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濱田, 英隆
九州共立大学数学科

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By
Hidetaka HAMADA

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Dedicated to Prof. Joji Kajiwara on his sixtieth birthday

1. Introduction

In this paper, we generalize the results of S. Nakano [11], H. Kazama [9], and J. P. Demailly [1]. Let X be a complex manifold of dimension n, and let E be a holomorphic vector bundle of rank r over X. Let $\Omega^p(E)$ denote the sheaf of germs of holomorphic p forms with values in E. S. Nakano [11] and H. Kazama [9] proved the following cohomology vanishing theorem.

THEOREM 1. *If X is weakly 1-complete and E is positive in the sense of S. Nakano, then*

$$H^q(X, \Omega^n(E)) = 0 \quad \textit{for} \quad q \geq 1.$$

As a generalization of the above result, J. P. Demailly [1] proved the following theorem.

THEOREM 2. *If X is a weakly pseudoconvex Kähler manifold and E is s-positive, then*

$$H^q(X, \Omega^n(E)) = 0$$

for $q \geq \max(1, n-s+1)$, *or for* $q \geq 1$ *if* $s \geq r$.

As a corollary he also obtained the following results.

THEOREM 3. *If X is a weakly pseudoconvex Kähler manifold and E is positive in the sense of P. A. Griffiths, then*

$$H^q(X, \Omega^n(E \otimes \det E)) = 0 \quad \textit{for} \quad q \geq 1.$$

THEOREM 4. *Assume that* $rs > 1$. *If* X *is a weakly pseudoconvex Kähler manifold and* E *is positive in the sense of P. A. Griffiths, then*

$$H^q(X, \Omega^n(E^* \otimes (\det E)^s)) = 0$$

for $q \geq \max(1, n-s+1)$, *or for* $q \geq 1$ *if* $s \geq r$.

In this paper, we prove some cohomology vanishing theorems for complex manifolds exhausted by complete Kähler domains. Since a weakly pseudoconvex Kähler manifold is exhausted by complete Kähler domains, our results are generalizations of the above results.

2. Definitions and preliminaries

Let X be a complex manifold. We consider the following condition (C.K.).

(C.K.) There exists a sequence $\{X_k\,;\, k = 1, 2, 3, \cdots\}$ of open submanifolds of X such that $X = \cup X_k$, $X_k \subset\subset X_{k+1}$ for every k, and each X_k has a complete Kähler metric.

DEFINITION 1. If X satisfies the condition (C.K.), we say that X is exhausted by complete Kähler domains.

DEFINITION 2. A complex manifold X is said to be weakly pseudoconvex if there exists a plurisubharmonic function ϕ on X such that the open set $X_c = \{z \in X\,;\, \phi(z) < c\}$ is relatively compact in X for every real number c.

If X is weakly pseudoconvex, X_c is weakly pseudoconvex for every real number c (consider the function $1/(c - \phi(z))$). Thus we have the following proposition by using Theorem 1.3 of J.P.Demailly [1].

PROPOSITION 1. *If a weakly pseudoconvex manifold* X *has a Kähler metric, then* X *satisfies the condition* (C.K.).

Let ds^2 be a hermitian metric on a complex manifold X of dimension n, and let $E \to X$ be a holomorphic vector bundle of rank r with a hermitian metric h along the fibers. Let $\Theta_h = \overline{\partial}(h^{-1}\partial h)$ be the curvature form for h. Then Θ_h is a $\mathrm{Hom}(E, E)$-valued $(1,1)$ form. Let TX (resp. T^*X) be the holomorphic tangent bundle (resp. holomorphic cotangent bundle) of X. Let $(s_1, s_2, \cdots, s_n)$ be an orthonormal frame of T^*X and let $(e_1, e_2, \cdots, e_r)$ be that of E in a neighborhood U of $z_0 \in X$. We can write

$$\Theta_h = \sum c_{jklm} s_j \wedge \overline{s_k} \otimes e_l^* \otimes e_m$$

with $c_{jklm} = \overline{c_{kjml}}$, where $(e_1^*, e_2^*, \cdots, e_r^*)$ is the dual frame of $(e_1, e_2, \cdots, e_r)$. The functions c_{jklm} are continuous on U. We define a hermitian form θ_h on $TX \otimes E$ by

$$\theta_h(x, y) = \sum c_{jklm}(z)\, x_{jl}\, \overline{y_{km}} \quad \text{for } x,\ y \in T_zX \otimes E_z\ (z \in U),$$

where $x_{jl} = (s_j \otimes e_l^*)(x)$, $y_{km} = (s_k \otimes e_m^*)(y)$.
(This definition is independent of the choice of $(s_1, s_2, \cdots, s_n)$ and $(e_1, e_2, \cdots, e_r)$.)

DEFINITION 3. ([1]) Let T and E be complex vector spaces of dimension n and r respectively. A tensor $x \in T \otimes E$ is said to be rank s if s is the smallest nonnegative integer such that we can write

$$x = \sum_{j=1}^{s} t_j \otimes e_j, \quad t_j \in T,\ e_j \in E.$$

DEFINITION 4. ([1]) We say that a holomorphic vector bundle E over X is s-positive if there exists a hermitian metric h along the fibers such that $\theta_h(x, x) > 0$ for any non zero tensor $x \in T_zX \otimes E_z$ of rank $\leq s$ at each point $z \in X$.

REMARK 1. If $s \geq \min(n, r)$, the s-positiveness is equivalent to the positiveness in the sense of S. Nakano [11]. If $s = 1$, the 1-positiveness is equivalent to the positiveness in the sense of P. A. Griffiths [5].

3. Cohomology vanishing theorems

Let $\langle \cdot\,, \cdot \rangle$ be the pointwise inner product with respect to ds^2 and h. Let L be the multiplication of $\sqrt{-1}\omega$ from the left, where ω is the (1,1) form associated to ds^2. Let Λ be the adjoint of L. We denote by $e(\Theta_h)$ the multiplication of Θ_h from the left. Let $z_0 \in X$ be fixed and let U be a neighborhood of z_0 such that there exist orthonormal frames $(s_1, s_2, \cdots, s_n)$ and $(e_1, e_2, \cdots, e_r)$. We can write

$$\omega = \frac{1}{2} \sum s_j \wedge \overline{s_j}$$

on U. For any vector $\boldsymbol{a} = (a_{J,l}) \in \mathbf{C}^m \quad (m = \text{rank } \wedge^{n,q} T^*X \otimes E)$, we put

$$\alpha = \sum_J{}' \sum_l a_{J,l}\, s_1 \wedge \cdots \wedge s_n \wedge \overline{s_J} \otimes e_l.$$

Then, we have

$$\Lambda\alpha = -2\sqrt{-1}\sum_{K}{}'\sum_{j,l}(-1)^{n+j}a_{jK,l}\, s_1\wedge\cdots\wedge\hat{s_j}\wedge\cdots\wedge s_n\wedge\overline{s_K}\otimes e_l$$

and

$$\langle\sqrt{-1}e(\Theta_h)\Lambda\alpha\,,\,\alpha\rangle(z) = (C_{n,q})^2\sum_{K}{}'\sum_{jklm} c_{jklm}(z)\, a_{jK,l}\overline{a_{kK,m}}$$

on U, where $C_{n,q}$ is a positive constant which depends only on n and q. Assume that E is $(n-q+1)$-positive (with respet to h). If we fix the multiindex K, $|K| = q-1$, the rank of the matrix $(a_{jK,l})_{j,l}$ is at most $n-q+1$. Let V be a neighborhood of z_0 with $V\subset\subset U$. Then

$$\langle\sqrt{-1}e(\Theta_h)\Lambda\alpha\,,\,\alpha\rangle(z) > 0$$

for any $z\in\overline{V}$ and for any $\boldsymbol{a}\in\mathbf{C}^m\setminus\{0\}$. Since $\langle\sqrt{-1}e(\Theta_h)\Lambda\alpha\,,\,\alpha\rangle(z)$ is a continuous function on the compact set $H=\overline{V}\times\{\boldsymbol{a}\in\mathbf{C}^m\,;\,\|\boldsymbol{a}\|=1\}$, there exists a positive constant C_1 such that

$$\langle\sqrt{-1}e(\Theta_h)\Lambda\alpha\,,\,\alpha\rangle(z) > C_1$$

on H. For any $z\in V$, let $\beta\in\wedge^{n,q}T_z^*X\otimes E_z$ with $\langle\beta\,,\,\beta\rangle = 1$. We can write

$$\beta = (C_{n,q})^{-1}\sum_{J}{}'\sum_{l}\beta_{J,l}\, s_1\wedge\cdots\wedge s_n\wedge\overline{s_J}\otimes e_l.$$

Then $(z,(\beta_{J,l}))\in H$ and we have

$$\langle\sqrt{-1}e(\Theta_h)\Lambda\beta\,,\,\beta\rangle \geq C_1(C_{n,q})^{-2}.$$

So we have the following proposition.

PROPOSITION 2. *Let X be a complex manifold of dimension n with a hermitian metric ds^2 and let Y be a relatively compact open subset of X. Let E be a holomorphic vector bundle over X with a hermitian metric h along the fibers. Let $\langle\cdot\,,\,\cdot\rangle$ be the pointwise inner product with respect to ds^2 and h. Assume that E is $(n-q+1)$-positive with respect to h. Then there exists a positive constant C such that*

$$\langle\sqrt{-1}e(\Theta_h)\Lambda\beta\,,\,\beta\rangle(z) \geq C\langle\beta\,,\,\beta\rangle(z)$$

for any $\beta\in\wedge^{n,q}T^*Y\otimes E$ and for any $z\in Y$.

Let X be a complex manifold, and let $E \to X$ be a holomorphic vector bundle over X. We set $L^{p,q}_{loc}(X,E) = \{$ locally square integrable E-valued (p,q) forms $\}$.

PROPOSITION 3. *Let X be a complex manifold of dimension n with a Kähler metric ds^2 and let Y be a relatively compact open subset of X with a complete Kähler metric $d\sigma^2$. Let E be a holomorphic vector bundle over X with a hermitian metric h along the fibers. Assume that E is $(n-q+1)$-positive with respect to h. Then for any $g \in L^{n,q}_{loc}(X,E)$ with $\overline{\partial} g = 0$ on Y, there exists an $f \in L^{n,q-1}_{loc}(Y,E)$ such that $\overline{\partial} f = g$ on Y.*

PROOF Let $\langle \cdot\,, \cdot \rangle$ be the pointwise inner product with respect to ds^2 and h, and let dv be the volume form on Y with respect to ds^2. For any $\beta \in \wedge^{n,q} T^*Y \otimes E$,

$$|\langle g\,, \beta \rangle|^2 \leq \langle g\,, g \rangle \langle \beta\,, \beta \rangle \ \leq \langle g\,, g\rangle\, C^{-1} \langle \sqrt{-1}(\Theta_h)\Lambda\beta\,, \beta\rangle$$

from Proposition 2. Since

$$\int_Y \langle g\,, g\rangle\, dv < \infty, \qquad \int_Y \langle g\,, g\rangle\, C^{-1} dv < \infty,$$

the proposition follows from Theorem 4.1 of J. P. Demailly [1].

The idea of the proof of the following theorem owes to S.Nakano and T. S. Rhai [12].

THEOREM 5. *Let X be a complex manifold of dimension n which is exhausted by complete Kähler domains $\{X_k\}$. Let $E \to X$ be a holomorphic vector bundle of rank r over X. Assume that $E|X_k$ is s-positive for each k, and that one of the following conditions is satisfied:*

(1) $q \geq \max(2, n-s+2)$,

(2) $s \geq r$ *and* $q \geq 2$.

Then for any $\varphi \in L^{n,q}_{loc}(X,E)$ with $\overline{\partial}\varphi = 0$, there exists a $\psi \in L^{n,q-1}_{loc}(X,E)$ such that $\overline{\partial}\psi = \varphi$ on X.

PROOF We take real valued C^∞ functions χ_k $(k = 2,3,\cdots)$ such that $0 \leq \chi_k \leq 1$, $\chi_k = 1$ on a neighborhood of $\overline{X_{k-1}}$, $\mathrm{supp}\chi_k \subset X_k$. By Proposition 3, we have $\varphi = \overline{\partial}\psi_3$ on X_3, where $\psi_3 \in L^{n,q-1}_{loc}(X_3,E)$. Set $\varphi_3 = \varphi - \overline{\partial}(\chi_3\psi_3)$. Then $\varphi_3 \in L^{n,q}_{loc}(X,E)$, $\overline{\partial}\varphi_3 = 0$, $\mathrm{supp}\varphi_3 \subset X \setminus X_2$, and $\mathrm{supp}(\chi_3\psi_3) \subset X_3$.

By Proposition 3, we have $\varphi_3 = \overline{\partial}\psi'_4$ on X_4, where $\psi'_4 \in L^{n,q-1}_{loc}(X_4, E)$. Then we have $\overline{\partial}\psi'_4 = 0$ on X_2. By Proposition 3, there exists an $\eta_4 \in L^{n,q-2}_{loc}(X_2, E)$ such that $\psi'_4 = \overline{\partial}\eta_4$. Putting $\psi_4 = \psi'_4 - \overline{\partial}(\chi_2\eta_4)$, we have $\varphi_3 = \overline{\partial}\psi_4$ on X_4, $\psi_4 \in L^{n,q-1}_{loc}(X_4, E)$, $\psi_4 = 0$ on a neighborhood of $\overline{X_1}$. As for $\varphi_4 = \varphi_3 - \overline{\partial}(\chi_4\psi_4)$, we have that $\varphi_4 \in L^{n,q}_{loc}(X, E)$, $\overline{\partial}\varphi_4 = 0$, $\mathrm{supp}\varphi_4 \subset X \setminus X_3$, and $\mathrm{supp}(\chi_4\psi_4) \subset X_4 \setminus X_1$. Thus proceeding, we obtain $\varphi_k \in L^{n,q}_{loc}(X, E)$ and $\psi_k \in L^{n,q-1}_{loc}(X_k, E)$ such that $\varphi_k = \varphi_{k-1} - \overline{\partial}(\chi_k\psi_k)$, $\mathrm{supp}\varphi_k \subset X \setminus X_{k-1}$, $\mathrm{supp}(\chi_k\psi_k) \subset X_k \setminus X_{k-3}$ for $k \geq 3$ ($X_0 = \emptyset$, $\varphi_2 = \varphi$). Hence putting $\psi = \sum_{k>2} \chi_k\psi_k$, we see that $\psi \in L^{n,q-1}_{loc}(X, E)$ and $\varphi = \overline{\partial}\psi$. This completes the proof.

Let $\Omega^n(E)$ denote the sheaf of germs of holomorphic n forms with values in E. Since

$$H^q(X, \Omega^n(E)) \cong \frac{\{f \in L^{n,q}_{loc}(X, E)\,;\, \overline{\partial}f = 0\}}{\{f \in L^{n,q}_{loc}(X, E)\,;\, \overline{\partial}g = f \quad \text{for some} \quad g \in L^{n,q-1}_{loc}(X, E)\}}$$

(cf. L. Hörmander [8], T. Ohsawa [13]), we have the following cohomology vanishing theorem.

THEOREM 6. *Let X be a complex manifold of dimension n which is exhausted by complete Kähler domains $\{X_k\}$, and let $E \to X$ be a holomorphic vector bundle of rank r. Assume that $E|X_k$ is s-positive for each k. Then,*

$$H^q(X,\ \Omega^n(E)) = 0$$

for $q \geq \max(2, n - s + 2)$, or for $q \geq 2$ if $s \geq r$.

COROLLARY 1. *Let X be a complex manifold of dimension n which is exhausted by complete Kähler domains $\{X_k\}$, and let $E \to X$ be a holomorphic vector bundle over X. Assume that $E|X_k$ is positive in the sense of S. Nakano for each k. Then,*

$$H^q(X, \Omega^n(E)) = 0 \quad \textit{for} \quad q \geq 2.$$

Let E be a holomorphic vector bundle of rank r over a complex manifold X. We put $\det E = \wedge^r E$. Assume that E is positive in the sense of P. A. Griffiths. Then $E \otimes \det E$ is positive in the sense of S. Nakano. Moreover $E^* \otimes (\det E)^s$ is s-positive if $rs > 1$ [1]. Therefore we have the following corollaries.

COROLLARY 2. *Let X be a complex manifold of dimension n which is exhausted by complete Kähler domains $\{X_k\}$, and let $E \to X$ be a holomorphic* *vector bundle over X. Assume that $E|X_k$ is positive in the sense of P. A. Griffiths for each k. Then,*

$$H^q(X,\,\Omega^n(E\otimes\det E))=0 \quad \text{for} \quad q\geq 2.$$

COROLLARY 3. *Let X be a complex manifold of dimension n which is exhausted by complete Kähler domains $\{X_k\}$, and let $E\to X$ be a holomorphic vector bundle of rank r over X. Assume that $E|X_k$ is positive in the sense of P. A. Griffiths for each k and $rs>1$. Then,*

$$H^q(X,\,\Omega^n(E^*\otimes(\det E)^s))=0$$

for $q\geq\max(2,n-s+2)$, or for $q\geq 2$ if $s\geq r$.

THEOREM 7. *Let X be a complex manifold of dimension n which is exhausted by complete Kähler domains $\{X_k\}$. Assume that, for each k, there exists a C^∞ strictly plurisubharmonic function on X_k. Then for any holomorphic vector bundle E over X,*

$$H^q(X,\mathcal{O}(E))=0 \quad \text{for} \quad q\geq 2.$$

PROOF Let $F=K^{-1}\otimes E$, where K is the canonical bundle over X. Since there exists a C^∞ strictly plurisubharmonic function on X_{k+1}, there exists a metric along the fibers of $F|X_k$ such that $F|X_k$ is positive in the sense of S. Nakano. By Corollary 1, we have $H^q(X,\,\Omega^n(F))=0$ for $q\geq 2$. Since $\mathcal{O}(E)\cong\Omega^n(K^{-1}\otimes E)=\Omega^n(F)$, we have $H^q(X,\mathcal{O}(E))=0$ for $q\geq 2$.

REMARK 2. Let X be a complex manifold which is the union of an increasing sequence of Stein open submanifolds $\{S_i\}$ such that X is not Stein ([2],[3],[4]). Since each S_i satisfies the condition (C.K.) by Proposition 1, X also satisfies the condition (C.K.). Let $X=\cup X_k$, $X_k\subset\subset X_{k+1}$, and each X_k is a complete Kähler manifold. For each k, there exists a C^∞ strictly plurisubharmonic function on X_k, since X_k is contained in some S_i. Therefore X satisfies the assumption of Theorem 7. If Theorem 5 holds when $q=1$, we can show that $H^1(X,\mathcal{O}(E))=0$ for any holomorphic vector bundle E over X as in the proof of Theorem 7. Let E be the trivial line bundle, then $H^1(X,\mathcal{O})=0$. Since X is the union of an increasing sequence of Stein open submanifolds and $H^1(X,\mathcal{O})=0$, by the result of A. Markoe [10], X is a Stein manifold. This is a contradiction.

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Hidetaka Hamada
Department of Mathematics
Kyushu Kyoritsu University
Jiyugaoka, Yahata-nishiku, Kitakyushu 807
Japan