Kernel functions for domains of dimension infinite

本田, 竜広

西原, 賢
九州大学教養部数学科

大貝, 聖子
九州大学教養部数学科

梶原, 壌二
九州大学理学部数学科

他

https://doi.org/10.15017/1449076

出版情報：九州大学教養部数学雑誌. 18 (2), pp.51-64, 1992-12. 九州大学教養部数学教室

Math. Rep.
18-2, 1992.

Kernel functions for domains of dimension infinite

Tatsuhiro HONDA, Joji KAJIWARA, Jong-Jin KIM*, Lin LI,
Masaru NISHIHARA, Seiko OHGAI, Noriko SUGAWARA

Dedicated to Professor Katsumi Shiratani on his sixtieth birthday

(Received June 24, 1992)

0. Introduction

Let D be a subset of a measurable space and $\mathscr{H}$ be a Hilbert space of functions on D. According to the texts S. Hitotumatu [15] or K. Yosida [30], a function $K(z, w)$ is called a *reproducing kernel* of $\mathscr{H}$ if, for any $w \in D$ the function $K(z, w)$ of a variable z belongs to the function space $\mathscr{H}$ and, for any $f \in \mathscr{H}$, the inner product of $f(z)$ and $K(z, w)$ in the Hilbert space $\mathscr{H}$ coincides with $f(w)$. There exists a kernel function if and only if the mapping, which attaches to $f \in \mathscr{H}$ the value $f(z)$ at z, is bounded. According to N. Aronszaijn [1], J. Mercer [22] is the first who characterized the kernel fuction $K(z, w)$ as a function on $D \times D$. S. Bergman [2] and [3] represented kernel functions as the sum $K(z, w) = \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(w)}$ for a complete orthonormal sequence $\{\varphi_j ; j=1, 2, 3, \ldots\}$ of the said function space $\mathscr{H}$. Bergman's kernel function is invariant under biholomorphic mappings. C. Fefferman [7] gave assymptotic expansions of the kernel fuctions of smooth strongly pseudoconvex domains near boundary points and proved smoothness on the boundary of biholomorphic mappings between smooth strongly pseudoconvex domains. Fefferman's results play important roles in the investigation of analytic automorphisms. The first and second anthors [16] want to extend Kodama's results [20] to cases of dimension infinite. This is the reason why the authors go to investigate kernel functions in infinite dimensional domains.

Let $f(z_1, z_2, \ldots, z_n)$ be a holomorphic function in a neighborhood of the origin

---

* Research was supported by KOSEF

in the n-dimensional complex space $\mathbf{C}^n$. The theorem of mean value gives the equality

$$f(0,0,\ldots,0)=\frac{1}{(2\pi)^n}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} f(r_1 e^{i\theta_1}+r_2 e^{i\theta_2}+\ldots+r_n e^{i\theta_n})\,d\theta_1\,d\theta_2\,\ldots\,d\theta_n.$$

By taking the absolute value of the square, multiplying an exponential function and applying Schwarz' inequality, we obtain the inequality

$$|f(0)|^2\leq\frac{1}{(2\pi)^n}\int_{|z_1|<R}\int_{|z_2|<R}\cdots\int_{|z_n|<R}|f(z)|^2\exp\left(-\frac{\sum_{j=1}^{n}|z_j|^2}{2}\right)d\lambda(z_1)\,d\lambda(z_2)\,\ldots\,d\lambda(z_n).$$

A pass to infinite dimensional reproducing kernels has two heights. The first one concerns the infinite dimensional integration and the second one concerns the translation of infinite dimensionalization of the above inequality so that the left hand side is $|f(z)|^2$ instead of $|f(0)|^2$.

Since a^n tends, respectively, to 0 or ∞ as $n\to\infty$ when $|a|<1$ or $|a|>1$, we should let the total measure of each component spaces to be 1.

Let $\Delta=\left\{z\in\mathbf{C}\,;\,|z|<\frac{1}{\sqrt{\pi}}\right\}$ be the disc in the complex plane $\mathbf{C}$. The reproducing kernel for the n-dimensional polydisc Δ^n is represented by

$$K(z,w)=\prod_{j=1}^{n}\frac{1}{(1-\pi z_j\overline{w}_j)^2}.$$

Let $\mathbf{N}$ be the set of positive integers and $\sum_{j\in\mathbf{N}}\mathbf{C}$ be the direct sum of countable copies of $\mathbf{C}$. A suspected reproducing kernel for the domain $(\sum_{j\in\mathbf{N}}\mathbf{C})\cap\Delta^{\mathbf{N}}$ is formally calculated as

$$K(z,w)=\prod_{j\in\mathbf{N}}\frac{1}{(1-\pi z_j\overline{w}_j)^2}$$

and exhibits weakness of the pseudoconvexity of the boundary of the domain in its own way. But we can not introduce σ-additive measure in the domain. In order to let the measure of each whole component spaces be 1, we use the Gaussian measure. Moreover, in order to obtain σ-additive measure and to apply the measure theory, we adopt the theory of abstract Wiener space which is explained, for example, in L. Gross [8], [9], [10] and [11]. For a Hilbert-Schmidt injective self adjoint operator T of a separable Hilbert space $\mathscr{H}$ into $\mathscr{H}$, we regard $(\mathscr{H}, T, \mathscr{H})$ as an *abstract Wiener space*. Early, Cameron-Martin [5] investigated translation $\zeta \to \zeta - z$ and proved that the translation contributes in the majoration of integrations, which give norms, as multiplications of exponential functions of the norm of the parallel vector z of the translation. The measure associated to the translation is absolutely continuous with respect to the original one. Skorohod [29] gave the derivative, i.e., the density function as an exponential function.

We identify a separable Hilbert space $\mathscr{H}$ with the sequence space $\mathscr{L}^2$, consider an abstract Wiener space $(\mathscr{H}, T, \mathscr{H})$, associate with a domain D in $\mathscr{H}$ a function space $\mathscr{A}^2_{\mathrm{G}}(D, \mathrm{d}\mu_T)$, use the results of Skorohod [29], for $z \in D$, derive an estimate which assures the continuity of the mapping, which attaches $f(z) \in \mathbf{C}$ to any $f \in \mathscr{A}^2_{\mathrm{G}}(D, \mathrm{d}\mu_T)$, and shows the existence of the reproducing kernel of the function space $\mathscr{A}^2_{\mathrm{G}}(D, \mathrm{d}\mu_T)$. We regard this function $K(z, w)$ as the *Bergman's kernel function* of D.

The authors would like to express their hearty gratitute to Professor P. Lelong who commanded the second author in Paris to study Gross' results and transfered Infinite Dimensional Complex Analysis to Asia in this way.

1. Gauss-premeasures

Let $\mathscr{H}$ be a separable real Hibert space equipped with the inner product $<,>$ and the norm $\| \ \|$. Let $\mathscr{L}$ be the family of finite dimensional subspaces of $\mathscr{H}$. A system $\{\mu_L ; L \in \mathscr{L}\}$ of measures on $\mathscr{L}$ is called the *Gaussian premeasure* on $\mathscr{H}$ if, for any $L \in \mathscr{L}$ with real dimension n and for any Borel set B of L, which is equipped with the norm $\| \ \|$ induced by the inner product in $\mathscr{H}$ and is regarded as the Euclidean n-space $\mathbf{R}^n$ with the Lebesgue measure $\mathrm{d}\sigma_n$, there holds

$$\mu_L(B) = \int_B \exp\left(-\frac{\| x \|^2}{2}\right) \frac{\mathrm{d}\sigma_n(x)}{(\sqrt{2\pi})^n}.$$

Let T be a Hilbert-Schmidt injective self-adjoint operator on $\mathscr{H}$. For any $L \in \mathscr{L}$, let $L^\perp$ be the *orthogonal complement* of L in $\mathscr{H}$. For any $L \in \mathscr{L}$ and any Borel set B in L, we put $L' = (T^{-1}(L^\perp))^\perp$ and

$$\mu_{T,L}(B)=\mu_{L'}(T^{-1}(B)\cap L').$$

The system $\{\mu_{T,L}(B)\ ;\ L\in\mathscr{L}\}$ of measures on $L\in\mathscr{L}$ defines a Borel measure μ_T on $\mathscr{H}$. The measure μ_T is not invariant under the translation $w\rightarrow w-z$ in $\mathscr{H}$. For any $z\in T\mathscr{H}$ and $w\in\mathscr{H}$, we set

$$\rho_T(z,w)=\exp\left(-\frac{1}{2}(\|T^{-1}z\|^2-2<T^{-1}z,\ T^{-1}w>\right).$$

Skorohod [29] proved that the translated measure $\mu_{T,z}$, defined by $\mu_{T,z}(B)=\mu_T(B-z)$ for any Borel set B of H, is equivalent to the original measure μ_T with the above function $\rho_T(z,w)$ as density, i.e,

$$\frac{d\mu_{T,z}}{d\mu_T}(w)=\rho_T(z,w)$$

for any $z\in T\mathscr{H}$ and $w\in\mathscr{H}$.

In the present paper, we discuss exclusively those in a complex Hilbert space $\mathscr{H}$. In this case, the integration is done in the real underlying space and the density function $\rho_T(z,w)$ is modified as follows:

$$\rho_T(z,w)=\exp\left(-\frac{1}{2}(\|T^{-1}z\|^2-2\mathrm{Re}(<T^{-1}z,T^{-1}w>)\right).$$

We denote by $L^2(\mathscr{H},\mu_T)$ the Hilbert space of classes of functions on $\mathscr{H}$ square integrable with respect to μ_T and introduce results of J.F. Colombeau [6] in the following lemma.

LEMMA 1. *For any $z\in T\mathscr{H}$, the function $\rho_T(z,w)$ belongs to $L^2(\mathscr{H},\mu_T)$ and satisfies*

$$\int_{\mathscr{H}}\rho_T(z,w)^2\,d\mu_T(w)\leq\exp(\|T^{-1}z\|^2).$$

2. Majoration of each values of holomorphic functions by the norm

For any $p\geq 1$, let $\mathscr{L}^p$ be the Banach space of sequences $\{\lambda_n\ ;\ n=1,2,3,\ldots\}$ of complex numbers with $\sum_{n=1}^{\infty}|\lambda_n|^p<\infty$.

Let $\mathscr{H}$ be a separable complex Hilbert space and $T:\mathscr{H}\rightarrow\mathscr{H}$ be a nuclear injective self adjoint operator on $\mathscr{H}$. By the spectral decomposition theorem, there exists an orthonormal basis $\{e_n ; n=1, 2, 3, \ldots\}$ of $\mathscr{H}$ and $\lambda=\{\lambda_n ; n=1, 2, 3, \ldots\} \in \mathscr{L}^1$ such that

$$Tz=\sum_{n=1}^{\infty}\lambda_n<z,\ e_n>e_n.$$

A nuclear operator is a Hilbert-Schmidt operator. For a Hilbert-Schmidt self adjoint operator T on $\mathscr{H}$, we have a similar decomposition too, making an amendment: the above $\{\lambda_n ; n=1, 2, 3, \ldots\} \in \mathscr{L}^2$. Let D be a domain in $\mathscr{H}$, we put $D_T=D\cap T\mathscr{H}$ and $\| z \|_T=\| T^{-1} z \|$ for $z\in D_T$.

Let $\mathscr{H}$ be a separable complex Hilbert space equipped with the inner product $<, >$ and the norm $\| \|$ and D be a domain, that is, connected open set in $\mathscr{H}$. A complex valued function f on D is said to be *Gateaux-holomorphic* if, for any $z\in D$ and any $w\in\mathscr{H}$, $f(z+tw)$ is a holomorphic function of one complex variable t in the open set $\{t\in\mathbf{C} ; z+tw\in D\}$ of the complex plane $\mathbf{C}$. A continuous and Gateaux holomorphic complex valued function in D is said to be *holomorphic* in D. For an oparator T in $\mathscr{H}$, we use the notations $D_T=D\cap T\mathscr{H}$ and $z_T=T^{-1} z$ for $z\in D_T$.

We put $d_D(z)=\inf\{\| z-w \| ; w\notin D\}$ for any $z\in D$ and call it the *boundary distance* of z with respect to the domain D. Let A be a subset of D. We put $d_D(A)=\inf\{d_D(z) ; z\in A\}$ and call it the *distance of* A from the boundary of D. A is said to be *D-bounded* if A is bounded in $\mathscr{H}$ with norm $\| \|$ and the distance $d_D(A)$ is positive. A complex valued function on D is said to be *D-bounded* if it is bounded on any D-bounded subset of D. Let a be a point of $\mathscr{H}$ and δ be a positive number. The open ball with center a and radius δ is denoted by $\mathrm{B}(a, \delta)$.

We consider the Hilbert space $\mathscr{L}^2$. For any positive integer n, let $\{\mathrm{e}_n\}$ be the sequence of numbers such that its m-th number is 0 for $m\neq n$ and the n-th number is 1. Let $\lambda=\{\lambda_n ; n=1, 2, 3, \ldots\}$ be an element of $\mathscr{L}^1$ with $\lambda_n>0$ for $n=1, 2, 3, \ldots$. We define an injective self-adjoint nuclear mapping $T: \mathscr{L}^2 \rightarrow \mathscr{L}^2$ by putting

$$T(\sum_{n=1}^{\infty} z_n\, \mathrm{e}_n)=\sum_{n=1}^{\infty}\lambda_n\, z_n\, \mathrm{e}_n.$$

Let D be a domain of $\mathscr{L}^2$. We consider the domain $D_T=D\cap T\mathscr{L}^2$ and denote by $\mathrm{L}^2(D, \mathrm{d}\mu_T)$ the Hilbert space of classes of square integrable functions with respect to $\mathrm{d}\mu_T$. Since $\mu_T(\mathscr{L}^2-T\mathscr{L}^2)=0$, the measure μ_T naturally induces a measure on D_T and there holds

$$\int_D f \mathrm{d}\mu_T = \int_{D_T} f | D_T \,\mathrm{d}\mu_T$$

for any μ_T-integrable function on D. In other words, the Hilbert space $\mathrm{L}^2(D_T, \mathrm{d}\mu_T)$ coincides with the set of restrictions $f | D_T$ of all f belonging to $\mathrm{L}^2(D, \mathrm{d}\mu_T)$. Let $\mathscr{H}_\mathrm{b}(D)$ be the space of functions which are holomorphic on D and are bounded on any D-bounded subset of D. For any positive integer p, we put

$$D_p = \left\{ z \in D \,;\, \| z \| \leq p, \ \ \mathrm{d}_D(z) \geq \frac{1}{p} \right\}$$

and

$$\| f \|_{D_p} = \sup \{ | f(z) | \,;\, z \in D_p \}$$

for any $f \in \mathscr{H}_\mathrm{b}(D)$. The family $\{ \| \, \|_{D_p} \,;\, p = 1, 2, 3, \ldots \}$ of seminorms defines the structure of a Fréchet space on the vector space $\mathscr{H}_\mathrm{b}(D)$. We denote by $\mathscr{H}_\mathrm{b}(D_T)$ the set of restrictions $f | D_T$ of all $f \in \mathscr{H}_\mathrm{b}(D)$, by $\mathscr{H}^2_\mathrm{b}(D_T, \mathrm{d}\mu_T)$ the intersection of $\mathscr{H}_\mathrm{b}(D_T)$ and $\mathrm{L}^2(D_T, \mathrm{d}\mu_T)$, and by $\mathscr{A}^2_\mathrm{b}(D_T, \mathrm{d}\mu_T)$ the closure of the linear subspace $\mathscr{H}^2_\mathrm{b}(D_T, \mathrm{d}\mu_T)$ in the Hilbert space $\mathrm{L}^2(D_T, \mathrm{d}\mu_T)$. We also put

$$\| f \|_{D_T, \mathrm{d}\mu_T} = \sqrt{\int_{D_T} | f(z) |^2 \mathrm{d}\mu_T(z)}$$

for $f \in \mathscr{A}^2_\mathrm{b}(D_T, \mathrm{d}\mu_T)$.

Under the above notations, we give the following estimations:

LEMMA 2. *For any point $z_0 \in D_T$, there exists a positive constant $C = C(z_0)$ such that*

$$| f(z_0) | \leq C \exp\left(\frac{1}{2} \| T^{-1} z_0 \|^2 \right) \| f \|_{D_T, \mathrm{d}\mu_T}$$

for any $f \in \mathscr{A}^2_\mathrm{b}(D_T, \mathrm{d}\mu_T)$.

PROOF. It suffices to show the lemma for $f \in \mathscr{H}^2_\mathrm{b}(D_T, \mathrm{d}\mu_T)$. Then there exists $\mathrm{F} \in \mathscr{H}_\mathrm{b}(D)$ whose restriction $F | D_T$ on D_T coincides with f. We put $r = d_D(z_0)/2$. Let n be any positive integer, $\langle \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n \rangle$ be the n-dimensional subspace of $\mathscr{H}$ spanned by $\mathrm{e}_1, \mathrm{e}_2, \ldots$ and e_n. Let $\mathrm{P}_n : \mathscr{H} \to \langle \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n \rangle$ be the canonical projection. Since $\mathrm{P}_n(\mathrm{B}(0, r)) \subset \mathrm{B}(0, r)$ and since F is bounded on $\mathrm{B}(z_0, r)$, open ball with center z_0 and radius r, $\{F(z_0+\mathrm{P}_n z)\}$ is a uniform bounded sequence of holomorphic functions on $\mathrm{B}(0, r)$. For any positive integer n and two n-tuples of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $r_1, r_2, \ldots, r_n$ with $\lambda_1^2 r_1^2+\lambda_2^2 r_2^2+\ldots+\lambda_n^2 r_n^2 \leq r^2$, by the theorem of mean value of n-complex variables, we have

$$F(z_0)=\frac{1}{(2\pi)^n}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} F(z_0+\lambda_1 r_1 \mathrm{e}^{\mathrm{i}\theta_1}+\lambda_2 r_2 \mathrm{e}^{\mathrm{i}\theta_2}+\ldots+\lambda_n r_n \mathrm{e}^{\mathrm{i}\theta_n})\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n,$$

by multiplication and integration of both sides of which, we have the following majoration concerning the integral over the positive quadrant of the real n-dimensional ellipsoid

$$\mathrm{E}(\lambda_1, \lambda_2, \ldots, \lambda_n)=\{(r_1, r_2, \ldots, r_n)\in\mathbf{R}^n ;\ \sum_{j=1}^{n}\lambda_j^2 r_j^2 \leq r^2,\ r_1, r_2, \ldots, r_n>0\} :$$

$$\int_{\mathrm{E}(\lambda_1,\lambda_2,\cdots,\lambda_n)} F(z_0)\, r_1 r_2 \ldots r_n \exp\Big(-\frac{1}{2}\sum_{j=1}^{n} r_j^2\Big)\mathrm{d}r_1\, \mathrm{d}r_2 \ldots \mathrm{d}r_n=$$

$$\int_{\mathrm{E}(\lambda_1,\lambda_2,\cdots\lambda_n)}\Big(\frac{1}{(2\pi)^n}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} F(z_0+\lambda_1 r_1 \mathrm{e}^{\mathrm{i}\theta_1}+\lambda_2 r_2 \mathrm{e}^{\mathrm{i}\theta_2}+\ldots+\lambda_n r_n \mathrm{e}^{\mathrm{i}\theta_n})$$

$$\mathrm{d}\theta_1\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n\Big) r_1 r_2 \ldots r_n \exp\Big(-\frac{1}{2}\sum_{j=1}^{n} r_j^2\Big)\mathrm{d}r_1\, \mathrm{d}r_2 \ldots \mathrm{d}r_n.$$

Introducing n complex variables $z_j=x_j+\mathrm{i}y_j$ $(j=1, 2, \ldots, n)$, we see that both sides of the above equalities coincide with the integration over the quadrant of the real $2n$-dimensional ellipsoid

$$\mathrm{C}_n(\lambda_1, \lambda_2, \ldots, \lambda_n)=\{(z_1, z_2, \ldots, z_n)\in\mathbf{C}^n ;\ \sum_{j=1}^{n}\lambda_j^2 | z_j |^2 \leq r^2\}$$

and we define the corresponding infinite dimensional one

$$\mathrm{C}_\infty(\lambda)=\{\sum_{n=1}^{\infty} z_n \mathrm{e}_n ;\ \sum_{j=1}^{\infty}\lambda_j^2 | z_j |^2 \leq r^2\}.$$

We regard the left hand side as $F(z_0)$ times the positive constant

$$\mathrm{L}_n(\lambda_1, \lambda_2, \ldots, \lambda_n, r)=\int_{\mathrm{E}(\lambda_1,\lambda_2,\cdots,\lambda_n)} r_1 r_2 \ldots r_n \exp\Big(-\frac{1}{2}\sum_{j=1}^{n} r_j^2\Big)\mathrm{d}r_1\, \mathrm{d}r_2 \ldots \mathrm{d}r_n$$

and we obtain the following inequality:

$$| F(z_0)|\, \mathrm{L}_n(\lambda_1, \lambda_2, \ldots, \lambda_n, r) \leq \frac{1}{(2\pi)^n}\int_{\mathrm{C}n(\lambda_1,\lambda_2,\cdots,\lambda_n)} F(z_0+\sum_{j=1}^{n}\lambda_1 z_j \mathrm{e}_j)\, \mathrm{d}x_1\, \mathrm{d}y_1\, \mathrm{d}x_2\, \mathrm{d}y_2 \ldots \mathrm{d}x_n\, \mathrm{d}y_n.$$

We introduce the infinite dimensional ellipsoid

$$C_{\infty-n}(\lambda)=\{\sum_{j=n+1}^{\infty} z_j e_j \in <e_1, e_2, \ldots, e_n>^{\perp}; \sum_{j=n+1}^{\infty} |\lambda_j z_j|^2 \leq r^2\}$$

and

$$C_{n,\infty-n}=\{\sum_{j=1}^{\infty} z_j e_j \in \mathcal{L}^2; \sum_{j=1}^{n} |\lambda_j z_j|^2 \leq r^2, \sum_{j=n+1}^{\infty} |\lambda_j z_j|^2 \leq r^2\}$$

too. Then the right hand side of the above inequality coincides with the left hand side of the following inequality:

$$\frac{1}{(2\pi)^n}\Big(\int_{C_n(\lambda_1,\lambda_2,\cdots\lambda_n)} \Big| F(z_0+\sum_{j=1}^{n}\lambda_j z_j e_j)\Big| \exp\Big(-\frac{1}{2}\sum_{j=1}^{\infty}|z_j|^2 dx_1 dy_1 dx_2 dy_2 \ldots dx_n$$

$$dy_n\Big)=$$

$$\Big(\frac{1}{(2\pi)^n}\int_{C_n(\lambda_1,\lambda_2,\cdots,\lambda_n)} \Big| F(z_0+\sum_{j=1}^{n}\lambda_j z_j e_j)\Big| \exp\Big(-\frac{1}{2}\sum_{j=1}^{\infty}|z_j|^2 dx_1 dy_1 dx_2 dy_2 \ldots$$

$$dx_n dy_n \times$$

$$\int_{C_{\infty-n}(\lambda_1,\lambda_2,\cdots,\lambda_n)} d\mu_T |<e_1, e_2, \ldots, e_n>^{\perp}\Big)/\int_{C_{\infty-n}(\lambda_1,\lambda_2,\cdots,\lambda_n)} d\mu_T |<e_1, e_2, \ldots, e_n>^{\perp}$$

$$=\int_{C_{n,\infty-n}(\lambda_1,\lambda_2,\cdots,\lambda_n)} |F(z_0+P_n z)|\, d\mu_T(z)/\int_{C_{\infty-n}(\lambda_1,\lambda_2,\cdots,\lambda_n)} d\mu_T |<e_1, e_2, \ldots, e_n>^{\perp} \leq$$

$$\int_{B(0,\sqrt{2}r)\cap T(\mathcal{L}^2)} |F(z_0+P_n z)|\, d\mu_T(z)/\int_{C_{\infty-n}(\lambda_1,\lambda_2,\cdots,\lambda_n)} d\mu_T |<e_1, e_2, \ldots, e_n>^{\perp}.$$

For a positive integer j, we also consider the real two-dimensional ellipse $E(0, r)=\{(x_j, y_j)\in \mathbf{R}^2; |x_j|^2+|y_j|^2\leq r^2\}$. Since $\{\sum_{j=n+1}^{\infty} z_j e_j \in <e_1, e_2, \ldots, e_n>^{\perp}; |z_j| \leq r/\sqrt{|\lambda_j| \|\lambda\|_1}\} \subset C_{\infty-n}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have the minoration

$$\int_{C_{\infty-n}(\lambda_1,\lambda_2,\cdots,\lambda_n)} d\mu_T |<e_1, e_2, \ldots, e_n>^{\perp} \geq \prod_{j=n+1}^{\infty}\Big(\frac{1}{2\pi}\int_{E(\lambda, r/\sqrt{(|\lambda_j| \|\lambda\|_1)}} dx_j dy_j\Big)$$

$$=\prod_{j=n+1}^{\infty}\Big(1-\exp\Big(-\frac{r^2}{|\lambda| \|\lambda\|_1}\Big)\Big)>0$$

because there holds $\exp\Big(-\frac{r^2}{|\lambda_j| \|\lambda\|_1}\Big)\leq \frac{|\lambda_j| \|\lambda\|_1}{r^2}$ for any positive integer j and hence $\Big\{\exp\Big(-\frac{r^2}{|\lambda_j| \|\lambda\|_1}\Big); j=1, 2, 3, \ldots\Big\} \in \mathcal{L}^1$. Since $\Big\{(r_1, r_2, \ldots, r_n) \in \mathbf{R}^n; 0\leq \frac{r}{|\lambda_j| \|\lambda\|_1}\Big\} \subset E(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have

$$L_n(\lambda_1, \lambda_2, \ldots, r) \geq \prod_{j=1}^{n}\int_0^{\frac{r}{(|\lambda_j| \|\lambda\|_1)}} r_j \exp\Big(-\frac{r_j^2}{2}\Big) dr_j \geq \prod_{j=1}^{\infty}\Big(1-\exp\Big(-\frac{r^2}{|\lambda_j| \|\lambda\|_1}\Big)\Big).$$

For

$$C(r)=\prod_{j=1}^{\infty}\left(1-\frac{r^2}{|\lambda_j|\,\|\lambda\|_1}\right)^{-2},$$

we have

$$|F(z_0)|\leq C(r)\int_{\mathrm{B}(0,r)}|F(z_0+\mathrm{P}_n z)|\,\mathrm{d}\mu_T(z).$$

For

$M=\sup\{|F(z_0+\mathrm{P}_n z)|\,;z\in\mathrm{B}(0,r)\}$, any positive number r and any $z\in$ $\mathrm{B}(0,r)$, by Lebesgue's dominated convergence theorem, we have

$$|F(z_0)|\leq C(r)\int_{\mathrm{B}(0,r)}|F(z_0+z)|\,\mathrm{d}\mu_T(z).$$

By Lemma 1 and Schwarz' Lemma, we have

$$\int_{\mathrm{B}(0,r)}|F(z_0+z)|\,\mathrm{d}\mu_T(z)=\int_{\mathrm{B}(z_0,r)}|F(z)|\,\mathrm{d}\mu_{T,z_0}(z)=$$

$$\int_{\mathrm{B}(z_0,r)}|F(z)|\frac{\mathrm{d}\mu_{T,z_0}}{\mathrm{d}\mu_T}\mathrm{d}\mu_T(z)=\int_{\mathrm{B}(z_0,r)}|F(z)|\,\rho_T(z_0,z)\,\mathrm{d}\mu_T(z)\leq$$

$$\int_D|F(z)|\,\rho_T(z_0,z)\,\mathrm{d}\mu_T(z)\leq\sqrt{\int_D|F(z)|^2d\mu_T(z)}\ \sqrt{\int_D\rho_T(z,z_0)^2\,\mathrm{d}\mu_T(z)}$$

$$=\sqrt{\int_{D_T}|f(z)|^2\,\mathrm{d}\mu_T(z)}\ \exp\left(\frac{\|T^{-1}z_0\|^2}{2}\right).$$

We define a norm $\|\ \|_T$ on $T(\mathscr{L}^2)$ by putting

$$\|z\|_T=\|T^{-1}z\|$$

for $z\in T^{-1}(\mathscr{L}^2)$. The mapping T of the Hilbert space $\mathscr{L}^2$ equipped with the norm $\|\ \|$ onto the Hilbert space $T(\mathscr{L}^2)$ equipped with the norm $\|\ \|_T$ is isometric. D_T is a domain in the Hilbert space $(T(\mathscr{L}^2),\|\ \|_T)$. Let $\mathscr{H}(D_T)$ be the space of holomorphic functions on the domain D_T. By Lemma 2, for any Cauchy sequence $\{f_p\,;p=1,2,3,\ldots\}$ of $\mathscr{H}^2_{\mathrm{b}}(D_T,d\mu_T)$, there exists a function $f(z)$ on D_T such that the sequence $\{f_p(z)\,;p=1,\ 2,\ 3,\ \ldots\}$ converges to $f(z)$ uniformly on the neighborhood $\{\|z-z_0\|_T<\mathrm{d}_D(z_0)/4\}$ of any point z_0 of D_T. Hence the limit function $f(z)$ is holomorphic on D_T. The closure $\mathscr{A}^2_{\mathrm{b}}(D_T,\mathrm{d}\mu_T)$ of $\mathscr{H}^2_{\mathrm{b}}(D_T,\mathrm{d}\mu_T)$ in $\mathrm{L}^2(D_T,d\mu_T)$ is contained in $\mathscr{H}_{\mathrm{b}}(D_T)$.

LEMMA 3. *The function space* $\mathscr{A}^2_{\mathrm{b}}(D_T,,\mathrm{d}\mu_T)$ *has a reproducing kernel* $K(z,w)$ *and it is separables.*

PROOF. By Lemma 2 the function space $\mathscr{A}^2_b(D_T, d\mu_T)$ has a reproducing kernel $K(z, w)$. Since D_T is separable, there exists a countable dence subset $\{w_\nu ; \nu=1, 2, 3, \dots\}$ of D_T. Let f be a function of $\mathscr{A}^2_b(D_T, d\mu_T)$ orthogonal to the countable subset $\{K(z, w_\nu) ; \nu=1, 2, 3, \dots\}$ of $\mathscr{A}^2_b(D_T, d\mu_T)$. By the reproducing property, we have

$$f(w_\nu) = \int_{D_T} f(z) \overline{K(z, w_\nu)} d\mu_T(z) = 0$$

for $\nu \geq 1$. Since f is continuous and $\{w_\nu ; \nu \geq 1\}$ is dense in D_T, f is identically zero in D_T. This means that the closure of the linear span of the countable set $\{K(z, w_\nu) ; \nu \geq 1\}$ coincides with the Hilbert space $\mathscr{A}^2_b(D_T, d\mu_T)$.

We summerize the above results in the following theorem.

THEOREM 1. *Let D be a domain in $\mathscr{L}^2$, $\lambda = (\lambda_j)$ be an element of $\mathscr{L}^1$ with $\lambda_j > 0$ for $j=1, 2, 3, \dots$ and $T: \mathscr{L}^2 \to \mathscr{L}^2$ be the operator defined by $T(\sum_{j=1}^{\infty} z_j e_j) = \sum_{j=1}^{\infty} \lambda_j z_j e_j$ for $\sum_{j=1}^{\infty} z_j e_j \in \mathscr{L}^2$. Let $\mathscr{H}_b(D_T)$ be the space of restrictions on $D_T = D \cap T(\mathscr{L}^2)$ of functions holomorphic on D and bounded on any D-bounded subset of D. Let $\mathscr{A}^2_b(D_T, d\mu_T)$ be the closure in $L^2(D_T, d\mu_T)$ of $\mathscr{H}_b(D_T) \cap L^2(D_T, d\mu_T)$. Then the Hilbert space $\mathscr{A}^2_b(D_T, d\mu_T)$ has a reproducing kernel $K(z, w)$. It is separable and embedded canonically in the Fréchet space $\mathscr{H}_b(D_T)$ of holomorphic functions in D_T.*

$$K(z, w) = \sum_{\nu=1}^{\infty} \varphi_\nu(z) \overline{\varphi_\nu(w)}$$

for any $(z, w) \in D_T \times D_T$.

3. Reproducing kernel of Reinhardt domains

A domain D in $\mathscr{L}^2$ is called a *Reinhardt domain* if for any $z = (z_j) \in D$ and $\theta = (\theta_j) \in \mathbf{R}^\infty$, $e^{i\theta} z = (e^{i\theta_j} z_j) \in D$. A domain D in $\mathscr{L}^2$ is said to be *pseudoconvex* if, for any finite dimensional linear subspace F of $\mathscr{L}^2$, $D \cap F$ is pseudoconvex in the space F of dimension finite. By a theorem P. Lelong [21] and S. Hitotumatu [14], D is pseudoconvex if, for any two dimensional linear subspace F of $\mathscr{L}^2$, $D \cap F$ is pseodoconvex in the space $F \cong \mathbf{C}^2$.

For $m \leq n$, let $\pi_{n,m} : \mathbf{C}^n \to \mathbf{C}^n$ be the projection defined by $\pi_{n,m}$

$(z_1, z_2, \ldots, z_n) = (z_1, z_2, \ldots, z_m, 0, 0, \ldots, 0) \in \mathbf{C}^n$ for any $(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$.

LEMMA 4. *Let m and n be integers with $1 \leq m \leq n$ and D be a pseudoconvex Reinhardt domain in $\mathbf{C}^n$ containing the origin. Then $\pi_{n,m}(D) \subset D$.*

PROOF. Since the domain of holomorphy D is complete and logarithmically convex, the projection $\pi_{n,m}(D)$ is contained in D.

For non negative integer m, let $\pi_{\infty,m} : \mathscr{L}^2 \longrightarrow \mathscr{L}^2$ be the projection defined by $\pi_{\infty,m}(\sum_{k=1}^{\infty} z_k \mathrm{e}_k) = \sum_{k=1}^{m} z_k \mathrm{e}_k \in \mathscr{L}^2$ for any $\sum_{k=1}^{\infty} z_k \mathrm{e}_k \in \mathscr{L}^2$.

LEMMA 5. *Let m be a positive integer and D be a pseudoconvex Reinhardt domain in $\mathscr{L}^2$ containing the origin. Then $\pi_{\infty,m}(D) \subset D$.*

PROOF. For any integers m and n with $m \leq n$, we put $\tilde{\pi}_{n,m}(\sum_{k=1}^{n} z_k \mathrm{e}_k) = \sum_{k=1}^{m} z_k \mathrm{e}_k \in \mathscr{L}^2$ for $\sum_{k=1}^{n} z_k \mathrm{e}_k \in \langle \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n \rangle$. By Lemma 4, $\pi_{\infty,m}(D) = \cup_{\nu \geq m} \tilde{\pi}_{\nu,m}$ $(\mathrm{D} \cap \langle \mathrm{e}_1, \mathrm{e}_2, \ldots, , \mathrm{e}_\nu \rangle) \subset D$.

LEMMA 6. *Let D be a pseudoconvex Reinhardt domain D containing the origin in $\mathscr{L}^2$. Then the linear span $\mathscr{M}$ of all monomials $a_\alpha z^\alpha = a_{\alpha_1 \alpha_2 \cdots \alpha_l} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_l^{\alpha_l}$ $(l \geq 0)$ is dense in $\mathscr{A}^2_{\mathfrak{b}}(D_T, \mathrm{d}\mu_T)$.*

PROOF. Let f be a function belonging to $\mathscr{H}^2_{\mathfrak{b}}(D_T, \mathrm{d}\mu_T)$ and

$$f(z) = \sum a_{\alpha_1 \alpha_2 \cdots \alpha_l} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_l^{\alpha_l}$$

be its Taylor expansion in the l-dimensional complete Reinhardt domain $D \cap \langle \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_l \rangle$ containing the origin.

For any positive integer p, we put

$$D_p = \left\{ z \in D \, ; \| z \|_2 < p, \ \mathrm{d}_D(z) > \frac{1}{p} \right\}.$$

The monotonically non-decreasing sequence $\{D_p \, ; p = 1, 2, 3, \ldots, \}$ of pseudoconvex domains containing the origin converges to D and we have

$$\int_D f(z)\overline{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_l^{\alpha_l}}\, d\mu_T =$$

$$\lim_{p\to\infty}\int_{D_p} f(z)\overline{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_l^{\alpha_l}}\, d\mu_T =$$

$$\lim_{p\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\int_{\pi\infty,m(D_p)} f\ (\lambda_1 z_1, \lambda_2 z_2, \dots, \lambda_m z_m, 0, 0, \dots, 0, \dots)$$

$$\overline{\lambda_1^{\alpha_1} z_1^{\alpha_1} \lambda_2^{\alpha_2} z_2^{\alpha_2} \dots \lambda_m^{\alpha_m} z_m^{\alpha_m}}\,\frac{1}{(2\pi i)^k}\exp\Bigl(-\frac{\sum_{j=1}^{k}|z_j|^2}{2}\Bigr)dx_1\, dy_1\, dx_2\, dy_2 \dots dx_k\, dy_k =$$

$$\lim_{p\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\int_{(r_1,r_2,\cdots,r_k,0,0,\cdots,0)\in\pi_{\infty,m}(D_p)}\prod_{j=1}^{k} r_j\, dr_1\, dr_2 \dots dr_k\frac{1}{(2\pi i)^k}$$

$$\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}(\sum_\beta a_{\beta_1\beta_2\cdots\beta_l}(\lambda_1 r_1)^{\alpha_1+\beta_1}(\lambda_2 r_2)^{\alpha_2+\beta_2}\dots\ (\lambda_l r_l)^{\alpha_l+\beta_l}$$

$$\exp\Bigl(-\frac{\sum_{j=1}^{k} r_j^2}{2}+i(\sum_{j=1}^{l}(\beta_j-\alpha_j)+\beta_{l+1}\,\theta_{l+1}+\beta_{l+2}\,\theta_{l+2}+\dots+\beta_m\,\theta_m)\Bigr)d\theta_1\, d\theta_2\ \dots\ d\theta_k$$

$$=\lim_{p\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\int_{(r_1,r_2,\cdots,r_k,0,0,\cdots,0)\in\pi_{\infty,m}(D_p),r_j\geq 0} a_{\alpha_1\alpha_2\cdots\alpha_l}(\lambda_1 r_1)^{2\alpha_1}(\lambda_2 r_2)^{2\alpha_2}\dots$$

$$(\lambda_l r_l)^{2\alpha_l}\exp\Bigl(-\frac{\sum_{j=1}^{k} r_j^2}{2}\Bigr)\prod_{j=1}^{k} r_j\, dr_1\, dr_2\ \dots\ dr_k =$$

$$a_{\alpha_1\alpha_2\cdots\alpha_l}\int_D |z^\alpha|^2\, d\mu_T.$$

Hence we have the formula

$$a_{\alpha_1\alpha_2\cdots\alpha_l}=\frac{\int_D f(z)\overline{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_l^{\alpha_l}}\, d\mu_T}{\int_D |z^\alpha|^2\, d\mu_T}$$

which completes the proof of the lemma.

THEOREM 2. *Let D be a pseudoconvex Reinhardt domain containing the origin in the Hilbert space $\mathscr{L}^2$. Then the function $K(z, w)$ on $D_T\times D_T$ defined by*

$$K(z, w)=\sum_\alpha\frac{z^\alpha\,\overline{w^\alpha}}{\int_D |z^\alpha|^2\, d\mu_T}$$

for any $(z, w)\in D_T\times D_T$ is the reproducing kernel of the function space $\mathscr{A}_b^2(D_T, d\mu_T)$.

References

[1] N. ARONSZAIJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337-404.

[2] S. BERGMAN, Sur les fonctions orthogonales de plusieurs variables complexes, avec les application à la théorie des fonctions analytiques, Mémoriales des Sciences Math. **106** (1947), Gauthies-Villars (Paris).

[3] S. BERGMAN, Sur la fonction-noyaus d'un domaine et ses applications dans la théorie des transformations pseudo-conformes, Mémoriales des Sciences Math. **108** (1948), Gauthies-Villars (Paris).

[4] H.J. BREMERMANN, *Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen*, Math. Ann. **128** (1954), 68-91.

[5] R.H. CAMERON-W.T. MARTIN, *Transformation of Wiener Integrals under translations*, Annals of Math. **45**-1 (1944), 386-396.

[6] J.F. COLOMBEAU, Differential Calculus and Holomorphy, North-Holland Math. Series, **64** (1982).

[7] C. FEFFERMAN, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Inventiones math. **26** (1974), 1-65.

[8] L. GROSS, Harmonic analysis on Hilbert spaces, Mem. Amer. Math. Soc. **46**, 1963.

[9] L. GROSS, *Abstract Wiener spaces*, in Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability **2**-1 (1965), 31-42.

[10] L. GROSS, *Potential theory on Hilbert space*, J. Functional Analysis **1** (1967), 123-181.

[11] L. GROSS, *Abstract Wiener measure and infinite dimensional potential theory in Lectures in Modern Analysis and Applications II.*, Lecture-Notes in Mathematics, New York (Springer) **140** (1970), 84-116.

[12] L. GRUMAN, *The Levi problem in certain infinite dimension vector spaces*, Illinois J. Math., **18** (1974), 20-26.

[13] M. HERVÉ, Analyticity in Infinite Dimensional Spaces, de Gruyter Studies in Mathematics **10** (1989).

[14] S. HITOTUMATU, *On some conjectures concerning pseudo-convex domains*, J. Math. Jap. **6** (1954), 177-195.

[15] S. HITOTUMATU, Theory of analytic functions of several complex variables, Baihukan 1960.

[16] T. HONDA and J. KAJIWARA, Sur un théorème de type d'Akio KODAMA, Mem. Fac. Sci. Kyushu Univ. **44**-1 (1990),61-66.

[17] J.M. ISSIDRO and L.L. STACHO, Holomorphic automorphism Groups in Banach spaces, North-Holland Mathematics Studies **105** (1984).

[18] J. KAJIWARA, *Opérateur d″ dans les espace de Hilbert avec croissance polynomiale*, Lectur Note in Mathematics (Springer) 474, Séminaire Pière Lelong (Analyse) Annee 1973/74, 91-108.

[19] J. KAJIWARA, M. NISHIHARA, S.OHGAI and N. SUGAWARA, *Uniform approximation of continuous functions and* C^∞*-functions in nuclear spaces by entire functions*, Portugaliae Math. **48**-3 (1991), 327-344.

[20] A. KODAMA, *Characterizations of certain weakly pseudoconvex domains* E (k, l), Tohoku Math. J. **40**-3 (1988), 343-365.

[21] P. LELONG, *Domaines convexes par rapport aux fonctions plurisousharmoniques*, J. d'Analyse Math. **2** (1952), 178-208.

[22] J. MERCER, *Functions of positive and negative type and their connection with the theory of integral equations*, Philos. Trans. Roy. Soc. London Ser. A **209** (1909), 415-446.

[23] F. NORGUET, *Sur les domaines d'holomorphie des fonctions uniformes de plusieurs variables complexes*, Bull. Soc. France 82, 139-159.

[24] K. OKA, *Sur les fonctions analytiques de plusieurs variables complexes:* IX Domaines finis sans point critique intérieur, Jap. J. Math. **23** (1953), 97-155.

[25] M. SCHOTTENLOHER, *Analytic continuation and regular classes in locally convex Hausdorff spaces*, Portugal. Math. **33** (1974), 219-250.

[26] M. SCHOTTENLOHER, *The Levi problem for domains spread over locally convex spaces with a finite dimensional Schauder decomposition*, Ann. Inst. Fourier **26** (1976), 207-237.
[27] M. SCHOTTENLOHER, *Michael problem and algebras of holomorphic functions*, Arch. Math. **37** (1981), 241-247.
[28] M. SCHOTTENLOHER, *Spectrum and envelope of holomorphy for infinite dimensional Riemann domains*, Math. Ann. **263** (1983), 213-219.
[29] A.V. SKOROHOD, Integration in Hilbert spaces, Erg. der Math. 79, Springer Verlag (1974).
[30] K. YOSIDA, Theory of Hilbert spaces, Kyoritu Zensho 49 (1953).
[31] K. YOSIDA, Functional Analysis, Springer Verlag Forth Edition (1974).

Address

Tatsuhiro HONDA: Ariake College of Technology, Ômuta, 836 Japan
Joji KAJIWARA: Department of Mathematics, Faculty of Science, Kyushu University 33, Fukoka, 812 Japan
Jong-Jin KIM: Department of Mathematics, Jeon-Bug National University, Jeon-Ju 560-756, Republic of Korea
Lin LI: Department of Mathematics, Jinan Television University, Jinan, Shandon, People's Republic of China
Masaru NISHIHARA: Departmant of Mathematics, Fukuoka Institute of Technology, Fukuoka, 811-02 Japan
Seiko OHGAI: Meijigakuen High School, Sensuicho 5-1, Tobata-ku, Kitakyushu, 804 Japan
Noriko SUGAWARA: Department of Mathematics, Fukuoka Institute of Technology, Fukuoka, 811-02 Japan

Facsimile to Professor Kajiwara: Japan(=81)-092-632-2737