On Gauss sums with Dirichlet characters

白谷, 克己
九州大学理学部数学科

内山, 成憲
九州大学理学部数学科

Pei, Xiaowun
Shenyang University of Workers

https://doi.org/10.15017/1449079

出版情報：九州大学教養部数学雑誌. 19, pp.1-7, 1993-12. 九州大学教養部数学教室
バージョン：
権利関係：

Math. Rep.
19, 1993.

On Gauss sums with Dirichlet characters

By

Katsumi SHIRATANI*, Shigenori UCHIYAMA, Xiaowun PEI

Dedicated to Prof. Joji Kajiwara on his sixtieth birthday

(Received July 5, 1993)

1. Introduction

Let $f \geq 1$ be a natural number and ζ_f a fixed primitive f-th root of unity. Let χ be any primitive Dirichlet character of conductor f.

A Gauss sum attached to the character χ is defined by

$$g(\chi) = \sum_{x \bmod f} \chi(x)\, \zeta_f^x,$$

where x runs over a complete system of representatives in the rational integer ring $\mathbf{Z}$ modulo f. This Gauss sum depends naturally on the selections of χ and ζ_f. From the definition we see easily that the number $g(\chi)$ is an integer in the cyclotomic field $\mathbf{Q}(\zeta_{\phi(f)}, \zeta_f)$. Here $\mathbf{Q}$ means the rational number field and ϕ denotes the Euler function. In the case where $f = 2^n$ the sum is an integer in the field $\mathbf{Q}(\zeta_{2^n})$.

It is known that in the case where $f = p$ there are other kinds of Gauss sums with power residue characters in algebraic number fields under certain conditions. For these Gauss sums the so-called Stickelberger theorem gives us the prime ideal decompositions [1]. More strictly speaking, from the Gross-Koblitz formula [2] together with the continuity of p-adic Γ-function, higher congruences can be obtained in principle.

Moreover we remark that there are Gauss sums of another kind, called exponential Gauss sums, which we here do not deal with.

In this note we give first a slightly more precise congruence than the Stickelberger formula in the case where $f = p$ is any odd prime from an elementary view-point, because two kinds of above sums coincide in this case.

*Partly supported by a Grant-in-Aid for General Scientific Research no.05640050 from Ministry of Education, Science and Culture, Japan.

In the second section we give a theorem on the p-adic values of the sums $g(\chi)$ in higher cases where $f = p^n$.

We point out that it is very important to determine the correct p-adic magnitudes of Gauss sums with Dirichlet characters for some applications of p-adic L-functions [4], [5].

2. Congruences

Let $\mathbf{Q}_p$ be the rational p-adic number field and $\mathbf{Z}_p$ the rational p-adic integer ring. Then we consider a fixed imbedding all algebraic numbers, hence especially ζ_f, in the algebraic closure $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. Therefore the values of χ and $g(\chi)$ are integers in the field $\bar{\mathbf{Q}}_p$. Then the number $g(\chi)$ belongs to the integer ring $O_{\mathbf{Q}_p(\zeta_{p^n})}$ of the p-adic cyclotomic field $\mathbf{Q}_p(\zeta_{p^n})$.

Now, we consider the case where $f = p^n$ $(n \geq 1)$. Then, expanding the Gauss sum by a prime element $\zeta_{p^n} - 1$ in the field $\mathbf{Q}_p(\zeta_{p^n})$ we have

$$\begin{aligned} g(\chi) &= \sum_{x=0}^{p^n-1} \chi(x)\,\zeta_{p^n}^x \\ &= \sum_{x=0}^{p^n-1} \chi(x) \sum_{j=0}^{x} \binom{x}{j} (\zeta_{p^n} - 1)^j \\ &= \sum_{j=1}^{p^n-1} \sum_{x=j}^{p^n-1} \chi(x) \frac{(x)_j}{j!} (\zeta_{p^n} - 1)^j \\ &= \sum_{j=1}^{p^n-1} \frac{(\zeta_{p^n} - 1)^j}{j!} \sum_{x=1}^{p^n-1} \chi(x)(x)_j, \end{aligned}$$

where $(x)_j$ denotes the Jordan factorial $x(x-1)\cdots(x-j+1)$.

First we treat the case $n = 1$, hence $p \neq 2$. Let ω be the Teichmüller character, namely $\omega(x) = \lim_{\rho\to\infty} x^{p^\rho}$ for $x \in \mathbf{Z}_p^* = \mathbf{Z}_p - p\mathbf{Z}_p$. Then any Dirichlet character χ with conductor p is a power of ω. Namely $\chi = \omega^r$ for some r $(1 \leq r \leq p-2)$. From the congruences $\omega(x) \equiv x \pmod{p}$ for $x \in \mathbf{Z}_p^*$ we have

$$\begin{aligned} \sum_{x=1}^{p-1} \chi(x)(x)_j &= \sum_{x=1}^{p-1} \omega^r(x) \sum_{m=1}^{j} S(j,\, m) x^m \\ &= \sum_{m=1}^{j} S(j,\, m) \sum_{x=1}^{p-1} \omega^r(x) x^m \\ &\equiv \sum_{m=1}^{j} S(j,\, m) \sum_{x=1}^{p-1} x^{r+m} \pmod{p} \\ &\equiv -S(j,\, p-1-r) \pmod{p}. \end{aligned}$$

Herein $S(j, m)$ means the Stirling number of the first kind defined by $(X)_j = \sum_{m=1}^{j} S(j, m)X^m$, and it is a rational integer. Therefore we have

$$g(\chi) \equiv -\sum_{j=1}^{p-1} \frac{1}{j!}(\zeta_p - 1)^j S(j, p-1-r) \pmod{p(\zeta_p - 1)}.$$

In the following we denote the prime ideal in the field $\mathbf{Q}_p(\zeta_{p^n})$ by $\mathfrak{p}_n$. Then we know $(p) = \mathfrak{p}_n^{p^{n-1}(p-1)}$ from the elementary ramification theory in the cyclotomic field $\mathbf{Q}_p(\zeta_{p^n})$. Therefore we see

$$g(\omega^r) \equiv -\sum_{j=1}^{p-1} \frac{1}{j!}(\zeta_p - 1)^j S(j, p-1-r) \pmod{\mathfrak{p}_1^p}.$$

As it holds that $S(j, p-1-r) = 0$ for $1 \leq j \leq p-2-r$, we obtain

$$g(\omega^r) \equiv -\frac{(\zeta_p - 1)^{p-1-r}}{(p-1-r)!} \pmod{\mathfrak{p}_1^{p-r}}.$$

Consequently we have a refinement of the Stickelberger theorem in this particular case.

THEOREM 1. *It holds that*

$$g(\omega^r) \equiv -\sum_{j=p-1-r}^{p-1} \frac{1}{j!} S(j, p-1-r)(\zeta_p - 1)^j \pmod{\mathfrak{p}_1^p}.$$

In particular

$$g(\omega^r) \equiv -\frac{1}{(p-1-r)!}(\zeta_p - 1)^{p-1-r} \pmod{\mathfrak{p}_1^{p-r}}.$$

Especially we have $\mathfrak{p}_1^{p-1-r} \parallel g(\omega^r)$.

By the way, take a prime element $\varpi = \sqrt[p-1]{-p}$ in $\mathbf{Q}_p(\zeta_p)$ such that $\varpi \equiv \zeta_p - 1 \pmod{\mathfrak{p}_1^2}$ holds. Then we know the congruences [3]

$$\varpi \equiv \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i}(\zeta_p - 1)^i \pmod{\mathfrak{p}_1^{p+1}},$$

$$\zeta_p \equiv \sum_{i=0}^{p-1} \frac{1}{i!}\varpi^i \pmod{\mathfrak{p}_1^p}.$$

From these congruences we have for $1 \le j \le p-1$

$$\frac{1}{j!}(\zeta_p - 1)^j \equiv \sum_{t=j}^{p-1} \frac{1}{t!}\mathfrak{S}(t, j)\varpi^t \pmod{\mathfrak{p}_1^p},$$

where $\mathfrak{S}(t, j) = \sum_{P_j} \frac{t!}{a_1!a_2!\cdots(1!)^{a_1}(2!)^{a_2}\cdots}$ means the Stirling number of the second kind. In the summation P_j runs through all such partitions $P_j : a_1 + a_2 + \cdots = j,\ a_1 + 2a_2 + \cdots = t$.

Hence we see

$$\begin{aligned} g(\omega^r) &\equiv -\sum_{j=1}^{p-1} S(j, p-1-r) \sum_{t=j}^{p-1} \frac{1}{t!}\mathfrak{S}(t,j)\varpi^t \pmod{\mathfrak{p}_1^p} \\ &\equiv -\sum_{t=1}^{p-1} \frac{1}{t!} \sum_{j=1}^{t} \mathfrak{S}(t,j)S(j, p-1-r)\varpi^t \pmod{\mathfrak{p}_1^p}. \end{aligned}$$

By the orthogonality relations of the Stirling numbers we obtain

$$g(\omega^r) \equiv -\frac{1}{(p-1-r)!}\varpi^{p-1-r} \pmod{\mathfrak{p}_1^p}.$$

This is a truncated one of the Gross-Koblitz formula [2].

COROLLARY. *For the prime element ϖ in the above we have the congruence*

$$g(\omega^r) \equiv -\frac{1}{(p-1-r)!}\varpi^{p-1-r} \pmod{\mathfrak{p}_1^p}.$$

To the cases where $n = 2, 3$ we can proceed with this method, but calculations are somewhat complicated. Hence we state only the results. In the case of $n=2$, p odd, set $\chi(1+p) = \zeta_p^{-a_\chi}$ with $1 \le a_\chi \le p-1$. Then we have

$$g(\chi) = p\chi(a_\chi)\zeta_{p^2}^{a_\chi}.$$

When $p = 2$ we have $g(\chi) = 2\zeta_4$. In the case where $n = 3$, p is odd, we set similarly $\chi(1+p^2) = \zeta_p^{-a_\chi}$ with $1 \le a_\chi \le p-1$. Then it holds that

$$g(\chi) \equiv -p\chi(a_\chi)\zeta_{p^3}^{a_\chi} a_\chi^{-\frac{p-1}{2}} 2^{-\frac{p-1}{2}} (\zeta_p - 1)^{\frac{p-1}{2}} \frac{1}{(\frac{p-1}{2})!} \pmod{p\mathfrak{p}_1^{\frac{p+1}{2}}}$$

3. The general case

In this section we treat higher cases in a unified manner. We discuss general case with $n \geq 2$, namely any Gauss sum with Dirichlet character with prime power conductor p^n, mainly for p odd.

First we take $x = x_0 + py$, $1 \leq x_0 \leq p-1$, $0 \leq y \leq p^{n-1}-1$ as a system of the representatives, which run in the summation of the Gauss sum.

$$\begin{aligned} g(\chi) &= \sum_{x \bmod p^n} \chi(x)\,\zeta_{p^n}^{x} \\ &= \sum_{x_0=1}^{p-1} \chi(x_0)\,\zeta_{p^n}^{x_0} \sum_{y \bmod p^{n-1}} \chi(1+px_0^{-1}y)\,\zeta_{p^n}^{py}, \end{aligned}$$

because we have $x \equiv x_0(1+px_0^{-1}y) \pmod{p^n}$.

Now, we set $h(x_0, \chi) = \sum_{y \bmod p^{n-1}} \chi(1+px_0^{-1}y)\zeta_{p^{n-1}}^{y}$. Let σ_{-1} be the element of the Galois group $G(\mathbf{Q}_p(\zeta_{p^{n-1}})/\mathbf{Q}_p)$ such that $\sigma_{-1} : \zeta_{p^{n-1}} \to \zeta_{p^{n-1}}^{-1}$. Because $\chi(1+px_0^{-1}y)$ is a p^{n-1}-th root of unity in $\mathbf{Q}_p(\zeta_{p^{n-1}})$, we have

$$h(x_0,\chi)^{\sigma_{-1}} = \sum_{y \bmod p^{n-1}} \chi(1+px_0^{-1}y)^{-1}\zeta_{p^{n-1}}^{-y},$$

hence we see

$$\begin{aligned} h(x_0,\chi)h(x_0,\chi)^{\sigma_{-1}} &= \sum_{y_1 \bmod p^{n-1}} \chi(1+px_0^{-1}y_1)\,\zeta_{p^{n-1}}^{y_1} \sum_{y_2 \bmod p^{n-1}} \chi(1+px_0^{-1}y_2)^{-1}\,\zeta_{p^{n-1}}^{-y_2} \\ &= \sum_{y_1,y_2} \chi\Big(\frac{1+px_0^{-1}y_1}{1+px_0^{-1}y_2}\Big)\,\zeta_{p^{n-1}}^{y_1-y_2} \end{aligned}$$

By changing the variables $y_1 = t + y_2$ we have

$$h(x_0,\chi)h(x_0,\chi)^{\sigma_{-1}} = \sum_{t \bmod p^{n-1}} \zeta_{p^{n-1}}^{t} \sum_{y_2 \bmod p^{n-1}} \chi\Big(1+\frac{px_0^{-1}t}{1+px_0^{-1}y_2}\Big).$$

By the way, we see easily by virtue of the orthogonality relations of characters together with the definition of conductor that

$$\begin{aligned} \sum_{y_2 \bmod p^{n-1}} \chi\Big(1+\frac{px_0^{-1}t}{1+px_0^{-1}y_2}\Big) &= p^{\nu_p(t)+1}\chi(1+px_0^{-1}t) \sum_{c \bmod p^{n-2-\nu_p(t)}} \chi(1+p^{\nu_p(t)+2}c) \\ &= \begin{cases} p^{n-1}\chi(1+px_0^{-1}t) & \text{if } \nu_p(t)+2 \geq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the above $\nu_p(t)$ means the exponential p-adic valuation of t, namely $p^{\nu_p(t)} \parallel t$. Therefore we have

$$\begin{aligned} h(x_0,\chi)h(x_0,\chi)^{\sigma_{-1}} &= \sum_{s=0}^{p-1} \zeta_p^s p^{n-1} \chi(1+p^{n-1}x_0^{-1}s) \\ &= \begin{cases} p^n & \text{if} \quad \chi(1+p^{n-1}) = \zeta_p^{-x_0}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Because it holds that $\nu_p(h(x_0,\chi)) = \nu_p(h(x_0,\chi)^{\sigma_{-1}})$ we conclude the following congruence. There exists a unit $\varepsilon(\chi) \in O_{\mathbf{Q}_p(\zeta_{p^{n-1}})}$ such that

$$g(\chi) \equiv \chi(x_0)\zeta_{p^n}^{x_0}\varepsilon(\chi)(\zeta_p - 1)^{\frac{p-1}{2}n} \pmod{\mathfrak{p}_1^{\frac{p-1}{2}n}\mathfrak{p}_{n-1}}.$$

By making use of the same notation for x_0 as before, namely $x_0 = a_\chi$ we obtain the following main theorem.

THEOREM 2. *For any Dirichlet character χ with conductor p^n (p odd, $n \geq 2$) the Gauss sum $g(\chi)$ satisfies the congruence*

$$g(\chi) \equiv \varepsilon(\chi)\chi(a_\chi)\zeta_{p^n}^{a_\chi}(\zeta_p - 1)^{\frac{p-1}{2}n} \pmod{\mathfrak{p}_1^{\frac{p-1}{2}n}\mathfrak{p}_{n-1}}$$

*for some unit $\varepsilon(\chi)$ in the field $\mathbf{Q}_p(\zeta_{p^{n-1}})$**.

Especially we have $\nu_p(g(\chi)) = \dfrac{n}{2}$.

In the case $p = 2$ we can derive similar congruences, in which we have only to use 2^{n-2} instead of p^{n-1}. In general cases of conductor f we know that any Dirichlet character χ is decomposed into three components, namely the Teichmüller component, character of p-prime conductor, and character of the second kind. According to this decomposition the Gauss sum can be factored in a similar way. Using this fact together with our Theorems 1, 2 we can obtain an information of the p-adic magnitudes of Gauss sums of general type.

*We can also express the result as an equality $g(\chi) = \varepsilon(\chi)\chi(a_\chi)\zeta_{p^n}^{a_\chi}(\zeta_p - 1)^{\frac{p-1}{2}n}$ with some unit $\varepsilon(\chi)$ in $\mathbf{Q}_p(\zeta_{p^{n-1}})$. It seems to be interesting to determine $\varepsilon(\chi)$ exactly.

References

[1] H. Davenport and H. Hasse: Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen, J. reine angew. Math., **17** (1936), 151–182

[2] R. Coleman: The Gross-Koblitz formula, Adv. Stud. in Pure Math., **12** (1987), 21–52

[3] K. Shiratani and M. Ishibashi: On explicit formulas for the norm residue symbol in prime cyclotomic fields, Mem. Fac. Sci. Kyushu Univ., **38** (1984), 203–231

[4] K. Shiratani: An application of p–adic zeta functions to some cyclotomic congruences, to appear

[5] L. C. Washington: Kummer's lemma for prime power cyclotomic fields, J. Number Theory, **40** (1992), 165–173

Katsumi Shiratani
Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812
Japan

Shigenori Uchiyama
Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812
Japan

Xiaowun Pei
Shenyang University of Workers
Shenyang
China