2_{H^m}\right)|u_\infty|^2_{H^k_{\ell-1}}\Big\} \end{aligned} \tag{7.13}$$

on $(0,T')$, where ϵ is any positive number; C is a positive constant independent of T' and ϵ.

Proof. It suffices to prove that

$$\zeta_R u_\infty \to u_\infty \text{ in } C([0,T']; H^k_\ell) \cap L^2(0,T'; H^k_\ell \times H^{k+1}_\ell)$$

as $R \to \infty$. We prove this by induction on ℓ.

We first observe that it holds that

$$\zeta_R u_\infty \to u_\infty \text{ in } C([0,T']; H^k) \cap L^2(0,T'; H^k \times H^{k+1}) \tag{7.14}$$

as $R \to \infty$, since $u_\infty \in C([0,T']; H^k) \cap L^2(0,T'; H^k \times H^{k+1})$. We also note that since $\mathrm{supp}\,(\zeta_R - \zeta_{R'}) \subset N_{R,R'} = \{x \in \mathbb{R}^n; R \leq |x| \leq 2R'\}$ for $R' > R$, it holds that

$$\|\zeta_R u_\infty - \zeta_{R'} u_\infty\|_{H^k_\ell} \leq C\|u_\infty\|_{H^k_\ell(N_{R,R'})}$$

for $R' > R \geq 1$.

Set

$$\begin{aligned} \varphi_{\ell,R,R'}(t) &= |\zeta_R u_\infty(t) - \zeta_{R'} u_\infty(t)|^2_{H^k_\ell}, \\ b(t) &= 1 + \|\tilde w(t)\|^2_{H^m} + \|\nabla \tilde w(t)\|_{H^m}\ \in\ L^1(0,T'), \end{aligned}$$

$$a_{\ell,R,R'}(t) = |\zeta_R u_{0\infty} - \zeta_{R'} u_{0\infty}|^2_{H^k_\ell} + \int_0^t |\zeta_R F_\infty - \zeta_{R'} F_\infty|^2_{H^k_\ell \times H^{k-1}_\ell} \, d\tau + \int_0^t (1 + \|\tilde{w}(t)\|^2_{H^m}) \|u_\infty\|^2_{H^k_{\ell-1}(N_{R,R'}) \times H^{k+1}_{\ell-1}(N_{R,R'})} \, d\tau.$$

Let us prove Proposition 7.4 for $\ell = 1$. By (7.2), we have

$$\varphi_{1,R,R'}(t) + \int_0^t D^k_1[\zeta_R u_\infty - \zeta_{R'} u_\infty] \, d\tau \leq C\Big\{a_{1,R,R'}(T') + \int_0^t b(\tau)\varphi_{1,R,R'}(\tau) \, d\tau\Big\} \tag{7.15}$$

for $t \in [0, T']$, where C is a constant depending on ϵ. By the Gronwall inequality, we obtain

$$\varphi_{1,R,R'}(t) \leq C a_{1,R,R'}(T') e^{C \int_0^{T'} b(\tau) \, d\tau} \tag{7.16}$$

for $t \in [0, T']$. Since $a_{1,R,R'}(T') \to 0$ as $R, R' \to \infty$, we see that $\sup_{0 \leq t \leq T'} \varphi_{1,R,R'}(t) \to 0$ as $R, R' \to \infty$. This, together with (7.15), yields that $\int_0^{T'} D^k_1[\zeta_R u_\infty - \zeta_{R'} u_\infty] \, d\tau \to 0$ as $R, R' \to \infty$. In view of (7.14), we thus conclude that $\{\zeta_R u_\infty\}$ is Cauchy in $C([0, T']; H^k_1) \cap L^2(0, T'; H^k_1 \times H^{k+1}_1)$ and

$$\zeta_R u_\infty \to u_\infty \text{ in } C([0, T']; H^k_1) \cap L^2(0, T'; H^k_1 \times H^{k+1}_1)$$

as $R \to \infty$. Letting $R \to \infty$ in (7.2) with $\ell = 1$, we have the desired estimate in Proposition 7.4 with $\ell = 1$. Proposition 7.4 thus holds for $\ell = 1$.

We next suppose that Proposition 7.4 holds for $\ell = p$. We will prove that it also holds for $\ell = p + 1$. By (7.2) and Remark 7.3, we have

$$\varphi_{p+1,R,R'}(t) + \int_0^t D^k_{p+1}[\zeta_R u_\infty - \zeta_{R'} u_\infty] \, d\tau \leq C\Big\{a_{p+1,R,R'}(T') + \int_0^t b(\tau)\varphi_{p+1,R,R'}(\tau) \, d\tau\Big\} \tag{7.17}$$

for $t \in [0, T']$, where C is a constant depending on ϵ and p. By the Gronwall inequality, we obtain

$$\varphi_{p+1,R,R'}(t) \leq C a_{p+1,R,R'}(T') e^{C \int_0^{T'} b(\tau) \, d\tau} \tag{7.18}$$

for $t \in [0, T']$. By the induction assumption, we see that $a_{p+1,R,R'}(T') \to 0$ as $R, R' \to \infty$, and hence, by (7.18), $\sup_{0 \leq t \leq T'} \varphi_{p+1,R,R'}(t) \to 0$ as $R, R' \to \infty$. This, together with (7.17), yields that $\int_0^{T'} D^k_{p+1}[\zeta_R u_\infty - \zeta_{R'} u_\infty] \, d\tau \to 0$ as $R, R' \to \infty$. It then follows that $\{\zeta_R u_\infty\}$ is Cauchy in $C([0, T']; H^k_{p+1}) \cap L^2(0, T'; H^k_{p+1} \times H^{k+1}_{p+1})$ and

$$\zeta_R u_\infty \to u_\infty \text{ in } C([0, T']; H^k_{p+1}) \cap L^2(0, T'; H^k_{p+1} \times H^{k+1}_{p+1})$$

as $R \to \infty$. It is not difficult to see that $\frac{d}{dt} E^k_\ell[u_\infty] = G^k_\ell$ on $(0, T')$ for some $G^k_\ell \in L^1(0, T')$, and, thus, $E^k_\ell[u_\infty](t)$ is absolutely continuous in $t \in [0, T']$. Letting $R \to \infty$ in (7.2) with

$\ell = p+1$, we have the desired estimate in Proposition 7.4 with $\ell = p+1$. Proposition 7.4 thus holds for $\ell = p+1$. This completes the proof. $\square$

We are now in a position to prove Proposition 6.7.

Proof of Proposition 6.7. Let $U = {}^\top(\Phi, W) \in C([0,T']; H^k_\ell) \cap L^2(0,T'; H^k_\ell \times H^{k+1}_\ell)$. Then, by Lemma 4.7, we see that

$$\|P_1 B[\tilde{u}]U\|_{H^k_\ell} \le C\|\tilde{w}\|_\infty \|\nabla\Phi\|_{L^2_\ell} \le C\delta\|U\|_{H^k_\ell}.$$

It then follows from Remark 7.12 and Proposition 7.4 that there exists a unique solution $U_\infty \in C([0,T']; H^k_\ell \times H^k_\ell) \cap L^2(0,T'; H^k_\ell \times H^{k+1}_\ell)$ of

$$\partial_t U_\infty + AU_\infty + B[\tilde{u}]U_\infty = F_\infty + P_1 B[\tilde{u}]U, \quad U_\infty|_{t=0} = u_{0\infty}, \tag{7.19}$$

and U_∞ satisfies

$$\begin{aligned} &\|U_\infty(t)\|_{H^k_\ell} + \int_0^t \|\nabla U_\infty\|^2_{H^{k-1}{}_\ell \times H^k_\ell}\, d\tau \\ &\le C_0\Big\{\|u_{0\infty}\|^2_{H^k_\ell} + \int_0^t \|F_\infty\|^2_{H^k_\ell \times H^{k-1}_\ell}\, d\tau \\ &\qquad + \delta^2\int_0^t \|U\|^2_{H^k_\ell}\, d\tau + \int_0^t b(\tau)\|U_\infty\|^2_{H^k_{\ell-1}}\, d\tau\Big\}. \end{aligned} \tag{7.20}$$

Here $b(\tau) = 1 + \|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}$.

We set $U^{(0)}_\infty = 0$ and define $U^{(j)}_\infty$ $(j = 1, 2, \cdots)$ inductively by the solution of (7.19) with $U = U^{(j-1)}_\infty$. Applying the Gronwall inequality to (7.20) with $U_\infty = U^{(1)}_\infty$ and $U = 0$, we have

$$\|U^{(1)}_\infty(t)\| \le A_0$$

for $t \in [0, T']$, where

$$A_0 = C_0\Big\{\|u_{0\infty}\|^2_{H^k_\ell} + \int_0^{T'} \|F_\infty\|^2_{H^k_\ell \times H^{k-1}_\ell}\, d\tau\Big\} e^{C_0\|b\|_{L^1(0,T')}}.$$

Similarly, using (7.20) with $U_\infty = U^{(j)}_\infty - U^{(j-1)}_\infty$ and $U = U^{(j-1)}_\infty - U^{(j-2)}_\infty$ for $j = 2, 3, \cdots$, one can inductively see that

$$\|U^{(j)}_\infty(t) - U^{(j-1)}_\infty(t)\|^2_{H^k_\ell} \le \frac{A_0(C_0K_0\delta^2 t)^{j-1}}{(j-1)!},$$

$$\int_0^t \|\nabla(U^{(j)}_\infty - U^{(j-1)}_\infty)\|^2_{H^{k-1}_\ell \times H^k_\ell}\, d\tau \le \frac{A_0}{K_0}\Big\{\frac{(C_0K_0\delta^2 t)^{j-1}}{(j-1)!} + \frac{\|b\|_{L^1(0,T')}}{\delta^2}\frac{(C_0K_0\delta^2 t)^j}{j!}\Big\}.$$

Here $K_0 = 1 + \|b\|_{L^1(0,T')} e^{C_0\|b\|_{L^1(0,T')}}$. It then follows that $U^{(j)}_\infty$ converges to a function U_∞ in $C([0,T']; H^k_\ell) \cap L^2(0,T'; H^k_\ell \times H^{k+1}_\ell)$ as $j \to \infty$. One can easily see that U_∞ satisfies

(7.19) with $U = U_\infty$, i.e., U_∞ is a solution of (6.1), and $U_\infty(t) \in H^k_{(\infty)}$ for all $t \in [0, T']$. By the uniqueness of solutions of (6.1) (see Proposition 6.4), we see that $U_\infty = u_\infty$.

Applying Remark 7.2 and Proposition 7.4 with F_∞ replaced by $F_\infty + P_1 B[\tilde{u}]u_\infty$, we have

$$
\begin{aligned}
&\frac{d}{dt}E^k_j[u_\infty] + dD^k_j[u_\infty] \\
&\quad \le \; C\Big\{\epsilon|u_\infty|^2_{L^2_j} + \Big(\big(1+\frac{1}{\epsilon}\big)\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|^2_{H^k_j} \\
&\qquad + \Big(1+\frac{1}{\epsilon}\Big)|F_\infty|^2_{H^k_j \times H^{k-1}_j} \\
&\qquad + j^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m}\right)|u_\infty|^2_{H^k_{j-1}}\Big\}
\end{aligned}
$$

for $j = 0, 1, \cdots, \ell$. Using Lemma 4.4 (ii) and Lemma 4.7, we see that

$$
\begin{aligned}
&\frac{d}{dt}E^k_j[u_\infty] + 2d_1|u_\infty|^2_{H^k_j \times H^{k+1}_j} \\
&\quad \le \; C\Big\{\epsilon|u_\infty|^2_{L^2_j} + \Big(\big(1+\frac{1}{\epsilon}\big)\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|^2_{H^k_j} \\
&\qquad + \Big(1+\frac{1}{\epsilon}\Big)|F_\infty|^2_{H^k_j \times H^{k-1}_j} + j^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m}\right)|u_\infty|^2_{H^k_{j-1}}\Big\}
\end{aligned}
$$

for $j = 0, 1, \cdots, \ell$, with some constant $d_1 > 0$. Taking $\epsilon > 0$ suitably small, we obtain

$$
\begin{aligned}
&\frac{d}{dt}E^k_j[u_\infty] + d_1|u_\infty|^2_{H^k_j \times H^{k+1}_j} \\
&\quad \le \; C\Big\{\Big(\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|^2_{H^k_j} + |F_\infty|^2_{H^k_j \times H^{k-1}_j} \\
&\qquad + j^2\left(1+\|\tilde{w}\|^2_{H^m}\right)|u_\infty|^2_{H^k_{j-1}}\Big\}
\end{aligned}
\tag{7.21}
$$

for $j = 0, 1, \cdots, \ell$.

We now prove (6.15) by induction on ℓ. When $\ell = 0$, inequality (7.21) with $j = 0$ is nothing but (6.15) with $\ell = 0$. Assume that (6.15) holds for $\ell = j - 1$. Then by adding $\frac{d}{2Cj^2(1+\delta^2)} \times$(7.21) to (6.15) with $\ell = j - 1$, we obtain the desired inequality (6.15) for $\ell = j$. This completes the proof. □

8 Proof of Theorem 3.1

In this section we prove Theorem 3.1.

We first establish the estimates for nonlinear and inhomogeneous terms $F_1(u, g)$ and $F_\infty(u, g)$:

$$
F_1(u,g) = \begin{pmatrix} F^0_1(u) \\ \tilde{F}_1(u,g) \end{pmatrix}
$$

$$= -P_1 \begin{pmatrix} \gamma(\phi \mathrm{div} + w \cdot \nabla \phi) \\ \phi \partial_t w + \gamma((1+\phi) w \cdot \nabla w) + \frac{\rho_*}{\gamma}(\nabla(P^{(1)}(\rho_* \phi)\phi^2)) - \frac{1}{\gamma}(1+\phi) g \end{pmatrix},$$

$$\begin{aligned} F_\infty(u,g) &= \begin{pmatrix} F^0_\infty(u) \\ \tilde{F}_\infty(u,g) \end{pmatrix} \\ &= -P_\infty \begin{pmatrix} \gamma(\phi \mathrm{div} + w \cdot \nabla \phi_1) \\ \phi \partial_t w + \gamma((1+\phi) w \cdot \nabla w) + \frac{\rho_*}{\gamma}(\nabla(P^{(1)}(\rho_* \phi)\phi^2)) - \frac{1}{\gamma}(1+\phi) g \end{pmatrix}, \end{aligned}$$

where

$$u = u_1 + u_\infty, \quad u = \begin{pmatrix} \phi \\ w \end{pmatrix}, \quad u_j = \begin{pmatrix} \phi_j \\ w_j \end{pmatrix}, \quad (j = 1, \infty).$$

We first state the estimates for $F_1(u,g)$ and $F_\infty(u,g)$.

Proposition 8.1. *There hold the estimates*

(i) $$\|F^0_1(u)\|_{L^1_1} \le C(\|\phi\|_{L^2}\|\mathrm{div} w\|_{L^2_1} + \|w\|_{L^2}\|\nabla \phi\|_{L^2_1}),$$

(ii) $$\begin{aligned} \|\tilde{F}_1(u,g)\|_{L^1_1} \le C(&\|\phi\|_{L^2}\|\partial_t w\|_{L^2_1} + \|w\|_{L^2}\|\nabla w\|_{L^2_1} \\ &+ \|\phi\|_{L^2}\|\nabla \phi\|_{L^2_1} + \|\phi\|_{L^2}\|g\|_{L^2_1} + \|g\|_{L^1_1}), \end{aligned}$$

(iii) $$\|F^0_\infty(u)\|_{H^m_1} \le C(\|\phi\|_{H^m}\|\mathrm{div} w\|_{H^m_1} + \|w\|_{H^m}\|\nabla \phi_1\|_{H^m_1}),$$

(iv) $$\begin{aligned} \|\tilde{F}_\infty(u,g))\|_{H^{m-1}_1} \le C\{&\|w\|_{H^m}\|\nabla w\|_{H^{m-1}_1} + \|\phi\|_{H^m}\|\nabla \phi\|_{H^{m-1}_1} \\ &+ \|\phi\|_{H^m}\|\partial_t w\|_{H^{m-1}_1} + (1 + \|\phi\|_{H^m})\|g\|_{H^{m-1}_1}\} \end{aligned}$$

uniformly for $u = {}^\top(\phi, w) = u_1 + u_\infty$ *with* $u_k = {}^\top(\phi_k, w_k)$ $(k = 1, \infty)$ *satisfying* $\|\phi\|_{L^\infty} \le \frac{1}{2}$ *and* $\|u\|_{H^m} \le 1$.

Proposition 8.1 directly follows from Lemmas 2.1 and 2.3.

We next estimate $F_j(u^{(1)}, g) - F_j(u^{(2)}, g)$ $(j = 1, \infty)$.

Proposition 8.2. *There hold the estimates*

(i) $$\begin{aligned} &\|F^0_1(u^{(1)}) - F^0_1(u^{(2)})\|_{L^1_1} \\ &\quad \le C\{\|\phi^{(1)} - \phi^{(2)}\|_{L^2}\|\mathrm{div} w^{(1)}\|_{L^2_1} + \|\phi^{(2)}\|_{L^2}\|\mathrm{div}(w^{(1)} - w^{(2)})\|_{L^2_1} \\ &\qquad + \|w^{(1)} - w^{(2)}\|_{L^2}\|\nabla \phi^{(1)}\|_{L^2_1} + \|w^{(2)}\|_{L^2}\|\nabla(\phi^{(1)} - \phi^{(2)})\|_{L^2_1}\}, \end{aligned}$$

(ii) $$\|\tilde F_1(u^{(1)},g)-\tilde F_1(u^{(2)},g)\|_{L^1_1}$$
$$\le C\{\|w^{(1)}-w^{(2)}\|_{L^2}\|\nabla w^{(1)}\|_{L^2_1}+\|w^{(2)}\|_{L^2}\|\nabla(w^{(1)}-w^{(2)})\|_{L^2_1}$$
$$+\|\phi^{(1)}-\phi^{(2)}\|_{L^2}(\|w^{(1)}\|_{L^\infty}\|\nabla w^{(1)}\|_{L^2_1}+\|\partial_t w^{(1)}\|_{L^2_1}+\|g\|_{L^2_1})$$
$$+\|\phi^{(2)}\|_{L^2}\|\partial_t(w^{(1)}-w^{(2)})\|_{L^2_1}$$
$$+(\|\nabla\phi^{(1)}\|_{L^2_1}+\|\nabla\phi^{(2)}\|_{L^2_1})\|\phi^{(1)}-\phi^{(2)}\|_{L^2}+\|\phi^{(1)}\|_{L^2}\|\nabla(\phi^{(1)}-\phi^{(2)})\|_{L^2_1}\},$$

(iii) $$\|F^0_\infty(u^{(1)})-F^0_\infty(u^{(2)}))\|_{H^{m-1}_1}$$
$$\le C\{\|\mathrm{div} w^{(1)}\|_{H^m_1}\|\phi^{(1)}-\phi^{(2)}\|_{H^{m-1}}+\|\phi^{(2)}\|_{H^m}\|\mathrm{div}(w^{(1)}-w^{(2)})\|_{H^{m-1}_1}$$
$$+\|\nabla\phi^{(1)}_1\|_{H^m_1}\|w^{(1)}-w^{(2)}\|_{H^{m-1}}+\|w^{(2)}\|_{H^m}\|\nabla(\phi^{(1)}_1-\phi^{(2)}_1)\|_{H^{m-1}_1}\},$$

(iv) $$\|\tilde F_\infty(u^{(1)},g)-\tilde F_\infty(u^{(2)},g))\|_{H^{m-2}_1}$$
$$\le C\{\|\phi^{(1)}-\phi^{(2)}\|_{H^{m-1}}(\|w^{(1)}\|_{H^m}\|\nabla w^{(1)}\|_{H^{m-1}_1}+\|\partial_t w^{(1)}\|_{H^{m-1}_1}+\|g\|_{H^{m-1}_1})$$
$$+\|w^{(1)}-w^{(2)}\|_{H^{m-1}}\|\nabla w^{(1)}\|_{H^{m-1}_1}+\|w^{(2)}\|_{H^m}\|\nabla(w^{(1)}-w^{(2)})\|_{H^{m-2}_1}$$
$$+\|\phi^{(2)}\|_{H^m}\|\partial_t(w^{(1)}-w^{(2)})\|_{H^{m-2}_1}$$
$$+(\|\nabla\phi^{(1)}\|_{H^{m-1}_1}+\|\nabla\phi^{(2)}\|_{H^{m-1}_1})\|\phi^{(1)}-\phi^{(2)}\|_{H^{m-1}}$$
$$+\|\phi^{(1)}\|_{H^m}\|\nabla(\phi^{(1)}-\phi^{(2)})\|_{H^{m-2}_1}\}$$

uniformly for $u^{(j)}={}^\top(\phi^{(j)},w^{(j)})=u^{(j)}_1+u^{(j)}_\infty$ *with* $u^{(j)}_k={}^\top(\phi^{(j)}_k,w^{(j)}_k)$ $(k=1,\infty)$ *satisfying* $\|\phi^{(j)}\|_{L^\infty}\le\frac12$ *and* $\|u^{(j)}\|_{H^m}\le 1$ $(j=1,2)$.

Proposition 8.2 directly follows from Lemmas 2.1–2.3.

To prove Theorem 3.1, we next show the existence of a solution $\{u_1,u_\infty\}$ of (4.1)-(4.2) on $[0,T]$ satisfying $u_j(0)=u_j(T)$ $(j=1,\infty)$ by an iteration argument.

For $p=0$, we define $u^{(0)}_1={}^\top(\phi^{(0)}_1,w^{(0)}_1)$ and $u^{(0)}_\infty={}^\top(\phi^{(0)}_\infty,w^{(0)}_\infty)$ by

$$\begin{cases} u^{(0)}_1(t)=S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}\mathbb{G}_1]+\mathscr{S}_1(t)[\mathbb{G}_1],\\ u^{(0)}_\infty(t)=S_{\infty,0}(t)(I-S_{\infty,0}(T))^{-1}\mathscr{S}_{\infty,0}(T)[\mathbb{G}_\infty]+\mathscr{S}_{\infty,0}(t)[\mathbb{G}_\infty],\end{cases}\tag{8.1}$$

where $t\in[0,T]$, $\mathbb{G}={}^\top(0,\frac1\gamma g(x,t))$, $\mathbb{G}_1=P_1\mathbb{G}$ and $\mathbb{G}_\infty=P_\infty\mathbb{G}$. Note that $u^{(0)}_j(0)=u^{(0)}_j(T)\quad(j=1,\infty)$.

For $p\ge1$, we define $u^{(p)}_1={}^\top(\phi^{(p)}_1,w^{(p)}_1)$ and $u^{(p)}_\infty={}^\top(\phi^{(p)}_1,w^{(p)}_1)$, inductively, by

$$\begin{cases} u^{(p)}_1(t)&=S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}F_1(u^{(p-1)},g)]+\mathscr{S}_1(t)[F_1(u^{(p-1)},g)],\\ u^{(p)}_\infty(t)&=S_{\infty,u^{(p-1)}}(t)(I-S_{\infty,u^{(p-1)}}(T))^{-1}\mathscr{S}_{\infty,u^{(p-1)}}(T)[F_\infty(u^{(p-1)},g)]\\ &\quad+\mathscr{S}_{\infty,u^{(p-1)}}(t)[F_\infty(u^{(p-1)},g)],\end{cases}\tag{8.2}$$

where $t\in[0,T]$ and $u^{(p-1)}=u^{(p-1)}_1+u^{(p-1)}_\infty$. Note that $u^{(p)}_j(0)=u^{(p)}_j(T)$ for $j=1,\infty$ and $p\ge1$.

Proposition 8.3. *There exists a constant $\delta_0 > 0$ such that if $[g]_m \leq \delta_0$, then there holds the estimates*

$$\text{(i)} \qquad \|\{u_1^{(p)}, u_\infty^{(p)}\}\|_{\mathscr{X}^m(0,T)} \leq C_1[g]_m$$

for all $p \geq 0$, and

$$\text{(ii)} \qquad \begin{aligned} &\|\{u_1^{(p+1)} - u_1^{(p)}, u_\infty^{(p+1)} - u_\infty^{(p)}\}\|_{\mathscr{X}^{m-1}(0,T)} \\ &\leq \ C_1[g]_m\|\{u_1^{(p)} - u_1^{(p-1)}, u_\infty^{(p)} - u_\infty^{(p-1)}\}\|_{\mathscr{X}^{m-1}(0,T)} \end{aligned}$$

for $p \geq 1$. Here C_1 is a constant independent of g and p.

Proof. The estimate (i) follows from Propositions 5.4, 6.8, 8.1 and Lemma 4.3 (ii).

Let us consider the estimate of the difference between $u^{(p+1)}$ and $u^{(p)}$. For $p \geq 0$, we set $\bar{\phi}_j^{(p)} = \phi_j^{(p+1)} - \phi_j^{(p)}$ and $\bar{w}_j^{(p)} = w_j^{(p+1)} - w_j^{(p)}$ for $j = 1, \infty$. Then by using (8.1) and (8.2), we see that $\bar{\phi}_j^{(p)}$ and $\bar{w}_j^{(p)}$ $(p \geq 1)$ satisfy

$$\begin{cases} \partial_t\bar{\phi}_1^{(p)} + \gamma\text{div}\bar{w}_1^{(p)} = F_{11}(\bar{u}^{(p-1)}), \\ \partial_t\bar{w}_1^{(p)} - \nu\triangle\bar{w}_1^{(p)} - \tilde{\nu}\nabla\text{div}\bar{w}_1^{(p)} + \gamma\nabla\bar{\phi}_1^{(p)} = F_{12}(\bar{u}^{(p-1)}, g), \end{cases} \tag{8.3}$$

$$\begin{cases} \partial_t\bar{\phi}_\infty^{(p)} + \gamma P_\infty(w^{(p)} \cdot \nabla\bar{\phi}_\infty^{(p)}) + \gamma\text{div}\bar{w}_\infty^{(p)} = F_{\infty1}(\bar{u}^{(p-1)}), \\ \partial_t\bar{w}_\infty^{(p)} - \nu\triangle\bar{w}_\infty^{(p)} - \tilde{\nu}\nabla\text{div}\bar{w}_\infty^{(p)} + \gamma\nabla\bar{\phi}_\infty^{(p)} = F_{\infty2}(\bar{u}^{(p-1)}, g), \end{cases} \tag{8.4}$$

where

$$\begin{aligned} &F_{11}(\bar{u}^{(p-1)}) = F_1^0(u^{(p)}) - F_1^0(u^{(p-1)}), \\ &F_{12}(\bar{u}^{(p-1)}, g) = \tilde{F}_1(u^{(p)}, g) - \tilde{F}_1(u^{(p-1)}, g), \\ &F_{\infty1}(\bar{u}^{(p-1)}) = F_\infty^0(u^{(p)}) - F_\infty^0(u^{(p-1)}) - \gamma P_\infty((w^{(p)} - w^{(p-1)}) \cdot \nabla\phi_\infty^{(p)}), \\ &F_{\infty2}(\bar{u}^{(p-1)}, g) = \tilde{F}_\infty(u^{(p)}, g) - \tilde{F}_\infty(u^{(p-1)}, g). \end{aligned}$$

The desired inequality (ii) can be obtained by applying Propositions 5.4, 6.8, 8.2, 8.3 (i) and Lemma 4.3 (ii). This completes the proof. □

We introduce a notation. We denote by $B_{\mathscr{X}^k(a,b)}(r)$ the closed unit ball of $\mathscr{X}^k(a,b)$ centered at 0 with radius r, i.e.,

$$B_{\mathscr{X}^k(a,b)}(r) = \left\{\{u_1, u_\infty\} \in \mathscr{X}^k(a,b); \|\{u_1, u_\infty\}\|_{\mathscr{X}^k(a,b)} \leq r\right\}.$$

Proposition 8.4. *There exists a constant $\delta_1 > 0$ such that if $[g]_m \leq \delta_1$, then the system* (4.1)-(4.2) *has a unique solution $\{u_1, u_\infty\}$ on $[0,T]$ in $B_{\mathscr{X}^m(0,T)}(C_1[g]_m)$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$. The uniqueness of solutions of* (4.1)-(4.2) *on $[0,T]$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ holds in $B_{\mathscr{X}^m(0,T)}(C_1\delta_1)$.*

Proof. Let $\delta_1 = \min\{\delta_0, \frac{1}{2C_1}\}$ with δ_0 given in Propositions 8.3. By Propositions 8.3, we see that if $[g]_m \le \delta_1$, then $u_j^{(p)} = {}^\top(\phi_j^{(p)}, w_j^{(p)})$ $(j = 1, \infty)$ converges to $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$ in the sense

$$\{u_1^{(p)}, u_\infty^{(p)}\} \ \to \ \{u_1, u_\infty\} \text{ in } \ \mathscr{X}^{m-1}(0, T),$$
$$u_\infty^{(p)} = {}^\top(\phi_\infty^{(p)}, w_\infty^{(p)}) \ \to \ u_\infty = {}^\top(\phi_\infty, w_\infty) \ *\text{-weakly in } L^\infty(0, T; H^m_{(\infty),1}),$$
$$w_\infty^{(p)} \ \to \ w_\infty \text{ weakly in } L^2(0, T; H^{m+1}_{(\infty),1}) \cap H^1(0, T; H^{m-1}_{(\infty),1}).$$

It is not difficult to see that $\{u_1, u_\infty\}$ is a solution of (4.1)-(4.2) satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

It remains to prove $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0, T]; H_1^m)$, which implies $\{u_1, u_\infty\} \in B_{\mathscr{X}^m(0,T)}(C_1[g]_m)$ with $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

As for w_∞, since $L^2(0, T; H^{m+1}) \cap H^1(0, T; H^{m-1}) \subset C([0, T]; H^m)$, we find that $w_\infty \in C([0, T]; H^m)$.

As for ϕ_∞, note that $\phi_\infty \in C([0, T]; H^1)$ and ϕ_∞ is the solution of

$$\begin{cases} \partial_t \phi_\infty + w \cdot \nabla \phi_\infty = g_\infty^0, \\ \phi_\infty|_{t=0} = \phi_{0\infty}, \end{cases} \tag{8.5}$$

where

$$g_\infty^0 \in L^2(0, T; H^m), \ \phi_{0\infty} \in H^m.$$

On the other hand, by Lemma 6.1, we see that there exists a solution of (8.5) which belongs to $C([0, T]; H^m)$ and that the uniqueness of solutions of (8.5) holds in $C([0, T]; H^1)$. Therefore, we find that

$$\phi_\infty \in C([0, T]; H^m).$$

To prove that $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0, T]; H_1^m)$, we note that u_∞ is written as

$$u_\infty(t) = S_{\infty,u}(t)(I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u, g)] + \mathscr{S}_{\infty,u}(t)[F_\infty(u, g)]$$

with $u = u_1 + u_\infty$. By Proposition 8.1 and Lemma 4.3 (ii), we see that $F_\infty(u, g) \in L^2(0, T; H^m_{(\infty),1} \times H^{m-1}_{(\infty),1})$. It then follows from Proposition 6.5 that if δ_1 is small such that $C_1\delta_1 \le \delta$, then $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0, T]; H_1^m)$. This completes the proof. □

To complete the construction of a time periodic solution of (1.5), we use the following proposition on the unique existence of solutions to the initial value problem.

Proposition 8.5. *Let $s \in \mathbb{R}$ and let $U_0 = U_{01} + U_{0\infty}$ with $U_{01} \in H^1_{(1),1}$ and $U_{0\infty} \in H^m_{(\infty),1}$. Then there exist constants $\delta_2 > 0$ and $C_2 > 0$ such that if*

$$M(U_{01}, U_{0\infty}, g) := \|U_{01}\|_{H^1_{(1),1}} + \|U_{0\infty}\|_{H^m_{(\infty),1}} + [g]_m \le \delta_2,$$

there exists a solution $\{U_1, U_\infty\}$ of the initial value problem for (4.1)-(4.2) on $[s, s+T]$ in $B_{\mathscr{X}^m(s,s+T)}(C_2 M(U_{01}, U_{0\infty}, g))$ satisfying the initial condition $U_j|_{t=s} = U_{0j}$ $(j = 0, \infty)$. The uniqueness for this initial value problem holds in $B_{\mathscr{X}^m(s,s+T)}(C_2\delta_2)$.

Proof. Let $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ be a given function in $C([s, s+T]; H^m) \cap L^2(s, s+T; H^{m+1})$. We define $S_{\infty,\tilde{u}}(t,s)u_{0\infty}$ and $\mathscr{S}_{\infty,\tilde{u}}(t,s)F_\infty$ by the solution operators for

$$\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \quad u_\infty|_{t=s} = u_{0\infty}.$$

with $F_\infty = 0$ and $u_{0\infty} = 0$, respectively. As in the proof of Proposition 6.5, one can see that if $\tilde{w}$ satisfies

$$\|\tilde{w}\|_{C([s,s+T];H^m)\cap L^2(s,s+T;H^{m+1})} \leq \delta, \tag{8.6}$$

then it holds that $S_{\infty,\tilde{u}}(t,s)$ and $\mathscr{S}_{\infty,\tilde{u}}(t,s)$ satisfy the estimates

$$\|S_{\infty,\tilde{u}}(t,s)U_{0\infty}\|_{H^k_{(\infty),1}} \leq Ce^{-a(t-s)}\|U_{0\infty}\|_{H^k_{(\infty),1}}, \tag{8.7}$$

$$\|\mathscr{S}_{\infty,\tilde{u}}(t,s)[F_\infty]\|_{H^k_{(\infty),1}} \leq C\left\{\int_s^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^k_{(\infty),1}\times H^{k-1}_{(\infty),1}}\,d\tau\right\}^{\frac{1}{2}} \tag{8.8}$$

for $t \in [s, s+T]$, $F_{0\infty} \in H^k_{(\infty),1}$, $F_\infty \in L^2(s, s+T; H^k_{(\infty),1} \times H^{k-1}_{(\infty),1})$ and $k = m-1$ or m with $C = C(\delta, T) > 0$ uniformly for $s \in \mathbb{R}$ and $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfying (8.6).

To prove Proposition 8.5, it now suffices to show the unique existence of the solution $\{U_1, U_\infty\} \in B_{\mathscr{X}^m(s,s+T)}(C_2M(U_{01}, U_{0\infty}, g))$ of

$$\begin{cases} U_1(t) &= S_1(t-s)U_{01} + \mathscr{S}_1(t-s)[F_1(U,g)], \\ U_\infty(t) &= S_{\infty,U}(t,s)u_{0\infty} + \mathscr{S}_{\infty,U}(t,s)[F_\infty(U,g)], \end{cases} \tag{8.9}$$

with $U = U_1 + U_\infty$ for a constant $C_2 > 0$, provided that $M(U_{01}, U_{0\infty}, g)$ is sufficiently small. We solve this problem by an iteration argument as in the proof of Proposition 8.4.

For $p = 0$, we define $U_j^{(0)} = {}^\top(\Phi_j^{(0)}, W_j^{(0)})$ $(j = 1, \infty)$ by

$$\begin{cases} U_1^{(0)}(t) = S_1(t-s)U_{01} + \mathscr{S}_1(t-s)[\mathbb{G}_1], \\ U_\infty^{(0)}(t) = S_{\infty,0}(t,s)U_{0\infty} + \mathscr{S}_{\infty,0}(t,s)[\mathbb{G}_\infty], \end{cases}$$

where $t \in [s, s+T]$, $\mathbb{G} = {}^\top(0, \frac{1}{\gamma}g(x,t))$, $\mathbb{G}_1 = P_1\mathbb{G}$ and $\mathbb{G}_\infty = P_\infty\mathbb{G}$.

For $p \geq 1$, we define $U_j^{(p)} = {}^\top(\Phi_j^{(p)}, W_j^{(p)})$ $(j = 1, \infty)$, inductively, by

$$\begin{cases} U_1^{(p)}(t) &= S_1(t-s)U_{01} + \mathscr{S}_1(t-s)[F_1(U^{(p-1)}, g)], \\ U_\infty^{(p)}(t) &= S_{\infty,U^{(p-1)}}(t,s)U_{0\infty} + \mathscr{S}_{\infty,U^{(p-1)}}(t,s)[F_\infty(U^{(p-1)}, g)], \end{cases}$$

where $t \in [s, s+T]$ and $U^{(p-1)} = U_1^{(p-1)} + U_\infty^{(p-1)}$.

As in the proof of Proposition 8.3, by using Proposition 5.1, (8.7), (8.8), Propositions 8.1, 8.2 and Lemma 4.3 (ii), we can inductively show that if $M(U_{01}, U_{0\infty}, g)$ is sufficiently small, then there hold the estimates

$$\|\{U_1^{(p)}, U_\infty^{(p)}\}\|_{\mathscr{X}^m(s,s+T)} \leq C_2M(U_{01}, U_{0\infty}, g)$$

for all $p \geq 0$, and

$$\begin{aligned}
&\|\{U_1^{(p+1)} - U_1^{(p)}, U_\infty^{(p+1)} - U_\infty^{(p)}\}\|_{\mathscr{X}^{m-1}(s,s+T)} \\
&\leq \ C_2 M(U_{01}, U_{0\infty}, g)\|\{U_1^{(p)} - U_1^{(p-1)}, U_\infty^{(p)} - U_\infty^{(p-1)}\}\|_{\mathscr{X}^{m-1}(s,s+T)}
\end{aligned}$$

for all $p \geq 1$. Hence, in a similar manner to the proof of Proposition 8.4, we see that there exists a solution $\{U_1, U_\infty\} \in B_{\mathscr{X}^m(s,s+T)}(C_2 M(U_{01}, U_{0\infty}, g))$ of (4.1)-(4.2) satisfying $U_j|_{t=s} = U_{0j}$ $(j = 0, \infty)$, provided that $M(U_{01}, U_{0\infty}, g) \leq \delta_2$ for a small constant $\delta_2 > 0$. In view of the iteration argument, we can see that the uniqueness holds in $B_{\mathscr{X}^m(s,s+T)}(C_2\delta_2)$. This completes the proof. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By Proposition 8.4, we see that if $[g]_m \leq \delta_1$, then (4.1)-(4.2) has a unique solution $\{u_1^{(0)}, u_\infty^{(0)}\} \in B_{\mathscr{X}^m(0,T)}(C_1[g]_m)$ satisfying $u_j^{(0)}(0) = u_j^{(0)}(T)$ $(j = 1, \infty)$. In particular, it holds that

$$\sup_{t\in[0,T]} \left\{\|u_1^{(0)}(t)\|_{H^1_{(1),1}} + \|u_\infty^{(0)}(t)\|_{H^m_{(\infty),1}}\right\} \leq C_1[g]_m. \tag{8.10}$$

Therefore, if g satisfies $(C_1 + 1)[g]_m \leq \delta_2$, then, by Proposition 8.5, we see that there exists a unique solution $\{u_1^{(1)}, u_\infty^{(1)}\} \in B_{\mathscr{X}^m(T,2T)}(C_2(C_1 + 1)[g]_m)$ of (4.1)-(4.2) satisfying $u_j^{(1)}|_{t=T} = u_j^{(0)}(T) = u_j^{(0)}(0)$ $(j = 1, \infty)$.

We introduce $\tilde{u}_j^{(1)}$ $(j = 1, \infty)$ and $\tilde{u}^{(1)}$ by

$$\tilde{u}_j^{(1)}(t) = u_j^{(1)}(t + T), \quad \tilde{u}^{(1)}(t) = \tilde{u}_1^{(1)}(t) + \tilde{u}_\infty^{(1)}(t) \ \text{ for } \ t \in [0, T].$$

Then we find that

$$\tilde{u}_1^{(1)}(0) = u_1^{(1)}(T) = u_1^{(0)}(T) = u_1^{(0)}(0),$$

$$\begin{aligned}
\partial_t \tilde{u}_1^{(1)}(t) + A\tilde{u}_1^{(1)}(t) = \partial_t u_1^{(1)}(t + T) + Au_1^{(1)}(t + T) \ &= \ F_1(u^{(1)}(t + T), g(t + T)) \\
&= F_1(\tilde{u}^{(1)}(t), g(t)).
\end{aligned}$$

Similarly, we see that

$$\tilde{u}_\infty^{(1)}(0) = u_\infty^{(0)}(0),$$

$$\partial_t \tilde{u}_\infty^{(1)}(t) + A\tilde{u}_\infty^{(1)}(t) + P_\infty(B[\tilde{u}^{(1)}(t)]\tilde{u}^{(1)}(t)) = F_\infty(\tilde{u}^{(1)}(t), g(t)).$$

Therefore, if $[g]_m \leq \delta_3 := \min\{\delta_1, \frac{C_2\delta_2}{C_1}, (C_1+1)\delta_2\}$, then, by the uniqueness of the solution, we find that $\{\tilde{u}_1^{(1)}(t), \tilde{u}_\infty^{(1)}(t)\} = \{u_1^{(0)}(t), u_\infty^{(0)}(t)\}$ for $t \in [0, T]$. Consequently, we have $\{u_1^{(1)}(t), u_\infty^{(1)}(t)\} = \{u_1^{(0)}(t - T), u_\infty^{(0)}(t - T)\}$ for $t \in [T, 2T]$.

We define $\{u_1(t), u_\infty(t)\}$ $(t \in [0, 2T])$ by $\{u_1(t), u_\infty(t)\} = \{u_1^{(k)}(t), u_\infty^{(k)}(t)\}$ for $t \in [kT, (k + 1)T]$, $k = 0, 1$. It then follows that $\{u_1(t + T), u_\infty(t + T)\} = \{u_1(t), u_\infty(t)\}$ for $t \in [0, T]$. Furthermore, we see from Proposition 8.5 and (8.10) that there exists a unique solution $\{v_1, v_\infty\} \in B_{\mathscr{X}^m(\frac{T}{2},\frac{3T}{2})}(C_2(C_1+1)[g]_m)$ of (4.1)-(4.2) on $[\frac{T}{2}, \frac{3T}{2}]$ satisfying $v_j|_{t=\frac{T}{2}} = u_j^{(0)}(\frac{T}{2})$ $(j = 1, \infty)$. By the uniqueness, it follows that $\{v_1, v_\infty\} = \{u_1, u_\infty\}$ on $[\frac{T}{2}, \frac{3T}{2}]$, which implies that $\{u_1, u_\infty\}$ is a solution of (4.1)-(4.2) on $[0, 2T]$ in $\mathscr{X}^m(0, 2T)$. Repeating this argument on intervals $[kT, (k+1)T]$ for $k = \pm 1, \pm 2 \cdots$, we obtain a solution $\{u_1, u_\infty\}$ of (4.1)-(4.2) in $\mathscr{X}^m_{per}(\mathbb{R})$ satisfying $\|\{u_1, u_\infty\}\|_{\mathscr{X}^m(0,T)} \leq C_1[g]_m$ that gives a time periodic solution u of (1.5) by setting $u = u_1 + u_\infty$.

In view of the iteration argument in Propositions 8.3 and 8.4, one can see that the uniqueness of time periodic solutions for (1.5) holds in $\{u = {}^\top(\phi, w); \{P_1 u, P_\infty u\} \in \mathscr{X}^m_{per}(\mathbb{R}), \|\{P_1 u, P_\infty u\}\|_{\mathscr{X}^m(0,T)} \leq C_1\delta_3\}$ if $[g]_m \leq \delta_3$. This completes the proof. □

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