Kernel functions for domains of dimension infinite

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Kernel functions for domains of dimension infinite

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Dedicated to Professor Katsumi Shiratani on his sixtieth birthday
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0. Introduction

Let $D$ be a subset of a measurable space and $\mathcal{H}$ be a Hilbert space of functions on $D$. According to the texts S. Hitotumatu [15] or K. Yosida [30], a function $K(z, w)$ is called a reproducing kernel of $\mathcal{H}$ if, for any $w \in D$ the function $K(z, w)$ of a variable $z$ belongs to the function space $\mathcal{H}$ and, for any $f \in \mathcal{H}$, the inner product of $f(z)$ and $K(z, w)$ in the Hilbert space $\mathcal{H}$ coincides with $f(w)$. There exists a kernel function if and only if the mapping, which attaches to $f \in \mathcal{H}$ the value $f(z)$ at $z$, is bounded. According to N. Aronszajn [1], J. Mercer [22] is the first who characterized the kernel function $K(z, w)$ as a function on $D \times D$. S. Bergman [2] and [3] represented kernel functions as the sum $K(z, w) = \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(w)}$ for a complete orthonormal sequence $\{\varphi_j; j=1, 2, 3, \ldots\}$ of the said function space $\mathcal{H}$. Bergman's kernel function is invariant under biholomorphic mappings. C. Fefferman [7] gave asymptotic expansions of the kernel functions of smooth strongly pseudoconvex domains near boundary points and proved smoothness on the boundary of biholomorphic mappings between smooth strongly pseudoconvex domains. Fefferman's results play important roles in the investigation of analytic automorphisms. The first and second authors [16] want to extend Kodama's results [20] to cases of dimension infinite. This is the reason why the authors go to investigate kernel functions in infinite dimensional domains.

Let $f(z_1, z_2, \ldots, z_n)$ be a holomorphic function in a neighborhood of the origin
in the n-dimensional complex space $\mathbb{C}^n$. The theorem of mean value gives the equality
\[
 f(0, ..., 0) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f(r_1 e^{i\theta_1} + r_2 e^{i\theta_2} + \cdots + r_n e^{i\theta_n}) \, d\theta_1 \, d\theta_2 \cdots d\theta_n.
\]

By taking the absolute value of the square, multiplying an exponential function and applying Schwarz' inequality, we obtain the inequality
\[
|f(0)|^2 \leq \frac{1}{(2\pi)^n} \int_{|z_1|<R} \int_{|z_2|<R} \cdots \int_{|z_n|<R} |f(z)|^2 \exp \left( -\frac{\sum_{j=1}^{n} |z_j|^2}{2} \right) \, d\lambda(z_1) \, d\lambda(z_2) \cdots d\lambda(z_n).
\]

A pass to infinite dimensional reproducing kernels has two heights. The first one concerns the infinite dimensional integration and the second one concerns the translation of infinite dimensionalization of the above inequality so that the left hand side is $|f(z)|^2$ instead of $|f(0)|^2$.

Since $a^n$ tends, respectively, to 0 or $\infty$ as $n \to \infty$ when $|a| < 1$ or $|a| > 1$, we should let the total measure of each component spaces to be 1.

Let $\Delta = \{z \in \mathbb{C} : |z| < \frac{1}{\sqrt{n}}\}$ be the disc in the complex plane $\mathbb{C}$. The reproducing kernel for the n-dimensional polydisc $\Delta^n$ is represented by
\[
 K(z, w) = \prod_{j=1}^{n} \frac{1}{(1 - \pi z_j \bar{w}_j)^2}.
\]

Let $\mathbb{N}$ be the set of positive integers and $\sum_{j \in \mathbb{N}} \mathbb{C}$ be the direct sum of countable copies of $\mathbb{C}$. A suspected reproducing kernel for the domain $(\sum_{j \in \mathbb{N}} \mathbb{C}) \cap \Delta^n$ is formally calculated as
\[
 K(z, w) = \prod_{j \in \mathbb{N}} \frac{1}{(1 - \pi z_j \bar{w}_j)^2}
\]
and exhibits weakness of the pseudoconvexity of the boundary of the domain in its own way. But we cannot introduce $\sigma$-additive measure in the domain. In order to let the measure of each whole component spaces be 1, we use the Gaussian measure. Moreover, in order to obtain $\sigma$-additive measure and to apply the measure theory, we adopt the theory of abstract Wiener space which is
exemplified, for example, in L. Gross [8], [9], [10] and [11]. For a Hilbert-Schmidt injective self adjoint operator $T$ of a separable Hilbert space $\mathcal{H}$ into $\mathcal{H}$, we regard $(\mathcal{H}, T, \mathcal{H})$ as an abstract Wiener space. Early, Cameron-Martin [5] investigated translation $\zeta \mapsto \zeta - z$ and proved that the translation contributes in the majoration of integrals, which give norms, as multiplications of exponential functions of the norm of the parallel vector $z$ of the translation. The measure associated to the translation is absolutely continuous with respect to the original one. Skorohod [29] gave the derivative, i.e., the density function as an exponential function.

We identify a separable Hilbert space $\mathcal{H}$ with the sequence space $\ell^2$, consider an abstract Wiener space $(\mathcal{H}, T, \mathcal{H})$, associate with a domain $D$ in $\mathcal{H}$ a function space $\mathcal{A}_d^2(D, d\mu_T)$, use the results of Skorohod [29], for $z \in D$, derive an estimate which assures the continuity of the mapping, which attaches $f(z) \in \mathcal{C}$ to any $f \in \mathcal{A}_d^2(D, d\mu_T)$, and shows the existence of the reproducing kernel of the function space $\mathcal{A}_d^2(D, d\mu_T)$. We regard this function $K(z, w)$ as the Bergman's kernel function of $D$.

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1. Gauss-premeasures

Let $\mathcal{H}$ be a separable real Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $\mathcal{L}$ be the family of finite dimensional subspaces of $\mathcal{H}$. A system $(\mu_L ; L \in \mathcal{L})$ of measures on $\mathcal{L}$ is called the Gaussian premeasure on $\mathcal{H}$ if, for any $L \in \mathcal{L}$ with real dimension $n$ and for any Borel set $B$ of $L$, which is equipped with the norm $\| \cdot \|$ induced by the inner product in $\mathcal{H}$ and is regarded as the Euclidean $n$-space $\mathbb{R}^n$ with the Lebesgue measure $d\sigma_n$, there holds

$$\mu_L(B) = \int_B \exp\left( -\frac{\| x \|^2}{2} \right) \, d\sigma_n(x).$$

Let $T$ be a Hilbert-Schmidt injective self-adjoint operator on $\mathcal{H}$. For any $L \in \mathcal{L}$, let $L^\perp$ be the orthogonal complement of $L$ in $\mathcal{H}$. For any $L \in \mathcal{L}$ and any Borel set $B$ in $L$, we put $L' = (T^{-1}(L^\perp))^\perp$ and
The system \( \{ \mu_{TL}(B) \mid L \in \mathcal{L} \} \) of measures on \( L \in \mathcal{L} \) defines a Borel measure \( \mu_T \) on \( \mathcal{H} \). The measure \( \mu_T \) is not invariant under the translation \( w \to w - z \) in \( \mathcal{H} \). For any \( z \in T \mathcal{H} \) and \( w \in \mathcal{H} \), we set
\[
\rho_T(z, w) = \exp\left( -\frac{1}{2} \langle T^{-1} z, T^{-1} w \rangle \right).
\]
Skorohod [29] proved that the translated measure \( \mu_{T,z} \) defined by \( \mu_{T,z}(B) = \mu_T(B - z) \) for any Borel set \( B \) of \( H \), is equivalent to the original measure \( \mu_T \) with the above function \( \rho_T(z, w) \) as density, i.e.,
\[
\frac{d\mu_{T,z}}{d\mu_T}(w) = \rho_T(z, w)
\]
for any \( z \in T \mathcal{H} \) and \( w \in \mathcal{H} \).

In the present paper, we discuss exclusively those in a complex Hilbert space \( \mathcal{H} \). In this case, the integration is done in the real underlying space and the density function \( \rho_T(z, w) \) is modified as follows:
\[
\rho_T(z, w) = \exp\left( -\frac{1}{2} \langle T^{-1} z, T^{-1} w \rangle \right).
\]
We denote by \( L^2(\mathcal{H}, \mu_T) \) the Hilbert space of classes of functions on \( \mathcal{H} \) square integrable with respect to \( \mu_T \) and introduce results of J.F. Colombeau [6] in the following lemma.

**Lemma 1.** For any \( z \in T \mathcal{H} \), the function \( \rho_T(z, w) \) belongs to \( L^2(\mathcal{H}, \mu_T) \) and satisfies
\[
\int_{\mathcal{H}} \rho_T(z, w)^2 \, d\mu_T(w) \leq \exp(\|T^{-1} z\|^2).
\]

2. **Majoration of each values of holomorphic functions by the norm**

For any \( p \geq 1 \), let \( \mathcal{L}^p \) be the Banach space of sequences \( \{ \lambda_n ; n = 1, 2, 3, \ldots \} \) of complex numbers with \( \sum_{n=1}^{\infty} |\lambda_n|^p < \infty \).

Let \( \mathcal{H} \) be a separable complex Hilbert space and \( T : \mathcal{H} \to \mathcal{H} \) be a nuclear injective self adjoint operator on \( \mathcal{H} \). By the spectral decomposition theorem,
there exists an orthonormal basis \( \{ e_n ; n = 1, 2, 3, \ldots \} \) of \( \mathcal{H} \) and \( \lambda = \{ \lambda_n ; n = 1, 2, 3, \ldots \} \in \mathbb{L}^1 \) such that

\[
Tz = \sum_{n=1}^{\infty} \lambda_n < z, e_n > e_n.
\]

A nuclear operator is a Hilbert-Schmidt operator. For a Hilbert-Schmidt self adjoint operator \( T \) on \( \mathcal{H} \), we have a similar decomposition too, making an amendment: the above \( \{ \lambda_n ; n = 1, 2, 3, \ldots \} \in \mathbb{L}^2 \). Let \( D \) be a domain in \( \mathcal{H} \), we put \( D_T = D \cap T\mathcal{H} \) and \( \| z \|_T = \| T^{-1} z \| \) for \( z \in D_T \).

Let \( \mathcal{H} \) be a separable complex Hilbert space equipped with the inner product \( <, > \) and the norm \( \| \| \) and \( D \) be a domain, that is, connected open set in \( \mathcal{H} \). A complex valued function \( f \) on \( D \) is said to be Gateaux-holomorphic if, for any \( z \in D \) and any \( w \in \mathcal{H} \), \( f(z + tw) \) is a holomorphic function of one complex variable \( t \) in the open set \( \{ t \in \mathbb{C} ; z + tw \in D \} \) of the complex plane \( \mathbb{C} \). A continuous and Gateaux holomorphic complex valued function in \( D \) is said to be holomorphic in \( D \).

For an operator \( T \) in \( \mathcal{H} \), we use the notations \( D_T = D \cap T\mathcal{H} \) and \( z_T = T^{-1} z \) for \( z \in D_T \).

We put \( d_\partial(z) = \inf \{ \| z - w \| ; w \notin D \} \) for any \( z \in D \) and call it the boundary distance of \( z \) with respect to the domain \( D \). Let \( A \) be a subset of \( D \). We put \( d_\partial(A) = \inf \{ d_\partial(z) ; z \in A \} \) and call it the distance of \( A \) from the boundary of \( D \). \( A \) is said to be \( D \)-bounded if \( A \) is bounded in \( \mathcal{H} \) with norm \( \| \| \) and the distance \( d_\partial(A) \) is positive. A complex valued function on \( D \) is said to be \( D \)-bounded if it is bounded on any \( D \)-bounded subset of \( D \). Let \( a \) be a point of \( \mathcal{H} \) and \( \delta \) be a positive number. The open ball with center \( a \) and radius \( \delta \) is denoted by \( B(a, \delta) \).

We consider the Hilbert space \( \mathbb{L}^2 \). For any positive integer \( n \), let \( \{ e_n \} \) be the sequence of numbers such that its \( m \)-th number is 0 for \( m \neq n \) and the \( n \)-th number is 1. Let \( \lambda = \{ \lambda_n ; n = 1, 2, 3, \ldots \} \) be an element of \( \mathbb{L}^1 \) with \( \lambda_n > 0 \) for \( n = 1, 2, 3, \ldots \). We define an injective self-adjoint nuclear mapping \( T : \mathbb{L}^2 \to \mathbb{L}^2 \) by putting

\[
T(\sum_{n=1}^{\infty} z_n e_n) = \sum_{n=1}^{\infty} \lambda_n z_n e_n.
\]

Let \( D \) be a domain of \( \mathbb{L}^2 \). We consider the domain \( D_T = D \cap T\mathbb{L}^2 \) and denote by \( L^2(D, d\mu_T) \) the Hilbert space of classes of square integrable functions with respect to \( d\mu_T \). Since \( \mu_T(\mathbb{L}^2 - T\mathbb{L}^2) = 0 \), the measure \( \mu_T \) naturally induces a
measure on $D_T$ and there holds
\[ \int_B f \, d\mu_T = \int_{D_T} f \, |D_T| \, d\mu_T \]
for any $\mu_T$-integrable function on $D$. In other words, the Hilbert space $L^2(D_T, d\mu_T)$ coincides with the set of restrictions $f|D_T$ of all $f$ belonging to $L^2(D, d\mu_T)$. Let $\mathcal{H}_b(D)$ be the space of functions which are holomorphic on $D$ and are bounded on any $D$-bounded subset of $D$. For any positive integer $p$, we put
\[ D_p = \{ z \in D ; \| z \| \leq p, \quad d_0(z) \geq \frac{1}{p} \} \]
and
\[ \| f \|_{D_p} = \sup \{ |f(z)| ; z \in D_p \} \]
for any $f \in \mathcal{H}_b(D)$. The family $\{ \| \|_{D_p} ; p = 1, 2, 3, \ldots \}$ of seminorms defines the structure of a Fréchet space on the vector space $\mathcal{H}_b(D)$. We denote by $\mathcal{H}_b(D_T)$ the set of restrictions $f|D_T$ of all $f \in \mathcal{H}_b(D)$, by $\mathcal{H}_b^2(D_T, d\mu_T)$ the intersection of $\mathcal{H}_b(D_T)$ and $L^2(D_T, d\mu_T)$, and by $\mathcal{H}_b^2(D_T, d\mu_T)$ the closure of the linear subspace $\mathcal{H}_b^2(D_T, d\mu_T)$ in the Hilbert space $L^2(D_T, d\mu_T)$. We also put
\[ \| f \|_{D_T, d\mu_T} = \sqrt{\int_{D_T} |f(z)|^2 \, d\mu_T(z)} \]
for $f \in \mathcal{H}_b^2(D_T, d\mu_T)$.

Under the above notations, we give the following estimations:

**Lemma 2.** For any point $z_0 \in D_T$, there exists a positive constant $C = C(z_0)$ such that
\[ |f(z_0)| \leq C \exp \left( \frac{1}{2} \| T^{-1} z_0 \|^2 \right) \| f \|_{D_T, d\mu_T} \]
for any $f \in \mathcal{H}_b^2(D_T, d\mu_T)$.

**Proof.** It suffices to show the lemma for $f \in \mathcal{H}_b(D_T, d\mu_T)$. Then there exists $F \in \mathcal{H}_b(D)$ whose restriction $F|D_T$ on $D_T$ coincides with $f$. We put $r = d_0(z_0)/2$. Let $n$ be any positive integer, $\langle e_1, e_2, \ldots, e_n \rangle$ be the $n$-dimensional subspace of $\mathcal{H}$ spanned by $e_1, e_2, \ldots, e_n$. Let $P_n : \mathcal{H} \to \langle e_1, e_2, \ldots, e_n \rangle$ be the canonical projection. Since $P_n(B(0, r)) \subset B(0, r)$ and since $F$ is bounded on $B(z_0, r)$.
open ball with center $z_0$ and radius $r$, \( \{F(z_0 + P_n z)\} \) is a uniform bounded sequence of holomorphic functions on $B(0, r)$. For any positive integer $n$ and two $n$-tuples of real numbers $\lambda_1, \lambda_2, ..., \lambda_n$ and $r_1, r_2, ..., r_n$ with $\lambda_1^2 r_1^2 + \lambda_2^2 r_2^2 + ... + \lambda_n^2 r_n^2 \leq r^2$, by the theorem of mean value of $n$-complex variables, we have

\[
F(z_0) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} F(z_0 + \lambda_1 r_1 e^{i\theta_1} + \lambda_2 r_2 e^{i\theta_2} + ... + \lambda_n r_n e^{i\theta_n}) d\theta_1 d\theta_2 \cdots d\theta_n.
\]

by multiplication and integration of both sides of which, we have the following majoration concerning the integral over the positive quadrant of the real $n$-dimensional ellipsoid

\[
E(\lambda_1, \lambda_2, ..., \lambda_n, r) = \{ (r_1, r_2, ..., r_n) \in \mathbb{R}^n; \sum_{j=1}^{n} \lambda_j^2 r_j^2 \leq r^2, \ r_1, r_2, ..., r_n > 0 \}:
\]

\[
\int_{E(\lambda_1, \lambda_2, ..., \lambda_n, r)} F(z_0) r_1 r_2 \cdots r_n \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \lambda_j^2 r_j^2\right) dr_1 dr_2 \cdots dr_n =
\]

\[
\int_{E(\lambda_1, \lambda_2, ..., \lambda_n, r)} \left(\frac{1}{(2\pi)^n}\right) \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} F(z_0 + \lambda_1 r_1 e^{i\theta_1} + \lambda_2 r_2 e^{i\theta_2} + ... + \lambda_n r_n e^{i\theta_n})
\]

\[
d\theta_1 d\theta_2 \cdots d\theta_n r_1 r_2 \cdots r_n \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \lambda_j^2 r_j^2\right) dr_1 dr_2 \cdots dr_n.
\]

Introducing $n$ complex variables $z_j = x_j + iy_j$ ($j=1, 2, ..., n$), we see that both sides of the above equalities coincide with the integration over the quadrant of the real $2n$-dimensional ellipsoid

\[
C_n(\lambda_1, \lambda_2, ..., \lambda_n) = \{ (z_1, z_2, ..., z_n) \in \mathbb{C}^n; \sum_{j=1}^{n} \lambda_j^2 |z_j|^2 \leq r^2 \}
\]

and we define the corresponding infinite dimensional one

\[
C_\infty(\lambda) = \{ \sum_{n=1}^{\infty} z_n e_n; \sum_{j=1}^{\infty} \lambda_j^2 |z_j|^2 \leq r^2 \}.
\]

We regard the left hand side as $F(z_0)$ times the positive constant

\[
L_n(\lambda_1, \lambda_2, ..., \lambda_n, r) = \int_{E(\lambda_1, \lambda_2, ..., \lambda_n, r)} r_1 r_2 \cdots r_n \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \lambda_j^2 r_j^2\right) dr_1 dr_2 \cdots dr_n
\]

and we obtain the following inequality:

\[
|F(z_0)| L_n(\lambda_1, \lambda_2, ..., \lambda_n, r) \leq \frac{1}{(2\pi)^n} \int_{C_n(\lambda_1, \lambda_2, ..., \lambda_n)} F(z_0 + \sum_{j=1}^{n} \lambda_j z_j e_j) dx_1 dy_1
\]

\[
dx_2 dy_2 \cdots dx_n dy_n.
\]

We introduce the infinite dimensional ellipsoid
\[ C_{n,-n}(\lambda) = \left\{ \sum_{j=n+1}^{\infty} z_j e_j \in \mathbb{C}^n \mid \sum_{j=n+1}^{\infty} |\lambda_j z_j|^2 \leq r^2 \right\} \]

and

\[ C_{n,-n} = \left\{ \sum_{j=n+1}^{\infty} z_j e_j \in \mathbb{C}^n \mid \sum_{j=n+1}^{\infty} |\lambda_j z_j|^2 \leq r^2 \right\} \]

too. Then the right hand side of the above inequality coincides with the left hand side of the following inequality:

\[
\frac{1}{(2\pi)^n} \left( \int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} \left| F(z_0 + \sum_{j=1}^{n} \lambda_j z_j e_j) \right| \exp\left( -\frac{1}{2} \sum_{j=1}^{\infty} |z_j|^2 \right) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \, ... \, dx_n \, dy_n \right)
\]

\[
\left( \frac{1}{(2\pi)^n} \int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} \left| F(z_0 + \sum_{j=1}^{n} \lambda_j z_j e_j) \right| \exp\left( -\frac{1}{2} \sum_{j=1}^{\infty} |z_j|^2 \right) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \, ... \, dx_n \, dy_n \right) \times
\]

\[
\int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} d\mu_T \left| e_1, e_2, ..., e_n > \right| \int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} d\mu_T \left| e_1, e_2, ..., e_n > \right|
\]

\[
= \int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} d\mu_T \left| e_1, e_2, ..., e_n > \right| \int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} d\mu_T \left| e_1, e_2, ..., e_n > \right|
\]

For a positive integer \( j \), we also consider the real two-dimensional ellipse

\[ E(0, r) = \{(x_1, y_1) \in \mathbb{R}^2 \mid |x_1|^2 + |y_1|^2 \leq r^2 \}. \]

Since \( \left\{ \sum_{j=n+1}^{\infty} z_j e_j \in \mathbb{C}^n \mid |z_j| \leq r/\sqrt{\lambda_j \|\lambda\|_1} \right\} \subset C_{n,-n} (\lambda_1, \lambda_2, ..., \lambda_n) \), we have the minoration

\[
\int_{C_{n,-n}(\lambda_1, \lambda_2, ..., \lambda_n)} d\mu_T \left| e_1, e_2, ..., e_n > \right| \geq \prod_{j=n+1}^{\infty} \left( \frac{1}{2\pi} \int_{E(0, r) \cap \mathbb{R}^1} dx_j \, dy_j \right)
\]

\[
= \prod_{j=n+1}^{\infty} \left( 1 - \exp\left( -\frac{r^2}{\lambda_j \|\lambda\|_1} \right) \right) > 0
\]

because there holds \( \exp\left( -\frac{r^2}{\lambda_j \|\lambda\|_1} \right) \leq \frac{1}{\lambda_j \|\lambda\|_1} \) for any positive integer \( j \)

and hence \( \left\{ \exp\left( -\frac{r^2}{\lambda_j \|\lambda\|_1} \right) ; j = 1, 2, 3, ... \right\} \in \mathbb{L}^1 \). Since \( \{(r_1, r_2, ..., r_n) \in \mathbb{R}^n ; 0 \leq \frac{r}{\lambda_j \|\lambda\|_1} \} \subset E(\lambda_1, \lambda_2, ..., \lambda_n) \), we have

\[
L_n(\lambda_1, \lambda_2, ..., r) \geq \prod_{j=1}^{n} \int_0^{\|\lambda_j\|_1} r_j \exp\left( -\frac{r_j^2}{2\lambda_j \|\lambda\|_1} \right) dr_j \geq \prod_{j=1}^{\infty} \left( 1 - \exp\left( -\frac{r^2}{2\lambda_j \|\lambda\|_1} \right) \right).
\]
For
\[ C(r) = \prod_{j=1}^{\infty} \left(1 - \frac{r^2}{\lambda_j \| \lambda \|} \right)^{-2}. \]
we have
\[ |F(z_0)| \leq C(r) \int_{B(0,r)} |F(z_0 + P_n z)| \, d\mu_T(z). \]

For any positive number \( r \) and any \( z \in B(0,r) \), by Lebesgue's dominated convergence theorem, we have
\[ |F(z_0)| \leq C(r) \int_{B(0,r)} |F(z_0 + z)| \, d\mu_T(z). \]

By Lemma 1 and Schwarz' Lemma, we have
\[ \int_{B(0,r)} |F(z_0 + z)| \, d\mu_T(z) = \int_{B(z_0,r)} |F(z)| \, d\mu_{T,z_0}(z) = \]
\[ \int_{B(z_0,r)} |F(z)| \frac{d\mu_{T,z_0}}{d\mu_T}(z) = \int_{B(z_0,r)} |F(z)| \rho_T(z_0, z) \, d\mu_T(z) \leq \]
\[ \int_D |F(z)| \rho_T(z_0, z) \, d\mu_T(z) \leq \sqrt{\int_D |F(z)|^2 \, d\mu_T(z)} \sqrt{\int_D \rho_T(z_0, z)^2 \, d\mu_T(z)} = \]
\[ = \sqrt{\int_D |f(z)|^2 \, d\mu_T(z)} \exp \left( \frac{\| T^{-1} z_0 \|^2}{2} \right). \]

We define a norm \( \| \cdot \|_T \) on \( T(\mathcal{L}^2) \) by putting
\[ \| z \|_T = \| T^{-1} z \| \]
for \( z \in T^{-1}(\mathcal{L}^2) \). The mapping \( T \) of the Hilbert space \( \mathcal{L}^2 \) equipped with the norm \( \| \cdot \| \) onto the Hilbert space \( T(\mathcal{L}^2) \) equipped with the norm \( \| \cdot \|_T \) is isometric. \( D_T \) is a domain in the Hilbert space \( (T(\mathcal{L}^2), \| \cdot \|_T) \). Let \( \mathcal{H}(D_T) \) be the space of holomorphic functions on the domain \( D_T \). By Lemma 2, for any Cauchy sequence \( \{ f_p ; p = 1, 2, 3, \ldots \} \) of \( \mathcal{H}_b(D_T, d\mu_T) \), there exists a function \( f(z) \) on \( D_T \) such that the sequence \( \{ f_p(z) ; p = 1, 2, 3, \ldots \} \) converges to \( f(z) \) uniformly on the neighborhood \( \{ \| z - z_0 \|_T < |d_0(z_0)/4| \} \) of any point \( z_0 \) of \( D_T \). Hence the limit function \( f(z) \) is holomorphic on \( D_T \). The closure \( \mathcal{A}_b(D_T, d\mu_T) \) of \( \mathcal{H}_b(D_T, d\mu_T) \) in \( L^2(D_T, d\mu_T) \) is contained in \( \mathcal{H}_b(D_T) \).

**Lemma 3.** The function space \( \mathcal{A}_b(D_T, d\mu_T) \) has a reproducing kernel \( K(z, w) \) and it is separable.
PROOF. By Lemma 2 the function space \( \mathcal{H}_b^2(D_T, d\mu_T) \) has a reproducing kernel \( K(z, w) \). Since \( D_T \) is separable, there exists a countable dense subset \( \{w_v; \nu=1, 2, 3, \ldots\} \) of \( D_T \). Let \( f \) be a function of \( \mathcal{H}_b^2(D_T, d\mu_T) \) orthogonal to the countable subset \( \{K(z, w_v); \nu=1, 2, 3, \ldots\} \) of \( \mathcal{H}_b^2(D_T, d\mu_T) \). By the reproducing property, we have

\[
f(w_v) = \int_{D_T} f(z) \overline{K(z, w_v)} d\mu_T(z) = 0
\]

for \( \nu \geq 1 \). Since \( f \) is continuous and \( \{w_v; \nu \geq 1\} \) is dense in \( D_T \), \( f \) is identically zero in \( D_T \). This means that the closure of the linear span of the countable set \( \{K(z, w_v); \nu \geq 1\} \) coincides with the Hilbert space \( \mathcal{H}_b^2(D_T, d\mu_T) \).

We summarize the above results in the following theorem.

**Theorem 1.** Let \( D \) be a domain in \( \mathbb{L}^2 \), \( \lambda = (\lambda_j) \) be an element of \( \mathbb{L}^1 \) with \( \lambda_j > 0 \) for \( j=1, 2, 3, \ldots \) and \( T: \mathbb{L}^2 \rightarrow \mathbb{L}^2 \) be the operator defined by \( T(\sum_{j=1}^{\infty} z_j e_j) = \sum_{j=1}^{\infty} \lambda_j z_j e_j \) for \( \sum_{j=1}^{\infty} z_j e_j \in \mathbb{L}^2 \). Let \( \mathcal{H}_b(D_T) \) be the space of restrictions on \( D_T = D \cap T(\mathbb{L}^2) \) of functions holomorphic on \( D \) and bounded on any \( D \)-bounded subset of \( D \). Let \( \mathcal{H}_b^2(D_T, d\mu_T) \) be the closure in \( L^2(D_T, d\mu_T) \) of \( \mathcal{H}_b(D_T) \cap L^2(D_T, d\mu_T) \). Then the Hilbert space \( \mathcal{H}_b^2(D_T, d\mu_T) \) has a reproducing kernel \( K(z, w) \). It is separable and embedded canonically in the Fréchet space \( \mathcal{H}_b(D_T) \) of holomorphic functions in \( D_T \).

\[
K(z, w) = \sum_{\nu=1}^{\infty} \varphi_\nu(z) \overline{\varphi_\nu(w)}
\]

for any \((z, w) \in D_T \times D_T \).

### 3. Reproducing kernel of Reinhardt domains

A domain \( D \) in \( \mathbb{L}^2 \) is called a **Reinhardt domain** if for any \( z = (z_j) \in D \) and \( \theta = (\theta_j) \in \mathbb{R}^n \), \( e^{i\theta} z = (e^{i\theta_j} z_j) \in D \). A domain \( D \) in \( \mathbb{L}^2 \) is said to be **pseudoconvex** if, for any finite dimensional linear subspace \( F \) of \( \mathbb{L}^2 \), \( D \cap F \) is pseudoconvex in the space \( F \) of dimension finite. By a theorem P. Lelong [21] and S. Hitotumatu [14], \( D \) is pseudoconvex if, for any two dimensional linear subspace \( F \) of \( \mathbb{L}^2 \), \( D \cap F \) is pseudoconvex in the space \( F \cong \mathbb{C}^2 \).

For \( m \leq n \), let \( \pi_{n,m}: \mathbb{C}^n \rightarrow \mathbb{C}^m \) be the projection defined by \( \pi_{n,m} \).
\begin{align*}
(z_1, z_2, ..., z_n) &= (z_1, z_2, ..., z_m, 0, 0, ..., 0) \in \mathbb{C}^n \text{ for any } (z_1, z_2, ..., z_n) \in \mathbb{C}^n.
\end{align*}

**Lemma 4.** Let \( m \) and \( n \) be integers with \( 1 \leq m \leq n \) and \( D \) be a pseudoconvex Reinhardt domain in \( \mathbb{C}^n \) containing the origin. Then \( \pi_{n,m}(D) \subset D \).

**Proof.** Since the domain of holomorphy \( D \) is complete and logarithmically convex, the projection \( \pi_{n,m}(D) \) is contained in \( D \).

For non negative integer \( m \), let \( \pi_{n,m} : \mathcal{L}^2 \to \mathcal{L}^2 \) be the projection defined by

\[
\pi_{n,m}(\sum_{k=1}^n z_k e_k) = \sum_{k=1}^m z_k e_k \in \mathcal{L}^2 \quad \text{for any} \quad \sum_{k=1}^m z_k e_k \in \mathcal{L}^2.
\]

**Lemma 5.** Let \( m \) be a positive integer and \( D \) be a pseudoconvex Reinhardt domain in \( \mathcal{L}^2 \) containing the origin. Then \( \pi_{n,m}(D) \subset D \).

**Proof.** For any integers \( m \) and \( n \) with \( m \leq n \), we put \( \pi_{n,m}(\sum_{k=1}^n z_k e_k) = \sum_{k=1}^m z_k e_k \in \mathcal{L}^2 \) for \( \sum_{k=1}^n z_k e_k \in \langle e_1, e_2, ..., e_n \rangle \). By Lemma 4, \( \pi_{n,m}(D) = \bigcup_{\nu \geq m} \pi_{\nu,m}(D \cap \langle e_1, e_2, ..., e_n \rangle) \subset D \).

**Lemma 6.** Let \( D \) be a pseudoconvex Reinhardt domain containing the origin in \( \mathcal{L}^2 \). Then the linear span \( \mathcal{M} \) of all monomials \( a^a z_1^{a_1} z_2^{a_2} \cdots z_l^{a_l} \) \((l \geq 0)\) is dense in \( \mathcal{A}_T(\mathcal{L}^2, d\mu_T) \).

**Proof.** Let \( f \) be a function belonging to \( \mathcal{H}_T(\mathcal{L}^2, d\mu_T) \) and

\[
f(z) = \sum a_{a_1a_2...a_l} z_1^{a_1} z_2^{a_2} \cdots z_l^{a_l}
\]

be its Taylor expansion in the \( l \)-dimensional complete Reinhardt domain \( D \cap \langle e_1, e_2, ..., e_l \rangle \) containing the origin.

For any positive integer \( p \), we put

\[
D_p = \left\{ z \in D : \|z\|_2 < p, \ d_p(z) > \frac{1}{p} \right\}.
\]

The monotonically non-decreasing sequence \( \{D_p : p = 1, 2, 3, ... \} \) of pseudoconvex domains containing the origin converges to \( D \) and we have
\[
\int_D f(z) z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \, d\mu_T = \\
\lim_{p \to \infty} \int_{D_p} f(z) z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \, d\mu_T = \\
\lim_{p \to \infty} \lim_{m \to \infty} \lim_{k \to \infty} \int_{\pi^{a_m}(D_p)} f(\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_m z_m, 0, 0, \ldots, 0, \ldots) \\
\frac{1}{(2\pi)^k} \exp \left( -\frac{\sum_{j=1}^{k} |z_j|^2}{2} \right) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \, \ldots \, dx_k \, dy_k = \\
\lim_{p \to \infty} \lim_{m \to \infty} \lim_{k \to \infty} \int_{(r_1, r_2, \ldots, r_k, 0, 0, \ldots, 0) \in \pi^{a_m}(D_p), r_i > 0} \prod_{j=1}^{k} r_j \, dr_1 \, dr_2 \, \ldots \, dr_k \frac{1}{(2\pi)^k} \\
\int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \sum_{\beta} a_{\beta_1, \beta_2, \ldots, \beta_k}(\lambda_1 r_1)^{a_{11}+\beta_1} (\lambda_2 r_2)^{a_{22}+\beta_2} \cdots (\lambda_k r_k)^{a_{kk}+\beta_k} \right) \, d\theta_1 \, d\theta_2 \, \ldots \, d\theta_k \\
\exp \left( -\frac{\sum_{j=1}^{k} r_j^2}{2} + i \left( \sum_{j=1}^{k} (\beta_j - \alpha_j) + \beta_{i+1} \theta_{i+1} + \beta_{i+2} \theta_{i+2} + \ldots + \beta_m \theta_m \right) \right) \, d\theta_1 \, d\theta_2 \, \ldots \, d\theta_k = \\
\lim_{p \to \infty} \lim_{m \to \infty} \lim_{k \to \infty} \int_{(r_1, r_2, \ldots, r_k, 0, 0, \ldots, 0) \in \pi^{a_m}(D_p), r_i > 0} a_{a_1, a_2, \ldots, a_k}(\lambda_1 r_1)^{2a_1} (\lambda_2 r_2)^{2a_2} \cdots \\
(\lambda_1 r_1)^{2\alpha_1} \exp \left( -\frac{\sum_{j=1}^{k} r_j^2}{2} \right) \prod_{j=1}^{k} r_j \, dr_1 \, dr_2 \, \ldots \, dr_k = \\
a_{a_1, a_2, \ldots, a_k} \int_D |z^\alpha|^2 \, d\mu_T.
\]

Hence we have the formula
\[
a_{a_1, a_2, \ldots, a_k} = \frac{\int_D f(z) z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \, d\mu_T}{\int_D |z^\alpha|^2 \, d\mu_T}
\]

which completes the proof of the lemma.

**Theorem 2.** Let \( D \) be a pseudoconvex Reinhardt domain containing the origin in the Hilbert space \( \mathcal{L}^2 \). Then the function \( K(z, w) \) on \( D_T \times D_T \) defined by
\[
K(z, w) = \sum_\alpha \frac{z^\alpha \overline{w^\alpha}}{\alpha! \int_0^\infty |z^\alpha|^2 \, d\mu_T}
\]
for any \( (z, w) \in D_T \times D_T \) is the reproducing kernel of the function space \( \mathcal{A}^2_\delta(D_T, d\mu_T) \).
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