Existence and stability of time periodic solution to the compressible Navier-Stokes equation for time periodic external force with symmetry

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Abstract

Time periodic problem for the compressible Navier-Stokes equation on the whole space is studied. The existence of a time periodic solution is proved for sufficiently small time periodic external force with some symmetry when the space dimension is greater than or equal to 3. The proof is based on the spectral properties of the time-$T$ map associated with the linearized problem around the motionless state with constant density in some weighted Sobolev space. The stability of the time periodic solution is also proved and the decay estimate of the perturbation is established.

1 Introduction

This paper studies time periodic problem of the following compressible Navier-Stokes equation in $\mathbb{R}^n$ ($n \geq 3$):

\begin{equation}
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho v) = 0, \\
\rho(\partial_t v + (v \cdot \nabla)v) - \mu \Delta v - (\mu + \mu') \nabla (\nabla \cdot v) + \nabla P(\rho) = \rho g.
\end{cases}
\end{equation}

Here $\rho = \rho(x,t)$ and $v = (v_1(x,t), \cdots, v_n(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^n$; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of $\rho$ satisfying

$$P'(\rho_*) > 0$$

for a given positive constant $\rho_*$; $\mu$ and $\mu'$ are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \frac{2}{n} \mu + \mu' \geq 0;$$

and $g = g(x,t)$ is a given external force periodic in $t$. We assume that $g = g(x,t)$ satisfies the conditions

\begin{equation}
g(x, t + T) = g(x, t) \quad (x \in \mathbb{R}^n, t \in \mathbb{R}), \\
g(-x, t) = -g(x, t) \quad (x \in \mathbb{R}^n, t \in \mathbb{R})
\end{equation}

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for some constant \( T > 0 \).

The purpose of this paper is to investigate the existence and stability of a time periodic solution of system (1.1) around the constant state \((\rho_*, 0)\).

Concerning the time periodic problem for the compressible Navier-Stokes equations, Valli ([11]) proved the existence and (exponential) stability of time periodic solutions on a bounded domain of \( \mathbb{R}^3 \) for sufficiently small time periodic external forces. On the other hand, for large time periodic external forces, the existence of time periodic solutions on a bounded domain of \( \mathbb{R}^3 \) was proved by Feireisl, Matušu-Necasová, Petzeltová and Straškraba ([2]) and Feireisl, Mucha, Novotný and Pokorný ([3]) in the framework of weak solutions. As for the time periodic problem on unbounded domains, Ma, Ukai, and Yang ([8]) studied the existence and stability of time periodic solutions on the whole space \( \mathbb{R}^n \). It was shown in [8] that if \( n \geq 5 \), there exists a time periodic solution \((\rho_{\text{per}}, v_{\text{per}})\) for a sufficiently small \( g \in C^0(\mathbb{R}; H^{N-1} \cap L^1) \) with \( g(x, t + T) = g(x, t) \), where \( N \in \mathbb{Z} \) satisfying \( N \geq n + 2 \). Furthermore, the time periodic solution is stable under sufficiently small perturbations and there holds the estimate

\[
\|(\rho(t), v(t)) - (\rho_{\text{per}}(t), v_{\text{per}}(t))\|_{H^{N-1}} \leq C(1 + t)^{-\frac{n}{2}} \|(\rho_0, v_0) - (\rho_{\text{per}}(t_0), v_{\text{per}}(t_0))\|_{H^{N-1} \cap L^1},
\]

where \( t_0 \) is a certain initial time and \((\rho, v)|_{t=t_0} = (\rho_0, v_0)\). Here \( H^k \) denotes the \( L^2 \)-Sobolev space on \( \mathbb{R}^n \) of order \( k \).

In this paper, we will show that for \( n \geq 3 \), if the external force \( g \) satisfies the oddness condition (1.2) and is small enough in some weighted Sobolev space, then (1.1) has a time periodic solution \((\rho_{\text{per}}, v_{\text{per}})\) and \( u_{\text{per}}(t) = (\rho_{\text{per}}(t) - \rho_*, v_{\text{per}}(t))\) satisfies

\[
\sup_{t \in [0, T]} (\|u_{\text{per}}(t)\|_{L^2} + \|x \nabla u_{\text{per}}(t)\|_{L^2}) \leq \|g\|_{C([0, T]; L^1 \cap L^2)} + \|x g\|_{C([0, T]; L^1 \cap L^2)} + \|g\|_{L^2(0, T; H^{m-1})} + \|x g\|_{L^2(0, T; H^{m-1})}. \tag{1.3}
\]

In addition, we will prove that the time periodic solution is stable under sufficiently small initial perturbation, and that the perturbation satisfies

\[
\|(\rho(t), v(t)) - (\rho_{\text{per}}(t), v_{\text{per}}(t))\|_{L^2} = O(t^{-\frac{n}{4}}) \text{ as } t \to \infty. \tag{1.4}
\]

The precise statements are given in Theorem 3.1 and Theorem 3.2 below.

The proof of the existence of a time periodic solution is given by an iteration argument by using the time-\( T \)-map associated with the linearized problem around \((\rho_*, 0)\). Substituting \( \phi = \frac{\rho - \rho_*}{\rho_*} \) and \( w = \frac{\psi}{\gamma} \) with \( \gamma = \sqrt{\rho'(\rho_*)} \) into (1.1), we see that (1.1) is rewritten as

\[
\partial_t u + A u = -B[u]u + G(u, g), \tag{1.5}
\]

where

\[
A = \begin{pmatrix} 0 & \gamma \nabla \\ \gamma \nabla & -\nu \Delta - \tilde{\nu} \nabla \nabla \end{pmatrix}; \quad \nu = \frac{\mu}{\rho_*}; \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_*}, \tag{1.6}
\]

\[
B[u]u = \gamma \begin{pmatrix} \tilde{w} \cdot \nabla \phi \\ 0 \end{pmatrix} \text{ for } u = \begin{pmatrix} \phi, w \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} \tilde{\phi}, \tilde{w} \end{pmatrix}. \tag{1.7}
\]
and
\[ G(u, g) = \left( \frac{f^0(u)}{f(u, g)} \right), \quad (1.8) \]
\[ f^0(u) = -\gamma \phi \text{div} w, \quad (1.9) \]
\[ \tilde{f}(u, g) = -\gamma (1 + \phi)(w \cdot \nabla w) - \phi \partial_t w - \nabla (P^{(1)}(\phi) \phi^2) + \frac{1 + \phi}{\gamma} g, \quad (1.10) \]
\[ P^{(1)}(\phi) = \frac{\rho}{\gamma} \int_0^1 (1 - \theta) P''(\rho_*(1 + \theta \phi)) d\theta. \]

To solve the time periodic problem for (1.5), we decompose \( u \) into a low frequency part \( u_1 \) and a high frequency part \( u_\infty \). Then \( u_1 \) and \( u_\infty \) satisfy
\[ \partial_t u_1 + Au_1 = F_1(u, g), \quad (1.11) \]
\[ \partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty(u, g), \quad (1.12) \]
where
\[ F_1(u, g) = P_1[-B[\tilde{u}]u + G(u, g)], \]
\[ F_\infty(u, g) = P_\infty[-B[\tilde{u}]u_1 + G(u, g)] \]
and
\[ \tilde{u} = u = u_1 + u_\infty, \quad u_j = P_j u \quad (j = 1, \infty). \]

Here \( P_1 \) and \( P_\infty \) are bounded linear operators from \( L^2 \) into a low frequency part and a high frequency part, respectively, satisfying \( P_1 + P_\infty = I \). (See sections 3 and 4 for the definitions and properties of \( P_1 \) and \( P_\infty \).)

We rewrite (1.11)-(1.12) as
\[ u_1(t) = S_1(t)u_{01} + \mathcal{J}_1(t)F_1(u, g), \quad (1.13) \]
\[ u_\infty(t) = S_{\infty, \tilde{u}}(t)u_{0\infty} + \mathcal{J}_{\infty, \tilde{u}}(t)F_\infty(u, g), \quad (1.14) \]
where
\[ u_{01} = (I - S_1(T))^{-1}\mathcal{J}_1(T)F_1(u, g), \]
\[ u_{0\infty} = (I - S_{\infty, \tilde{u}}(T))^{-1}\mathcal{J}_{\infty, \tilde{u}}(T)F_\infty(u, g) \]
with
\[ \tilde{u} = u = u_1 + u_\infty. \]

Here \( S_1(t) \) is the solution operator for the linear initial value problem for (1.11) with the inhomogeneous term \( F_1(u, g) \equiv 0 \) under the initial condition \( u_1|_{t=0} = u_{01} \); \( \mathcal{J}_1(t) \) is the one for (1.11) with a given inhomogeneous term \( F_1(u, g) \) under the initial condition \( u_1|_{t=0} = 0 \); and \( S_{\infty, \tilde{u}}(t) \) and \( \mathcal{J}_{\infty, \tilde{u}}(t) \) are similarly defined by the solution operators for the linear initial value problem for (1.12). We will investigate properties of \( S_1(t) \), \( \mathcal{J}_1(t) \), \( S_{\infty, \tilde{u}}(t) \) and \( \mathcal{J}_{\infty, \tilde{u}}(t) \) in weighted Sobolev spaces. The necessary estimates for \( S_1(t) \) and \( \mathcal{J}_1(t) \) will be obtained by the explicit formulas for these operators through the Fourier
transform, while those for $S_{\infty, \mathcal{A}}(t)$ and $\mathcal{S}_{\infty, \mathcal{A}}(t)$ will be established by a weighted energy method.

The stability of the time periodic solution will be also shown by a decomposition method associated with the spectral properties of the linearized operator which, in this case, is a decomposition into low and high frequency parts (cf., [4, 10]). Based on the estimate (1.3) for $u_{\text{per}}(t) = \tilde{\tau}(\rho_{\text{per}}(t) - \rho_\ast, v_{\text{per}}(t))$, we can apply the Hardy inequality to show the stability of the time periodic solution $\tilde{\tau}(\rho_{\text{per}}(t), v_{\text{per}}(t))$ under sufficiently small initial perturbations and the decay estimate (1.4) in a similar manner to [10]. In contrast to the problem in [10], the terms $v_{\text{per}} \cdot \nabla \phi + \phi \div v_{\text{per}}$ appear in the transport equation for the perturbation. These terms can be handled by using the energy method and the boundedness properties of the projection onto the low frequency part as in [4], together with the Hardy inequality. (See also [1]).

This paper is organized as follows. In section 2, we introduce notations and auxiliary lemmas used in this paper. In section 3, we state main results of this paper. In section 4, we reformulate the problem. Section 5 is devoted to studying $S_1(t)$ and $\mathcal{S}_1(t)$ and we derive the estimates for low frequency part. In section 6, we state some spectral properties of $S_{\infty, \mathcal{A}}(t)$ and $\mathcal{S}_{\infty, \mathcal{A}}(t)$ and derive the estimates for high frequency part. In section 7, we establish the weighted energy estimate for high frequency part which gives spectral information on $S_{\infty, \mathcal{A}}(t)$ and $\mathcal{S}_{\infty, \mathcal{A}}(t)$. In section 8, we estimate nonlinear terms and then give a proof of the existence of a time periodic solution.

## 2 Preliminaries

In this section we first introduce some notations which will be used throughout this paper. We then introduce some auxiliary lemmas which will be useful in the proof of the main results.

For a given Banach space $X$, the norm on $X$ is denoted by $\| \cdot \|_X$.

Let $1 \leq p \leq \infty$. $L^p$ stands for the usual $L^p$ space over $\mathbb{R}^n$. The inner product of $L^2$ is denoted by $(\cdot, \cdot)$. For a nonnegative integer $k$, $H^k$ stands for the usual $L^2$-Sobolev space of order $k$. (As usual, $H^0 = L^2$.)

The set of all vector fields $w = \tau(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in L^p$ ($j = 1, \cdots, n$), i.e., $(L^p)^n$, is simply denoted by $L^p$; and the norm $\| \cdot \|_{(L^p)^n}$ on it is denoted by $\| \cdot \|_{L^p}$ if no confusion will occur. Similarly, for a function space $X$, the set of all vector fields $w = \tau(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in X$ ($j = 1, \cdots, n$), i.e., $X^n$, is simply denoted by $X$; and the norm $\| \cdot \|_{X^n}$ on it is denoted by $\| \cdot \|_X$ if no confusion will occur. (For example, $(H^k)^n$ is simply denoted by $H^k$ and the norm $\| \cdot \|_{(H^k)^n}$ is denoted by $\| \cdot \|_{H^k}$.)

For $u = \tau(\phi, w)$ with $\phi \in H^k$ and $w = \tau(w_1, \cdots, w_n) \in H^n$, we define the norm $\|u\|_{H^k \times H^n}$ of $u$ on $H^k \times H^n$ by

$$\|u\|_{H^k \times H^n} = (\|\phi\|^2_{H^k} + \|w\|^2_{H^n})^{\frac{1}{2}}.$$  

When $m = k$, we simply write $H^k \times (H^k)^n$ as $H^k$, and, also, $\|u\|_{H^k \times (H^k)^n}$ as $\|u\|_{H^k}$ if no
confusion will occur:

\[ H^k := H^k \times (H^k)^n, \quad \|u\|_{H^k} := \|u\|_{H^k \times (H^k)^n} \quad (u = \tau(\phi, w)). \]

Similarly, when \( u = \tau(\phi, w) \in X \times Y \) with \( w = \tau(w_1, \ldots, w_n) \) for function spaces \( X \) and \( Y \), we denote its norm \( \|u\|_{X \times Y} \) by

\[ \|u\|_{X \times Y} = \left( \|\phi\|_X^2 + \|w\|_Y^2 \right)^{\frac{1}{2}} \quad (u = \tau(\phi, w)). \]

When \( Y = X^n \), we simply write \( X \times X^n \) as \( X \), and also its norm \( \|u\|_{X \times X^n} \) as \( \|u\|_X \):

\[ X := X \times X^n, \quad \|u\|_X := \|u\|_{X \times X^n} \quad (u = \tau(\phi, w)). \]

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi-index. We use the following notation

\[ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{j=1}^n \alpha_j. \]

For any integer \( l \geq 0 \), \( \nabla^l f \) denotes \( x \)-derivatives of order \( l \) of a function \( f \).

For \( 1 \leq p < \infty \), \( L^p((1 + |x|^p) dx) \) stands for the weighted \( L^p \) space over \( \mathbb{R}^n \) defined by

\[ L^p((1 + |x|^p) dx) = \left\{ f; \int_{\mathbb{R}^n} |f(x)|^p (1 + |x|^p) dx < +\infty \right\}. \]

In particular, we denote \( L^1((1 + |x|) dx) \) by \( L_1^1 \).

We denote by \( \hat{f} \) or \( \mathcal{F}[f] \) the Fourier transform of \( f \):

\[ \hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} dx \quad (\xi \in \mathbb{R}^n). \]

The inverse Fourier transform of \( f \) is denoted by \( \mathcal{F}^{-1}[f] \):

\[ \mathcal{F}^{-1}[f](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi)e^{ix \cdot \xi} d\xi \quad (x \in \mathbb{R}^n). \]

For a nonnegative integer \( k \) and positive constants \( r_1 \) and \( r_\infty \) with \( r_1 < r_\infty \), \( H^k_1 \) denotes the set of all \( f \in H^k \) satisfying \( \text{supp} \; \hat{f} \subset \{ |\xi| \geq r_1 \} \), and \( H^{(1)}_k \) denotes the set of all \( f \in H^k \) satisfying \( \text{supp} \; \hat{f} \subset \{ |\xi| \leq r_\infty \} \).

For a nonnegative integers \( k \) and \( \ell \), the spaces \( H^k_\ell \) and \( H^k_{(\infty),\ell} \) are defined by

\[ H^k_\ell = \{ f \in H^k; \|f\|_{H^k_\ell} < +\infty \}, \]

where

\[ \|f\|_{H^k_\ell} = \left( \sum_{j=0}^\ell \|f\|_{H^j_k}^2 \right)^{\frac{1}{2}}. \]
\[ |f|_{H^k} = \left( \sum_{|\alpha| \leq k} \| x^\alpha \partial_x f \|_{L^2}^2 \right)^{\frac{1}{2}}, \]

and

\[ H_{(\infty),t}^k = \{ f \in H_{(\infty)}^k; \| f \|_{H^k_t} < +\infty \}. \]

The space \( L^2_{(1),1} \) is defined by

\[ L^2_{(1),1} = \{ f \in L^2_1; f \in L^2_{(1)} \}; \]

and the space \( H^1_{(1),1} \) is defined by

\[ H^1_{(1),1} = \{ f \in H^1_{(1)}; \| f \|_{H^1_{(1),1}} < +\infty \}, \]

where

\[ \| f \|_{H^1_{(1),1}} = (\| f \|_{L^2}^2 + \| x \nabla f \|_{L^2}^2)^{\frac{1}{2}}. \]

We note that \( H^k_{(\infty),t} \) and \( H^1_{(1),1} \) are closed subspaces of \( H^k_t \) and \( H^1_1 \), respectively.

We will consider the time periodic problem in function spaces with some symmetry. We define \( \Gamma \) by

\[ (\Gamma u)(x) = \top(\phi(-x),-w(-x)) \quad (u(x) = \top(\phi(x),w(x)), \ x \in \mathbb{R}^n). \]

We indicate function spaces satisfying the symmetric condition \( \Gamma u = u \) by the subscript \( _{sym} \). More precisely, We denote by \( X_{sym} \) the set of all \( u = \top(\phi,w) \in X \) satisfying the symmetric conditions \( \Gamma u = u \), i.e., \( \phi(-x) = \phi(x) \) and \( w(-x) = -w(x) \) \( (x \in \mathbb{R}^n) \):

\[ X_{sym} = \{ u = \top(\phi,w) \in X; \Gamma u = u \}. \]

Let \(-\infty \leq a < b \leq \infty \). We denote by \( C^k([a,b];X) \) the set of all \( C^k \) functions on \([a,b]\) with values in \( X \). The Bochner space on \((a,b)\) is denoted by \( L^p(a,b;X) \) and the \( L^2\)-Bochner-Sobolev space of order \( k \) is denoted by \( H^k(a,b;X) \).

Let \( k \) be a nonnegative integer satisfying \( k \geq 1 \). The space \( \mathcal{Y}^k(a,b) \) is defined by

\[ \mathcal{Y}^k(a,b) = \mathcal{Y}_1(a,b) \times \mathcal{Y}_\infty^k(a,b) \]

equipped with the norm

\[ \| \{ u_1, u_\infty \} \|_{\mathcal{Y}^k(a,b)} = \left( \| u_1 \|_{\mathcal{Y}_1(a,b)}^2 + \| u_\infty \|_{\mathcal{Y}_\infty^k(a,b)}^2 \right)^{\frac{1}{2}}, \]

where

\[ \mathcal{Y}_1(a,b) = \{ u_1 = \top(\phi_1,w_1) \in C^1([a,b];(H^1_{(1),1})_{sym}); \partial_t w_1 \in C([a,b];L^2_t) \}, \]

\[ \mathcal{Y}_\infty^k(a,b) = \{ u_\infty \in L^\infty([a,b];X) \}. \]
\[ \mathcal{Y}_k(a, b) = \{ u_\infty = \tau(\phi_\infty, w_\infty) \in C([a, b]; (H_{(\infty, 1)}^k)_\text{sym}); w_\infty \in L^2(a, b; H_{(\infty, 1)}^{k+1}) \cap H^1(a, b; H_{(\infty, 1)}^{k-1}) \}, \]

\[ \| u_1 \|_{\mathcal{Y}_1(a, b)} = \left( \| u_1 \|_{C^1([a, b]; H_{(\infty, 1)}^1)}^2 + \| \partial_t w_1 \|_{C([a, b]; L^2)}^2 \right)^{\frac{1}{2}}, \]

\[ \| u_\infty \|_{\mathcal{Y}_\infty(a, b)} = \left( \| u_\infty \|_{C([a, b]; H_{(\infty, 1)}^k)}^2 + \| w_\infty \|_{L^2(a, b; H_{(\infty, 1)}^{k+1}) \cap H^1(a, b; H_{(\infty, 1)}^{k-1})}^2 \right)^{\frac{1}{2}}. \]

We also introduce function spaces of \( T \)-periodic functions in \( t \). \( C_{\text{per}}(\mathbb{R}; X) \) denotes the set of all \( T \)-periodic continuous functions with values in \( X \) equipped with the norm \( \| \cdot \|_{C([0, T]; X)} \); and \( L^2_{\text{per}}(\mathbb{R}; X) \) denotes the set of all \( T \)-periodic locally square integrable functions with values in \( X \) equipped with the norm \( \| \cdot \|_{L^2(0, T; X)} \). Similarly, \( H^1_{\text{per}}(\mathbb{R}; X) \) and \( \mathcal{A}^k_{\text{per}}(\mathbb{R}) \), and so on, are defined.

Let \( X \) be a Banach space and let \( P \) be a bounded linear operator on \( X \). We denote by \( r_X(P) \) the spectral radius of \( P \).

For operators \( A \) and \( B \), \([A, B]\) denotes the commutator of \( A \) and \( B \) :

\[ [A, B]f = A(Bf) - B(Af). \]

We next state some lemmas which will be used in the proof of the main results.

**Lemma 2.1.** Let \( n \geq 3 \) and let \( m \geq \left[ \frac{n}{2} \right] + 1 \). Then there holds the inequality

\[ \| f \|_{L^\infty} \leq C \| \nabla f \|_{H^{m-1}} \]

for \( f \in H^m \).

Lemma 2.1 is proved as follows. Let \( n \geq 3 \) and set \( 2^* := \frac{2n}{n-2} \). Since \( m \geq \left[ \frac{n}{2} \right] + 1 \), we see that \( m - 1 \geq \frac{n}{2^*} \). It then follows from the Sobolev inequalities that

\[ \| f \|_{L^\infty} \leq C \| f \|_{W^{m, 2^*}} \leq C \| \nabla f \|_{H^{m-1}}, \]

which shows Lemma 2.1.

**Lemma 2.2.** Assume \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \left[ \frac{n}{2} \right] + 1 \). Let \( m_j \) and \( \mu_j \) (\( j = 1, \ldots, \ell \)) satisfy \( 0 \leq |\mu_j| \leq m_j \leq m + |\mu_j| \), \( \mu = \mu_1 + \cdots + \mu_\ell \), \( m = m_1 + \cdots + m_\ell \geq (\ell - 1)m + |\mu| \). Then there holds

\[ \| \partial_t^{\mu_1} f_1 \cdots \partial_t^{\mu_\ell} f_\ell \|_{L^2} \leq C \prod_{1 \leq j \leq \ell} \| f_j \|_{H^{m_j}}. \]

See, e.g., [7], for the proof of Lemma 2.2.
Lemma 2.3. Let \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \left\lceil \frac{n}{2} \right\rceil + 1 \). Suppose that \( F \) is a smooth function on \( I \), where \( I \) is a compact interval of \( \mathbb{R} \). Then for a multi-index \( \alpha \) with \( 1 \leq |\alpha| \leq m \), there hold the estimates
\[
\| \partial_x^\alpha F(f_1)f_2 \|_{L^2} \leq C \| F \|_{C^{[\alpha]}(I)} \left( 1 + \| \nabla f_1 \|_{m-1} \right) \| \nabla f_2 \|_{H^m} \| f_2 \|_{H^{|\alpha|}}
\]
for \( f_1 \in H^m \) with \( f_1(x) \in I \) for all \( x \in \mathbb{R}^n \) and \( f_2 \in H^{[\alpha]} \); and
\[
\| \partial_x^\alpha F(f_1)f_2 \|_{L^2} \leq C \| F \|_{C^{[\alpha]}(I)} \left( 1 + \| \nabla f_1 \|_{m-1} \right) \| \nabla f_2 \|_{H^m} \| f_2 \|_{H^{|\alpha|-1}}.
\]
for \( f_1 \in H^{m+1} \) with \( f_1(x) \in I \) for all \( x \in \mathbb{R}^n \) and \( f_2 \in H^{[\alpha]-1} \).

See, e.g., [5], for the proof of Lemma 2.3.

3 Main results

In this section we state our results on the existence and stability of a time periodic solution for system (1.1).

We begin with the existence of a time periodic solution. To state the existence result, we introduce operators which decompose a function into its low and high frequency parts.

We define operators \( P_1 \) and \( P_\infty \) on \( L^2 \) by
\[
P_j f = F^{-1} \hat{x}_j F[f] \quad (f \in L^2, j = 1, \infty),
\]
where
\[
\hat{x}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j = 1, \infty), \quad 0 \leq \hat{x}_j \leq 1 \quad (j = 1, \infty),
\]
\[
\hat{x}_1(\xi) = \left\{ \begin{array}{ll}
1 & (|\xi| \leq r_1), \\
0 & (|\xi| \geq r_\infty),
\end{array} \right.
\]
\[
\hat{x}_\infty(\xi) = 1 - \hat{x}_1(\xi),
\]
\[
0 < r_1 < r_\infty.
\]

We fix \( 0 < r_1 < r_\infty < \frac{2\pi}{\nu \nu_0} \) so that (5.4) in Lemma 5.5 below holds for \( |\xi| \leq r_\infty \).

Theorem 3.1. Let \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \left\lceil \frac{n}{2} \right\rceil + 1 \). Assume that \( g(x,t) \) satisfies (1.2) and \( g \in C_{\text{per}}(\mathbb{R}; L^1_1 \cap L^2_2) \cap L^2_{\text{per}}(\mathbb{R}; H^{m-1}_1) \). Set
\[
[g]_m = \| g \|_{C([0,T]; L^1_1 \cap L^2_2)} + \| g \|_{L^2(0,T; H^{m-1}_1)}.
\]
Then there exist constants \( \delta > 0 \) and \( C > 0 \) such that if \( [g]_m \leq \delta \), then the system (1.5) has a time periodic solution \( u = u_1 + u_\infty \) satisfying \( \{u_1, u_\infty\} \in \mathcal{A}^{m}_{\text{per}}(\mathbb{R}) \) with \( \| \{u_1, u_\infty\} \| \mathcal{A}^{m}(0,T) \leq C[g]_m \).

Furthermore, the uniqueness of time periodic solutions of (1.5) holds in the class \( \{u = \tau(\phi, w); \{P_1 u, P_\infty u\} \in \mathcal{A}^{m}_{\text{per}}(\mathbb{R}), \| \{P_1 u, P_\infty u\} \| \mathcal{A}^{m}(0,T) \leq C\delta \} \).

8
Our next issue is to study the stability of the time periodic solution obtained in Theorem 3.1.

Let \( \bar{\tau}(\rho_{\text{per}}, v_{\text{per}}) \) be the time periodic solution given in Theorem 3.1. We denote the perturbation by \( u = \bar{\tau}(\phi, w) \), where \( \phi = \rho - \rho_{\text{per}}, w = v - v_{\text{per}} \). Substituting \( \rho = \phi + \rho_{\text{per}} \) and \( v = w + v_{\text{per}} \) into (1.1), we see that the perturbation \( u = \bar{\tau}(\phi, w) \) is governed by

\[
\begin{cases}
\partial_t \phi + v_{\text{per}} \cdot \nabla \phi + \phi \Delta v_{\text{per}} + \rho_{\text{per}} \Delta v + w \cdot \nabla \rho_{\text{per}} = F^0, \\
\partial_t w + v_{\text{per}} \cdot \nabla w + w \cdot \nabla v_{\text{per}} - \frac{\mu}{\rho_{\text{per}}} \Delta w - \frac{\mu + \mu'}{\rho_{\text{per}}} \nabla \Delta v_{\text{per}} = \tilde{F},
\end{cases}
\tag{3.1}
\]

where

\[
F^0 = -\text{div}(\phi w),
\]
\[
\tilde{F} = -w \cdot \nabla w - \frac{\phi}{\rho_{\text{per}}(\rho_{\text{per}} + \phi)}(\mu \Delta w + (\mu + \mu') \Delta \nabla w)
\]
\[
+ \frac{\phi}{\rho_{\text{per}}(\rho_{\text{per}} + \phi)}(\mu \Delta v_{\text{per}} + (\mu + \mu') \nabla \Delta v_{\text{per}})
\]
\[
+ \frac{\phi^2}{\rho_{\text{per}}^2} \nabla (P(\rho_{\text{per}}) \phi) + \frac{\phi}{\rho_{\text{per}}(\rho_{\text{per}} + \phi)} \nabla (P'(\rho_{\text{per}}) \phi) + \frac{1}{\rho_{\text{per}}} \nabla (P(\rho_{\text{per}}) \phi^2),
\]
\[
P''(\rho_{\text{per}}, \phi) = \int_0^1 P'((\rho_{\text{per}} + \theta \phi) d\theta,
\]
\[
P(\rho_{\text{per}}, \phi) = \int_0^1 (1 - \theta) P''(\rho_{\text{per}} + \theta \phi) d\theta.
\]

We consider the initial value problem for (3.1) under the initial condition

\[ u|_{t=0} = u_0 = \bar{\tau}(\phi_0, w_0). \tag{3.2} \]

Our result on the stability of the time periodic solution is stated as follows.

**Theorem 3.2.** Let \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \left[ \frac{n}{2} \right] + 1 \). Assume that \( g(x, t) \) satisfies (1.2) and \( g \in C_\text{per}(\mathbb{R}; L^1 \cap L^2) \cap L^\infty(\mathbb{R}; H^m) \). Let \( \bar{\tau}(\rho_{\text{per}}, v_{\text{per}}) \) be the time periodic solution obtained in Theorem 3.1 and let \( u_0 = \bar{\tau}(\phi_0, w_0) \in H^m \cap L^1 \). Then there exist constants \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) such that if

\[ g|_{m+1} < \epsilon_1, \quad \|u_0\|_{H^m \cap L^1} < \epsilon_2, \]

there exists a unique global solution \( u = \bar{\tau}(\phi, w) \in C([0, \infty); H^m) \) of (3.1)-(3.2) and \( u \) satisfies

\[ \|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{2}} \quad (t \in [0, +\infty), \quad k = 0, 1). \]

Theorem 3.2 follows from the same argument as that in [10]; and we omit the details. In contrast to the problem in [10], several linear terms with coefficients including \( v_{\text{per}} \) appear
in the equations for the perturbation. In the transport equation for the perturbation, there appear the terms \( v_{\text{per}} \cdot \nabla \phi + \phi \text{div} v_{\text{per}} \) and these terms can be handled by using the energy method and the boundedness properties of the projection onto the low frequency part as in [4, 10], together with the Hardy inequality; the linear terms including \( v_{\text{per}} \) in the equation of motion for the perturbation can be handled by using the Hardy inequality. (See also [1]).

4 Reformulation of the problem

In this section we reformulate the time periodic problem for (1.5).

To solve the time periodic problem for (1.5), we decompose \( u \) into a low frequency part \( u_1 \) and a high frequency part \( u_\infty \); and we rewrite the problem into a system of equations for \( u_1 \) and \( u_\infty \).

Set

\[ u_1 = P_1 u, \quad u_\infty = P_\infty u. \]

Applying the operators \( P_1 \) and \( P_\infty \) to (1.5), we obtain,

\[
\begin{align*}
\partial_t u_1 + A u_1 &= F_1(u_1 + u_\infty, g), \quad (4.1) \\
\partial_t u_\infty + A u_\infty + P_\infty (B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \quad (4.2)
\end{align*}
\]

Here

\[
\begin{align*}
F_1(u_1 + u_\infty, g) &= P_1[-B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g)], \\
F_\infty(u_1 + u_\infty, g) &= P_\infty[-B[u_1 + u_\infty]u_1 + G(u_1 + u_\infty, g)].
\end{align*}
\]

Suppose that (4.1) and (4.2) are satisfied by some functions \( u_1 \) and \( u_\infty \). Then, since \( P_1 + P_\infty = I \), by adding (4.1) to (4.2), we obtain

\[
\partial_t (u_1 + u_\infty) + A(u_1 + u_\infty) = -P_\infty(B[u_1 + u_\infty]u_\infty) + (P_1 + P_\infty)F(u_1 + u_\infty, g) = -B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g).
\]

Set \( u = u_1 + u_\infty \), then we have

\[
\partial_t u + A u + B[u]u = G(u, g).
\]

Consequently, if we show the existence of a pair of functions \( \{u_1, u_\infty\} \) satisfying (4.1)-(4.2), then we can obtain a solution \( u \) of (1.5). Therefore, we will consider (4.1)-(4.2) to solve the time periodic problem for (1.5).

The following two lemmas are concerned with symmetry of (1.5) and (4.1)-(4.2). We recall that \( \Gamma \) is defined by

\[
(\Gamma u)(x) = ^T(\phi(-x), -w(-x)) \quad (u(x) = ^T(\phi(x), w(x)), \quad x \in \mathbb{R}^n).
\]
Lemma 4.1. Set $g(x, t) = \top(0, g(x, t))$ and assume that $\Gamma g(x, t) = g(x, t) (x \in \mathbb{R}^n, t \in \mathbb{R})$.

(i) If $u = \top(\phi, w)$ is a solution of (1.5), then $\Gamma u$ is also a solution of (1.5).

(ii) If $\{u_1, u_\infty\}$ is a solution of (4.1)-(4.2), then $\{\Gamma u_1, \Gamma u_\infty\}$ is also a solution of (4.1)-(4.2).

Lemma 4.2. Assume that $\Gamma g(x, t) = g(x, t) (x \in \mathbb{R}^n, t \in \mathbb{R})$.

(i) If $(\Gamma u)(x, t) = u(x, t) (x \in \mathbb{R}^n, t \in \mathbb{R})$, then

$$\Gamma(\partial_t u + A u + B[u]u - G(u, g))(x, t) = [\partial_t u + A u + B[u]u - G(u, g)](x, t)$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$.

(ii) If $\{\Gamma u_1(x, t), \Gamma u_\infty(x, t)\} = \{u_1(x, t), u_\infty(x, t)\} (x \in \mathbb{R}^n, t \in \mathbb{R})$, then

$$\Gamma(\partial_t u_1 + A u_1 - F_1(u_1 + u_\infty, g))(x, t) = [\partial_t u_1 + A u_1 - F_1(u_1 + u_\infty, g)](x, t)$$

and

$$\Gamma(\partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) - F_\infty(u_1 + u_\infty, g))(x, t) = [\partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) - F_\infty(u_1 + u_\infty, g)](x, t)$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$.

Lemma 4.1 (i) and Lemma 4.2 (i) can be verified by direct computations. As for Lemma 4.1 (ii) and Lemma 4.2 (ii), by using the facts $\hat{f}(-\xi) = \hat{f}(-\cdot)(\xi)$ and $\chi_j(-\xi) = \chi_j(\xi) (j = 1, \infty)$, we see that $\Gamma P_j = P_j \Gamma (j = 1, \infty)$. Based on these relations, Lemma 4.1 (ii) and Lemma 4.2 (ii) can be proved by a straightforward computation.

By Lemma 4.1 and Lemma 4.2, one can consider (4.1)-(4.2) in space of functions satisfying $\{\Gamma u_1, \Gamma u_\infty\} = \{u_1, u_\infty\}$, i.e., $u_j = \top(\phi_j(x, t), w_j(x, t)) = \top(\phi_j(-x, t), -w_j(-x, t)) (j = 1, \infty)$.

We look for a time periodic solution $\{u_1, u_\infty\}$ for the system (4.1)-(4.2). To solve the time periodic problem for (4.1)-(4.2), we introduce solution operators for the following linear problems:

$$\begin{align*}
\left\{ \begin{array}{l}
\partial_t u_1 + A u_1 = F_1, \\
u_{|t=0} = u_{01},
\end{array} \right. \quad (4.3)
\end{align*}$$

and

$$\begin{align*}
\left\{ \begin{array}{l}
\partial_t u_\infty + A u_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \\
u_{|t=0} = u_{0\infty},
\end{array} \right. \quad (4.4)
\end{align*}$$

where $\tilde{u} = \top(\tilde{\phi}, \tilde{w}), u_{01}, u_{0\infty}, F_1$ and $F_\infty$ are given functions.
As a result, for any \( L \) from Lemma 4.3.

To formulate the time periodic problem, we denote by \( S_1(t) \) the solution operator for (4.3) with \( F_1 = 0 \), and by \( \mathcal{J}_1(t) \) the solution operator for (4.3) with \( u_0 = 0 \). We also denote by \( S_{\infty,u}(t) \) the solution operator for (4.4) with \( F_{\infty} = 0 \) and by \( \mathcal{J}_{\infty,u}(t) \) the solution operator for (4.4) with \( u_{\infty} = 0 \). (The precise definition of these operators will be given later.)

If \( \{ u_1, u_{\infty} \} \) satisfies (4.1)-(4.2), then \( u_1(t) \) and \( u_{\infty}(t) \) are written as

\[
\begin{align*}
u_1(t) & = S_1(t)u_1(0) + \mathcal{J}_1(t)[F_1(u, g)], \\
u_{\infty}(t) & = S_{\infty,u}(t)u_{\infty}(0) + \mathcal{J}_{\infty,u}(t)[F_{\infty}(u, g)]
\end{align*}
\]

with \( u = u_1 + u_{\infty} \).

Suppose that \( \{ u_1, u_{\infty} \} \) is a \( T \)-time periodic solution of (4.5)-(4.6). Then, since \( u_1(T) = u_1(0) \) and \( u_{\infty}(T) = u_{\infty}(0) \), we see that

\[
\begin{align*}
\begin{cases}
(I - S_1(T))u_1(0) = \mathcal{J}_1(T)[F_1(u, g)], \\
(I - S_{\infty,u}(T))u_{\infty}(0) = \mathcal{J}_{\infty,u}(T)[F_{\infty}(u, g)], \\
u_1 + u_{\infty}.
\end{cases}
\end{align*}
\]

Therefore if \( (I - S_1(T)) \) and \( (I - S_{\infty,u}(T)) \) are invertible in a suitable sense, then one obtains

\[
\begin{align*}
\begin{cases}
u_1(t) = S_1(t)u_1(0) + \mathcal{J}_1(t)[F_1(u, g)], \\
u_{\infty}(t) = S_{\infty,u}(t)u_{\infty}(0) + \mathcal{J}_{\infty,u}(t)[F_{\infty}(u, g)]
\end{cases}
\end{align*}
\]

with

\[
\begin{align*}
\begin{cases}
u_1 = u_1 + u_{\infty}, \\
u_0 = (I - S_1(T))^{-1}\mathcal{J}_1(T)[F_1(u, g)], \\
u_{\infty} = (I - S_{\infty,u}(T))^{-1}\mathcal{J}_{\infty,u}(T)[F_{\infty}(u, g)].
\end{cases}
\end{align*}
\]

Therefore, to obtain a \( T \)-time periodic solution of (4.1)-(4.2), we look for a pair of functions \( \{ u_1, u_{\infty} \} \) satisfying (4.7)-(4.8). We will investigate the solution operators \( S_1(t), \mathcal{J}_1(t), S_{\infty,u}(t) \) and \( \mathcal{J}_{\infty,u}(t) \) in sections 5 and 6.

Next, we introduce some lemmas which will be used in the proof of Theorem 3.1. We first derive some inequalities for the low frequency part.

**Lemma 4.3.** (i) Let \( k \) be a nonnegative integer. Then \( P_1 \) is a bounded linear operator from \( L^2 \) to \( H^k \). In fact, it holds that

\[
\| \nabla^k P_1 f \|_{L^2} \leq C_k \| f \|_{L^2} \quad (f \in L^2).
\]

As a result, for any \( 2 \leq p \leq \infty \), \( P_1 \) is bounded from \( L^2 \) to \( L^p \).

(ii) Let \( k \) be a nonnegative integer. Then there hold the estimates

\[
\| \nabla^k f_1 \|_{L^2} + \| f_1 \|_{L^p} \leq C_{k,p} \| f_1 \|_{L^2} \quad (f \in L^2_{(1)}),
\]
where $2 \leq p \leq \infty$,
\[
\|f_1\|_{H^k} \leq C_k \|f_1\|_{L^2} \quad (f \in L^2_{(1),1}),
\]
\[
\|\nabla f_1\|_{H^k} \leq C_k \|f_1\|_{H^1_{(1),1}} \quad (f \in H^1_{(1),1}),
\]
\[
\|f_1\|_{L^2} + \|f_1\|_{H^1_{(1),1}} \leq C_k \|f_1\|_{L^2} \quad (f \in L^2_{(1),1} \cap L^1).
\]

**Proof.** The boundedness of $P_1$ from $L^2$ to $H^k$ can be easily verified by using the Plancherel theorem, since $\text{supp} \ P_1 f \subset \{\xi; |\xi| \leq r_\infty\}$; and, then, the boundedness of $P_1$ from $L^2$ to $L^p$ with $2 \leq p \leq \infty$ follows from the Sobolev inequality.

As for (ii), the first inequality can be obtained as in the same reason for (i). The second inequality is obtained by (i) and the following computation. For $0 \leq |\alpha| \leq k$ and $f_1 \in L^2_{(1),1}$, we see that
\[
\|x^\alpha \partial_x^\alpha f_1\|_{L^2} = (2\pi)^{-\frac{n}{2}} \|\partial_x (\xi^\alpha \hat{f}_1)\|_{L^2(|\xi| \leq r_\infty)}
\]
\[
\leq C \{ \|\xi^{|\alpha|+1} \hat{f}_1\|_{L^2(|\xi| \leq r_\infty)} + \|\partial_x \xi^{|\alpha|} \hat{f}_1\|_{L^2(|\xi| \leq r_\infty)} \}
\]
\[
\leq C \{ \|\hat{f}_1\|_{L^2(|\xi| \leq r_\infty)} + \|\partial_x \xi^{|\alpha|} \hat{f}_1\|_{L^2(|\xi| \leq r_\infty)} \}
\]
\[
\leq C \|f_1\|_{L^2}.
\]
The third inequality follows from the second inequality with $f_1$ replaced by $\nabla f_1$, since, by the first inequality, we have $\|\nabla f_1\|_{L^2} \leq C \|f_1\|_{H^1_{(1),1}}$. As for the last inequality, we have
\[
\|f_1\|_{L^2}^2 = (2\pi)^{-n} \left\{ \|\hat{f}_1\|_{L^2(|\xi| \leq r_\infty)}^2 + \|\partial_x \xi \hat{f}_1\|_{L^2(|\xi| \leq r_\infty)}^2 \right\}
\]
\[
\leq C \left\{ \sup_{|\xi| \leq r_\infty} (|\hat{f}_1(\xi)| + |\partial_x \xi \hat{f}_1(\xi)|) \right\}^2
\]
\[
\leq C \|f_1\|_{L^1}^2,
\]
and, likewise, we can obtain $\|f_1\|_{H^1_{(1),1}} \leq C \|f_1\|_{L^1}$. This completes the proof. □

As for the high frequency part, we have the following inequalities.

**Lemma 4.4.** (i) Let $k$ be a nonnegative integer. Then $P_\infty$ is a bounded linear operator on $H^k$.

(ii) There hold the inequalities
\[
\|P_\infty f\|_{L^2} \leq C \|\nabla f\|_{L^2} \quad (f \in H^1),
\]
\[
\|f_\infty\|_{L^2} \leq C \|\nabla f_\infty\|_{L^2} \quad (f_\infty \in H^1_{(\infty)}).
\]
Lemma 4.4 (i) immediately follows from the definition of $P_\infty$ by using the Plancherel theorem; and, similarly, inequalities in (ii) can be easily seen since $\text{supp} \widehat{\check{P}_\infty f} \subset \{ \xi; |\xi| \geq r_1 \}$ and $\text{supp} \check{f}_\infty \subset \{ \xi; |\xi| \geq r_1 \}$ for $f_\infty \in H^1_{(\infty)}$. We omit the proof.

We next introduce a cut-off function $\zeta_R$. Let $\tilde{\zeta} \in C^\infty([0, \infty))$ be a nonincreasing function satisfying

$$0 \leq \tilde{\zeta} \leq 1, \quad \tilde{\zeta}(r) = \begin{cases} 1 & (|r| \leq 1), \\ 0 & (|r| > 2). \end{cases}$$

Set

$$\zeta_R(x) = \tilde{\zeta}\left(\frac{|x|}{R}\right).$$

(4.9)

Then, $\zeta_R \in C^\infty_0(\mathbb{R}^n)$, $0 \leq \zeta_R \leq 1$, and

$$\zeta_R(x) = \begin{cases} 1 & (|x| \leq R), \\ 0 & (|x| > 2R). \end{cases}$$

An elementary computation gives the following lemma.

**Lemma 4.5.** Let $\zeta_R$ be defined in (4.9). For a nonnegative integer $\ell$ and a multi-index $\alpha$, there holds

$$||x|^{\ell}|\partial_\alpha^\ell(\zeta_R(x))| \leq C_{\alpha, \ell} R^{\ell-|\alpha|} \quad (x \in \mathbb{R}^n).$$

If $|\alpha| \geq 1$, then it holds that

$$\text{supp}(\partial_\alpha^\ell \zeta_R) \subset \{ R \leq |x| \leq 2R \}.$$

**Lemma 4.6.** Let $\chi_1$ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then for a nonnegative integer $\ell$, there holds

$$||x|^{\ell}|\chi_1 * f| \leq C\{||x|^{\ell}|\chi_1|_{L^1}||f||_{L^2} + ||\chi_1||_{L^1}||x|^{\ell}|f||_{L^2}\} \quad (f \in L^2_{\ell}).$$

Here $C$ is a positive constant depending only on $\ell$.

**Proof.** Let $\chi_1$ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then

$$||x|^{\ell}|\chi_1 * f| \leq |x|^{\ell}\int_{\mathbb{R}^n} |\chi_1(x - y)f(y)|dy$$

$$\leq C \int_{\mathbb{R}^n} |x - y|^{\ell}|\chi_1(x - y)||f(y)|dy + C \int_{\mathbb{R}^n} |\chi_1(x - y)||y|^{\ell}|f(y)|dy.$$

Therefore, the Young inequality gives

$$||x|^{\ell}|\chi_1 * f||_{L^2} \leq C\{||x|^{\ell}|\chi_1||_{L^1}||f||_{L^2} + ||\chi_1||_{L^1}||x|^{\ell}|f||_{L^2}\} \quad (f \in L^2_{\ell}).$$

This completes the proof. □
Lemma 4.7. Let $\ell \in \mathbb{N}$. Then there exists a positive constant $C$ depending only on $\ell$ such that
\[
\| x^\ell \nabla f_\infty \|_{L^2}^2 \geq \frac{r_1^2}{2} \| x^{\ell-1} f_\infty \|_{L^2}^2 - C \| x^{\ell-1} f_\infty \|_{L^2}^2 \quad (f_\infty \in H^1_{(\infty,\ell)}).
\]

Proof. By the multinomial theorem, we have
\[
\| x^\ell \nabla f_\infty \|_{L^2}^2 = \int_{\mathbb{R}^n} |x_1^2 + \cdots + x_n^2| |\nabla f_\infty|^2 \, dx
= \int_{\mathbb{R}^n} \sum_{|\alpha| = \ell} \binom{|\alpha|}{\alpha} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} |\nabla f_\infty|^2 \, dx
= \sum_{|\alpha| = \ell} \binom{|\alpha|}{\alpha} \| x^{\alpha_1} \cdots x^{\alpha_n} \nabla f_\infty \|_{L^2}^2,
\]
where
\[
\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}, \quad \alpha! = \alpha_1! \cdots \alpha_n!.
\]
Therefore, it follows from the Plancherel theorem that
\[
\| x^\ell \nabla f_\infty \|_{L^2}^2 \geq \frac{1}{2} \sum_{|\alpha| = \ell} \binom{|\alpha|}{\alpha} \| \nabla((x_1^{\alpha_1} \cdots x_n^{\alpha_n}) f_\infty) \|_{L^2}^2 - C \| x^{\ell-1} f_\infty \|_{L^2}^2
= \frac{1}{2} (2\pi)^{-n} \sum_{|\alpha| = \ell} \binom{|\alpha|}{\alpha} \| \hat{\xi}(\partial_{\xi}^{\alpha_1} \cdots \partial_{\xi}^{\alpha_n}) \hat{f}_\infty \|_{L^2}^2 - C \| x^{\ell-1} f_\infty \|_{L^2}^2.
\]
Since $\text{supp} \hat{f}_\infty \subset \{ |\xi| \geq r_1 \}$, we see that
\[
\| x^\ell \nabla f_\infty \|_{L^2}^2 \geq \frac{r_1^2}{2} (2\pi)^{-n} \sum_{|\alpha| = \ell} \binom{|\alpha|}{\alpha} \| \hat{\xi}(\partial_{\xi}^{\alpha_1} \cdots \partial_{\xi}^{\alpha_n}) \hat{f}_\infty \|_{L^2}^2 - C \| x^{\ell-1} f_\infty \|_{L^2}^2
\geq \frac{r_1^2}{2} \| x^{\ell-1} f_\infty \|_{L^2}^2 - C \| x^{\ell-1} f_\infty \|_{L^2}^2.
\]
This completes the proof. \qed

5 Properties of $S_1(t)$ and $\mathcal{A}_1(t)$

In this section we investigate $S_1(t)$ and $\mathcal{A}_1(t)$ and establish an estimate for a solution $u_1$ of
\[
\partial_t u_1 + Au_1 = F_1 \quad (5.1)
\]
satisfying $u_1(0) = u_1(T)$. 15
We consider the restriction of $A$ on $L^2_{(1)}$. By Lemma 4.3 (ii), we see that $\|Au_1\|_{L^2} \leq C\|u_1\|_{L^2}$ for $u_1 \in L^2_{(1)}$.

Let

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top \xi \\ i\gamma \xi & \nu |\xi|^2 I_n + \bar{\beta} \xi^\top \xi \end{pmatrix} \quad (\xi \in \mathbb{R}^n).$$

Then, since $Au_1 = \mathcal{F}^{-1}\hat{A}_\xi \hat{u}_1$, we see that $\text{supp } \hat{A}_\xi \hat{u}_1 \subset \{\xi; |\xi| \leq r_\infty\}$ for $u_1 \in L^2_{(1)}$.

Therefore, the restriction of $A$ on $L^2_{(1)}$ is a bounded linear operator on $L^2_{(1)}$.

We denote by $A_1$ the restriction of $A$ on $L^2_{(1)}$. Then $A_1$ is a bounded linear operator on $L^2_{(1)}$ and it satisfies $\|A_1 u_1\|_{L^2} \leq C\|u_1\|_{L^2}$ for $u_1 \in L^2_{(1)}$ and

$$A_1 u_1 = \mathcal{F}^{-1}\hat{A}_\xi \mathcal{F} u_1 \quad (u_1 \in L^2_{(1)}).$$

Furthermore, $-A_1$ generates a uniformly continuous semigroup $S_1(t) = e^{-tA_1}$ that is given by

$$S_1(t)u_1 = \mathcal{F}^{-1}e^{-t\hat{A}_\xi} \mathcal{F} u_1 \quad (u_1 \in L^2_{(1)});$$

and it holds that $S_1(t)$ satisfies $S_1(\cdot)u_1 \in C^1([0, \infty); L^2_{(1)})$ for each $u_1 \in L^2$ and

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 (=-AS_1(t)u_1), \quad S_1(0)u_1 = u_1 \quad \text{for } u_1 \in L^2_{(1)},$$

$$\|\partial_t^k S_1(t)u_1\|_{L^2} \leq \|A_1\|^k \|u_1\|_{L^2} \quad \text{for } u_1 \in L^2_{(1)}, \; t \geq 0, \; k = 0, 1,$$

where $\|A_1\|$ denotes the operator norm of $A_1$. The estimates can be obtained by the energy method based on the relation

$$(Au, u) = \nu \|\nabla u\|_{L^2}^2 + \bar{\nu} \|\nabla \cdot u\|_{L^2}^2.$$
Proposition 5.1. (i) $A_1$ is a bounded linear operator on $H^1_{(1,1)}$ and $S_1(t) = e^{-tA_1}$ is a uniformly continuous semigroup on $H^1_{(1,1)}$. Furthermore, it holds that $S_1(\cdot)u_1 \in C^1([0,T]; H^1_{(1,1)})$, $\partial_t S_1(\cdot)u_1 \in C([0,T]; L^2_1)$ for each $u_1 \in H^1_{(1,1)}$ and all $T' > 0$,

$$\|\partial_t^k S_1(t)u_1\|_{H^1_{(1,1)}} \leq C\|u_1\|_{H^1_{(1,1)}} \text{ for } u_1 \in H^1_{(1,1)}, \ t \in [0,T'], \ k = 0,1,$$

and

$$\|\partial_t S_1(t)u_1\|_{L^2_1} \leq C\|u_1\|_{H^1_{(1,1)}} \text{ for } u_1 \in H^1_{(1,1)}, \ t \in [0,T'],$$

where $T' > 0$ is any given positive number and $C$ is a positive constant depending on $T'$.

(ii) $\mathcal{S}_1(\cdot)$ satisfies that $\mathcal{S}_1(\cdot)[F_1] \in C^1([0,T]; H^1_{(1,1)})$ for each $F_1 \in C([0,T]; H^1_{(1,1)})$ and

$$\|\partial_t^k \mathcal{S}_1(t)[F_1]\|_{H^1_{(1,1)}} \leq C\|F_1\|_{C([0,T]; H^1_{(1,1)})} \text{ for } F_1 \in C([0,T]; H^1_{(1,1)}), \ t \in [0,T], \ k = 0,1,$$

where $C$ is a positive constant depending on $T$. If, in addition, $F_1 \in C([0,T]; L^2_2)$, then $\partial_t \mathcal{S}_1(t)[F_1] \in C([0,T]; L^2_2)$ and

$$\|\partial_t \mathcal{S}_1(t)[F_1]\|_{L^2_2} \leq C\|F_1\|_{C([0,T]; L^2_2)} \text{ for } F_1 \in C([0,T]; H^1_{(1,1)} \cap L^2_2), \ t \in [0,T],$$

where $C$ is a positive constant depending on $T$.

(iii) It holds that

$$S_1(t)\mathcal{S}_1(t')[F_1] = \mathcal{S}_1(t')[S_1(t)F_1]$$

for any $t \geq 0$, $t' \in [0,T]$ and $F_1 \in C([0,T]; X)$, where $X = L^2_1 \cap H^1_{(1,1)}$.

(iv) It holds that $\Gamma S_1(t) = S_1(t)\Gamma$ and $\Gamma \mathcal{S}_1(t) = \mathcal{S}_1(t)\Gamma$. Consequently, the assertions (i)–(iii) above hold with function spaces $L^2_2 \cap H^1_{(1,1)}$ and $L^2_2$ replaced by $(L^2_1)_{\text{sym}}$, $(H^1_{(1,1)})_{\text{sym}}$ and $(L^2_1)_{\text{sym}}$, respectively.

The proof of Proposition 5.1 will be given later.

We next investigate invertibility of $I - S_1(T)$.

Proposition 5.2. Let $F_1 = \mathcal{T}(F_1^0(x), \tilde{F}_1(x)) \in L^2_1 \cap L^1_1$ and suppose that $\tilde{F}_1(-x) = -\tilde{F}_1(x)$ for $x \in \mathbb{R}^n$. Then there uniquely exists $u \in H^1_{(1,1)}$ that satisfies

$$(I - S_1(T))u = F_1 \quad \text{and} \quad \|u\|_{H^1_{(1,1)}} \leq C\|F_1\|_{L^1_1}. \quad (5.2)$$

Furthermore, if $\Gamma F_1 = F_1$, then $\Gamma u = u$.

The proof of Proposition 5.2 will be given later.

In view of Proposition 5.2, $I - S_1(T)$ has a bounded inverse $(I - S_1(T))^{-1}: (L^2_1 \cap L^1_1)_{\text{sym}} \rightarrow (H^1_{(1,1)})_{\text{sym}}$ and it holds that

$$\|(I - S_1(T))^{-1}F_1\|_{H^1_{(1,1)}} \leq C\|F_1\|_{L^1_1}.$$
Using Proposition 5.1 (ii) and Proposition 5.2, we can obtain the following estimate for \( S_1(T)(I - S_1(T))^{-1} \).

**Proposition 5.3.** For \( F_1 \in C([0, T]; (L^2 \cap L^1_{sym})_1) \), it holds that \( S_1(T)(I - S_1(T))^{-1}F_1 \in (H^1_{(1), 1})_{sym} \) and

\[
\| S_1(T)(I - S_1(T))^{-1}F_1 \|_{H^1_{(1), 1}} \leq C \| F_1 \|_{C([0, T]; L^1)}.
\]

We are now in a position to give an estimate for a solution of (5.1) satisfying \( u_1(0) = u_1(T) \).

**Proposition 5.4.** Set

\[
u_1(t) = S_1(t)S_1(T)(I - S_1(T))^{-1}F_1 + S_1(t)[F_1]
\]

for \( F_1 = (F_1^0(x, t), \tilde{F}_1(x, t)) \in C([0, T]; (L^2 \cap L^1_{sym})_1) \). Then \( u_1 \) is a solution of (5.1) in \( \mathcal{Y}_1(0, T) \) satisfying \( u_1(0) = u_1(T) \) and

\[
\| u_1 \|_{\mathcal{Y}_1(0, T)} \leq C \| F_1 \|_{C([0, T]; L^1)}.
\]

**Proof.** We find from Proposition 5.1 (iii) and Proposition 5.2 that \( u_1(0) = u_1(T) \). As for the estimate for \( u_1 \), the first term on the right-hand side of (5.3) is estimated by using Proposition 5.1 (i) and Proposition 5.3. The second term on the right-hand side of (5.3) is estimated by using Proposition 5.1 (ii) and Lemma 4.3 (ii). Hence, we obtain the desired estimate. This completes the proof. \( \square \)

In the rest of this section we will give proofs of Proposition 5.1 and Proposition 5.2.

**Lemma 5.5.** ([9]) (i) The set of all eigenvalues of \(-\hat{A}_\xi\) consists of \( \lambda_j(\xi) (j = 1, \pm) \), where

\[
\left\{ \begin{array}{l}
\lambda_1(\xi) = -\nu|\xi|^2, \\
\lambda_\pm(\xi) = -\frac{1}{2}(\nu + \tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}.
\end{array} \right.
\]

If \( |\xi| < \frac{2\nu}{\nu + \tilde{\nu}} \), then

\[
\text{Re} \lambda_\pm = -\frac{1}{2}(\nu + \tilde{\nu})|\xi|^2, \quad \text{Im} \lambda_\pm = \pm \gamma|\xi| \sqrt{1 - \frac{(\nu + \tilde{\nu})^2}{4\gamma^2}}|\xi|^2.
\]

(ii) \( e^{-t\hat{A}_\xi} \) has the spectral resolution

\[
e^{-t\hat{A}_\xi} = \sum_{j=1, \pm} e^{\lambda_j(\xi) t}\Pi_j(\xi),
\]
Lemma 5.6. Let \( \alpha \) be a multi-index. Then the following estimates hold true uniformly
for \( \xi \) with \( |\xi| \leq r_\infty \) and \( t \in [0, T] \).

(i) \( |\partial_\xi^\alpha \lambda_1| \leq C_\alpha |\xi|^{2-|\alpha|}, |\partial_\xi^\alpha \lambda_\pm| \leq C_\alpha |\xi|^{1-|\alpha|} \) (\( |\alpha| \geq 0 \)).

(ii) \( |(\partial_\xi^\alpha \Pi_1) \hat{F}| \leq C_\alpha |\xi|^{-|\alpha|} |\hat{F}_1|, |(\partial_\xi^\alpha \Pi_\pm) \hat{F}| \leq C_\alpha |\xi|^{-|\alpha|} |\hat{F}_1| \) (\( |\alpha| \geq 0 \)), where \( F = \hat{F}(F_1^0, \hat{F}_1) \).

(iii) \( |\partial_\xi^\alpha (e^{\lambda_1 t})| \leq C_{\alpha, T} |\xi|^{2-|\alpha|} \) (\( |\alpha| \geq 1 \)).

(iv) \( |\partial_\xi^\alpha (e^{\lambda_{\pm} t})| \leq C_{\alpha, T} |\xi|^{1-|\alpha|} \) (\( |\alpha| \geq 1 \)).

(v) \( |(\partial_\xi^\alpha e^{-tA_\xi}) \hat{F}| \leq C_{\alpha, T} (|\xi|^{-|\alpha|} |\hat{F}_1| + |\xi|^{-|\alpha|} |\hat{F}_1|) \) (\( |\alpha| \geq 1 \)), where \( F = \hat{F}(F_1^0, \hat{F}_1) \).

(vi) \( |\partial_\xi^\alpha (I - e^{\lambda_1 t})^{-1}| \leq C_{\alpha, T} |\xi|^{-2-|\alpha|} \) (\( |\alpha| \geq 0 \)).

(vii) \( |\partial_\xi^\alpha (I - e^{\lambda_{\pm} t})^{-1}| \leq C_{\alpha, T} |\xi|^{-1-|\alpha|} \) (\( |\alpha| \geq 0 \)).

Lemma 5.6 can be verified by direct computations based on Lemma 5.5.

Let us prove Proposition 5.1.

Proof of Proposition 5.1. We see from Lemma 4.3 (ii) that

\[ \|A_1 u_1\|_{H_{1,1}^1} \leq C \|\nabla u_1\|_{H_1^1} \leq C \|u_1\|_{H_{1,1}^1} \quad (u_1 \in H_{1,1}^1), \]

and so, \( A_1 \) is bounded on \( H_{1,1}^1 \). It then follows that \( S_1(\cdot) u_1 \in C([0, T'); H_{1,1}^1) \) for each \( u_1 \in H_{1,1}^1 \) and

\[ \|\partial_t^k S_1(t) u_1\|_{H_{1,1}^1} \leq C \|u_1\|_{H_{1,1}^1} \quad \text{for} \quad u_1 \in H_{1,1}^1, \quad t \in [0, T'], \quad k = 0, 1, \]

where \( \Pi_j(\xi) \) is eigenprojections for \( \lambda_j(\xi) \) (\( j = 1, \pm \)), and \( \Pi_j(\xi) \) (\( j = 1, \pm \)) satisfy

\[
\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \xi^{\top} \xi \end{pmatrix}, \\
\Pi_+(\xi) = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_+ & i\gamma \xi \\
-i\gamma \xi & \lambda_+ - \xi^{\top} \xi \end{pmatrix}, \\
\Pi_-(\xi) = -\frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_+ & i\gamma \xi \\
-i\gamma \xi & \lambda_- - \xi^{\top} \xi \end{pmatrix}.
\]

Furthermore, if \( 0 < r_\infty < \frac{2\omega}{\nu + \beta} \), then there exist a constant \( C > 0 \) such that the estimates

\[ \|\Pi_j(\xi)\| \leq C (j = 1, \pm) \] (5.4)

hold for \( |\xi| \leq r_\infty \).

Hereafter we fix \( 0 < r_1 < r_\infty < \frac{2\omega}{\nu + \beta} \) so that (5.4) in Lemma 5.5 holds for \( |\xi| \leq r_\infty \).
where $T' > 0$ is any given positive number and $C$ is a positive constant depending on $T'$. Since $\|A_1 u_1\|_{L^2} \leq C \|\nabla u_1\|_{H^1} \leq C \|u_1\|_{H^1}^{(i),1}$ for $u_1 \in H^1_{(i),1}$ by Lemma 4.3 (ii), we see from the relation $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ that $\partial_t S_1(t)u_1 \in C([0, T'); L^2]$ and

$$\|\partial_t S_1(t)u_1\|_{L^2} \leq \|S_1(t)u_1\|_{H^1}^{(i),1} \leq C \|u_1\|_{H^1}^{(i),1}.$$  

The assertion (ii) follows from (i) and the relation $\partial_t S_1(t)F_1 = -A_1 S_1(t)F_1 + F_1(t)$. The assertion (iii) easily follows from the definitions of $S_1(t)$ and $S_1(t)$. As for (iv), we observe that $\Gamma A_1 = A_1 \Gamma$, from which we find that $\Gamma S_1(t) = S_1(t)\Gamma$, and hence, $\Gamma S_1(t) = S_1(t)\Gamma$. This completes the proof. \(\square\)

Let us finally prove Proposition 5.2.

**Proof of Proposition 5.2.** We define a function $u$

$$u = \mathcal{F}^{-1}(I - e^{-T\hat{\lambda}_t})^{-1}\hat{F}_1$$

for $F_1 = \hat{\mathcal{F}}(F_0, \hat{F}_1)$. It suffices to show that $\|u\|_{H^1} \leq C \|F_1\|_{L^1}$. By the Plancherel theorem, we see that

$$\|u\|_{L^2} = (2\pi)^{-\frac{n}{2}} \|(I - e^{-T\hat{\lambda}_t})^{-1}\hat{F}_1\|_{L^2(\|\xi\| \leq r_\infty)}$$

$$\leq (2\pi)^{-\frac{n}{2}} \left\{ \|(I - e^{-T\lambda_1})^{-1}\Pi_1\hat{F}_1\|_{L^2(\|\xi\| \leq r_\infty)} + \|(I - e^{-T\lambda_\infty})^{-1}\Pi_1\hat{F}_1\|_{L^2(\|\xi\| \leq r_\infty)} \right\}$$

$$=: I_1 + I_2 + I_3.$$  

Observe that $\Pi_1\hat{F}_1$ depends only on $\hat{F}_1$ but not on $\hat{F}_0$.

By using Lemma 5.5, Lemma 5.6 and the fact $\hat{F}_1(0) = 0$, we see that

$$I_1 \leq C \left\| \frac{1}{|\xi|^2} \hat{F}_1 \right\|_{L^2(\|\xi\| \leq r_\infty)}$$

$$\leq C \left\| \frac{1}{|\xi|} \right\|_{L^2(\|\xi\| \leq r_\infty)} \|x\hat{F}_1\|_{L^1}.$$  

Since

$$\left\| \frac{1}{|\xi|} \right\|_{L^2(\|\xi\| \leq r_\infty)} < +\infty$$

for $n \geq 3$, we find that

$$I_1 \leq C \|x\hat{F}_1\|_{L^1}.$$  

Similarly, we can obtain $I_2 + I_3 \leq C \|F_1\|_{L^1}$, and hence, we see that

$$\|u\|_{L^2} \leq C\{\|F_1\|_{L^1} + \|x\hat{F}_1\|_{L^1}\}. \quad (5.5)$$

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Next, by the Plancherel theorem, it follows that
\[
\|x\nabla u\|_{L^2_{(1)}} = (2\pi)^{-\frac{3}{2}} \left\| (i\xi) \left( (I - e^{-T\hat{A}})^{-1} \hat{F}_1 \right) \right\|_{L^2(\xi \leq r_\infty)} \\
\leq C \left\{ \| (I - e^{-T\hat{A}})^{-1} \hat{F}_1 \|_{L^2(\xi \leq r_\infty)} + \| i\xi \hat{\partial}_\xi \left( (I - e^{-T\hat{A}})^{-1} \right) \hat{F}_1 \|_{L^2(\xi \leq r_\infty)} \right\}.
\]

The first term on right-hand side has already been estimated and it is bounded by the right-hand side of (5.5). As for the second and third terms on the right-hand side, similarly to above, one can find from Lemma 5.6 that
\[
\| i\xi \hat{\partial}_\xi \left( (I - e^{-T\hat{A}})^{-1} \right) \hat{F}_1 \|_{L^2(\xi \leq r_\infty)} + \| i\xi (I - e^{-T\hat{A}})^{-1} \partial_\xi \hat{F}_1 \|_{L^2(\xi \leq r_\infty)} \leq C\| F_1 \|_{L^1_1}.
\]
We thus obtain
\[
\|x\nabla u\|_{L^2_{(1)}} \leq C\| F_1 \|_{L^1_1}.
\]
Finally, we see from Proposition 5.1 (iv) that if \( \Gamma F_1 = F_1 \), then \( \Gamma u = u \). This completes the proof.

6 Properties of \( S_\infty, \tilde{u}(t) \) and \( S_\infty, \tilde{\phi}(t) \)

In this section we investigate \( S_\infty, \tilde{u}(t) \) and \( S_\infty, \tilde{\phi}(t) \).

Let us consider the following initial value problem
\[
\begin{align*}
\partial_t u_\infty + Au_\infty + P_\infty (B[\tilde{u}]u_\infty) &= F_\infty, \\
u|_{t=0} &= u_0|_\infty,
\end{align*}
\]
(6.1)
where
\[
F_\infty = \begin{pmatrix} F_\infty^0 \\ F_\infty^\infty \end{pmatrix}, \quad P_\infty (B[\tilde{u}]u) = \begin{pmatrix} \gamma P_\infty (\tilde{w} \cdot \nabla \phi) \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} \phi \\ \tilde{w} \end{pmatrix}, \quad \tilde{\phi} = \begin{pmatrix} \tilde{\phi} \\ \tilde{w} \end{pmatrix}.
\]

We begin with the solvability of (6.1). Let us first consider the following system:
\[
\begin{align*}
\partial_t \phi + \gamma (\tilde{w} \cdot \nabla \phi) &= f^0, \\
\phi|_{t=0} &= \phi_0.
\end{align*}
\]
(6.2)

**Lemma 6.1.** ([6, Theorem 4.1].) Let \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \). Set \( k = m - 1 \) or \( m \). Assume that \( \tilde{w} \in C([0, T^*]; H^m) \cap L^2(0, T'; H^{m+1}), f^0 \in L^2(0, T'; H^k) \) and \( \phi_0 \in H^k \). Here \( T' \) is a given positive number. Then (6.2) has a unique solution \( \phi \in C([0, T^*]; H^k) \) and \( \phi \) satisfies
\[
\| \phi(t) \|^2_{H^k} \leq C \left\{ \| \phi_0 \|^2_{H^k} + \int_0^t \| \tilde{w} \|^2_{H^{m+1}} \| \phi \|^2_{H^k} ds + \int_0^t \| f^0 \|_{H^k} \| \phi \|_{H^k} ds \right\}
\]
By Lemma 4.3 (i), we have

\[ \|\phi(t)\|_{H^k}^2 \leq Ce^{C\int_0^t(1+\|\tilde{\omega}\|_{H^{m+1}})ds} \left\{ \|\phi_0\|_{H^k}^2 + \int_0^t \|f^0\|_{H^k}^2 ds \right\} \]

for \( t \in [0, T'] \). Moreover, the solution is unique in \( C([0, T']; H^1) \).

We next consider the following system:

\[
\begin{cases}
\partial_t \phi_\infty + \gamma P_\infty (\tilde{\omega} \cdot \nabla \phi_\infty) = F_\infty^0, \\
\phi_\infty|_{t=0} = \phi_\infty^0.
\end{cases}
\]

(6.3)

Note that (6.3) is rewritten as

\[
\partial_t \phi_\infty + \gamma (\tilde{\omega} \cdot \nabla \phi_\infty) = F_\infty^0 + \gamma P_1(\tilde{\omega} \cdot \nabla \phi_\infty).
\]

(6.4)

As for the solvability of (6.3), we have the following lemma.

**Lemma 6.2.** Let \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \left[ \frac{n}{2} \right] + 1 \). Set \( k = m-1 \) or \( m \).

Assume that \( \tilde{\omega} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1}) \), \( F_\infty^0 \in L^2(0, T'; H^k) \) and \( \phi_\infty^0 \in H^k \). Here \( T' \) is a given positive number. Then (6.3) has a unique solution \( \phi_\infty \in C([0, T']; H^k) \) and \( \phi_\infty \) satisfies

\[
\|\phi_\infty(t)\|_{H^k}^2 \leq C \left\{ \|\phi_0\|_{H^k}^2 + \int_0^t \left( \|\tilde{\omega}\|_{H^{m+1}} + \|\tilde{\omega}\|_{H^m}^2 \right) \|\phi_\infty\|_{H^k}^2 ds \right. \\
+ \int_0^t \|F_\infty^0\|_{H^k} \|\phi_\infty\|_{H^k} ds \right\}
\]

and

\[
\|\phi_\infty(t)\|_{H^k}^2 \leq Ce^{C\int_0^t(1+\|\tilde{\omega}\|_{H^{m+1}}+\|\tilde{\omega}\|_{H^m}^2)ds} \left\{ \|\phi_0\|_{H^k}^2 + \int_0^t \|F_\infty^0\|_{H^k}^2 ds \right\}
\]

for \( t \in [0, T'] \).

**Proof.** We define \( \{\phi_\infty^{(p)}\}_{p=0}^\infty \) as follows. For \( p = 0 \), \( \phi_\infty^{(0)} \) is the solution of

\[
\begin{cases}
\partial_t \phi_\infty^{(0)} + \gamma (\tilde{\omega} \cdot \nabla \phi_\infty^{(0)}) = F_\infty^0, \\
\phi_\infty^{(0)}|_{t=0} = \phi_\infty^0.
\end{cases}
\]

(6.5)

For \( p \geq 1 \), \( \phi_\infty^{(p)} \) is the solution of

\[
\begin{cases}
\partial_t \phi_\infty^{(p)} + \gamma (\tilde{\omega} \cdot \nabla \phi_\infty^{(p)}) = F_\infty^0 + \gamma P_1(\tilde{\omega} \cdot \nabla \phi_\infty^{(p-1)}), \\
\phi_\infty^{(p)}|_{t=0} = \phi_\infty^0.
\end{cases}
\]

(6.6)

By Lemma 4.3 (i), we have

\[
\|P_1(\tilde{\omega} \cdot \nabla \phi_\infty)\|_{H^m} \leq C\|\tilde{\omega}\|_{L^\infty} \|\nabla \phi_\infty\|_{L^2} \leq C\|\tilde{\omega}\|_{H^m} \|\phi_\infty\|_{H^k}
\]

(6.7)
since $m \geq \lfloor \frac{n}{2} \rfloor + 1 \geq 2$. In view of Lemma 6.1 and (6.7), we find by a standard argument that

$$\| \phi_{\infty}^{(p+1)}(t) - \phi_{\infty}^{(p)}(t) \|_{H^k} \leq A_0 \frac{(A_1 t)^{p+1}}{(p+1)!} \quad (p \geq 0),$$

where

$$A_0 = C e^{C \int_0^{T'} (1 + \| \hat{w} \|_{H^{m+1}} + \| \hat{w} \|_{H^m}) \, dr} \left\{ \| \phi_0 \|_{H^k}^2 + \int_0^{T'} \| F_0 \|_{H^k}^2 \, d\tau \right\},$$

$$A_1 = C \| \hat{w} \|_{C((0,T'];H^m)} e^{C \int_0^{T'} (1 + \| \hat{w} \|_{H^{m+1}}) \, dr}.$$

Therefore, one can see that $\phi_{\infty}^{(p)}$ converges in $C([0, T']; H^k)$ to a function $\phi_{\infty} \in C([0, T']; H^k)$ that satisfies

$$\begin{cases}
\partial_t \phi_{\infty} + \gamma (\hat{w} \cdot \nabla \phi_{\infty}) = F_0^0 + \gamma P_1 (\hat{w} \cdot \nabla \phi_{\infty}), \\
\phi_{\infty}|_{t=0} = \phi_{0\infty}.
\end{cases} \quad (6.8)$$

In view of (6.4), we see that $\phi_{\infty}$ is a solution of (6.3). The estimates for $\phi_{\infty}$ follows from Lemma 6.1 and (6.7).

It remains to prove $\operatorname{supp} \hat{\phi}_{\infty}(t) \subset \{ \| \xi \| \geq r_1 \}$ for $t \in [0, T']$. Let $\tilde{\chi}_{\infty} \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \tilde{\chi}_{\infty} \subset \{ \| \xi \| < r_1 \}$. Let us consider the Fourier transform of (6.3):

$$\partial_t \hat{\phi}_{\infty} + \gamma \hat{\chi}_{\infty} (\hat{w} \cdot \nabla \hat{\phi}_{\infty}) = \hat{F}_0, \quad \hat{\phi}_{\infty}|_{t=0} = \hat{\phi}_{0\infty}.$$ 

Taking the inner product of this equation with $\hat{\chi}_{\infty}^2 \hat{\phi}_{\infty}$, we have $\frac{d}{dt} \| \hat{\chi}_{\infty} \hat{\phi}_{\infty} \|_{L^2}^2 = 0$. We thus deduce that $\| \hat{\chi}_{\infty} \hat{\phi}_{\infty}(t) \|_{L^2}^2 = \| \hat{\chi}_{\infty} \hat{\phi}_{0\infty} \|_{L^2}^2 = 0$ for $t \in [0, T']$. It then follows that $\operatorname{supp} \hat{\phi}_{\infty}(t) \subset \{ \| \xi \| \geq r_1 \}$ for $t \in [0, T']$. This completes the proof. \qed

We next consider the following system:

$$\begin{cases}
\partial_t w_{\infty} - \nu \Delta w_{\infty} - \hat{u} \nabla \operatorname{div} w_{\infty} = \tilde{F}_\infty, \\
w_{\infty}|_{t=0} = w_{0\infty}.
\end{cases} \quad (6.9)$$

**Lemma 6.3.** (i) Let $n \geq 3$ and let $m$ be an integer satisfying $m \geq \lfloor \frac{n}{2} \rfloor + 1$. Set $k = m - 1$ or $m$. Assume that $\tilde{F}_\infty \in L^2(0, T'; H^{k-1})$ and $w_{0\infty} \in H^k$. Here $T'$ is a given positive number. Then (6.9) has a unique solution $w_{\infty} \in C([0, T']; H^k) \cap L^2(0, T'; H^{k+1}) \cap H^1(0, T'; H^{k-1})$ and

$$\| w_{\infty}(t) \|_{H^k}^2 + \int_0^t \| w_{\infty} \|_{H^{k+1}}^2 + \| \partial_t w_{\infty} \|_{H^{k-1}}^2 \, d\tau \leq C \left\{ \| w_{0\infty} \|_{H^k}^2 + \int_0^t \| \tilde{F}_\infty \|_{H^{k-1}}^2 \, ds \right\}$$

for $t \in [0, T']$ with $C = C(T') > 0$.

(ii) Assume, further, that $\tilde{F}_\infty \in L^2(0, T'; H^{k-1}_{(\infty)})$ and $w_{0\infty} \in H^k_{(\infty)}$. Then the solution $w_{\infty}$ satisfies

$$w_{\infty} \in C([0, T']; H^k_{(\infty)}) \cap L^2(0, T'; H^{k+1}_{(\infty)}) \cap H^1(0, T'; H^{k-1}_{(\infty)}).$$
Lemma 6.3 (i) follows from standard theory of parabolic equation. The assertion (ii) can be proved in a similar manner to the proof of Lemma 6.2. We omit the details.

By using Lemma 6.2 and Lemma 6.3, we show the solvability of (6.1).

**Proposition 6.4.** Let $n \geq 3$ and let $m$ be an integer satisfying $m \geq \left[ \frac{n}{2} \right] + 1$. Set $k = m - 1$ or $m$. Assume that

\[
\begin{align*}
\dot{w} &\in C([0, T'); H^m) \cap L^2(0, T'; H^{m+1}), \\
u_0 &= \tau(\phi_0, w_0) \in H^k, \\
F_0 &= \tau(F_0^0, F_0^1) \in L^2(0, T'; H^k_0) \times H^{k-1}_1).
\end{align*}
\]

Here $T'$ is a given positive number. Then there exists a unique solution $u_\infty = \tau(\phi_\infty, w_\infty)$ of (6.1) satisfying

\[
\phi_\infty \in C([0, T'); H^k_0), \quad w_\infty \in C([0, T'); H^k_0) \cap L^2(0, T'; H^{k+1}_0) \cap H^1(0, T'; H^{k-1}_0).
\]

**Proof.** We write (6.1) as

\[
\begin{align*}
\begin{cases}
\partial_t \phi_\infty + \gamma P_\infty (\dot{w} \cdot \nabla \phi_\infty) + \gamma \text{div} \, w_\infty &= F_0^0, \\
n_\infty - \nu \Delta \phi_\infty = \nu \text{div} \, w_\infty + \gamma \nabla \phi_\infty &= F_0^1, \\
\phi_\infty |_{t=0} &= \nu \phi_0, \quad w_\infty |_{t=0} = w_0 \infty.
\end{cases}
\end{align*}
\]

We define $u^{(p)} = \tau(\phi^{(p)}_\infty, w^{(p)}_\infty)$ ($p = 0, 1, \cdots$) as follows. For $p = 0$, $w^{(0)} = w_\infty$ and $\phi^{(0)}_\infty$ is the solution of

\[
\begin{align*}
\begin{cases}
\partial_t \phi^{(0)}_\infty + \gamma P_\infty (\dot{w} \cdot \nabla \phi^{(0)}_\infty) &= F_0^0, \\
\phi^{(0)}_\infty |_{t=0} &= \phi_0. \quad (6.11)
\end{cases}
\end{align*}
\]

For $p \geq 1$, $w^{(p)}_\infty$ is the solution of

\[
\begin{align*}
\begin{cases}
\partial_t w^{(p)}_\infty - \nu \Delta w^{(p)}_\infty = -\gamma \nabla \phi^{(p-1)}_\infty + F_0^1, \\
w^{(p)}_\infty |_{t=0} = w_0 \infty. \quad (6.12)
\end{cases}
\end{align*}
\]

and $\phi^{(p)}_\infty$ is the solution of

\[
\begin{align*}
\begin{cases}
\partial_t \phi^{(p)}_\infty + \gamma P_\infty (\dot{w} \cdot \nabla \phi^{(p)}_\infty) = -\gamma \text{div} \, w^{(p)}_\infty + F_0^0, \\
\phi^{(p)}_\infty |_{t=0} &= \phi_0. \quad (6.13)
\end{cases}
\end{align*}
\]

As in the proof of Lemma 6.2, by using Lemma 6.2 and Lemma 6.3, one can show that $u^{(p)} = \tau(\phi^{(p)}_\infty, w^{(p)}_\infty)$ converges to a pair of function $u_\infty = \tau(\phi_\infty, w_\infty)$ in $C([0, T'); H^k_0) \times [C([0, T'); H^k_0) \cap L^2(0, T'; H^{k+1}_0)]$. It is not difficult to see that $u_\infty = \tau(\phi_\infty, w_\infty)$ is a unique solution of (6.1). This completes the proof. □

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We now define $S_{\infty,\tilde{u}}(t)$ and $\mathcal{S}_{\infty,\tilde{u}}(t)$ formally introduced in section 4.

In the remaining of this section we fix an integer $m$ satisfying $m \geq \left[ \frac{n}{2} \right] + 1$ and a function $\tilde{u} = \mathcal{T}(\tilde{\phi}, \tilde{w})$ satisfying

$$\tilde{\phi} \in C_{\text{per}}(\mathbb{R}; H^m), \quad \tilde{w} \in C_{\text{per}}(\mathbb{R}; H^m) \cap L^2_{\text{per}}(\mathbb{R}; H^{m+1})$$

(6.14)

In view of Proposition 6.4, we define $S_{\infty,\tilde{u}}(t)$ ($t \geq 0$) and $\mathcal{S}_{\infty,\tilde{u}}(t)$ ($t \in [0, T]$) as follows.

Let $k = m - 1$ or $m$. The operator $S_{\infty,\tilde{u}}(t) : H_{(\infty)}^k \rightarrow H_{(\infty)}^k$ ($t \geq 0$) is defined by

$$u_{\infty}(t) = S_{\infty,\tilde{u}}(t) u_{0\infty} \quad \text{for} \quad u_{0\infty} = \mathcal{T}(\phi_{0\infty}, w_{0\infty}) \in H_{(\infty)}^k,$$

where $u_{\infty}(t)$ is the solution of (6.1) with $F_{\infty} = 0$; and the operator $\mathcal{S}_{\infty,\tilde{u}}(t) : L^2(0, T; H_{(\infty)}^k \times H_{(\infty)}^{k-1}) \rightarrow H_{(\infty)}^k$ ($t \in [0, T]$) is defined by

$$u_{\infty}(t) = \mathcal{S}_{\infty,\tilde{u}}(t)[F_{\infty}] \quad \text{for} \quad F_{\infty} = \mathcal{T}(F_{\infty}^0, \tilde{F}_{\infty}) \in L^2(0, T; H_{(\infty)}^k \times H_{(\infty)}^{k-1}),$$

where $u_{\infty}(t)$ is the solution of (6.1) with $u_{0\infty} = 0$.

The operators $S_{\infty,\tilde{u}}(t)$ and $\mathcal{S}_{\infty,\tilde{u}}(t)$ have the following properties in weighted Sobolev spaces.

**Proposition 6.5.** Let $n \geq 3$ and let $m$ be a nonnegative integer satisfying $m \geq \left[ \frac{n}{2} \right] + 1$. Let $k = m - 1$ or $m$ and let $\ell$ be a nonnegative integer. Assume that $\tilde{u} = \mathcal{T}(\tilde{\phi}, \tilde{w})$ satisfies (6.14). Then there exists a constant $\delta > 0$ such that if $\|\tilde{w}\|_{C([0, T]; H^m) \cap L^2(0, T; H^{m+1})} \leq \delta$, the following assertions hold true.

(i) It holds that $S_{\infty,\tilde{u}}(\cdot) u_{0\infty} \in C([0, \infty); H_{(\infty),\ell}^k)$ for each $u_{0\infty} = \mathcal{T}(\phi_{0\infty}, w_{0\infty}) \in H_{(\infty),\ell}^k$ and there exists a constant $a > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the estimate

$$\|S_{\infty,\tilde{u}}(t) u_{0\infty}\|_{H_{(\infty),\ell}^k} \leq C e^{-at} \|u_{0\infty}\|_{H_{(\infty),\ell}^k}$$

for all $t \geq 0$ and $u_{0\infty} \in H_{(\infty),\ell}^k$, with a constant $C = C(T) > 0$.

(ii) It holds that $\mathcal{S}_{\infty,\tilde{u}}(\cdot) F_{\infty} \in C([0, T]; H_{(\infty),\ell}^k)$ for each $F_{\infty} = \mathcal{T}(F_{\infty}^0, \tilde{F}_{\infty}) \in L^2(0, T; H_{(\infty),\ell}^k \times H_{(\infty),\ell}^{k-1})$ and $\mathcal{S}_{\infty,\tilde{u}}(t)$ satisfies the estimate

$$\|\mathcal{S}_{\infty,\tilde{u}}(t)[F_{\infty}]\|_{H_{(\infty),\ell}^k} \leq C \left\{ \int_0^t e^{-a(t-\tau)} \|F_{\infty}\|^2_{H_{(\infty),\ell}^k \times H_{(\infty),\ell}^{k-1}} d\tau \right\}^{\frac{1}{2}}$$

for $t \in [0, T]$ and $F_{\infty} \in L^2(0, T; H_{(\infty),\ell}^k \times H_{(\infty),\ell}^{k-1})$ with $C = C(T) > 0$.

(iii) It holds that $r_{H_{(\infty),\ell}^k}(S_{\infty,\tilde{u}}(T)) < 1$. 

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(iv) $I - S_{\infty,\hat{u}}(T)$ has a bounded inverse $(I - S_{\infty,\hat{u}}(T))^{-1}$ on $H_{(\infty,\ell)}^k$ and $(I - S_{\infty,\hat{u}}(T))^{-1}$ satisfies
\[
\| (I - S_{\infty,\hat{u}}(T))^{-1} u \|_{H_{(\infty,\ell)}^k} \leq C \| u \|_{H_{(\infty,\ell)}^k} \quad \text{for } u \in H_{(\infty,\ell)}^k.
\]

(v) If $\Gamma \hat{u} = \hat{u}$, then $\Gamma S_{\infty,\hat{u}}(t) = S_{\infty,\hat{u}}(t) \Gamma$ and $\Gamma S_{\infty,\hat{u}}(t) = S_{\infty,\hat{u}}(t) \Gamma$. Consequently, if $\Gamma \hat{u} = \hat{u}$, then the assertions (i)–(iv) above hold with function spaces $H_{(\infty,\ell)}^k$ and $H_{(\infty,\ell)}^k \times H_{(\infty,\ell)}^{k-1}$ replaced by $(H_{(\infty,\ell)}^k)_{sym}$ and $(H_{(\infty,\ell)}^k \times H_{(\infty,\ell)}^{k-1})_{sym}$, respectively.

**Remark 6.6.** In this paper we will apply Proposition 6.5 with $\ell = 1$ to prove Theorem 3.1. For the purpose of future use, we formulate and prove it for a general nonnegative integer $\ell$.

Proposition 6.5 will be proved by the weighted energy method. In fact, Proposition 6.5 follows from the weighted energy estimate in the following proposition.

**Proposition 6.7.** Let $n \geq 3$ and let $m$ be a nonnegative integer satisfying $m \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Let $k = m - 1$ or $m$ and let $\ell$ be a nonnegative integer. Assume that
\[
\begin{align*}
\phi_{0\infty} &= \tau(\phi_{0\infty}, w_{0\infty}) \in H_{(\infty,\ell)}^k, \\
F_{0\infty} &= \tau(F_{0\infty}, \tilde{F}_{0\infty}) \in L^2(0, T'; H_{(\infty,\ell)}^k \times H_{(\infty,\ell)}^{k-1})
\end{align*}
\]
for all $T' > 0$ and that $\hat{u} = \tau(\hat{\phi}, \hat{w})$ satisfies (6.14). Assume also that $u_{\infty} = \tau(\phi_{\infty}, w_{\infty})$ is the solution of (6.1) satisfying
\[
\phi_{\infty} \in C([0, T'; H_{(\infty)}^k), \quad w_{\infty} \in C([0, T'; H_{(\infty)}^k] \cap L^2(0, T'; H_{(\infty)}^{k+1})
\]
for all $T' > 0$.

Then there exist a positive constant $\delta$ and an energy functional $\mathcal{E}_\ell^k[u_{\infty}]$ such that if
\[
\| \hat{w} \|_{C([0, T]; H^m) \cap L^2(0, T; H^{m+1})} \leq \delta,
\]
there holds the estimate
\[
\frac{d}{dt} \mathcal{E}_\ell^k[u_{\infty}](t) + d(\| \phi_{\infty} \|_{H_{H_{T}^k}}^2 + \| w_{\infty} \|_{H_{T}^{k+1}}^2) \leq C \left\{ \| F_{\infty} \|_{L^2(H_{T}^k \times H_{T}^{k-1})}^2 + (\| \nabla \hat{w} \|_{H^m} + \| \nabla \bar{w} \|_{H^m}) \| \phi_{\infty} \|_{H_{T}^k}^2 \right\}
\]
(6.15)
on $(0, T')$ for all $T' > 0$. Here $d$ is a positive constant depending on $\ell$; $C$ is a positive constant depending on $T$ but not on $T'$; $\mathcal{E}_\ell^k[u_{\infty}]$ is equivalent to $\| u_{\infty} \|_{H_{T}^k}^2$, i.e,
\[
C^{-1} \| u_{\infty} \|_{H_{T}^k}^2 \leq \mathcal{E}_\ell^k[u_{\infty}] \leq C \| u_{\infty} \|_{H_{T}^k}^2,
\]
and $\mathcal{E}_\ell^k[u_{\infty}](t)$ is absolutely continuous in $t \in [0, T')$ for all $T' > 0$. 

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The proof of Proposition 6.7 will be given in section 7.

By using Proposition 6.7, we prove Proposition 6.5.

**Proof of Proposition 6.5.** Set

\[
\omega = \frac{1}{T} \int_0^T \left( \|\nabla \tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|_{H^m}^2 \right) dt,
\]

\[
z(t) = \left( \|\nabla \tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|_{H^m}^2 \right) - \omega,
\]

\[
Z(t) = \int_0^t z(\tau) d\tau.
\]

Observe that \(Z(t)\) satisfies \(Z(t + T) = Z(t)\) for any \(t \in \mathbb{R}\), and so it holds that

\[
\sup_{t \in \mathbb{R}} |Z(t)| \leq \sup_{\tau \in [0,T]} |Z(\tau)| \leq C(1 + \|\tilde{w}\|^2_{L^2(0,T;H^{m+1})}),
\]

where \(C = C(T) > 0\).

By Proposition 6.7 with \(F_\infty = 0\), we see that there exists a positive constant \(d_1\) such that

\[
\frac{d}{dt} \mathcal{E}_t^k[u_\infty](t) + d_1 \mathcal{E}_t^k[u_\infty](t) \leq C \omega \mathcal{E}_t^k[u_\infty](t) + C z(t) \mathcal{E}_t^k[u_\infty](t) \quad (t \geq 0).
\]

(6.16)

If \(\omega \leq \frac{d_1}{2T}\), then we find from (6.16) that

\[
\frac{d}{dt} \mathcal{E}_t^k[u_\infty](t) + \frac{d_1}{2} \mathcal{E}_t^k[u_\infty](t) \leq C z(t) \mathcal{E}_t^k[u_\infty](t) \quad (t \geq 0).
\]

We thus obtain

\[
\frac{d}{dt} \left( e^{\frac{d_1}{2} t} e^{-CZ(t)} \mathcal{E}_t^k[u_\infty](t) \right) \leq 0 \quad (t \geq 0),
\]

and hence,

\[
\mathcal{E}_t^k[u_\infty](t) \leq \mathcal{E}_t^k[u_\infty](0) e^{-\frac{d_1}{2} t} e^{-CZ(t)} \leq e^{C(1 + \|\tilde{w}\|^2_{L^2(0,T;H^{m+1})})} \mathcal{E}_t^k[u_\infty](0) e^{-\frac{d_1}{2} t} \quad (t \geq 0).
\]

Consequently, we have

\[
\|S_{\infty,\tilde{u}}(t) u_0\|_{H_{(\infty),\ell}^k} \leq C e^{-\frac{d_1}{2} t} \|u_0\|_{H_{(\infty),\ell}^k} \quad (t \geq 0).
\]

This proves (i). The assertion (ii) is proved similarly; and we omit the proof.

As for (iii), since \(\tilde{u} = \tilde{\tau}(\tilde{\phi}, \tilde{w}) \in C_{per}(\mathbb{R}; H^m)\), it follows from (i) that, for each \(j \in \mathbb{N}\),

\[
\|(S_{\infty,\tilde{u}}(T))^j u\|_{H_{(\infty),\ell}^k} = \|S_{\infty,\tilde{u}}(jT) u\|_{H_{(\infty),\ell}^k} \leq C e^{-d_2 j T} \|u\|_{H_{(\infty),\ell}^k},
\]

where \(d_2 = \frac{d_1}{T} > 0\). Hence, we have

\[
\|(S_{\infty,\tilde{u}}(T))^j\| \leq C e^{-d_2 j T}.
\]

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We thus obtain
\[ \lim_{j \to \infty} \|(S_{\infty, \hat{u}}(T))^j\|^2 \leq \lim_{j \to \infty} C^j e^{-d_2 T} = e^{-d_2 T} < 1. \]
This shows (iii). The assertion (iv) is an immediate consequence of (iii).

As for (v), we see that if \( \Gamma \hat{u} = \hat{u} \), then \( \Gamma P_\infty(B[\hat{u}]u_\infty) = P_\infty(B[\hat{u}]\Gamma u_\infty) \), and so,
\[ \Gamma(\partial_t u_\infty + A u_\infty + P_\infty(B[\hat{u}]u_\infty)) = \partial_t \Gamma u_\infty + A \Gamma u_\infty + P_\infty(B[\hat{u}]\Gamma u_\infty). \]
It then follows from the uniqueness of solutions of (6.1) that \( \Gamma S_{\infty, \hat{u}}(t) = S_{\infty, \hat{u}}(t) \Gamma \) and \( \Gamma \mathcal{S}_{\infty, \hat{u}}(t) = \mathcal{S}_{\infty, \hat{u}}(t) \Gamma \). This completes the proof.

We conclude this section with the estimate for a solution \( u_\infty \) of (6.1) satisfying \( u_\infty(0) = u_\infty(T) \).

**Proposition 6.8.** Let \( n \geq 3 \) and let \( m \) be a nonnegative integer satisfying \( m \geq \left[ \frac{n}{2} \right] + 1 \). Assume that
\[ F_\infty = \top(F^0_\infty, \hat{F}_\infty) \in L^2(0, T; (H_{(\infty),1}^k \times H_{(\infty),1}^{k-1})_{\text{sym}}) \]
with \( k = m - 1 \) or \( m \). Assume also that \( \hat{\varphi} = \top(\hat{\phi}, \hat{\varphi}) \) satisfies (6.14) and \( \Gamma \hat{u} = \hat{u} \). Then there exists a positive constant \( \delta \) such that if
\[ \|\hat{\varphi}\|_{C([0, T]; H^m)} \leq \delta, \]
the following assertion holds true.

The function
\[ u_\infty(t) := S_{\infty, \hat{u}}(t)(I - S_{\infty, \hat{u}}(T))^{-1}\mathcal{S}_{\infty, \hat{u}}(T)[F_\infty] + \mathcal{S}_{\infty, \hat{u}}(t)[F_\infty] \quad (6.17) \]
is a solution of (6.1) in \( \mathcal{Y}_{\infty}^k(0, T) \) satisfying \( u_\infty(0) = u_\infty(T) \) and the estimate
\[ \|u_\infty\|_{\mathcal{Y}_{\infty}^k(0, T)} \leq C\|F_\infty\|_{L^2(0, T; H_{(\infty),1}^k \times H_{(\infty),1}^{k-1})}. \]

**Proof.** By Proposition 6.7 and Proposition 6.5, we see that
\[
\begin{align*}
&\|u_\infty(t)\|_{H_1^k}^2 + \|w_\infty\|_{L^2(0, t; H_1^k)}^2 \\
&\leq C \int_0^T \left( \|\nabla \hat{\varphi}\|_{H^m} + \|\hat{\varphi}\|_{H^m}^2 \right) \|\phi_\infty\|_{H_1^k}^2 \, ds \\
&\leq C \left\{ \|F_\infty\|_{L^2(0, t; H_{(\infty),1}^k \times H_{(\infty),1}^{k-1})}^2 + \delta \|\phi_\infty\|_{C([0, T]; H_1^k)}^2 \right\}
\end{align*}
\]
for \( t \in [0, T] \). Therefore, if \( \delta \) is so small that \( C \delta \leq \frac{1}{2} \), then we obtain
\[ \|u_\infty\|_{C([0, T]; H_1^k)}^2 + \|w_\infty\|_{L^2(0, T; H_1^k)}^2 \leq C\|F_\infty\|_{L^2(0, T; H_{(\infty),1}^k \times H_{(\infty),1}^{k-1})}^2. \quad (6.18) \]

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Next, since $u_\infty = \top (\phi_\infty, w_\infty)$ satisfies the system of equations

\[
\begin{aligned}
\partial_t \phi_\infty + \gamma P_\infty (\hat{w} \cdot \nabla \phi_\infty) + \gamma \text{div} w_\infty &= F_0, \\
\partial_t w_\infty - \nu \Delta w_\infty - \hat{\nu} \text{div} w_\infty + \gamma \nabla \phi_\infty &= \hat{F}_\infty,
\end{aligned}
\]

we obtain

\[
\| \partial_t w_\infty \|_{H^{k-1}_\infty, 1} \leq C \{ \| w_\infty \|_{H^{k+1}_\infty, 1} + \| \phi_\infty \|_{H^{k}_\infty, 1} + \| \hat{F}_\infty \|_{H^{k-1}_\infty, 1} \}.
\]

Hence, it follows from (6.18) that

\[
\| \partial_t w_\infty \|_{L^2(0,T; H^{k-1}_\infty, 1)} \leq C \| F_\infty \|_{L^2(0,T; H^{k}_\infty, 1 \times H^{k-1}_\infty, 1)}.
\]

Consequently, we see from (6.18) and (6.19) that

\[
\| u_\infty \|_{B^k_\infty(0,T)} \leq C \| F_\infty \|_{L^2(0,T; H^k_\infty, 1 \times H^{k-1}_\infty, 1)}.
\]

This completes the proof.

\[
\square
\]

7 Weighted energy estimates for $P_\infty$ part

In this section we prove Proposition 6.7 by a weighted energy method.

We first consider the following equation.

\[
\begin{aligned}
\partial_t u_\infty + A u_\infty + B[\hat{u}] u_\infty &= F_\infty, \\
u |t| = u_0 &= u_{0\infty},
\end{aligned}
\]

where

\[
F_\infty = \begin{pmatrix} F_0 \\ \hat{F}_\infty \end{pmatrix}, B[\hat{u}] u = \begin{pmatrix} \gamma \hat{w} \cdot \nabla \phi \\ 0 \end{pmatrix}, u = \begin{pmatrix} \phi \\ w \end{pmatrix}, \hat{u} = \begin{pmatrix} \hat{\phi} \\ \hat{w} \end{pmatrix}.
\]

We introduce some notations. For nonnegative integers $k$ and $\ell$, we define $E^k_\ell [u_\infty]$ by

\[
E^k_\ell [u_\infty] = \kappa (|\phi_\infty|^2_{H^k_\ell} + |w_\infty|^2_{H^k_\ell}) + \sum_{|\alpha| \leq k-1} (\partial_\alpha^2 u_\infty, |x|^2 \partial_\alpha^2 \phi_\infty).
\]

Here $\kappa$ is a positive constant to be determined later.

Note that there exists a constant $\kappa_0 > 0$ such that if $\kappa \geq \kappa_0$, then $E^k_\ell [u_\infty]$ is equivalent to $|u_\infty|^2_{H^k_\ell}$, i.e.,

\[
C^{-1} |u_\infty|^2_{H^k_\ell} \leq E^k_\ell [u_\infty] \leq C |u_\infty|^2_{H^k_\ell}
\]

for some constant $C > 0$.

We also define $D^k_\ell [u_\infty]$ for integers $k \geq 1$ and $\ell \geq 0$ by

\[
D^k_\ell [u_\infty] = |\nabla \phi_\infty|^2_{H^{k-1}_\ell} + |\nabla w_\infty|^2_{H^k_\ell}.
\]
Proposition 7.1. Let $m$ be a nonnegative integer satisfying $m \geq \left[ \frac{n}{2} \right] + 1$ and let $\ell$ be a nonnegative integer. Assume that

$$u_{0\infty} = (\phi_{0\infty}, w_{0\infty}) \in H^k,$$

$$F_{\infty} = (F^0_{\infty}, F_{\infty}) \in L^2(0, T'; H^k \times H^{k-1})$$

for $k = m - 1$ or $k = m$. Here $T'$ is a given positive number. Assume also that $u_{\infty} = (\phi_{\infty}, w_{\infty})$ is the solution of (7.1) with $\tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1})$ and that $u_{\infty} = (\phi_{\infty}, w_{\infty})$ satisfies

$$\phi_{\infty} \in C([0, T']; H^k), \quad w_{\infty} \in C([0, T']; H^k) \cap L^2(0, T'; H^{k+1}).$$

Then there exist positive constants $\kappa \geq \kappa_0$ and $d > 0$ such that the estimate

$$\frac{d}{dt} E^k_t[\zeta_R u_{\infty}] + d D^k_t[\zeta_R u_{\infty}]$$

$$\leq C \left\{ \epsilon |\zeta_R u_{\infty}|^2_2 + \left( 1 + \frac{\ell^2}{\epsilon} \right) \|\tilde{w}\|_H^m + \|\nabla \tilde{w}\|_H^m \right\} \|\zeta_R \phi_{\infty}\|_H^k$$

$$+ \left( 1 + \frac{1}{\epsilon} \right) |\zeta_R F^0_{\infty}|^2_{H^k} + \left( 1 + \frac{1}{\epsilon} \right) |\zeta_R u_{\infty}|^2_{H^{k-1}}$$

$$+ \epsilon^2 \left( 1 + \frac{1}{\epsilon} \right) (1 + \|\tilde{w}\|^2_H^m) |\zeta_R u_{\infty}|^2_{H^k}$$

$$+ \left( 1 + \frac{1}{\epsilon} \right) (1 + \|\tilde{w}\|^2_H^{m(N)\infty}) |u_{\infty}|^2_{H^{k-1}(N) \times H^{k-1}(N)\infty} \right\} \quad (7.2)$$

holds on $(0, T')$, where $\epsilon$ is any positive number; $C$ is a positive constant independent of $T'$, $\epsilon$ and $R \geq 1$; and $N_R$ denotes the set $N_R = \{ x \in \mathbb{R}^n; R \leq |x| \leq 2R \}$.

**Proof.** By multiplying $\zeta_R$ to (7.1), we obtain

$$\begin{cases}
\partial_t (\zeta_R \phi_{\infty}) + \gamma \tilde{w} \cdot \nabla (\zeta_R \phi_{\infty}) + \gamma \text{div}(\zeta_R w_{\infty}) = \zeta_R F^0_{\infty} + K_1(\nabla \zeta_R), \\
\partial_t (\zeta_R w_{\infty}) - \nu \Delta (\zeta_R w_{\infty}) - \nu \nabla \text{div}(\zeta_R w_{\infty}) + \gamma \nabla (\zeta_R \phi_{\infty}) = \zeta_R F_{\infty} + K_2(\nabla \zeta_R),
\end{cases} \quad (7.3)$$

where

$$K_1(\nabla \zeta_R) = \gamma (w_{\infty} \cdot \nabla \zeta_R + \tilde{w} \cdot \nabla \zeta_R \phi_{\infty}),$$

$$K_2(\nabla \zeta_R) = -\nu (\zeta_R, \Delta |w_{\infty}) - \nu (|\zeta_R, \nabla \text{div}| w_{\infty}) + \gamma \nabla \zeta_R \phi_{\infty}.$$  

For a multi-index $\alpha$ satisfying $|\alpha| \leq k$, we take the inner product of $\partial_x^\alpha(7.3)_1$ with $|x|^{2\ell} \partial_x^\alpha(\zeta_R \phi_{\infty})$ to obtain

$$\frac{1}{2} \frac{d}{dt} \| |x|^{2\ell} \partial_x^\alpha(\zeta_R \phi_{\infty}) \|_L^2 + \gamma (\partial_x^\alpha \text{div}(\zeta_R w_{\infty}), |x|^{2\ell} \partial_x^\alpha (\zeta_R \phi_{\infty}))$$

$$= \sum_{j=1}^2 I^{(j)}_{\alpha, \ell, R} + \mathcal{P}^{(1)}_{\alpha, \ell}[\zeta_R u_{\infty}] + Q_{1, \alpha, \ell}(\nabla \zeta_R), \quad (7.4)$$
where

\[ I_{\alpha,t,R}^{(1)} = -\gamma \left\{ \frac{1}{2} \left( \text{div} \tilde{w}, |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|^2 \right) + \left( |\partial_x^\alpha \tilde{w}| \nabla (\zeta_R \phi_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)| \right) \right\}, \]

\[ I_{\alpha,t,R}^{(2)} = \left( \partial_x^\alpha (\zeta_R F^0_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)| \right), \]

\[ \mathcal{P}_{\alpha,t}^{(1)}[\zeta_R u_\infty] = \frac{\gamma}{2} (\tilde{w} \cdot \nabla (|x|^{2\ell}), |\partial_x^\alpha (\zeta_R \phi_\infty)|^2), \]

\[ Q_{1,\alpha,t}(\nabla \zeta_R) = (\partial_x^\alpha K_1 (\nabla \zeta_R), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|). \]

Here we used

\[
\begin{align*}
(\partial_x^\alpha (\gamma \tilde{w} \cdot \nabla (\zeta_R \phi_\infty)), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|) \\
= \gamma (\tilde{w} \cdot \nabla \partial_x^\alpha (\zeta_R \phi_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|) + \gamma (|\partial_x^\alpha \tilde{w}| \nabla (\zeta_R \phi_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|) \\
= \frac{1}{2} \gamma (|x|^{2\ell} \tilde{w}, |\partial_x^\alpha (\zeta_R \phi_\infty)|^2) + \gamma (|\partial_x^\alpha \tilde{w}| \nabla (\zeta_R \phi_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|) \\
= -\frac{1}{2} \gamma (|x|^{2\ell} \tilde{w}, |\partial_x^\alpha (\zeta_R \phi_\infty)|^2) - \frac{1}{2} \gamma (\tilde{w} \cdot \nabla (|x|^{2\ell}), |\partial_x^\alpha (\zeta_R \phi_\infty)|^2) \\
+ \gamma (|\partial_x^\alpha \tilde{w}| \nabla (\zeta_R \phi_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|) \\
= I_{\alpha,t,R}^{(1)} + \mathcal{P}_{\alpha,t}^{(1)}(\nabla (|x|^{2\ell})).
\end{align*}
\]

This calculation can be justified by using the standard Friedrichs commutator argument.

We take the inner product of \( \partial_x^\alpha (7.3)_2 \) with \( |x|^{2\ell} |\partial_x^\alpha (\zeta_R w_\infty)| \) and integrate by parts to obtain

\[
\frac{1}{2} \frac{d}{dt} \left( ||x|^\ell |\partial_x^\alpha (\zeta_R w_\infty)|^2 \right)_{L^2} + \nu \left( ||x|^\ell |\nabla \partial_x^\alpha (\zeta_R w_\infty)|^2 \right)_{L^2} + \nu ||x|^\ell \text{div} \partial_x^\alpha (\zeta_R w_\infty) ||_{L^2}^2 \\
- \gamma (|\partial_x^\alpha (\zeta_R \phi_\infty)|, |x|^{2\ell} |\partial_x^\alpha (\zeta_R w_\infty)|) \\
= I_{\alpha,t,R}^{(3)} + \mathcal{P}_{\alpha,t}^{(2)}[\zeta_R u_\infty] + Q_{2,\alpha,t}(\nabla \zeta_R),
\]

where

\[
I_{\alpha,t,R}^{(3)} = \left\{ \begin{array}{ll}
((\zeta_R F_\infty), |x|^{2\ell} (\zeta_R w_\infty)) & (\alpha = 0), \\
-(\partial_x^{\alpha-1} (\zeta_R F_\infty), |x|^{2\ell} \partial_x^{\alpha+1} (\zeta_R w_\infty)) & (|\alpha| \geq 1),
\end{array} \right.
\]

\[
\mathcal{P}_{\alpha,t}^{(2)}[\zeta_R u_\infty] = (\nu \partial_x^\alpha \nabla (\zeta_R w_\infty) + \tilde{\nu} \partial_x^\alpha \text{div} (\zeta_R w_\infty) + \gamma \partial_x^\alpha (\zeta_R \phi_\infty), |x|^{2\ell} |\partial_x^\alpha (\zeta_R w_\infty)|) \\
- (\partial_x^{\alpha-1} (\zeta_R F_\infty), \partial_x (|x|^{2\ell} |\partial_x^\alpha (\zeta_R w_\infty)|), \\
Q_{2,\alpha,t}(\nabla \zeta_R) = (\partial_x^\alpha (K_2 (\nabla \zeta_R)), |x|^{2\ell} |\partial_x^\alpha (\zeta_R \phi_\infty)|).
\]

By adding (7.4) to (7.5), we see that

\[
\frac{1}{2} \frac{d}{dt} \left( ||x|^\ell |\partial_x^\alpha (\zeta_R \phi_\infty)|^2 \right)_{L^2} + ||x|^\ell |\partial_x^\alpha (\zeta_R w_\infty)|^2 \right)_{L^2} 
\]
By using Lemma 2.2 and Lemma 2.3, we obtain

\[
\sum_{j=1}^{3} I_{\alpha,l,R}^{(j)} + \mathcal{P}_{\alpha,l}^{(1)}[\zeta R u_\infty] + \mathcal{P}_{\alpha,l}^{(2)}[\zeta R u_\infty] + Q_{1,\alpha,l}(\nabla \zeta R) + Q_{2,\alpha,l}(\nabla \zeta R). \tag{7.6}
\]

Taking \( \epsilon_2 > 0 \) suitably small, we have

\[
\frac{1}{2} \frac{d}{dt} |\zeta R u_\infty|_{H^k_{t}}^2 + \nu \frac{1}{2} |\nabla (\zeta R) u_\infty|_{H^k_{t}}^2 + \tilde{\nu} \frac{1}{2} |\text{div}(\zeta R w_\infty)|_{H^k_{t}}^2 \\
\leq \epsilon |\zeta R u_\infty|_{L^2_{t}}^2 + \epsilon_1 |\nabla (\zeta R) \phi_\infty|_{H^{k-1}_{t}}^2 + C \left( 1 + \frac{1}{\epsilon} \right) |\zeta R F|_{H^k_{t} \times H^{k-1}_{t}}^2 \\
+ C \|\nabla \tilde{w}\|_{H^m} |\zeta R \phi_\infty|_{H^k_{t} \times H^{k-1}_{t}}^2,
\]
We next estimate \( \|x|^{2k} \nabla \partial_x^n \phi_\infty \|_{L^2_x}^2 \) for \( \alpha \) with \( |\alpha| \leq k-1 \). For a multi-index \( \alpha \) satisfying \( |\alpha| \leq k-1 \), we take the inner product of \( \partial_x^n (7.3)_2 \) with \( |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty) \) to obtain

\[
(\partial_t \partial_x^n (\zeta_R w_\infty), |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty)) + \gamma |\nabla \partial_x^n (\zeta_R \phi_\infty)|_{L^2_x}^2 \\
= \sum_{i=1}^3 J_{\alpha,R}^{(i)} + (\partial_x^0 K_2(\nabla \zeta_R), |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty)),
\]

where

\[
J_{\alpha,R}^{(1)} = (\nu (\partial_x^n \Delta (\zeta_R w_\infty), |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty)),
J_{\alpha,R}^{(2)} = (\hat{\nu} (\partial_x^n (\nabla \div (\zeta_R w_\infty)), |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty)),
J_{\alpha,R}^{(3)} = (\partial_x^n (\zeta_R \bar{F}_\infty), |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty)).
\]

As for the first term on the left-hand side, we have

\[
(\partial_t \partial_x^n (\zeta_R w_\infty), |x|^{2k} \nabla \partial_x^n (\zeta_R \phi_\infty)) \\
= \frac{d}{dt} (\partial_x^n (\zeta_R w_\infty), |x|^{2k} \partial_x^n \nabla (\zeta_R \phi_\infty)) + (\partial_x^n (\zeta_R w_\infty), \nabla (|x|^{2k} \partial_x^n \partial_t (\zeta_R \phi_\infty)) \\
+ (\partial_x^n \div (\zeta_R w_\infty), |x|^{2k} \partial_x^n \partial_t (\zeta_R \phi_\infty)).
\]

By (7.3), we have

\[
\partial_t (\zeta_R \phi_\infty) = -\gamma \hat{w} \cdot \nabla (\zeta_R \phi_\infty) - \gamma \div (\zeta_R w_\infty) + \zeta_R F_0^0 + K_1 (\nabla \zeta_R).
\]

Substituting this into (7.9), we obtain

\[
\frac{d}{dt} (\partial_x^n (\zeta_R w_\infty), |x|^{2k} \partial_x^n \nabla (\zeta_R \phi_\infty)) \\
= \sum_{i=1}^6 J_{\alpha,R}^{(i)} - \partial_x^n (\zeta_R w_\infty), |x|^{2k} \partial_x^n \nabla (\zeta_R \phi_\infty)) \\
+ (\partial_x^n (\zeta_R w_\infty), \nabla (|x|^{2k} \partial_x^n K_1 (\nabla \zeta_R)) + (\partial_x^n \div (\zeta_R w_\infty), |x|^{2k} \partial_x^n K_1 (\nabla \zeta_R)),
\]

where

\[
J_{\alpha,R}^{(4)} = \gamma (\partial_x^n \div (\zeta_R w_\infty), |x|^{2k} \partial_x^n (\hat{w} \cdot \nabla (\zeta_R \phi_\infty))),
J_{\alpha,R}^{(5)} = \gamma (\partial_x^n \div (\zeta_R w_\infty), |x|^{2k} \partial_x^n (\div (\zeta_R w_\infty))),
J_{\alpha,R}^{(6)} = - (\partial_x^n \div (\zeta_R w_\infty), |x|^{2k} \partial_x^n (\zeta_R F_0^0)),
\]

and

\[
\partial_x^n (\zeta_R w_\infty) = \gamma (\partial_x^n (\zeta_R w_\infty), \nabla (|x|^{2k} \partial_x^n (\hat{w} \cdot \nabla (\zeta_R \phi_\infty))) \\
+ \gamma (\partial_x^n (\zeta_R w_\infty), \nabla (|x|^{2k} \partial_x^n \div (\zeta_R w_\infty))) \\
- (\partial_x^n (\zeta_R w_\infty), \nabla (|x|^{2k} \partial_x^n (\zeta_R F_0^0))).
\]
This, together with (7.8), gives
\[
\frac{d}{dt} \left( \partial^\alpha_x (\zeta_R w_\infty), |x|^{2\ell} \partial^\alpha_x \nabla (\zeta_R \phi_\infty) \right) + \gamma |\nabla \partial^\alpha_x (\zeta_R \phi_\infty)|^2_{L^2_t}
\]
\[
= \sum_{i=4}^6 J^{(i)}_{a,\ell,R} + \mathcal{P}_{a,\ell}^{(3)}[\zeta_R u_\infty] + Q_{3,\alpha,\ell}(\nabla \zeta_R),
\]  
(7.10)
where
\[
Q_{3,\alpha,\ell} = \left( \partial^\alpha_x K_2(\nabla \zeta_R), |x|^{2\ell} \partial^\alpha_x (\zeta_R \phi_\infty) \right)
\]
\[-\left( \partial^\alpha_x (\zeta_R w_\infty), \nabla(|x|^{2\ell} \partial^\alpha_x K_1(\nabla \zeta_R)) \right)
\]
\[-\left( \partial^\alpha_x \text{div}(\zeta_R w_\infty), |x|^{2\ell} \partial^\alpha_x K_1(\nabla \zeta_R) \right).
\]
By Lemma 2.2 and Lemma 2.3, we obtain
\[
\left| \sum_{|\alpha| \leq k-1} \sum_{i=1}^6 J^{(i)}_{a,\ell,R} \right| \leq \frac{\gamma}{4} |\nabla \partial^\alpha_x (\zeta_R \phi_\infty)|^2_{L^2_t} + C \left( \gamma + \frac{1}{\gamma} \right) |\nabla (\zeta_R w_\infty)|^2_{H^k_t}
\]
\[
+ \gamma \|\tilde{w}\|^2_{H^m} \|\nabla (\zeta_R \phi_\infty)\|^2_{H^{k-1}_t} + \frac{C}{\gamma} |\zeta_R F_\infty|^2_{H^{k-1}_t},
\]
\[
\left| \sum_{|\alpha| \leq k-1} \mathcal{P}_{a,\ell}^{(3)}[\zeta_R u_\infty] \right| \leq \tilde{\ell} |\zeta_R w_\infty|^2_{L^2_t} + \tilde{\ell} |\nabla (\zeta_R w_\infty)|^2_{H^{-1}_t}
\]
\[
+ \frac{C \ell}{\epsilon} \|\tilde{w}\|^2_{H^m} \|\nabla (\zeta_R \phi_\infty)\|^2_{H^{k-1}_t}
\]
\[
+ C \ell \left( 1 + \frac{1}{\epsilon} \right) |\zeta_R w_\infty|^2_{H^{k-1}_t} + C \ell |\zeta_R F_\infty|^2_{H^{k-1}_t},
\]
\[
\left| \sum_{|\alpha| \leq k-1} Q_{3,\alpha,\ell}(\nabla \zeta_R) \right| \leq \frac{\gamma}{4} |\nabla (\zeta_R \phi_\infty)|^2_{H^{k-1}_{t-1}(N_R)} + \tilde{\ell} |\zeta_R w_\infty|^2_{L^2_t(N_R)}
\]
\[
+ C |\nabla (\zeta_R w_\infty)|^2_{H^{k-1}_{t-1}(N_R)}
\]
\[
+ C \left( \frac{1}{\gamma} + \frac{1}{\epsilon} \right) (1 + \|\tilde{w}\|^2_{H^m(N_R)}) |u_\infty|^2_{H^{k-1}_{t-1}(N_R)}
\]
for any \( \tilde{\epsilon} > 0 \) with \( C > 0 \) independent of \( \tilde{\epsilon} \).

Combining these estimates with (7.8) and (7.10), we see that
\[
\frac{d}{dt} \sum_{|\alpha| \leq k-1} (\partial^\alpha_x (\zeta_R w_\infty), |x|^{2\ell} \partial^\alpha_x \nabla (\zeta_R \phi_\infty)) + \frac{\gamma}{2} |\nabla (\zeta_R \phi_\infty)|^2_{H^k_t}
\]
\[
\leq \tilde{\ell} |\zeta_R w_\infty|^2_{L^2_t}
\]
\[
+ C \left\{ |\nabla (\zeta_R w_\infty)|^2_{H^k_t} + \left( 1 + \frac{1}{\epsilon} \right) \|\tilde{w}\|^2_{H^m} \|\nabla (\zeta_R \phi_\infty)\|^2_{H^{k-1}_t} + |\zeta_R F_\infty|^2_{H^{k-1}_t} \right\}
\]
\[ + C \left( 1 + \frac{1}{\epsilon} \right) \left\{ \ell |\zeta_R^w \infty|^2_{H_t^{k-1}} + (1 + \| \tilde{w} \|^2_{H^m(N_R)} |u_\infty|^2_{H_{k-1}^k(N_R)} \right\} \] (7.11)

for any \( \epsilon > 0 \) with \( C > 0 \) independent of \( \epsilon \).

Consider now \( \kappa \times (7.7) + (7.11) \) with a constant \( \kappa > 0 \). Taking \( \kappa > 0 \) so large that \(|\nabla (\zeta_R^w \infty)|^2_{H_t^k} \) on the right-hand side is absorbed into the left-hand side and setting \( \epsilon_1 = \frac{\gamma}{4\kappa} \) and \( \tilde{\epsilon} = \epsilon^{-1} \epsilon \), we arrive at

\[ \frac{d}{dt} E_0^k[\zeta_R u_\infty](t) + dD_0^k[\zeta_R u_\infty] \leq \epsilon |\zeta_R u_\infty|_{L_t^2}^2 + C \left( 1 + \frac{\ell^2}{\epsilon} \right) \| \tilde{w} \|^2_{H^m} + \| \nabla \tilde{w} \|^2_{H^m} \| \zeta_R \phi_\infty \|^2_{H_t^k} \]

\[ + C \left( 1 + \frac{1}{\epsilon} \right) |\zeta_R F_\infty|_{H_t^k \times H_{t-1}^{k-1}}^2 \]

\[ + C \ell^2 \left( 1 + \frac{1}{\epsilon} \right) (1 + \| \tilde{w} \|^2_{H^m}) |\zeta_R u_\infty|^2_{H_{t-1}^k} \]

\[ + C \left( 1 + \frac{1}{\epsilon} \right) (1 + \| \tilde{w} \|^2_{H^m(N_R)}) |u_\infty|^2_{H_{t-1}^k(N_R)} \]

for any \( \epsilon > 0 \) with \( C > 0 \) independent of \( \epsilon \). This completes the proof. \( \square \)

**Remark 7.2.** Similarly to the proof of Proposition 6.4, one can prove that if

\( \tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1}), \)

\( u_{0 \infty} \in H^k, \)

\( F_\infty \in L^2(0, T'; H^k \times H^{k-1}), \)

then there exists a unique solution \( u_\infty = \uparrow (\phi_\infty, w_\infty) \) of (7.1) in \( C([0, T']; H^k) \cap L^2(0, T'; H^k \times H^{k+1}) \). Furthermore, by setting \( \ell = 0 \) and \( \zeta_R \equiv 1 \) in the proof of Proposition 7.1, one can see that \( E_0^k[u_\infty](t) \) is absolutely continuous in \( t \in [0, T'] \) and there holds the estimate

\[ \frac{d}{dt} E_0^k[u_\infty] + dD_0^k[u_\infty] \]

\[ \leq C \left\{ \epsilon \| u_\infty \|^2_2 + (\| \tilde{w} \|^2_{H^m} + \| \nabla \tilde{w} \|^2_{H^m}) \| \nabla \phi_\infty \|^2_{H_t^{k-1}} \right\} \]

\[ + (1 + \frac{1}{\epsilon}) \| F_\infty \|^2_{H_t^k \times H_{t-1}^{k-1}} \] (7.12)

on \( (0, T') \), where \( \epsilon \) is any positive number; and \( C \) is a positive constant independent of \( T' \) and \( \epsilon \).

**Remark 7.3.** One can easily see that (7.2) holds with \( \zeta_R \) and \( N_R \) replaced by \( \zeta_R - \zeta_R \) and \( N_{R,R'} \) for \( R' > R \geq 1 \), where \( N_{R,R'} \) denotes the set \( N_{R,R'} = \{ x \in \mathbb{R}^n ; R \leq |x| \leq 2R' \} \).

**Proposition 7.4.** Let \( m \) be a nonnegative integer satisfying \( m \geq \left[ \frac{n}{2} \right] + 1 \) and let \( \ell \) be an integer satisfying \( \ell \geq 1 \). Assume that

\( u_{0 \infty} = \uparrow (\phi_0, w_{0 \infty}) \in H^k_\ell, \)

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\[ F_\infty = \top(F_\infty^0, F_\infty) \in L^2(0, T'; H^k_\ell \times H^{k-1}_\ell) \]

for \( k = m - 1 \) or \( k = m \). Here \( T' \) is a given positive number. Assume also that \( u_\infty = \top(\phi_\infty, w_\infty) \) is the solution of (7.1) with \( \tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1}) \) and that \( u_\infty = \top(\phi_\infty, w_\infty) \) satisfies

\[ \phi_\infty \in C([0, T']; H^k), \quad w_\infty \in C([0, T']; H^k) \cap L^2(0, T'; H^{k+1}). \]

Then it holds that

\[ \phi_\infty \in C([0, T']; H^k_\ell), \quad w_\infty \in C([0, T']; H^k_\ell) \cap L^2(0, T'; H^{k+1}_\ell). \]

Furthermore, there exist positive constants \( \kappa \geq \kappa_0 \) and \( d > 0 \) such that \( E^k_\ell[u_\infty](t) \) is absolutely continuous in \( t \in [0, T'] \) and there holds the estimate

\[
\frac{d}{dt} E^k_\ell[u_\infty] + dD^k_\ell[u_\infty] \\
\leq C \left\{ \epsilon |u_\infty|^2_{L^2} + \left( 1 + \frac{\ell^2}{\epsilon} \right) \|\tilde{w}\|^2_{H^m} + \|\nabla \tilde{w}\|_{H^m} \right\} \|\phi_\infty\|^2_{H^k_\ell} \\
+ \left( 1 + \frac{1}{\epsilon} \right) |F_\infty|^2_{H^k_\ell \times H^{k-1}_\ell} \\
+ \ell^2 \left( 1 + \frac{1}{\epsilon} \right) \left( 1 + \|\tilde{w}\|^2_{H^m} \right) |u_\infty|^2_{H^{k-1}_\ell} \}
\]

(7.13)

on \((0, T')\), where \( \epsilon \) is any positive number; \( C \) is a positive constant independent of \( T' \) and \( \epsilon \).

**Proof.** It suffices to prove that

\[ \zeta_R u_\infty \to u_\infty \text{ in } C([0, T']; H^k_\ell) \cap L^2(0, T'; H^k_\ell \times H^{k+1}_\ell) \]

as \( R \to \infty \). We prove this by induction on \( \ell \).

We first observe that it holds that

\[ \zeta_R u_\infty \to u_\infty \text{ in } C([0, T']; H^k) \cap L^2(0, T'; H^k \times H^{k+1}) \]

(7.14)

as \( R \to \infty \), since \( u_\infty \in C([0, T']; H^k) \cap L^2(0, T'; H^k \times H^{k+1}) \). We also note that since \( \text{supp}(\zeta_R - \zeta_{R'}) \subset N_{R,R'} = \{ x \in \mathbb{R}^n : R \leq |x| \leq 2R' \} \) for \( R' > R \), it holds that

\[ \|\zeta_R u_\infty - \zeta_{R'} u_\infty\|_{H^k_\ell} \leq C \|u_\infty\|_{H^k_\ell(N_{R,R'})} \]

for \( R' > R \geq 1 \).

Set

\[ \varphi_{\ell,R,R'}(t) = |\zeta_R u_\infty(t) - \zeta_R u_\infty(t)|^2_{H^k_\ell}, \]

\[ b(t) = 1 + \|\tilde{w}(t)\|^2_{H^m} + \|\nabla \tilde{w}(t)\|_{H^m} \in L^1(0, T'), \]

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\[ a_{\ell,R,R'}(t) = |\zeta_R u_{0\infty} - \zeta_{R'} u_{0\infty}|^2_{H^k_t} + \int_0^t |\zeta_R F_{\infty} - \zeta_{R'} F_{\infty}|^2_{H^k_t \times H^{k-1}_t} \, d\tau \]

\[ + \int_0^t (1 + \|\dot{u}(t)\|^2_{H^\infty}) \|u_{\infty}\|^2_{H^k_{t-1}(N_{R,R'}) \times H^{k+1}_{t-1}(N_{R,R'})} \, d\tau. \]

Let us prove Proposition 7.4 for \( \ell = 1 \). By (7.2), we have

\[ \varphi_{1,R,R'}(t) + \int_0^t \mathcal{D}_k^1 \zeta_R u_{\infty} - \zeta_{R'} u_{\infty} \, d\tau \]

\[ \leq C \left\{ a_{1,R,R'}(T') + \int_0^t b(\tau) \varphi_{1,R,R'}(\tau) \, d\tau \right\} \quad (7.15) \]

for \( t \in [0,T'] \), where \( C \) is a constant depending on \( \epsilon \). By the Gronwall inequality, we obtain

\[ \varphi_{1,R,R'}(t) \leq C a_{1,R,R'}(T') e^{C \int_0^t b(\tau) \, d\tau} \quad (7.16) \]

for \( t \in [0,T'] \). Since \( a_{1,R,R'}(T') \to 0 \) as \( R, R' \to \infty \), we see that \( \sup_{0 \leq t \leq T'} \varphi_{1,R,R'}(t) \to 0 \) as \( R, R' \to \infty \). This, together with (7.15), yields that \( \int_0^{T'} \mathcal{D}_k^1 \zeta_R u_{\infty} - \zeta_{R'} u_{\infty} \, d\tau \to 0 \) as \( R, R' \to \infty \). In view of (7.14), we thus conclude that \( \{\zeta_R u_{\infty}\} \) is Cauchy in \( C([0,T'];H^k_t) \cap L^2(0,T';H^k_1 \times H^{k+1}_1) \)

as \( R \to \infty \). Letting \( R \to \infty \) in (7.2) with \( \ell = 1 \), we have the desired estimate in Proposition 7.4 with \( \ell = 1 \). Proposition 7.4 thus holds for \( \ell = 1 \).

We next suppose that Proposition 7.4 holds for \( \ell = p \). We will prove that it also holds for \( \ell = p + 1 \). By (7.2) and Remark 7.3, we have

\[ \varphi_{p+1,R,R'}(t) + \int_0^t \mathcal{D}_{p+1}^k \zeta_R u_{\infty} - \zeta_{R'} u_{\infty} \, d\tau \]

\[ \leq C \left\{ a_{p+1,R,R'}(T') + \int_0^t b(\tau) \varphi_{p+1,R,R'}(\tau) \, d\tau \right\} \quad (7.17) \]

for \( t \in [0,T'] \), where \( C \) is a constant depending on \( \epsilon \) and \( p \). By the Gronwall inequality, we obtain

\[ \varphi_{p+1,R,R'}(t) \leq C a_{p+1,R,R'}(T') e^{C \int_0^t b(\tau) \, d\tau} \quad (7.18) \]

for \( t \in [0,T'] \). By the induction assumption, we see that \( a_{p+1,R,R'}(T') \to 0 \) as \( R, R' \to \infty \), and hence, by (7.18), \( \sup_{0 \leq t \leq T'} \varphi_{p+1,R,R'}(t) \to 0 \) as \( R, R' \to \infty \). This, together with (7.17), yields that \( \int_0^{T'} \mathcal{D}_{p+1}^k \zeta_R u_{\infty} - \zeta_{R'} u_{\infty} \, d\tau \to 0 \) as \( R, R' \to \infty \). It then follows that \( \{\zeta_R u_{\infty}\} \) is Cauchy in \( C([0,T'];H^k_{p+1}) \cap L^2(0,T';H^k_{p+1} \times H^{k+1}_{p+1}) \)

as \( R \to \infty \). It is not difficult to see that \( \frac{d}{dt} E^k_{\ell}[u_{\infty}] = G^k_{\ell} \) on \( (0,T') \) for some \( G^k_{\ell} \in L^1(0,T') \), and, thus, \( E^k_{\ell}[u_{\infty}](t) \) is absolutely continuous in \( t \in [0,T'] \). Letting \( R \to \infty \) in (7.2) with
\( \ell = p + 1 \), we have the desired estimate in Proposition 7.4 with \( \ell = p + 1 \). Proposition 7.4 thus holds for \( \ell = p + 1 \). This completes the proof.

We are now in a position to prove Proposition 6.7.

**Proof of Proposition 6.7.** Let \( U = \begin{pmatrix} \Phi \end{pmatrix} \in C([-\infty, T]; H_\ell^2) \cap L^2((0, T'; H_\ell^k \times H_\ell^{k+1}) \). Then, by Lemma 4.7, we see that

\[
\| P_t B[\bar{u}] U \|_{H_\ell^2} \leq C \| \bar{u} \|_\infty \| \nabla \Phi \|_{L^2_\ell} \leq C \delta \| U \|_{H_\ell^2}.
\]

It then follows from Remark 7.12 and Proposition 7.4 that there exists a unique solution \( U_\infty \in C([-\infty, T'; H_\ell^k \times H_\ell^k) \cap L^2((0, T'; H_\ell^k \times H_\ell^{k+1}) \) of

\[
\partial_t U_\infty + A U_\infty + B[\bar{u}] U_\infty = F_\infty + P_t B[\bar{u}] U, \quad U_\infty|_{t=0} = u_0, \tag{7.19}
\]

and \( U_\infty \) satisfies

\[
\| U_\infty(t) \|_{H_\ell^2} + \int_0^t \| \nabla U_\infty \|_{H_\ell^{k-1} \times H_\ell^k}^2 d\tau \\ \leq C_0 \left\{ \| u_0 \|_{H_\ell^k}^2 + \int_0^t \| F_\infty \|_{H_\ell^k \times H_\ell^{k-1}}^2 d\tau + \delta^2 \int_0^t \| U \|_{H_\ell^k}^2 d\tau + \int_0^t b(\tau) \| U_\infty \|_{H_\ell^{k-1}}^2 d\tau \right\}. \tag{7.20}
\]

Here \( b(\tau) = 1 + \| \bar{u} \|_{H_m}^2 + \| \nabla \bar{u} \|_{H_m} \).

We set \( U_\infty(0) = 0 \) and define \( U_\infty^{(j)} \ (j = 1, 2, \ldots) \) inductively by the solution of (7.19) with \( U = U_\infty^{(j-1)} \). Applying the Gronwall inequality to (7.20) with \( U_\infty = U_\infty^{(1)} \) and \( U = 0 \), we have

\[
\| U_\infty^{(1)}(t) \| \leq A_0 \]

for \( t \in [0, T'] \), where

\[
A_0 = C_0 \left\{ \| u_0 \|_{H_\ell^k}^2 + \int_0^{T'} \| F_\infty \|_{H_\ell^k \times H_\ell^{k-1}}^2 d\tau \right\} e^{C_0 \| b \|_{L^1(0, T')}}.
\]

Similarly, using (7.20) with \( U_\infty = U_\infty^{(j)} - U_\infty^{(j-1)} \) and \( U = U_\infty^{(j-1)} - U_\infty^{(j-2)} \) for \( j = 2, 3, \ldots \), one can inductively see that

\[
\| U_\infty^{(j)}(t) - U_\infty^{(j-1)}(t) \|_{H_\ell^k}^2 \leq \frac{A_0(C_0 K_0 \delta^2 t)^{j-1}}{(j-1)!},
\]

\[
\int_0^t \| \nabla(U_\infty^{(j)} - U_\infty^{(j-1)}) \|_{H_\ell^{k-1} \times H_\ell^k}^2 d\tau \leq \frac{A_0}{K_0} \left\{ \frac{(C_0 K_0 \delta^2 t)^{j-1}}{(j-1)!} + \frac{\| b \|_{L^1(0, T')} (C_0 K_0 \delta^2 t)^j}{j!} \right\}.
\]

Here \( K_0 = 1 + \| b \|_{L^1(0, T')} e^{C_0 \| b \|_{L^1(0, T')}} \). It then follows that \( U_\infty^{(j)} \) converges to a function \( U_\infty \) in \( C([-\infty, T']; H_\ell^k) \cap L^2((0, T'; H_\ell^k \times H_\ell^{k+1}) \) as \( j \to \infty \). One can easily see that \( U_\infty \) satisfies
(7.19) with $U = U_\infty$, i.e., $U_\infty$ is a solution of (6.1), and $U_\infty(t) \in H^k_{(\infty)}$ for all $t \in [0, T']$. By the uniqueness of solutions of (6.1) (see Proposition 6.4), we see that $U_1 = u_1$.

Applying Remark 7.2 and Proposition 7.4 with $F_1$ replaced by $F_1 + P_1 B[\tilde{u}] u_\infty$, we have

$$d \frac{dE_j^k[u_\infty]}{dt} + DD_j^k[u_\infty] \leq C \left\{ \epsilon |u_\infty|^2_{H_j^k} + \left( 1 + \frac{1}{\epsilon} \right) \| \hat{w} \|^2_{H^m} + \| \nabla \hat{w} \|^2_{H^m} \right\}$$

$$\quad \quad + \left( 1 + \frac{1}{\epsilon} \right) |F_\infty|^2_{H_j^k \times H_{j-1}^{k-1}} + j^2 \left( 1 + \frac{1}{\epsilon} \right) \left( 1 + \| \hat{w} \|^2_{H^m} \right) |u_\infty|^2_{H_j^{k-1}}$$

for $j = 0, 1, \ldots, \ell$. Using Lemma 4.4 (ii) and Lemma 4.7, we see that

$$d \frac{dE_j^k[u_\infty]}{dt} + 2d_1 |u_\infty|^2_{H_j^k \times H_{j+1}^{k+1}}$$

$$\quad \quad \leq C \left\{ \epsilon |u_\infty|^2_{H_j^k} + \left( 1 + \frac{1}{\epsilon} \right) \| \hat{w} \|^2_{H^m} + \| \nabla \hat{w} \|^2_{H^m} \right\}$$

$$\quad \quad + \left( 1 + \frac{1}{\epsilon} \right) |F_\infty|^2_{H_j^k \times H_{j-1}^{k-1}} + j^2 \left( 1 + \frac{1}{\epsilon} \right) \left( 1 + \| \hat{w} \|^2_{H^m} \right) |u_\infty|^2_{H_j^{k-1}}$$

for $j = 0, 1, \ldots, \ell$, with some constant $d_1 > 0$. Taking $\epsilon > 0$ suitably small, we obtain

$$d \frac{dE_j^k[u_\infty]}{dt} + d_1 |u_\infty|^2_{H_j^k \times H_{j+1}^{k+1}}$$

$$\quad \quad \leq C \left\{ \left( \| \hat{w} \|^2_{H^m} + \| \nabla \hat{w} \|^2_{H^m} \right) \| \phi_\infty \|^2_{H_j^k} + |F_\infty|^2_{H_j^k \times H_{j-1}^{k-1}}$$

$$\quad \quad \quad + j^2 \left( 1 + \| \hat{w} \|^2_{H^m} \right) |u_\infty|^2_{H_j^{k-1}} \right\}$$

(7.21)

for $j = 0, 1, \ldots, \ell$.

We now prove (6.15) by induction on $\ell$. When $\ell = 0$, inequality (7.21) with $j = 0$ is nothing but (6.15) with $\ell = 0$. Assume that (6.15) holds for $\ell = j - 1$. Then by adding $\frac{d}{2c j^2(1 + \delta^2)} \times (7.21)$ to (6.15) with $\ell = j - 1$, we obtain the desired inequality (6.15) for $\ell = j$. This completes the proof.

\[ \square \]

8 Proof of Theorem 3.1

In this section we prove Theorem 3.1.

We first establish the estimates for nonlinear and inhomogeneous terms $F_1(u, g)$ and $F_\infty(u, g)$:

$$F_1(u, g) = \begin{pmatrix} F_0^0(u) \\ \tilde{F}_1(u, g) \end{pmatrix}$$

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Proposition 8.2. \[
\|u\| \text{ uniformly for } F
\]

We first state the estimates for \( F_1(u, g) \) and \( F_\infty(u, g) \).

**Proposition 8.1.** There hold the estimates

(i) \[
\|F_1^0(u)\|_{L^1_1} \leq C(\|\phi\|_{L^2_1}\|\text{div}w\|_{L^2_1} + \|w\|_{L^2_1}\|\nabla\phi\|_{L^2_1}),
\]

(ii) \[
\|F_1^0(u, g)\|_{L^1_1} \leq C(\|\phi\|_{L^2_1}\|\partial_tw\|_{L^2_1} + \|w\|_{L^2_1}\|\nabla w\|_{L^2_1}
+ \|\phi\|_{L^2_1}\|\nabla\phi\|_{L^2_1} + \|\phi\|_{L^2_1}\|g\|_{L^2_1} + \|g\|_{L^1_1}),
\]

(iii) \[
\|F_\infty^0(u)\|_{H^{m-1}_1} \leq C(\|\phi\|_{H^{m-1}_1}\|\text{div}w\|_{H^{m-1}_1} + \|w\|_{H^{m-1}_1}\|\nabla\phi\|_{H^{m-1}_1}),
\]

(iv) \[
\|F_\infty^0(u, g)\|_{H^{m-1}_1} \leq C\{\|\phi\|_{H^{m}_1}\|\nabla w\|_{H^{m-1}_1} + \|\phi\|_{H^{m}_1}\|\nabla\phi\|_{H^{m-1}_1}
+ \|\phi\|_{H^{m}_1}\|\partial_tw\|_{H^{m-1}_1} + (1 + \|\phi\|_{H^{m}_1})\|g\|_{H^{m-1}_1}\}
\]

uniformly for \( u = \bar{\tau}(\phi, w) = u_1 + u_\infty \) with \( u_k = \bar{\tau}(\phi_k, w_k) \) \((k = 1, \infty)\) satisfying \( \|\phi\|_{L^\infty} \leq \frac{1}{2} \) and \( \|u\|_{H^{m}} \leq 1 \).

Proposition 8.1 directly follows from Lemmas 2.1 and 2.3.

We next estimate \( F_j(u^{(1)}, g) - F_j(u^{(2)}, g) \) \((j = 1, \infty)\).

**Proposition 8.2.** There hold the estimates

(i) \[
\|F_1^0(u^{(1)}) - F_1^0(u^{(2)})\|_{L^1_1}
\leq C\{\|\phi^{(1)} - \phi^{(2)}\|_{L^2_1}\|\text{div}w^{(1)}\|_{L^2_1} + \|\phi^{(1)}\|_{L^2_1}\|\text{div}(w^{(1)} - w^{(2)})\|_{L^2_1}
+ \|w^{(1)} - w^{(2)}\|_{L^2_1}\|\nabla\phi^{(1)}\|_{L^2_1} + \|w^{(2)}\|_{L^2_1}\|\nabla(\phi^{(1)} - \phi^{(2)})\|_{L^2_1}\},
\]

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\[(ii) \| \tilde{F}_1(u^{(1)}, g) - \tilde{F}_1(u^{(2)}, g) \|_{L^1_t} \leq C \{ \| w^{(1)} - w^{(2)} \|_{L^2} \| \nabla w^{(1)} \|_{L^2_t} + \| w^{(2)} \|_{L^2} \| \nabla(w^{(1)} - w^{(2)}) \|_{L^2_t} \\
+ \| \phi^{(1)} - \phi^{(2)} \|_{L^2_t} (\| w^{(1)} \|_{L^\infty} \| \nabla w^{(1)} \|_{L^\infty_t} + \| \partial_t w^{(1)} \|_{L^\infty_t} + \| g \|_{L^\infty_t}) \\
+ \| \phi^{(2)} \|_{L^2_t} \| \partial_t (w^{(1)} - w^{(2)}) \|_{L^2_t} \\
+ (\| \nabla \phi^{(1)} \|_{L^2_t} + \| \nabla \phi^{(2)} \|_{L^2_t}) \| \phi^{(1)} - \phi^{(2)} \|_{L^2_t} + \| \phi^{(1)} \|_{L^2_t} \| \nabla(\phi^{(1)} - \phi^{(2)}) \|_{L^2_t} \}, \]

\[(iii) \| F^0_\infty(u^{(1)}) - F^0_\infty(u^{(2)}) \|_{H^{m-1}_t} \leq C \{ \| \text{div} w^{(1)} \|_{H^m} \| \phi^{(1)} - \phi^{(2)} \|_{H^{m-1}_t} + \| \phi^{(2)} \|_{H^m} \| \text{div}(u^{(1)} - u^{(2)}) \|_{H^{m-1}_t} \\
+ \| \nabla \phi^{(1)} \|_{H^m} \| w^{(1)} - w^{(2)} \|_{H^{m-1}_t} + \| w^{(2)} \|_{H^m} \| \nabla(\phi^{(1)} - \phi^{(2)}) \|_{H^{m-1}_t} \}, \]

\[(iv) \| \tilde{F}_\infty(u^{(1)}, g) - \tilde{F}_\infty(u^{(2)}, g) \|_{H^{m-2}_t} \leq C \{ \| \phi^{(1)} - \phi^{(2)} \|_{H^{m-2}_t} \| w^{(1)} \|_{H^m} \| \nabla w^{(1)} \|_{H^{m-1}_t} + \| \partial_t w^{(1)} \|_{H^{m-1}_t} + \| g \|_{H^{m-1}_t} \\
+ \| w^{(1)} - w^{(2)} \|_{H^{m-1}_t} \| \nabla w^{(1)} \|_{H^{m-1}_t} + \| w^{(2)} \|_{H^m} \| \nabla(w^{(1)} - w^{(2)}) \|_{H^{m-2}_t} \\
+ \| \phi^{(2)} \|_{H^m} \| \partial_t (w^{(1)} - w^{(2)}) \|_{H^{m-2}_t} \\
+ (\| \nabla \phi^{(1)} \|_{H^{m-1}_t} + \| \nabla \phi^{(2)} \|_{H^{m-1}_t}) \| \phi^{(1)} - \phi^{(2)} \|_{H^{m-1}_t} \\
+ \| \phi^{(1)} \|_{H^m} \| \nabla(\phi^{(1)} - \phi^{(2)}) \|_{H^{m-2}_t} \} \]

uniformly for \( u^{(j)} = \tau(\phi^{(j)}, w^{(j)}) = u^{(j)} + u^{(j)}_k \) with \( u^{(j)}_k = \tau(\phi^{(j)}_k, w^{(j)}_k) \) \((k = 1, \infty)\) satisfying \( \| \phi^{(j)} \|_{L^\infty} \leq \frac{1}{2} \) and \( \| u^{(j)} \|_{H^m} \leq 1 \) \((j = 1, 2)\).

Proposition 8.2 directly follows from Lemmas 2.1–2.3.

To prove Theorem 3.1, we next show the existence of a solution \( \{u_1, u_\infty\} \) of (4.1)-(4.2) on \([0, T]\) satisfying \( u_j(0) = u_j(T) \) \((j = 1, \infty)\) by an iteration argument.

For \( p = 0 \), we define \( u^{(0)}_1 = \tau(\phi^{(0)}_1, w^{(0)}_1) \) and \( u^{(0)}_\infty = \tau(\phi^{(0)}_\infty, w^{(0)}_\infty) \) by

\[
\begin{cases}
  u^{(0)}_1(t) = S_1(t) \mathcal{J}_1(T)[(I - S_1(T))^{-1}G_1] + \mathcal{J}_1(t)[G_1], \\
  u^{(0)}_\infty(t) = S_{\infty, 0}(t)(I - S_{\infty, 0}(T))^{-1}\mathcal{J}_{\infty, 0}(T)[G_\infty] + \mathcal{J}_{\infty, 0}(t)[G_\infty],
\end{cases}
\tag{8.1}
\]

where \( t \in [0, T], \mathcal{G} = \tau(0, \frac{1}{2} g(x, t)), G_1 = P_1 \mathcal{G} \) and \( G_\infty = P_\infty \mathcal{G} \). Note that \( u^{(0)}_j(0) = u^{(0)}_j(T) \) \((j = 1, \infty)\).

For \( p \geq 1 \), we define \( u^{(p)}_1 = \tau(\phi^{(p)}_1, w^{(p)}_1) \) and \( u^{(p)}_\infty = \tau(\phi^{(p)}_\infty, w^{(p)}_\infty) \), inductively, by

\[
\begin{cases}
  u^{(p)}_1(t) = S_1(t)\mathcal{J}_1(T)[(I - S_1(T))^{-1}F_1(u^{(p-1)}_1, g)] + \mathcal{J}_1(t)[F_1(u^{(p-1)}_1, g)], \\
  u^{(p)}_\infty(t) = S_{\infty, u^{(p-1)}_1}(t)(I - S_{\infty, u^{(p-1)}_1}(T))^{-1}\mathcal{J}_{\infty, u^{(p-1)}_1}(T)[F_\infty(u^{(p-1)}_1, g)] + \mathcal{J}_{\infty, u^{(p-1)}_1}(t)[F_\infty(u^{(p-1)}_1, g)],
\end{cases}
\tag{8.2}
\]

where \( t \in [0, T] \) and \( u^{(p-1)}_1 = u^{(p-1)}_1 + u^{(p-1)}_\infty \). Note that \( u^{(p)}_j(0) = u^{(p)}_j(T) \) for \( j = 1, \infty \) and \( p \geq 1 \).
Proposition 8.3. There exists a constant \(\delta_0 > 0\) such that if \([g]_m \leq \delta_0\), then there holds the estimates

\[
\|\{u_1^{(p)}, u_\infty^{(p)}\}\|_{\mathcal{D}^{m_0}(0,T)} \leq C_1 [g]_m
\]

for all \(p \geq 0\), and

\[
\|\{u_1^{(p+1)} - u_1^{(p)}, u_\infty^{(p+1)} - u_\infty^{(p)}\}\|_{\mathcal{D}^{m_1}(0,T)} \leq C_1 [g]_m \|\{u_1^{(p)} - u_1^{(p-1)}, u_\infty^{(p)} - u_\infty^{(p-1)}\}\|_{\mathcal{D}^{m_1}(0,T)}
\]

for \(p \geq 1\). Here \(C_1\) is a constant independent of \(g\) and \(p\).

**Proof.** The estimate (i) follows from Propositions 5.4, 6.8, 8.1 and Lemma 4.3 (ii).

Let us consider the estimate of the difference between \(u_1^{(p+1)}\) and \(u_1^{(p)}\). For \(p \geq 0\), we set \(\tilde{\phi}_j^{(p)} = \phi_j^{(p+1)} - \phi_j^{(p)}\) and \(\tilde{w}_j^{(p)} = w_j^{(p+1)} - w_j^{(p)}\) for \(j = 1, \infty\). Then by using (8.1) and (8.2), we see that \(\tilde{\phi}_j^{(p)}\) and \(\tilde{w}_j^{(p)}\) (\(p \geq 1\)) satisfy

\[
\begin{cases}
\partial_t \tilde{\phi}_j^{(p)} + \gamma \text{div} \tilde{w}_j^{(p)} = F_{11}(\tilde{u}^{(p-1)}), \\
\partial_t \tilde{w}_j^{(p)} - \nu \Delta \tilde{w}_j^{(p)} - \tilde{\nu} \text{div} \tilde{w}_j^{(p)} + \gamma \nabla \tilde{\phi}_j^{(p)} = F_{12}(\tilde{u}^{(p-1)}, g),
\end{cases}
\]

(8.3)

\[
\begin{cases}
\partial_t \tilde{\phi}_\infty^{(p)} + \gamma P_\infty(w_\infty^{(p)} \cdot \nabla \tilde{\phi}_\infty^{(p)}) + \gamma \text{div} \tilde{w}_\infty^{(p)} = F_{12}(\tilde{u}^{(p-1)}), \\
\partial_t \tilde{w}_\infty^{(p)} - \nu \Delta \tilde{w}_\infty^{(p)} - \tilde{\nu} \text{div} \tilde{w}_\infty^{(p)} + \gamma \nabla \tilde{\phi}_\infty^{(p)} = F_{22}(\tilde{u}^{(p-1)}, g),
\end{cases}
\]

(8.4)

where

\[
\begin{align*}
F_{11}(\tilde{u}^{(p-1)}) &= F_1^0(u^{(p)}) - F_1^0(u^{(p-1)}), \\
F_{12}(\tilde{u}^{(p-1)}, g) &= \tilde{F}_1(u^{(p)}, g) - \tilde{F}_1(u^{(p-1)}, g), \\
F_{12}(\tilde{u}^{(p-1)}) &= F_1^0(u^{(p)}) - F_0^0(u^{(p-1)}) - \gamma P_\infty((w^{(p)} - w^{(p-1)}) \cdot \nabla \tilde{\phi}_\infty^{(p)}), \\
F_{22}(\tilde{u}^{(p-1)}, g) &= \tilde{F}_\infty(u^{(p)}, g) - \tilde{F}_\infty(u^{(p-1)}, g).
\end{align*}
\]

The desired inequality (ii) can be obtained by applying Propositions 5.4, 6.8, 8.2, 8.3 (i) and Lemma 4.3 (ii). This completes the proof. \(\square\)

We introduce a notation. We denote by \(B_{\mathcal{D}^k(a,b)}(r)\) the closed unit ball of \(\mathcal{D}^k(a,b)\) centered at 0 with radius \(r\), i.e.,

\[
B_{\mathcal{D}^k(a,b)}(r) = \left\{\{u_1, u_\infty\} \in \mathcal{D}^k(a,b); \|\{u_1, u_\infty\}\|_{\mathcal{D}^k(a,b)} \leq r\right\}.
\]

Proposition 8.4. There exists a constant \(\delta_1 > 0\) such that if \([g]_m \leq \delta_1\), then the system (4.1)-(4.2) has a unique solution \(\{u_j, u_\infty\}\) on \([0, T]\) in \(B_{\mathcal{D}^{m_0}(0,T)}(C_1[g]_m)\) satisfying \(u_j(0) = u_j(T)\) (\(j = 1, \infty\)). The uniqueness of solutions of (4.1)-(4.2) on \([0, T]\) satisfying \(u_j(0) = u_j(T)\) (\(j = 1, \infty\)) holds in \(B_{\mathcal{D}^{m_0}(0,T)}(C_1 \delta_1)\).
**Proof.** Let $\delta_1 = \min\{\delta_0, \frac{1}{2\epsilon_1}\}$ with $\delta_0$ given in Propositions 8.3. By Propositions 8.3, we see that if $[g]_m \leq \delta_1$, then $u_j^{(p)} = \tau(\phi_j^{(p)}, w_j^{(p)})$ $(j = 1, \infty)$ converges to $u_j = \tau(\phi_j, w_j)$ $(j = 1, \infty)$ in the sense

$$
\{u_1^{(p)}, u_\infty\} \to \{u_1, u_\infty\} \text{ in } \mathcal{D}^{-1}(0, T),
$$

$$
u_\infty^{(p)} = \tau(\phi_\infty^{(p)}, w_\infty^{(p)}) \to u_\infty = \tau(\phi_\infty, w_\infty) *-\text{weakly in } L^\infty(0, T; H_m^\infty),
$$

$$w_\infty^{(p)} \to w_\infty \text{ weakly in } L^2(0, T; H_m^{m+1}) \cap H_1(0, T; H_m^{m-1}).$$

It is not difficult to see that $\{u_1, u_\infty\}$ is a solution of (4.1)-(4.2) satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

It remains to prove $u_\infty = \tau(\phi_\infty, w_\infty) \in C([0, T]; H_m)$, which implies $\{u_1, u_\infty\} \in B_{\mathcal{D}_m^{(0, T)}}(C_1[g]_m)$ with $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

As for $w_\infty$, since $L^2(0, T; H_m^{m+1}) \cap H_1(0, T; H_m^{m-1}) \subset C([0, T]; H_m)$, we find that $w_\infty \in C([0, T]; H_m)$.

As for $\phi_\infty$, note that $\phi_\infty \in C([0, T]; H_1)$ and $\phi_\infty$ is the solution of

$$
\begin{cases}
\partial_t \phi_\infty + w \cdot \nabla \phi_\infty = g_0^0, \\
\phi_\infty|_{t=0} = \phi_0^0,
\end{cases}
$$

where

$$g_0^0 \in L^2(0, T; H_m^m), \phi_0^0 \in H_m^m.$$

On the other hand, by Lemma 6.1, we see that there exists a solution of (8.5) which belongs to $C([0, T]; H_1)$ and that the uniqueness of solutions of (8.5) holds in $C([0, T]; H_1)$. Therefore, we find that

$$\phi_\infty \in C([0, T]; H_1^m).$$

To prove that $u_\infty = \tau(\phi_\infty, w_\infty) \in C([0, T]; H_1^m)$, we note that $u_\infty$ is written as

$$u_\infty(t) = S_{\infty,u}(t)(I - S_{\infty,u}(T))^{-1}\mathcal{S}_{\infty,u}(T)[F_\infty(u, g)] + \mathcal{S}_{\infty,u}(t)[F_\infty(u, g)]$$

with $u = u_1 + u_\infty$. By Proposition 8.1 and Lemma 4.3 (ii), we see that $F_\infty(u, g) \in L^2(0, T; H_m^m, H_m^{m-1})$. It then follows from Proposition 6.5 that if $\delta_1$ is small such that $C_1\delta_1 \leq \delta$, then $u_\infty = \tau(\phi_\infty, w_\infty) \in C([0, T]; H_1^m)$. This completes the proof. \(\square\)

To complete the construction of a time periodic solution of (1.5), we use the following proposition on the unique existence of solutions to the initial value problem.

**Proposition 8.5.** Let $s \in \mathbb{R}$ and let $U_0 = U_{01} + U_{0\infty}$ with $U_{01} \in H_{1, 1}^1$ and $U_{0\infty} \in H_{m, 1}^m$. Then there exist constants $\delta_2 > 0$ and $C_2 > 0$ such that if

$$M(U_{01}, U_{0\infty}, g) := ||U_{01}||_{H_{1, 1}^1} + ||U_{0\infty}||_{H_{m, 1}^m} + [g]_m \leq \delta_2,$$

there exists a solution $\{U_t, U_{0t}\}$ of the initial value problem for (4.1)-(4.2) on $[s, s + T]$ in $B_{\mathcal{D}_m^{(s, s + T)}}(C_2M(U_{01}, U_{0\infty}, g))$ satisfying the initial condition $U_j|_{t=s} = U_{0j}$ $(j = 0, \infty)$.

The uniqueness for this initial value problem holds in $B_{\mathcal{D}_m^{(s, s + T)}}(C_2\delta_2)$. 

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Proof. Let $\tilde{u} = \nabla (\tilde{\phi}, \tilde{w})$ be a given function in \( C([s, s + T]; H^m) \cap L^2(s, s + T; H^{m+1}) \). We define $S_{\infty, \tilde{u}}(t, s) u_{0\infty}$ and $\mathcal{J}_{\infty, \tilde{u}}(t, s) F_{\infty}$ by the solution operators for

$$
\partial_t u_{\infty} + A u_{\infty} + P_{\infty}(B[\tilde{u}] u_{\infty}) = F_{\infty}, \quad u_{\infty}|_{t=s} = u_{0\infty},
$$

with $F_{\infty} = 0$ and $u_{0\infty} = 0$, respectively. As in the proof of Proposition 6.5, one can see that if $\tilde{w}$ satisfies

$$
\|\tilde{w}\|_{C([s, s+T]; H^m) \cap L^2(s, s + T; H^{m+1})} \leq \delta,
$$

(8.6)

then it holds that $S_{\infty, \tilde{u}}(t, s)$ and $\mathcal{J}_{\infty, \tilde{u}}(t, s)$ satisfy the estimates

$$
\|S_{\infty, \tilde{u}}(t, s) U_{0\infty}\|_{H_{(\infty, 1)}^k} \leq C e^{-\alpha(t-s)} \|U_{0\infty}\|_{H_{(\infty, 1)}^k},
$$

(8.7)

$$
\|\mathcal{J}_{\infty, \tilde{u}}(t, s)[F_{\infty}]\|_{H_{(\infty, 1)}^k} \leq C \left\{ \int_s^t e^{-\alpha(t-r)} \|F_{\infty}\|_{H_{(\infty, 1)}^k \times H_{(\infty, 1)}^{k-1}} dr \right\}^{\frac{1}{2}}
$$

(8.8)

for $t \in [s, s + T]$, $F_{0\infty} \in H_{(\infty, 1)}^k$, $F_{\infty} \in L^2(s, s + T; H_{(\infty, 1)}^k \times H_{(\infty, 1)}^{k-1})$ and $k = m - 1$ or $m$ with $C = C(\delta, T) > 0$ uniformly for $s \in \mathbb{R}$ and $\tilde{u} = \nabla (\tilde{\phi}, \tilde{w})$ satisfying (8.6).

To prove Proposition 8.5, it now suffices to show the unique existence of the solution $\{U_1, U_{\infty}\} \in B_{\mathcal{H}^m(s, s+T)}(C_2 M(U_{01}, U_{0\infty}, g))$ of

$$
\begin{align*}
\left\{ \begin{array}{l}
U_1(t) = S_1(t-s) U_{01} + \mathcal{J}_1(t-s)[F_1(U, g)], \\
U_{\infty}(t) = S_{\infty, U}(t, s) U_{0\infty} + \mathcal{J}_{\infty, U}(t, s)[F_{\infty}(U, g)],
\end{array} \right.
\right.
\end{align*}
$$

(8.9)

for $U = U_1 + U_{\infty}$ for a constant $C_2 > 0$, provided that $M(U_{01}, U_{0\infty}, g)$ is sufficiently small. We solve this problem by an iteration argument as in the proof of Proposition 8.4.

For $p = 0$, we define $U_j^{(0)} = \nabla (\Phi_j^{(0)}, W_j^{(0)})$ ($j = 1, \infty$) by

$$
\begin{align*}
\left\{ \begin{array}{l}
U_1^{(0)}(t) = S_1(t-s) U_{01} + \mathcal{J}_1(t-s)[G_1], \\
U_{\infty}^{(0)}(t) = S_{\infty, U}(t, s) U_{0\infty} + \mathcal{J}_{\infty, U}(t, s)[G_{\infty}],
\end{array} \right.
\right.
\end{align*}
$$

where $t \in [s, s + T]$, $G = \nabla (0, \frac{1}{3} g(x, t))$, $G_1 = P_1 G$ and $G_{\infty} = P_{\infty} G$.

For $p \geq 1$, we define $U_j^{(p)} = \nabla (\Phi_j^{(p)}, W_j^{(p)})$ ($j = 1, \infty$), inductively, by

$$
\begin{align*}
\left\{ \begin{array}{l}
U_1^{(p)}(t) = S_1(t-s) U_{01} + \mathcal{J}_1(t-s)[F_1(U^{(p-1)}, g)], \\
U_{\infty}^{(p)}(t) = S_{\infty, U}(t, s) U_{0\infty} + \mathcal{J}_{\infty, U}(t, s)[F_{\infty}(U^{(p-1)}, g)],
\end{array} \right.
\right.
\end{align*}
$$

where $t \in [s, s + T]$ and $U^{(p-1)} = U_{1}^{(p-1)} + U_{\infty}^{(p-1)}$.

As in the proof of Proposition 8.3, by using Proposition 5.1, (8.7), (8.8), Propositions 8.1, 8.2 and Lemma 4.3 (ii), we can inductively show that if $M(U_{01}, U_{0\infty}, g)$ is sufficiently small, then there hold the estimates

$$
\left\| \{U_1^{(p)}, U_{\infty}^{(p)}\} \right\|_{\mathcal{H}^m(s, s+T)} \leq C_2 M(U_{01}, U_{0\infty}, g)
$$

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for all $p \geq 0$, and
\[
\|U_1^{(p)}(s,s+T) - U_1^{(p)}(0)\|_{\mathcal{F}^m} \leq C_2 M(U_{01}, U_{0\infty}, g) \|U_1^{(p)}(s) - U_1^{(p)}(0)\|_{\mathcal{F}^m} \]
for all $p \geq 1$. Hence, in a similar manner to the proof of Proposition 8.4, we see that there exists a solution $\{U_1, U_{21}\} \in B_{\mathcal{F}^m}((s,s+T))$ of (4.1)-(4.2) satisfying $U_j|_{t=s} = U_{0j}$ ($j = 0, \infty$), provided that $M(U_{01}, U_{0\infty}, g) \leq \delta_2$ for a small constant $\delta_2 > 0$. In view of the iteration argument, we can see that the uniqueness holds in $B_{\mathcal{F}^m}((s,s+T))$. This completes the proof. \hfill \Box

We are now in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 8.4, we see that if $[g]_m \leq \delta_1$, then (4.1)-(4.2) has a unique solution $\{u_1^{(0)}, u_1^{(1)}\} \in B_{\mathcal{F}^m}(0,T)$ satisfying $u_1^{(0)}(0) = u_1^{(0)}(T)$ ($j = 1, \infty$). In particular, it holds that
\[
\sup_{t \in [0,T]} \left\{\|u_1^{(0)}(t)\|_{H^1_{(1,1)}}, \|u_1^{(0)}(t)\|_{H^m_{(1,1)}}\right\} \leq C_1[g]_m. \quad (8.10)
\]
Therefore, if $g$ satisfies $(C_1 + 1)[g]_m \leq \delta_2$, then, by Proposition 8.5, we see that there exists a unique solution $\{u_1^{(0)}, u_1^{(1)}\} \in B_{\mathcal{F}^m}(0,T)$ of (4.1)-(4.2) satisfying $u_1^{(0)}|_{t=T} = u_1^{(0)}(T) = u_1^{(0)}(0)$ ($j = 1, \infty$).

We introduce $\tilde{u}_1^{(1)}$ ($j = 1, \infty$) and $\tilde{u}_1^{(1)}$ by
\[
\tilde{u}_1^{(1)}(t) = u_1^{(1)}(t + T), \quad \tilde{u}_1^{(1)}(t) = \tilde{u}_1^{(1)}(t) + \tilde{u}_1^{(1)}(t) \quad \text{for} \quad t \in [0,T].
\]
Then we find that
\[
\begin{align*}
\partial_t \tilde{u}_1^{(1)}(t) + A\tilde{u}_1^{(1)}(t) &= \partial_t u_1^{(1)}(t + T) + A u_1^{(1)}(t + T) = F_1(u(t + T), g(t + T)) \\
&= F_1(\tilde{u}_1^{(1)}(t), g(t)).
\end{align*}
\]
Similarly, we see that
\[
\begin{align*}
\tilde{u}_\infty^{(1)}(0) &= u_\infty^{(0)}(0), \\
\partial_t \tilde{u}_\infty^{(1)}(t) + A\tilde{u}_\infty^{(1)}(t) + P_{\infty}(B[\tilde{u}_1^{(1)}(t)]\tilde{u}_1^{(1)}(t)) &= F_\infty(\tilde{u}_1^{(1)}(t), g(t)).
\end{align*}
\]
Therefore, if $[g]_m \leq \delta_3 := \min\{\delta_1, \frac{C_2 \delta_2}{C_1}, (C_1 + 1)\delta_2\}$, then, by the uniqueness of the solution, we find that $\{\tilde{u}_1^{(1)}(t), \tilde{u}_\infty^{(1)}(t)\} = \{u_1^{(0)}(t), u_\infty^{(0)}(t)\}$ for $t \in [0,T]$. Consequently, we have $\{u_1^{(1)}(t), u_\infty^{(1)}(t)\} = \{u_1^{(0)}(t - T), u_\infty^{(0)}(t - T)\}$ for $t \in [T,2T]$. We define $\{u_1(t), u_\infty(t)\} \in B_{\mathcal{F}^m}(0,2T)$ by $\{u_1(t), u_\infty(t)\} = \{u_1^{(k)}(t), u_\infty^{(k)}(t)\}$ for $t \in [kT, (k + 1)T]$, $k = 0, 1$. It then follows that $\{u_1(t + T), u_\infty(t + T)\} = \{u_1(t), u_\infty(t)\}$ for $t \in [0,T]$. Furthermore, we see from Proposition 8.5 and (8.10) that there exists a
unique solution \( \{v_1, v_\infty\} \in B_\mathcal{X}^m \left( \left[ \frac{T}{2}, \frac{3T}{2} \right] \right) \) of (4.1)-(4.2) on \( \left[ \frac{T}{2}, \frac{3T}{2} \right] \) satisfying \( v_j \big|_{t=\frac{T}{2}} = u_j^0 \left( \frac{T}{2} \right) \) (\( j = 1, \infty \)). By the uniqueness, it follows that \( \{v_1, v_\infty\} = \{u_1, u_\infty\} \) on \( \left[ \frac{T}{2}, \frac{3T}{2} \right] \), which implies that \( \{u_1, u_\infty\} \) is a solution of (4.1)-(4.2) on \( [0, 2T] \) in \( \mathcal{X}^m \). Repeating this argument on intervals \( [kT, (k+1)T] \) for \( k = \pm 1, \pm 2 \cdots \), we obtain a solution \( \{u_1, u_\infty\} \) of (4.1)-(4.2) in \( \mathcal{X}^m_{\per}(\mathbb{R}) \) satisfying \( \|\{u_1, u_\infty\}\|_{\mathcal{X}^m(0,T)} \leq C_1[g]_m \) that gives a time periodic solution \( u \) of (1.5) by setting \( u = u_1 + u_\infty \).

In view of the iteration argument in Propositions 8.3 and 8.4, one can see that the uniqueness of time periodic solutions for (1.5) holds in \( \mathcal{X}^m_{\per}(\mathbb{R}) \), \( \|\{P_1 u, P_\infty u\}\|_{\mathcal{X}^m(0,T)} \leq C_1 \delta_3 \) if \( [g]_m \leq \delta_3 \). This completes the proof. \( \square \)

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