# Invariant plurisubharmonic exhaustion functions on complex Lie groups

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# Invariant plurisubharmonic exhaustion functions on complex Lie groups

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*Dedicated to Prof. Joji Kajiwara on his sixtieth birthday*

## 1. Introduction

On the Stein group $GL(n, \mathbf{C})$ there exists a natural strongly plurisubharmonic exhaustion function

$$\widehat{\varphi}(A) := \operatorname{trace}(A\,A^*) + \frac{1}{\det(A\,A^*)}.$$

This function $\widehat{\varphi}$ is invariant on the unitary subgroup

$$U(n) := \{\, A \in GL(n, \mathbf{C}) \mid A^{-1} = A^* \,\}$$

which is a maximal compact real Lie subgroup of $GL(n, \mathbf{C})$, i.e., for $A \in GL(n, \mathbf{C})$ and $X, Y \in U(n)$,

$$\widehat{\varphi}(X\,A\,Y) = \widehat{\varphi}(A).$$

Let $G$ be a connected complex Lie group of complex dimension $n$ and $G^0$ the maximal toroidal subgroup of $G$, i.e.,

$$G^0 = \{\, x \mid f(x) = f(e) \text{ for every holomorphic function } f \text{ on } G \,\}.$$

Let $\mathfrak{k}$ be the real Lie algebra of a maximal compact real Lie subgroup $K$ of $G$. Put $q := \dim_{\mathbf{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Then we construct a natural plurisubharmonic and strongly $(q+1)$-pseudoconvex exhaustion function on $G$ in the following theorem.

THOREM 1. *There exists a $C^\infty$ plurisubharmonic function $\varphi$ on $G$ satisfying* (1), (2) *and* (3).

(1) *$\varphi$ is $K$-invariant, that is,*

$$\varphi(x) = \varphi(yxz)$$

*for any $x \in G, y, z \in K$.*

(2) *The Levi form of $\varphi$ is semi positive and has $n-q$ positive eigenvalues on $G$, in other words, $\varphi$ is plurisubharmonic and strongly $(q+1)$-pseudoconvex on $G$.*

(3) *$\varphi$ is an exhaustion function on $G$, i.e., for any $c \in \mathbf{R}$.*

$$\{ x \in G \mid \varphi(x) < c \} \Subset G.$$

As an immediate consequence of the above theorem we obtain the following corollary which was proved by other methods in [11] and partially in [5].

COROLLARY 1. *$G$ is weakly 1-complete in the sense of Nakano* [10] *and strongly $(q+1)$-complete in the sense of Andreotti and Grauert* [1].

Using the result of [2,4] we can give another proof of the statement of the following corollary which was proved in [7].

COROLLARY 2. *If $\mathfrak{k} \cap \sqrt{-1}\mathfrak{k} = \{0\}$, then $G$ is a Stein group.*

## 2. In case of abelian Lie groups

Throughout this section we assume $G$ is a connected complex abelian Lie group of complex dimension $n$ and $G^0$ is the maximal toroidal subgroup

$$G^0 := \{ x \mid f(x) = f(e) \text{ for every holomorphic function } f \text{ on } G \}$$

of complex dimension $m$. Let $K^0$ be a maximal compact subgroup of $G^0$. From the result of [8] there exists a lattice $\Gamma$ of $\mathbf{C}^m$ such that $G^0 \cong \mathbf{C}^m/\Gamma$. We may assume $\Gamma = \mathbf{Z}\{e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_q\}$, where $e_i$ is the $i$-th unit vector of $\mathbf{C}^m$ and $e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_q$ are $\mathbf{R}$-linearly independent, and $K^0 \cong \mathbf{R}\{e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_q\}/\Gamma$. Let $v_i = (v_{i1}, v_{i2}, \ldots, v_{im})$ and $v_i = \Re v_i + \sqrt{-1}\Im v_i$ ($\Re v_i, \Im v_i \in \mathbf{R}^m$). Since $e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_q$ are $\mathbf{R}$-linearly independent, $\Im v_1, \ldots, \Im v_q$ are $\mathbf{R}$-linearly independent. Without loss of generality we may assume the $q \times q$ real matrix $[v_{ij} \ ; \ 1 \leq i, j \leq q\ ]$ is invertible. We put

$$v_{q+1} := \sqrt{-1}e_{q+1},\ v_{q+2} := \sqrt{-1}e_{q+2},\ \ldots,\ v_m := \sqrt{-1}e_m.$$

Then $e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_q, v_{q+1}, \ldots, v_m$ are $\mathbf{R}$-linearly independent vectors of $\mathbf{C}^m$. For any $z = (z_1, z_2, \ldots, z_m)$ there exists a unique vector $(t_1, t_2, \ldots, t_{2m}) \in \mathbf{R}^{2m}$ such that

$$z = \sum_{i=1}^{m} t_i e_i + \sum_{i=1}^{q} t_{m+i}\sqrt{-1}\Im v_i + \sum_{i=q+1}^{m} t_{m+i} v_i.$$

We define the exhaustion function

$$\Phi : \mathbf{C}^m/\Gamma \ni z + \Gamma \longmapsto \sum_{i=q+1}^{m} t_{m+i}^2 \in \mathbf{R}$$

on $\mathbf{C}^m/\Gamma$. We denote by $A = [\,a_{ij}\,]$ the inverse matrix of

$$\begin{bmatrix} \Im v_1 \\ \vdots \\ \Im v_q \\ e_{q+1} \\ \vdots \\ e_m \end{bmatrix}.$$

Let $x_i := \Re z_i$ and $y_i := \Im z_i$. Then we have $t_{m+i} = \sum_{k=1}^{m} y_k\, a_{ki}, \quad (i \geq q+1)$. The Levi form of $\Phi$ is given by

$$\left[\frac{\partial^2 \Phi(z+\Gamma)}{\partial z_i \partial \overline{z_j}}\right] = \frac{1}{4}\left[\frac{\partial^2 \Phi}{\partial y_i \partial y_j}\right] = \frac{1}{2} B\, B^t,$$

where $B$ is the matrix

$$\begin{bmatrix} 0 & \cdots & 0 & a_{1\,q+1} & \cdots & a_{1\,m} \\ 0 & \cdots & 0 & a_{2\,q+1} & \cdots & a_{2\,m} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & a_{m\,q+1} & \cdots & a_{m\,m} \end{bmatrix}.$$

Since $B$ is a real $(m,m)$-matrix of rank $m-q$, $\frac{1}{2}B\,B^t$ is positive semi definite and $m-q$ positive eigenvalues. By the definition of $\Phi$ we can see that $\Phi(z + z^* + \Gamma) = \Phi(z+\Gamma)$ for any $z \in \mathbf{C}^m$ and $z^* \in K^0$. This means $\Phi$ is $K^0$-invariant. By the result of [8] $G \cong G^0 \times \mathbf{C}^{*p} \times \mathbf{C}^r$ for some non-negative integers $p$ and $r$ with

$m+p+r=n$. Then we may assume $G=G^0\times \mathbf{C}^{*p}\times \mathbf{C}^r$. Since $G$ is abelian, $G$ has the unique maximal compact subgroup $K$. We take the function

$$\widetilde{\Phi}(z+\Gamma,\xi,\eta):=\Phi(z+\Gamma)+\sum_{i=1}^{p}\left(\frac{1}{|\,\xi_i\,|^2}+|\,\xi_i\,|^2\right)+\sum_{j=1}^{r}|\,\eta_j\,|^2$$

for $(z+\Gamma,\xi,\eta)\in G^0\times\mathbf{C}^{*p}\times\mathbf{C}^r$, where $\xi=(\xi_1,\dots,\xi_p)$ and $\eta=(\eta_1,\dots,\eta_r)$. Since $K=K^0\times\{\,\xi\mid\ |\,\xi_i\,|\ =1\,\}\times 0\subset G^0\times\mathbf{C}^{*p}\times\mathbf{C}^r$, it is easily to show that $\widetilde{\Phi}$ is $K$-invariant. Then we have the following

PROPOSITION 1. *$\widetilde{\Phi}$ is a $K$-invariant plurisubharmonic exhaustion function on $G^0\times\mathbf{C}^{*p}\times\mathbf{C}^r$ and its Levi form has $n-q$ positive eigen values on $G^0\times\mathbf{C}^{*p}\times\mathbf{C}^r$.*

## 3. Fibration of complex Lie groups

From now on we assume $G$ be a connected complex Lie group of complex dimension $n$, which is not necessarily abelian. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $K_C$ the complex Lie subgroup with the Lie subalgebra $\mathfrak{k}_C:=\mathfrak{k}+\sqrt{-1}\mathfrak{k}$. Then $K_C$ is a closed Lie subgroup of $G$ ([6]). Put $a:=\dim_C G/K_C$. Let $\psi: G\longrightarrow G/K_C$ be the projection which induces the $K_C$-principal bundle over $G/K_C$. There exists an open neighborhood $U$ of the unit element $e$ in $G$ and the canonical coordinate $(w_1,\dots,w_n)$ such that

$$U_1:=\{\,x\in U\mid w_1(x)=\cdots=w_a(x)=0\,\}$$

and

$$U_2:=\{\,x\in U\mid w_{a+1}(x)=\cdots=w_n(x)=0\,\}$$

satisfy $U=U_1\cdot U_2$, $U\cap K_C=U_2$, $U_1\cap U_2=\{\,e,\}$, $U\cdot K_C=U_1\cdot K_C$ and $(U_1^{-1}\cdot U_1)\cap K_C=\{\,e\,\}$. Then there exists a countable subset $\{\,a_\nu\mid a_\nu\in G,\ a_0=e,\ \nu=0,1,2,\dots\,\}$ such that $\{\,\psi(a_\nu U_1)\mid \nu=0,1,2,\dots\,\}$ is a locally finite covering of $G/K_C$. For any $\psi(x)\in\psi(a_\nu U_1)$ there exists a unique element $x_\nu\in U_1$ and $h_\nu\in K_C$ such that $\psi(x)=\psi(a_\nu x_\nu)$ and $x=a_\nu x_\nu h_\nu$. We put

$$\Psi_\nu\ :\ \psi^{-1}(\psi(a_\nu U_1))\ni x\longmapsto(h_\nu,\ \psi(x))\in K_C\times\psi(a_\nu U_1)$$

and

$$\Lambda_{\nu\mu}(\pi(x)):=(a_\nu x_\nu)^{-1}(a_\mu x_\mu)\ :\ \psi(a_\nu U_1)\cap\psi(a_\mu U_1)\longrightarrow K_C.$$

Then $(\{\psi(a_\nu U_1)\},\{\Lambda_{\nu\mu}\})$ gives a defining 1-cocyle of the bundle $G\longrightarrow G/K_C$. This means $\Psi_0:\ \psi^{-1}(\psi(e))\cong K_C\longrightarrow K_C\times\psi(e)$ is an isomorphism as a Lie group. From the result of [6] it follows that $G/K_C$ is isomorphic onto $\mathbf{C}^a$ and then the $K_C$-principal bundle $\psi : G \longrightarrow G/K_C$ is holomorphically trivial (in virtue of [3]). Then we have a 0-cochain $\{\Lambda_\nu : \psi(a_\nu U_1) \longrightarrow K_C\}$ satisfying $\Lambda_{\nu\mu} = \Lambda_\nu^{-1}\Lambda_\mu$ in $\psi(a_\nu U_1) \cap \psi(a_\mu U_1)$. Putting $\lambda_0 := \Lambda_0(\psi(e))$ and $\tilde{\Lambda}_\nu := \lambda_0^{-1}\Lambda_\nu$, we get the biholomorphic mapping

$$\alpha : G \supset \psi^{-1}(\psi(a_\nu U_1)) \ni x \longmapsto (\tilde{\Lambda}_\nu(\psi(x))h_\nu, \psi(x)) \in K_C \times G/K_C,$$

where $x = a_\nu x_\nu h_\nu, x_\nu \in U_1, h_\nu \in K_C$. And $\alpha \mid K_C : K_C \longrightarrow K_C \times \psi(e)$ is an isomorphism as a complex Lie group. From the above argument we may assume there exists a biholomorphic mapping $\alpha$ from $G$ onto $K_C \times \mathbf{C}^a$ such that $\alpha \mid K_C$ is an isomorphism from $K_C$ onto $K_C \times \{0\}$ as a complex Lie group. Further there exists a connected semi-simple and, then, Stein subgroup $S^0$ of $K_C$ such that, for the connected center $Z$ of $K_C$,

$$\rho_0 : Z \times S_0 \ni (x, y) \longmapsto xy \in K_C$$

is a finite covering homomorphism. By the result of [7,8] $G^0 \subset Z$ and $Z \cong G^0 \times \mathbf{C}^{*^p} \times \mathbf{C}^r$ for some non-negative integers $p$ and $r$. Then we may assume $Z = G^0 \times \mathbf{C}^{*^p} \times \mathbf{C}^r$. Then we obtain a Stein subgroup $S := \mathbf{C}^{*^p} \times \mathbf{C}^r \times S_0 \times \mathbf{C}^a$ of $Z \times S_0 \times \mathbf{C}^a$. We get a finite covering homomorphism

$$\rho : G^0 \times S \ni (x_0, x_1, x_2, x_3, x_4) \longmapsto (\rho_0((x_0, x_1, x_2), x_3), x_4) \in K_C \times \mathbf{C}^a$$

for $(x_0, x_1, x_2, x_3, x_4) \in G^0 \times \mathbf{C}^{*^p} \times \mathbf{C}^r \times S_0 \times \mathbf{C}^a$. From this homomorphism $\rho$ we can assume $K_C \times \mathbf{C}^a = G^0 \times S/\ker\rho$. Let $\pi_1 : G^0 \times S \to G^0$ and $\pi_2 : G^0 \times S \to S$ be the canonical projections. Since $\pi_2(\ker\rho)$ is a finite subgroup of $S$, $S/\pi_2(\ker\rho)$ is a Stein group. We obtain a homomorphism

$$\pi : G^0 \times S/\ker\rho \ni (a, b)\ker\rho \longmapsto b\pi_2(\ker\rho) \in S/\pi_2(\ker\rho)$$

for $a \in G^0, b \in S$. From this projection $\pi$, $G \cong G^0 \times S/\ker\rho$ is regard as a fiber bundle over the Stein group $S/\pi_2(\ker\rho)$ whose fiber is isomorphic onto $G^0$ and whose structure group is the finite subgroup $\pi_1(\ker\rho)$ of $G^0$. Let $K^0$ be the maximal compact subgroup of $G^0$. Since $G^0$ is abelian, we obtain $\pi_1(\ker\rho) \subset K^0$.

## 4. Proof of the theorem

We remark the following elementary lemma.

LEMMA 1. *Let $S$ be a connected closed complex Lie subgroup of $GL(n,\mathbf{C})$ and $L$ be a maximal compact subgroup of $S$. Then there exists a matrix $A \in GL(n,\mathbf{C})$ such that $A\ L\ A^{-1} \subset U(n)$.*

PROOF Since $L$ is a compact subgroup of $GL(n,\mathbf{C})$, there exists a maximal compact subgroup $L^*$ of $GL(n,\mathbf{C})$ such that $L \subset L^*$. It is known that all maximal compact subgroups are conjugate each other. This means $L^* = A\ U(n)\ A^{-1}$ for some $A \in GL(n,\mathbf{C})$. This proves the statement of the lemma.

Since $\ker\rho$ in **3** is finite subgroup of $G^0 \times S$ and $\mathbf{C}^a$ (and also $\mathbf{C}^r$) has no non-trivial finite subgroup, we may assume $\ker\rho \subset G^0 \times \mathbf{C}^{*p} \times S_0$ and $K_C \times \mathbf{C}^a = (G^0 \times \mathbf{C}^{*p} \times S_0)/\ker\rho \times \mathbf{C}^r \times \mathbf{C}^a$. Let $\Pi_1 : G^0 \times \mathbf{C}^{*p} \times S_0 \longrightarrow G^0$, $\Pi_2 : G^0 \times \mathbf{C}^{*p} \times S_0 \longrightarrow \mathbf{C}^{*p}$ and $\Pi_3 : G^0 \times \mathbf{C}^{*p} \times S_0 \longrightarrow S_0$ be the canonical projections. Then $\Pi_1(\ker\rho) \subset K^0$, $\Pi_2(\ker\rho) \subset \{(\xi_1,\dots,\xi_p) \mid |\xi_i| = 1\} \subset \mathbf{C}^{*p}$ and, for a maximal compact subgroup $L$ of $S_0$, $\Pi_3(\ker\rho) \subset L$. Since $S_0$ is semi-simple, by the result of [7] we may regard $S_0$ is a closed subgroup of $GL(n,\mathbf{C})$. By Lemma there exists a matrix $A \in GL(n,\mathbf{C})$ such that $A\ L\ A^{-1} \subset U(n)$. Using the strictly plurisubharmonic exhaustion function $\widehat{\varphi}$ on $GL(n,\mathbf{C})$ in the introduction of this paper, $\varphi_1(X) := (\widehat{\varphi}|S_0)(A\ X\ A^{-1})$ is a strictly plurisubharmonic $\Pi_3(\ker\rho)$-invariant exhaustion function on $GL(n,\mathbf{C})$. We take the maximal compact subgroup

$$K^* := (K^0 \times \{(\xi_1,\dots,\xi_p) \mid |\xi_i| = 1\} \times L)/\ker\rho \times (0,0),$$

where $(0,0) \in \mathbf{C}^r \times \mathbf{C}^a$. Since $\alpha|K_C$ is an isomorphism of $K_C$ onto $K_C \times 0$. Then $K := \alpha^{-1}(K^*)$ is a maximal compact subgroup of $G$. By Proposition in **2** there exists a $K^0$-invariant plurisubharmonic exhaustion function $\varphi_0 : G^0 \longrightarrow \mathbf{R}$. From the invariance of $\varphi_0$ and $\varphi_1$ it follows that

$$\varphi := \varphi_0(y_0) + \varphi_1(y_1) + \sum_{i=1}^{p}\left(\frac{1}{|\xi_i|^2} + |\xi_i|^2\right) + \sum_{j=1}^{r}|z_i|^2 + \sum_{k=1}^{a}|w_k|^2$$

($y_0 \in G^0, y_1 \in S_0, \xi \in \mathbf{C}^{*p}, z \in \mathbf{C}^r$, and $w \in \mathbf{C}^a$) is regarded as a plurisubharmonic function on $G$ and its Levi form has $(n-q)$ positive eigenvalues, where $\mathfrak{k}^0$ is the Lie algebra of the maximal compact subgroup $K^0$ of $G^0$ and $q = \dim_C(\mathfrak{k}^0 \cap \sqrt{-1}\mathfrak{k}^0)$. Let $\mathfrak{u}$ be the Lie algebra of $U(n)$. Since $\mathfrak{u}\cap\sqrt{-1}\mathfrak{u} = 0$, it is easy to show that $q = \dim_C(\mathfrak{k} \cap \sqrt{-1}\mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of $K$. For any maximal compact subgroup $\widetilde{K}$ of $G$ there exists $z_0 \in G$ such that $K = z_0\ \widetilde{K}\ z_0^{-1}$. We define the function $\widetilde{\varphi}(x) := \varphi(z_0^{-1}\ x\ z_0)$ on $G$. Since $\varphi$ is $K$-invariant, we obtain, for $\widetilde{y}, \widetilde{z} \in \widetilde{K}$, $\widetilde{\varphi}(\widetilde{y}x\widetilde{z}) = \varphi(z_0^{-1}\widetilde{y}z_0z_0^{-1}xz_0z_0^{-1}\widetilde{z}z_0) = \widetilde{\varphi}(x)$. This means $\widetilde{\varphi}$ is $\widetilde{K}$-invariant and satisfies the statements (2) and (3) of the theorem.

## References

[1] A. ANDREOTTI AND H. GRAUERT: Thèorémes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France, **90** (1962), 193–259

[2] F. DOCQUIER AND H. GRAUERT: Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann., **140** (1960),94–123

[3] H. GRAUERT: Analytische Faserungen über holomorph-vollständigen Räumen, Math. Ann., **135** (1958), 263–273

[4] H. GRAUERT: On Levi's Problem and the imbedding of real analytic manifold, Ann. Math., **68** (1958), 460–472

[5] H. KAZAMA: On pseudoconvexity of complex abelian Lie groups, J. Math. Soc. Japan, **25** (1973), 329–333

[6] Y.MATSUSHIMA: Espaces homogènes de Stein des groupes de Lie complexes, Nagoya Math. J., **16** (1960), 205–218

[7] Y. MATSUSHIMA AND A. MORIMOTO: Sur certains espaces fibrés holomorphes sur une variété de Stein, Bull. Soc. Math. France, **88** (1960), 137–155

[8] A. MORIMOTO: Non-compact complex Lie groups without non-constant holomorphic functions, Proc. Conf. on Complex Analysis, Springer 1965, Minneapolis 1964, 256–272

[9] A. MORIMOTO: On the classification of non-compact complex abelian Lie groups, Trans. Amer. Math. Soc., **123** (1966), 200–228

[10] S. NAKANO: Vanishing theorem for weakly 1-complete manifolds, Algebraic geometry and commutative algebra, Kinokuniya, Tokyo, 1973, 169–179

[11] S. TAKEUCHI: On completeness of holomorphic principal bundles, Nagoya Math. J., **56** (1975), 121–138

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