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Generic Torelli theorem for some hypersurfaces of the flag veriety

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Introduction

This is a continuation of a study of the generic Torelli problem for hypersurfaces of compact homogeneous Kähler manifolds, which we began in [**7**]. In this paper, we consider hypersurfaces of the simplest kind in the flag variety or the multiple lines $(\mathbf{P}^1)^N$.

Let $\mathbf{F}=\mathbf{F}_n$ denote the flag variety consisting of all flags $\ell=(\ell_1, \cdots, \ell_n)$ with $\ell_i \subset \mathbf{P}^n$ a linear subspace of dimension $i-1$, $1 \leq i \leq n$. As is well-known, we can find the generators of Pic $(\mathbf{F}_n)$ naturally associated with the fundamental weights of Lie algebra of type A_n. Hence, Pic $(\mathbf{F}_n) \simeq \mathbf{Z}^n$, and the linear equivalence class of a hypersurface X can be determined by an n-ple of integers $(d_1, \cdots, d_n)$. When all the d_i are the same, say d, we call X a hypersurface of type (d), and put $O_{\mathbf{F}}(d):=O_{\mathbf{F}}(X)$.

As for such hypersurfaces, we show the following:

THEOREM A. *Let M_d be the space parametrizing all the nonsingular hypersurfaces of type (d) of the flag variety $\mathbf{F}_n$ modulo* Aut $(\mathbf{F}_n, O_{\mathbf{F}}(d))$. *Let $P: M_d \rightarrow \Gamma\backslash D$ be the period map associated to the middle cohomology. Then P is generically injective in the following cases:*

(1) *d is odd and, $d \geq n+3$ or $(d, n)=(3, 2)$.*

(2) *d is even and $d \geq$* $\max(6, 2[(n+1)/2]+2)$, *where $[q]$ denotes the greatest integer not exceeding $q \in \mathbf{Q}$.*

Recall that a nonsingular hypersurface of type (3) of $\mathbf{F}_2$ appeared in the classification of canonical surfaces with $c_1^2=3p_g-6$ ([**9**, §4]). We know from [**9**, §4] that another kind of canonical surfaces with the same numerical characters $p_g=8$, $K^2=18$ can be realized as hypersurfaces of multidegree

(3, 3, 3) of $(\mathbf{P}^1)^3$. Therefore, in view of Theorem A, it is natural to ask whether the generic Torelli theorem also holds for such surfaces. The answer is affirmative by the following:

THEOREM B. *Let M_d denote the space parametrizing all the nonsingular hypersurfaces of nulti-degree $(d, d, \cdots, d)$ of $(\mathbf{P}^1)^N$, $N \geq 3$, modulo* $\mathrm{Aut}((\mathbf{P}^1)^N, O(d, \cdots, d))$. *Let $P: M_d \to \Gamma\backslash D$ be the period map associated to the middle cohomology. Then P is generically injective provided that $d \geq 5$ or $d = 3$.*

Our essential tool is the theory of the infinitesimal variation of Hodge structure developed in [**1**] etc., especially the Symmetrizer Lemma invented by Donagi [**3**] (see also [**6**]). Almost all the materials which we shall need for the proof can be found in our previous papers [**7**] and [**8**]. The infinitesimal Torelli theorem for general hypersurfaces of $\mathbf{F}_n$ can be found in [**11**, 3.2].

1. Preliminaries

In this section, we recall fundamental properties of the flag variety which can be found in [**10**] or [**11**].

NOTATION 1.1. Consider the complex simple Lie algebra of type A_n. We fix a Cartan subalgebra and a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ of the root system Φ. Then we have a decomposition of Φ into positive and negative roots: $\Phi = \Phi^+ \cup \Phi^-$. More precisely, we have

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j} : 1 \leq i \leq n, 0 \leq j, i+j \leq n\}. \tag{1.1}$$

We denote by $(\cdot, \cdot)$ an inner product induced by the Killing form. We can normalize it so that $(\alpha, \alpha) = 2$ holds for all $\alpha \in \Phi$. Moreover, we have for all Cartan integers

$$(\alpha, \beta) \in \{0, \pm 1\} \qquad \text{if } \alpha \neq \beta \in \Phi^+.$$

Let $\lambda_1, \cdots, \lambda_n$ be the fundamental dominant weights satisfying $(\lambda_i, \alpha_j) = \delta_{ij}$ and put $\delta = \sum_{i=1}^n \lambda_i$. If $\beta = \alpha_i + \cdots + \alpha_{i+j} \in \Phi^+$, then height of β is given by

$$ht(\beta) = j = (\delta, \beta). \tag{1.2}$$

Therefore, we have

LEMMA 1.2. *For every* $1 \leq i \leq n$,

$$\operatorname{card}\{\beta \in \Phi^+ : ht(\beta) \geq i\} = \binom{n-i+2}{2}.$$

NOTATION 1.3. Let $\mathbf{F} = \mathbf{F}_n$ be the flag variety of $\mathbf{P}^n$, $n \geq 2$. Then $\mathbf{F}_n = SL(n+1, \mathbf{C})/B$, where B denotes the subgroup of lower triangular matrices. Note that we have

$$N := \dim \mathbf{F}_n = \binom{n+1}{2}. \tag{1.3}$$

We denote by Pic $(\mathbf{F})$ the Picard group of $\mathbf{F}$. Let $L \in \operatorname{Pic}(\mathbf{F})$. It is a homogeneous line bundle which corresponds to an irreducible B-module with lowest weight $-\lambda$, $\lambda = \sum m_i \lambda_i$. This correspondence is bijective and we have $\operatorname{Pic}(\mathbf{F}) \simeq \bigoplus_{i=1}^n \mathbf{Z}\lambda_i$. Hence, we can identify L with $m = (m_1, \cdots, m_n) \in \mathbf{Z}^n$. Since we consider only line bundles whose m_i's are equal, say m, we denote the corresponding line bundle by $O_{\mathbf{F}}(m)$. Furthermore, we set $\boldsymbol{V}(m) = \boldsymbol{V} \otimes O_{\mathbf{F}}(m)$ for any vector bundle $\boldsymbol{V}$ on $\mathbf{F}$. With this notation, the canonical bundle $K_{\mathbf{F}}$ can be written as

$$K_{\mathbf{F}} = O_{\mathbf{F}}(-2). \tag{1.4}$$

Now we recall the following theorem due to Wehler [**11**, 1.12]:

THEOREM 1.4. *Let* $\mathbf{F}$ *denote the flag variety of* $\mathbf{P}^n$, *and let* $\Theta_{\mathbf{F}}$ *be its tangent sheaf. Choose* $s \in \mathbf{N}$. *Then, for* $a \in \mathbf{N}$ *with* $a \geq 2$,

$$H^j(\mathbf{F}, (\wedge^s \Theta_{\mathbf{F}})(-a)) = 0$$

provided that $0 \leq j < \operatorname{card}\{\beta \in \Phi^+ : ht(\beta) > (s+1)/(a-1)\}$.

LEMMA 1.5. *The following hold.*

(1) $H^q(\mathbf{F}_n, O_{\mathbf{F}}(a)) = 0$ *for any* $a \in \mathbf{Z}$ *if* $0 < q < N$.

(2) $H^q(\mathbf{F}_n, \Theta_{\mathbf{F}}) = 0$ *for* $q > 0$.

(3) $H^q(\mathbf{F}_n, \Omega^p_{\mathbf{F}}) \neq 0$ *if and only if* $p = q$, *where* $\Omega^p_{\mathbf{F}} = \wedge^p \Theta^*_{\mathbf{F}}$.

PROOF. See [**11**, 1.5] and the references there.

THEOREM 1.6. *Put* $S^a = H^0(\mathbf{F}, O_{\mathbf{F}}(a))$ *for* $a \geq 0$.

(1) *For any positive integer* a, *the line bundle* $O_{\mathbf{F}}(a)$ *is normally generated. In particular, for any nonnegative integers* b, c, *the multiplication map* $S^b \otimes S^c \to S^{b+c}$ *is surjective.*

(2) *The Koszul sequence*

$$\wedge^2 S^1 \otimes S^{a-1} \to S^1 \otimes S^a \to S^{a+1} \to 0$$

is exact if $a \geq n+1$, *or if* $n = 2$ *and* $a \geq 2$.

PROOF. (1) is a well-known fact (see, [**10**] or [**11**, 1.5]). As for (2), recall that $O_{\mathbf{F}}(1)$ corresponds to the weight $\delta = \sum \lambda_i$. Then, by the proof of Main Theorem in [**10**], we see that the natural map

$$\psi_a : S^1 \otimes \mathrm{Ker}(S^1 \otimes S^{a-1} \to S^a) \to \mathrm{Ker}(S^1 \otimes S^a \to S^{a+1})$$

is surjective for $a \geq n+1$. Moreover, it is known that the polarized manifold $(\mathbf{F}_2, O_{\mathbf{F}}(1))$ is of Δ-genus 1 (see, [**5**, (5.1.4)]). Therefore, ψ_a is surjective for $a \geq 2$ by [**4**, Theorem 4.1]. The rest follows from the proof of [**7**, (7.1.5)].

COROLLARY 1.7.

(1) *Let* $0 < a < b$. *Then the Koszul sequence*

$$\wedge^2 S^a \otimes S^{b-a} \to S^a \otimes S^b \to S^{a+b} \to 0$$

is exact if $b \geq n+1$, *or if* $n = 2$ *and* $b \geq 2$.

(2) *Consider the projective embedding* $\mathbf{F}_n \to \mathbf{P}^{n(a)}$ *defined by the linear system* $|O_{\mathbf{F}}(a)|$, $a \in \mathbf{N}$, *where* $n(a)+1 = \dim S^a$. *Then the defining ideal of* $\mathbf{F}$ *in* $\mathbf{P}^{n(a)}$ *is generated in degree* $\leq \max(2, [(n+1)/a])$. *In particular, if* $3a > n+1$, *then it is generated by quadrics. If* $n = 2$, *then it is generated by quadrics for any* $a > 0$.

PROOF. Analogous to [**7**, (7.1.6) and (7.1.7)].

2. The Jacobian ring and the duality theorem

NOTATION 2.1. Our reference here is [**6**] and [**7**, 6.1]. For a fixed positive integer d, we put $L = O_{\mathbf{F}}(d)$ and let Σ_L denote the sheaf of first order differential operators on sections of L. Then we have an exact sequence

$$0 \to O_{\mathbf{F}} \to \Sigma_L \to \Theta_{\mathbf{F}} \to 0 \tag{2.1}$$

whose extension class is $-2\pi\sqrt{-1}c_1(L) \in H^1(\Omega^1_{\mathbf{F}})$. An irreducible non-singular member X of $|L|$ will be called a smooth hypersurface of type (d). We let $x \in H^0(\mathbf{F}_n, L)$ define X, and let $j(x)$ denote its 1-jet extension. Then the contraction with $j(x)$ gives us an exact sequence

$$0 \to \Theta_{\mathbf{F}}(-\log X) \to \Sigma_L \xrightarrow{\cdot j(x)} L \to 0, \tag{2.2}$$

where $\Theta_{\mathbf{F}}(-\log X)$ is the logarithmic tangent sheaf. Tensoring (2.2) with $O_{\mathbf{F}}(a-d)$ and considering the derived cohomology long exact sequence, we get the natural map

$$j_a : H^0(\mathbf{F}_n, \Sigma_L(a-d)) \to S^a.$$

We put $J^a_X = \mathrm{Im}(j_a)$ and $R^a_X = \mathrm{Coker}(j_a)$. We call $J_X = \oplus_{a \geq 0} J^a_X$ and $R_X = \oplus_{a \geq 0} R^a_X$ the *Jacobian ideal* and the *Jacobian ring* of X, respectively. For simplicity, we sometimes suppress the subscript and write $R = R_X$, etc. We further put

$$\sigma = (N+1)d-4. \tag{2.3}$$

LEMMA 2.2. *Let J be the Jacobian ideal of a smooth hypersurface of type (d) of $\mathbf{F}_n$. Then it is generated in degree d. In particular, $R^a \simeq S^a$ for $0 \leq a \leq d$.*

PROOF. Let a be a nonnegative integer. It is known that $H^0(O_{\mathbf{F}}(a))$ and $H^0(\Theta_{\mathbf{F}})$ are irreducible $sl(n+1, \mathbf{C})$-modules. Therefore, the natural multiplication map

$$H^0(\Theta_{\mathbf{F}}) \otimes H^0(O_{\mathbf{F}}(a)) \to H^0(\Theta_{\mathbf{F}}(a))$$

is surjective by Schur's Lemma. By the argument in [**7**, (6.2.4) and (6.2.5)], we get the assertion.

LEMMA 2.3. *Let q be a positive integer.*

(1) $H^q(\mathbf{F}_n, \Theta_{\mathbf{F}}(a)) = 0$ *for any nonnegative integer a.*

(2) $H^1(\mathbf{F}_n, \Theta_{\mathbf{F}}(-1)) = 0$ *if $n \geq 3$, and* $H^1(\mathbf{F}_2, \Theta_{\mathbf{F}}(-1)) \simeq \mathbf{C}$.

(3) $H^q(\mathbf{F}_n, \Theta_{\mathbf{F}}(-2)) = 0$ *unless $q = N-1$.*

(4) $H^1(\mathbf{F}_n, \Theta_{\mathbf{F}}(-3)) = 0$.

PROOF. We shall use Wehler's result [**11**, 1.10 and 1.11] : he introduced a filtration on $\Theta_{\mathbf{F}}$ by subbundles whose successive quotients are direct sum of line bundles corresponding to positive roots of $sl(n+1, \mathbf{C})$.

(1) : By [**11**, 1.11] and Bott's theorem, it suffices to show that, for each $\alpha \in \Phi^+$, the weight $\delta + \alpha + a\delta = \alpha + (a+1)\delta$ is singular or regular with index 0, i.e., $(\alpha + (a+1)\delta, \beta) \geq 0$ for any $\beta \in \Phi^+$. Since $(\alpha, \beta) \geq -1$ and $ht(\beta) \geq 1$, we have

$$(\alpha + (a+1)\delta, \beta) = (\alpha, \beta) + (a+1)ht(\beta) \geq a.$$

This gives (1).

(2) : Assume first that $n \geq 3$. As in the proof of (1), it suffices to show that, for each $\alpha \in \Phi^+$, α is singular, i.e., there exists $\beta \in \Phi^+$ satisfying $(\alpha, \beta) = 0$. This can be checked by a direct calculation.

We next consider the case $n = 2$. Recall that $\mathbf{F}_2$ is isomorphic to the $\mathbf{P}^1$-bundle $\mathbf{P}(\Theta_{\mathbf{P}^2})$ and that $O_{\mathbf{F}}(1)$ is nothing but the relatively ample tautological line bundle. Let $\pi : \mathbf{F}_2 \to \mathbf{P} = \mathbf{P}^2$ denote the projection map. We have

$$0 \to O_{\mathbf{F}}(2) \otimes \pi^* K_{\mathbf{P}} \to \Theta_{\mathbf{F}} \to \pi^* \Theta_{\mathbf{P}} \to 0.$$

Tensoring this with $O_{\mathbf{F}}(-1)$ and considering the cohomology long exact sequence, we have

$$H^q(\Theta_{\mathbf{F}}(-1)) \simeq H^q(O_{\mathbf{F}}(1) \otimes \pi^* K_{\mathbf{P}}) \simeq H^q(\mathbf{P}^2, \Theta_{\mathbf{P}} \otimes K_{\mathbf{P}}) \simeq H^q(\Omega^1_{\mathbf{P}}),$$

since we have $H^p(\mathbf{F}, (\pi^* V)(-1)) = 0$ for any p and vector bundle V on $\mathbf{P}^2$. Therefore, we get (2).

(3) : Since $H^q(\Theta_{\mathbf{F}}(-2)) \simeq H^q(\Omega^{N-1}_{\mathbf{F}})$, this follows from Lemma 1.5, (3).

(4) : If $n \geq 3$, this follows from Theorem 1.4. If $n = 2$, then $\Phi^+ = \{\alpha_1, \alpha_2, \delta = \alpha_1 + \alpha_2\}$. As in the proof of (1), we only have to show that,

for each $\alpha \in \Phi^+$, the weight $\alpha - 2\delta$ is singular or index$\neq 1$. We have $(\alpha - 2\delta, \alpha) = 0$ for $\alpha = \alpha_i$ $(i = 1, 2)$. On the other hand, $\alpha_1 + \alpha_2 - 2\delta = -\delta$ is regular with index 3. Therefore, $H^1(\mathbf{F}_2, \Theta_{\mathbf{F}}(-3)) = 0$.

DUALITY THEOREM 2.4. *Let R be the Jacobian ring of a smooth hypersurface of type (d), $d \geq 3$. Then $R^{\sigma} \simeq \mathbf{C}$ holds for the integer σ in (2.3). For each integer a, $0 \leq a \leq d$, the natural map $R^{\sigma - a} \to (R^a)^*$ induced by the pairing*

$$R^a \otimes R^{\sigma - a} \to R^{\sigma} \simeq \mathbf{C}.$$

is surjective. Furthermore, it is injective except in the following cases:

(1) $n = 2$, $a = d - 1$.

(2) $n = 2$, $a = d = 3$.

PROOF. By [7, (6.3.3) and (6.3.4)] and the Serre duality, it suffices to check the following conditions:

(S.1) $H^{p-1}((\wedge^{p-1}\Theta_{\mathbf{F}})(a - pd)) = 0$ for $2 \leq p \leq N$.

(S.2) $H^{p-1}((\wedge^{p}\Theta_{\mathbf{F}})(a - pd)) = 0$ for $2 \leq p \leq N$.

(I.1) $H^{p}((\wedge^{p-1}\Theta_{\mathbf{F}})(a - pd)) = 0$ for $1 \leq p \leq N - 1$.

(I.2) $H^{p}((\wedge^{p}\Theta_{\mathbf{F}})(a - pd)) = 0$ for $1 \leq p \leq N - 1$.

Recall that $R^{\sigma - a} \to (R^a)^*$ is surjective (resp. injective) if (S.1) and (S.2) (resp. (I.1) and (I.2)) hold. Since the assertion follows from Wehler's vanishing theorem (Theorem 1.4) and Lemma 2.3, we do not reproduce the detail but note here the following: In practice, when $n = 2$, we need to rewrite the above conditions using the Serre duality, etc. For example, we have

$$H^1((\wedge^2\Theta_{\mathbf{F}})(-3)) \simeq H^1(\Omega^1_{\mathbf{F}}(-1)) \simeq H^2(\Theta_{\mathbf{F}}(-1))^*.$$

3. Symmetrizer lemma and the IVHS

Let X be a smooth hypersurface of type (d), $d \geq 3$, of the flag variety, and let R be its Jacobian ring.

LEMMA 3.1. *With the above notation, the following hold.*

(1) *If $0 \leq a \leq b$ and $b - a \leq d$, then*

$$R^{b-a} \to (R^a)^* \otimes R^b$$

is injective.

(2) *If* $0<a \leq b$, $a+b < \sigma$ *and* $\sigma - b \leq \max(d, n+1)$, *then*

$$\wedge^2 R^a \otimes R^{\sigma-(a+b)} \to R^a \otimes R^{\sigma-b} \to R^{\sigma-(b-a)} \to 0$$

is exact.

PROOF. By virtue of Lemmas 1.6, 2.2 and Duality Theorem 2.4, we can show (1) and (2) in the same way as in [**7**, (7.2.1) and (7.2.2)].

SYMMETRIZER LEMMA 3.2. *If* $0<a \leq b \leq d$, $a+b<\sigma$, $\sigma - b \geq \max(d, n+1)$, *the Koszul sequence*

$$0 \to R^{b-a} \to (R^a)^* \otimes R^b \to (\wedge^2 R^a)^* \otimes R^{a+b}$$

is exact except in the case : $n=2$, $b-a=d-1$. *In other words, the symmetrizer of the multiplication map* $R^a \otimes R^b \to R^{a+b}$ *is* $R^{b-a} \otimes R^a \to R^b$.

PROOF. Using Lemma 3.1 and Theorem 2.4, we get the assertion as in [**7**, (7.2.3)]. See also [**3**] for the notion of the symmetrizer of a bilinear map.

NOTATION 3.3. By means of the ample class $\omega = c_1(O_X(d))$, we can define the primitive part $H^{N-1}_{prim}(X)$ of the middle cohomology group $H^{N-1}(X)$:

$$H^{N-1}_{prim}(X) = \operatorname{Ker} \{\cup \omega : H^{N-1}(X) \to H^{N+1}(X)\} .$$

Then we get a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$, where

(1) $H_{\mathbf{Z}} = H^{N-1}(X) \cap H^{N-1}_{prim}(X, \mathbf{Q})$,

(2) $H^{p,q} = H^{p,q}(X) \cap H^{N-1}_{prim}(X, \mathbf{C})$, and

(3) $Q : H_{\mathbf{Z}} \otimes H_{\mathbf{Z}} \to H^{2N-2}(X, \mathbf{Z}) \simeq \mathbf{Z}$ is the cup-product map.

If we set

$$T_{proj} = \{v \in H^1(X, \Theta_X) : v \cdot \omega = 0\},$$

then the contraction map $\Theta_X \otimes \Omega^p_X \to \Omega^{p-1}_X$ induces a pairing

$$v_p : T_{proj} \otimes H^{p,q} \to H^{p-1,q+1}$$

or, equivalently, a linear map

$$v = \oplus v_p : T_{proj} \to \bigoplus_{p+q=N-1} \operatorname{Hom}_{\mathbf{C}}(H^{p,q}, H^{p-1,q+1})$$

which can be identified with the differential of the period map associated with the Kuranishi family of deformations of (X, ω). The data

$$\{H_{\mathbf{Z}}, H^{p,q}, Q, T_{proj}, v\}$$

is called the infinitesimal variation of Hodge structure (IVHS for short) of X. For the theory of IVHS, see [**1**].

LEMMA 3.4. *Let X be a smooth hypersurface of the flag variety $\mathbf{F}_n$, and assume that it is of type (d), $d \geq 3$.*

(1) $T_{proj} \simeq R^d \simeq H^1(X, \Theta_X)$.

(2) $H^{p,q} \simeq H^q(\mathbf{F}, \Omega_{\mathbf{F}}^{p+1}(\log X))$ *for $p+q = N-1$, where $\Omega_{\mathbf{F}}^{p+1}(\log X)$ is the sheaf of meromorphic $(p+1)$-forms with at most logarithmic pole along X. Furthermore, $H^{r-1}(\Omega_{\mathbf{F}}^{N-r+1}(\log X)) \simeq R^{rd-2}$ for $r = 1, 2$.*

(3) *The map $v_{N-1}: T_{proj} \otimes H^{N-1,0} \to H^{N-2,1}$ can be identified with the multiplication map $R^d \otimes R^{d-2} \to R^{2d-2}$.*

PROOF. (1) can be shown as in [**7**, (8.2.5)] using the exact sequences (2.1) and (2.2). Note that the completeness of deformations of X in $\mathbf{F}$ was shown by Wehler [**11**, 2.3].

The first half of (2) follows from [**8**, Lemma 4.5], since $H^0(\mathbf{F}, K_{\mathbf{F}}) = 0$. As for the last half of (2), by [**7**, (8.2.3) and (8.2.4)] and by the Serre duality, the following conditions imply $R^{rd-2} \simeq H^{r-1}(\Omega_{\mathbf{F}}^{N-r+1}(\log X))$:

(a) $H^t((\wedge^t \Theta_{\mathbf{F}})((r-t)d-2)) = 0$ for $1 \leq t \leq r-1$.

(b) $H^t((\wedge^t \Theta_{\mathbf{F}})((r-1-t)d-2)) = 0$ for $1 \leq t \leq r-2$.

(c) $H^{t-1}((\wedge^t \Theta_{\mathbf{F}})((r-t)d-2) = 0$ for $2 \leq t \leq r-1$.

For $r \leq 2$, we only have to show $H^1(\Theta_{\mathbf{F}}(-2)) = 0$, for which Lemma 2.3, (3) can be applied.

Now, (3) follows from (1) and (2) (see, [**7**, (8.2.5)]).

4. Proof of Theorem A

In this section, we show Theorem A.

NOTATION 4.1. Let $\mathbf{P}(S^d)_o$ denote the space parametrizing all the smooth hypersurfaces of type (d), $d \geq 3$, on which the group Aut $(\mathbf{F}, O_{\mathbf{F}}(d))$ acts naturally. We denote by M_d the factor space $\mathbf{P}(S^d)_o/\mathrm{Aut}(\mathbf{F}, O_{\mathbf{F}}(d))$. Then we can define a period map

$$(4.1) \qquad \tilde{P} : \mathbf{P}(S^d)_o \to \Gamma \backslash D$$

associated to the middle cohomology, where D denotes the Griffiths period domain and Γ is the monodromy group. From this, we get the period map in question :

$$(4.2) \qquad P : M_d \to \Gamma \backslash D$$

4.2. In view of [2], it suffices for our purpose to show the variational Torelli theorem. Let X and X' be two smooth generic hypersurfaces of type (d) and suppose that they have the same period, i.e., $\tilde{P}(X) = \tilde{P}(X')$. Then we can assume that their IVHS's are isomophic. What we must show is that X and X' is in the same Aut $(\mathbf{F}, O_{\mathbf{F}}(d))$ — orbit. Therefore, it suffices to show that $J_X^d = J_{X'}^d$ in S^d (see, [7, (9.19)] and the references there).

By Lemma 3.4, we have the following commutative diagram :

$$\begin{array}{ccccc} R_X^d & \otimes & R_X^{d-2} & \to & R_X^{2d-2} \\ \phi_d \downarrow \wr & & \wr\downarrow \phi_{d-2} & & \wr\downarrow\phi_{2d-2} \\ R_{X'}^d & \otimes & R_X^{d-2} & \to & R_{X'}^{2d-2}, \end{array}$$

where the ϕ 's are unknown isomorphisms of vector spaces. Applying Symmetrizer Lemma 3.2, we have the commutative diagram

$$\begin{array}{cccccc} & R_X^2 & \otimes & R_X^{d-2} & \to & R_X^d \\ (\mathrm{SD})_1 : & \phi_2 \downarrow \wr & & \wr\downarrow \phi_{d-2} & & \wr\downarrow \phi_d \\ & R_{X'}^2 & \otimes & R_X^{d-2} & \to & R_{X'}^d, \end{array}$$

Put $d = 2m + 2$ or $2m + 1$ according to whether d is even or odd. Analogously, applying 3.2 successively, we get a sequence of commutative diagrams

$$\begin{array}{cccccc} & R_X^2 & \otimes & R_X^{d-2i} & \to & R_X^{d-2i+2} \\ (\mathrm{SD})_i : & \phi_2 \downarrow \wr & & \wr\downarrow \phi_{d-2i} & & \wr\downarrow \phi_{d-2i+2} \\ & R_{X'}^2 & \otimes & R_{X'}^{d-2i} & \to & R_{X'}^{d-2i+2}, \end{array}$$

for $1 \leq i \leq m$. If d is even, then the terminal diagram $(\mathrm{SD})_m$ is

$$\begin{array}{ccccc} & R_X^2 & \otimes\ R_X^2 & \to & R_X^4 \\ (\mathrm{SD})_m : & \phi_2 \downarrow \wr & \wr \downarrow \phi_2' & & \wr \downarrow \phi_4 \\ & R_{X'}^2 & \otimes\ R_{X'}^2 & \to & R_{X'}^4 \end{array}$$

In d is odd, we can apply Symmetrizer Lemma to $(\mathrm{SD})_m$ once more and get

$$\begin{array}{ccccc} & R_X^1 & \otimes\ R_X^1 & \to & R_X^2 \\ (\mathrm{SD})_{m+1} : & \phi_1 \downarrow \wr & \wr \downarrow \phi_1' & & \wr \downarrow \phi_2 \\ & R_{X'}^1 & \otimes\ R_{X'}^1 & \to & R_{X'}^2 \end{array}$$

As in [**7**, (10.2.2)], we can show that ϕ_j' $(j = 1, 2)$ appeared above is a non-zero multiple of ϕ_j. Therefore, we can assume that $\phi_j' = \phi_j$. Recall that we have $R^a = S^a$ for $a < d$ by Lemma 2.2.

LEMMA 4.3. *With the above notation, assume that* $d \geq \max(6, 2[(n+1)/2]+2)$ *if d is even, and $d \geq n+3$ or $n = 2$, $d = 3$ if d is odd.* Then $J_X^d = J_{X'}^d$ *in* S^d.

PROOF. Since the proof for odd d is quite similar, we assume here that d is even, $d \geq \max(6, 2[(n+1)/2]+2)$. We first consider the case $n \geq 3$. By using diagrams $(\mathrm{SD})_i$, $m+2-[(n+1)/2] \leq i \leq m$, we get the following commutative diagram as in the proof of [**7**, (10.2.4)]:

$$\begin{array}{ccc} \mathrm{Sym}^{[(n+1)/2]}(S^2) & \to & S^{2[(n+1)/2]} \\ \mathrm{Sym}^{[(n+1)/2]}(\phi_2) \downarrow \wr & & \wr \downarrow \phi_{2[n/2]} \\ \mathrm{Sym}^{[(n+1)/2]}(S^2) & \to & S^{2[(n+1)/2]}, \end{array}$$

where $\mathrm{Sym}^k(V)$ denotes the k-th symmetric tensor product of a vector space V. Note that the kernel of $\mathrm{Sym}^{[(n+1)/2]}(S^2) \to S^{2[(n+1)/2]}$ is nothing but the degree $[(n+1/2)]$ piece $I^{[(n+1)/2]}$ of the defining ideal I of $\mathbf{F}_n$ in $\mathbf{P}^{n(2)}$. By Corollary 1.7, (2), it contains enough information to recover $\mathbf{F}_n$. Therefore, ϕ_2 comes from an element of $\mathrm{Aut}(\mathbf{F}, O_{\mathbf{F}}(2))$. Then, we see that all the ϕ in the diagrams $(\mathrm{SD})_i$ come from an automorphism of $\mathbf{F}$. Since the map $S^2 \otimes S^{d-2} \to R^d$ factors through S^d, we get the following commutative diagram by $(\mathrm{SD})_1$:

$$\begin{array}{ccccccc} 0 \to & J_X^d & \to & S^d & \to & R_X^d & \to 0 \\ & \downarrow\wr & & \downarrow\wr & & \wr\downarrow \phi_d & \\ 0 \to & J_{X'}^d & \to & S^d & \to & R_{X'}^d & \to 0, \end{array}$$

where $S^d \to S^d$ is induced by the known isomorphisms ϕ_1 and ϕ_{d-2}. Therefore, $J_X^d = J_{X'}^d$ up to an autmorphism of $\mathbf{F}_n$.

If $n = 2$, then we do not need a somewhat complicated argument as above. In this case, I is generated in degree 2. So we can recover I^2 by using $(\mathrm{SD})_m$ and see that ϕ_2 comes from an automorphism of $\mathbf{F}$. Therefore, we have no trouble in recovering J^d.

5. Proof of Theorem B

Since Theorem B can be shown along a line analogous to the case of the flag variety, we do not reproduce the detail and only list the facts which we need for the proof. Let N be an integer with $N \geq 3$, and put $W = (\mathbf{P}^1)^N$. We write $O_W(a) = O_W(a, a, \cdots, a)$ for simplicity. In particular, we have $K_W = O_W(-2)$. We call an irreducible nonsingular member of $|O_W(d)|$ a smooth hypersurface of type (d).

LEMMA 5.1. *The group $H^q(W, \Omega_W^p(a))$ vanishes except in the following cases*:

(1) $q = 0$, $a \geq 2$; $q = p$, $a = 0$; $q = N$, $a \leq -2$.

(2) $p = 0$, $q = 0$, $0 \leq a \leq 1$; $p = N$, $q = N$, $-1 \leq a \leq 0$.

PROOF. Use the Kunneth formula.

LEMMA 5.2. *Put $S^a = H^0(W, O_W(a))$. Then the following Koszul sequence is exact for $1 \leq a < b$ and $b \geq 2$*:

$$\wedge^2 S^a \otimes S^{b-a} \to S^a \otimes S^b \to S^{a+b} \to 0.$$

Furthermore, the defining ideal of W in $\mathbf{P}^{n(a)}$, $n(a) = \dim S^a - 1$, is generated by quadrics for any $a > 0$.

DUALITY THEOREM 5.3. *Let R (resp. J) be the Jacobian ring (resp. ideal) of a smooth hypersurface of type (d) of W, which can be defined similarly as in §3. Assume* that $d \geq 3$. *Then* J *is generated in degree* d *and* $R^a \simeq S^a$ *for* $0 \leq a < d$. *Furthermore,* $R^{\sigma} \simeq \mathbf{C}$, *where* $\sigma = (N+1)d-4$. *For each integer* $0 \leq a \leq \sigma$, *the pairing.*

$$R^a \otimes R^{\sigma-a} \rightarrow R^{\sigma} \simeq \mathbf{C}$$

is perfect except in the following cases :

(1) N *is odd and* $a = d(N+1)/2-2$.

(2) N *is even and* $a = dN/2-2,\ d(N+2)/2-2$.

SYMMETRIZER LEMMA 5.4. *Let* $d \geq 3$, *and let* R *be as in* Duality Theorem 5.3. *If* $0 < a \leq b$, $a+b < \sigma$ *and* $\sigma - b \geq d$, *then the Koszul sequence*

$$0 \rightarrow R^{b-a} \rightarrow (R^a)^* \otimes R^b \rightarrow (\wedge^2 R^a)^* \otimes R^{a+b}$$

is exact except in the cases : $b-a = \sigma/2$ *if* N *is odd, and* $b-a = (\sigma+d)/2$ *if* N *is even.*

LEMMA 5.5. *Let* X *be a smooth hypersurface of type* (d), $d \geq 3$, *of* W. *Then the cup-product map map* $H^1(\Theta_X) \otimes H^0(\Omega_X^{N-1}) \rightarrow H^1_{prim}(\Omega_X^{N-2})$ *can be identified with the multiplicaiton map* $R_X^d \otimes R_X^{d-2} \rightarrow R_X^{2d-2}$ *in a canonical way.*

The proof of Theorem B can be sketched as follows. We can apply Symmetrizer Lemma 5.4 to $R^d \otimes R^{d-2} \rightarrow R^{2d-2}$ successively. If we put $e =$ 1, 2 according as whether d is odd or even, then we finally arrive at $R^e \otimes R^e \rightarrow R^{2e}$, which is nothing but $S^e \otimes S^e \rightarrow S^{2e}$ under the condition $d \geq 3$ but $d \neq 4$. Since the defining ideal of W in $\mathbf{P}^{n(e)}$ is generated by quadrics, we can easily recover J^d up to an element of $\mathrm{Aut}(W, O_W(e))$ as in the proof of Lemma 4.3. Therefore, Theorem B follows.

REMARK 5.6. A general hypersurface of type (3) of $\mathbf{F}_2$ has Picard number 2, whereas that of $(\mathbf{P}^1)^3$ has Picard number 3 (see, [**11**, 5.5 and 5.6]).

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