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https://doi.org/10.15017/1449082

出版情報：九州大学教養部数学雑誌. 19, pp.27-30, 1993-12. 九州大学教養部数学教室

Math. Rep.
19, 1993.

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(Received Aug. 23, 1993)

Dedicated to Prof. Joji Kajiwara on his sixtieth birthday

1. Introduction

Every Stein manifold admits a complete Kähler metric, but the converse is not true [3]. On the other hand, every Stein manifold is exhausted by complete Kähler domains (Proposition 1), and the author [4] proved that the converse is true for domains over a Stein manifold (cf.[7]). The purpose of this paper is to extend the above result to the case of domains over $\mathbf{P}^n$. This result is also a generalization of Behnke-Stein's theorem [1].

In the proof, we use a theorem due to Kajiwara [5] which connects the theory of domains over $\mathbf{P}^n$ with that of domains over $\mathbf{C}^n$.

2. Preliminaries

DEFINITION 1. A complex manifold X is said to be exhausted by complete Kähler domains if there exists a sequence $\{X_k\,;\,k = 1, 2, 3, \cdots\}$ of open submanifolds of X such that $X = \cup X_k$, $X_k \subset\subset X_{k+1}$ for every k, and each X_k has a complete Kähler metric.

PROPOSITION 1. (i) *Every Stein manifold is exhausted by complete Kähler domains.*

(ii) *Let X and Y be complex manifolds exhausted by complete Kähler domains. Then $X \times Y$ is exhausted by complete Kähler domains.*

(iii) *Let X be a complex manifold exhausted by complete Kähler domains, and Y be a complex closed submanifold of X. Then Y is exhausted by complete Kähler domains.*

PROOF (i) Let S be a Stein manifold. There exists a strictly plurisubharmonic function φ on S such that $S_c = \{x \in S\,;\, \varphi(x) < c\}$ is relatively compact in S for any real number c. We may assume that $\inf\{\varphi(x)\,;\, x \in S\} = 0$. If we put

$$\psi(x) = \frac{1}{(c - \varphi(x))} \qquad (x \in S_c),$$

ψ is strictly plurisubharmonic on S_c and the set $\{x \in S_c\,;\, \psi(x) \leq k\}$ is compact for any real number k. Hence S_c is a Stein manifold. By Grauert [3], S_c has a complete Kähler metric. Therefore, S is exhausted by complete Kähler domains $\{S_i\,;\, i = 1, 2, 3, \cdots\}$.

(ii) Let X and Y be exhausted by complete Kähler domains $\{X_i; i = 1, 2, 3, \cdots\}$ and $\{Y_i\,;\, i=1, 2, 3, \cdots\}$ respectively. Let ds_i^2 be a complete Kähler metric on X_i and let $d\sigma_i^2$ be that on Y_i. We denote by π_i (resp. τ_i) the projection $X_i \times Y_i \to X_i$ (resp. $X_i \times Y_i \to Y_i$). If we put

$$dm_i^2 = (\pi_i)^*(ds_i^2) + (\tau_i)^*(d\sigma_i^2),$$

dm_i^2 is a complete Kähler metric on $X_i \times Y_i$. Therefore, $X \times Y$ is exhausted by complete Kähler domains $\{X_i \times Y_i\,;\, i = 1, 2, 3, \cdots\}$.

(iii) Let X be exhausted by complete Kähler domains $\{X_i\,;\, i = 1, 2, 3, \cdots\}$. Let ds_i^2 be a complete Kähler metric on X_i. We put

$$Y_i = Y \cap X_i,$$

and denote by α_i the inclusion map $Y_i \to X_i$. If we put

$$d\sigma_i^2 = (\alpha_i)^*(ds_i^2),$$

$d\sigma_i^2$ is a complete Kähler metric on Y_i. Therefore, Y is exhausted by complete Kähler domains $\{Y_i\,;\, i = 1, 2, 3, \cdots\}$.

Let D and M be complex manifolds.

DEFINITION 2. We say that (D, Φ) is an open set over M if Φ is a holomorphic mapping of D into M such that Φ is locally biholomorphic. Moreover, if D is connected, (D, Φ) is called a domain over M.

For domains over a Stein manifold, the author proved the following theorem [4].

THEOREM 1. *Let (D, Φ) be a domain over a Stein manifold S. If D is exhausted by complete Kähler domains, then D is a Stein manifold.*

3. Domains over $\mathbf{P}^n$

Let (D, Φ) be a domain over $\mathbf{P}^n$. We denote by π the canonical projection $X = \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n$, and define the set Y by

$$Y = \{(x, p) \in X \times D\,;\ \pi(x) = \Phi(p)\}.$$

Putting $\Phi'(x,p) = x$, $\pi'(x,p) = p$, (Y, Φ') becomes an open set over X with respect to the topology induced from the product topology of $X \times D$, and both Φ' and π' are holomorphic. For every $\boldsymbol{a} = (a_1, a_2, \cdots, a_{n+1}) \in \mathbf{C}^{n+1}$, $|\boldsymbol{a}| = 1$, we put

$$H_{\boldsymbol{a}} = \{x \in X\,;\ \textstyle\sum \overline{a_j} x_j = 1\},$$
$$Y_{\boldsymbol{a}} = (\Phi')^{-1}(H_{\boldsymbol{a}}),$$
$$L_{\boldsymbol{a}} = \{[u_1 : u_2 : \cdots : u_{n+1}] \in \mathbf{P}^n\,;\ \textstyle\sum \overline{a_j} u_j = 0\},$$
$$\omega_{\boldsymbol{a}} = D \setminus \Phi^{-1}(L_{\boldsymbol{a}}).$$

Then $(Y_{\boldsymbol{a}}, \Phi'|Y_{\boldsymbol{a}})$ is an open set over $H_{\boldsymbol{a}}$, and $(\omega_{\boldsymbol{a}}, \Phi|\omega_{\boldsymbol{a}})$ is a domain over $\mathbf{P}^n \setminus L_{\boldsymbol{a}}$. Since $\pi'|Y_{\boldsymbol{a}} : Y_{\boldsymbol{a}} \to \omega_{\boldsymbol{a}}$ is biholomorphic, $(Y_{\boldsymbol{a}}, \Phi'|Y_{\boldsymbol{a}})$ is a domain over $H_{\boldsymbol{a}}$. The following proposition is given in Kajiwara [5] (cf. Kiselman [6]).

PROPOSITION 2. *Let (D, Φ) and $Y_{\boldsymbol{a}}$ be as above. Assume that Φ is not a biholomorphic mapping onto $\mathbf{P}^n$. Then D is a Stein manifold if and only if $Y_{\boldsymbol{a}}$ is a Stein manifold for every $\boldsymbol{a} \in \mathbf{C}^{n+1}$, $|\boldsymbol{a}| = 1$.*

Using Proposition 1, Proposition 2, and Theorem 1, we have the following theorem.

THEOREM 2. *Let (D, Φ) be a domain over $\mathbf{P}^n$ such that Φ is not a biholomorphic mapping onto $\mathbf{P}^n$. If D is exhausted by complete Kähler domains, then D is a Stein manifold.*

PROOF We can write

$$Y_{\boldsymbol{a}} = \{(x, p) \in H_{\boldsymbol{a}} \times D\,;\ \pi(x) = \Phi(p)\}.$$

Since $H_{\boldsymbol{a}}$ is biholomorphic to $\mathbf{C}^n$, $H_{\boldsymbol{a}}$ is a Stein manifold. Since $Y_{\boldsymbol{a}}$ is a complex closed submanifold of $H_{\boldsymbol{a}} \times D$, $Y_{\boldsymbol{a}}$ is also exhausted by complete Kähler domains by Proposition 1. $Y_{\boldsymbol{a}}$ is a domain over a Stein manifold $H_{\boldsymbol{a}}$, so it is a Stein manifold by Theorem 1. By Proposition 2, D is a Stein manifold.

REMARK 1. Recently the author has obtained the result that a domain exhausted by complete Kähler domains over any complex manifold is p_4-convex in the sense of F. Docquier-H. Grauert [2].

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