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矢ケ部, 巌
九州大学教養部数学教室

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Iwao YAKABE

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1. Introduction

In their paper [1], Heatherly and Ligh introduced the notion of pseudo-distributive near-rings and studied the basic properties of pseudo-distributive near-rings. We are interested in quasi-ideals of pseudo-distributive near-rings. The purpose of this paper is to study some properties of pseudo-distributive near-rings in terms of quasi-ideals.

In Sections 2 and 3, we consider some elementary facts about quasi-ideals of pseudo-distributive near-rings. These facts will be used in the following sections. In Section 4, we deal with Q-simple pseudo-distributive near-rings and show that a pseudo-distributive near-ring is Q-simple if and only if it is either a division ring or a zero ring of prime order. In Sections 5 and 6, we characterize the regular elements and the regular duo elements of pseudo-distributive near-rings in terms of quasi-ideals, respectively.

For the basic terminology and notation we refer to [3].

2. Preliminaries

Let N be a near-ring, which always means right one throughout this paper.

If A, B and C are three non-empty subsets of N, then AB (ABC) denotes the set of all finite sums of the form $\Sigma a_k b_k$ with $a_k \in A$, $b_k \in B$ ($\Sigma a_k b_k c_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C$). Note that $ABC=(AB)C \subseteq A(BC)$ in general.

A *right N-subgroup* (*left N-subgroup*) of N is a subgroup H of $(N, +)$ such that $HN \subseteq H$ ($NH \subseteq H$). For every subgroup S of $(N, +)$, SN is a right N-subgroup of N. For a non-empty subset A of N, $(A)_r$ ($(A)_l$) denotes the right (left) N-subgroup of N generated by A.

A *quasi-ideal* of a zero-symmetric near-ring N is a subgroup Q of $(N,$

$+$) such that $NQ \cap QN \subseteq Q$ (see [6, Proposition 3]). Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal. For a non-empty subset A of N, $(A)_q$ denotes the quasi-ideal of N generated by A, and $[A]$ denotes the subgroup of $(N, +)$ generated by A.

3. Quasi-ideals of pseudo-distributive near-rings

A *pseudo-distributive near-ring* is a near-ring N such that:

(P1) For each $a, b, c, d \in N$, $ab+cd=cd+ab$.

(P2) For each positive integer k and for each $a, b_1, \dots, b_k, c_1, \dots, c_k \in N$,

$$a\left(\sum_{i=1}^{k} b_i c_i\right) = \sum_{i=1}^{k} a b_i c_i.$$

Let N be a pseudo-distributive near-ring. Then it follows from the property (P2) that for any non-empty subsets A, B and C of N,

$$ABC=(AB)C=A(BC).$$

So, for every subgroup S of $(N, +)$, NS is a left N-subgroup of N. Moreover, it follows from the definition of a pseudo-distributive near-ring that N^2 is a subring of N.

We note also that every pseudo-distributive near-ring N is zero-symmetric. In fact, for any element x of N, $x0$ is in N^2. So $x0=(x0)0=0$.

Lemma 3.1. *Let N be a pseudo-distributive near-ring.*

(1) *For any elements x, y and z of N, $(-x)yz=x(-y)z=x(-yz)$.*

(2) *For every element x of N and for every element w of N^2,*

$$x(-w)=(-x)w.$$

(3) *For every element x of N, $[x]Nx=\{xnx \mid n \in N\}$.*

Proof. (1) Since N is zero-symmetric, by (P2) we get

$$xyz+x(-y)z=x(yz+(-y)z)=x0=0,$$

whence $-xyz=x(-y)z$. So $(-x)yz=x(-y)z=x(-yz)$.

(2) Since w is an element of N^2, we can write in the form $w=\Sigma y_i z_i$ with $y_i, z_i \in N$. Then, by (P1), $-w=\Sigma(-y_i)z_i$. So, by (1) we get

$x(-w)=x(\Sigma(-y_i)z_i)=\Sigma x(-y_i)z_i=\Sigma(-x)y_iz_i=(-x)(\Sigma y_iz_i)=(-x)w.$

(3) Let M denote the set $\{xnx \mid n\in N\}$. Then the inclusion $M\subseteq[x]Nx$ is evident. On the other hand, any element a of $[x]Nx$ has the form $a=\Sigma xn_ix+\Sigma(-x)m_jx$ with n_i, $m_j\in N$. Moreover, by (1), $(-x)m_jx=x(-m_j)x$. So, we get

$$a=x(\Sigma n_ix+\Sigma(-m_j)x)=x(\Sigma n_i+\Sigma(-m_j))x\in M.$$

Hence $[x]Nx\subseteq M$.

Proposition 3.2. *Let N be a pseudo-distributive near-ring and a an element of N^2.*

(i) $N[a]=Na$.

(ii) $(a)_l=[a]+Na$ *and* $N(a)_l=Na$.

(iii) $(a)_r=[a]+[a]N$ *and* $(a)_rN=[a]N$.

Proof. (i) Since $a\in N^2$, by Lemma 3.1 (2) we get $N[a]\subseteq Na$. On the other hand, the inclusion $Na\subseteq N[a]$ always holds. Hence $N[a]=Na$.

(ii) Since $[a]\subseteq N^2$, by (P1), $[a]+Na$ is a subgroup of $(N, +)$. Moreover, by (P2) and (i), we get

$$N([a]+Na)=N[a]+NNa=Na\subseteq[a]+Na.$$

Hence $[a]+Na$ is a left N-subgroup of N containing a. So we get

$$(a)_l\subseteq[a]+Na\subseteq(a)_l.$$

Thus $(a)_l=[a]+Na$. The above argument also shows $N(a)_l=Na$.

Similarly, we have (iii).

Proposition 3.3. *Let e be an idempotent element of a pseudo-distributive near-ring N and R, L right and left N-subgroups of N, respectively. Then Re and eL are quasi-ideals of N such that*

$$Re=R\cap Ne \quad \text{and} \quad eL=eN\cap L.$$

Proof. It holds for arbitrary near-rings that Re is a quasi-ideal such that $Re=R\cap Ne$ (see [7, Lemma 1]). So we show only the rest.

First we show that eN is a right N-subgroup of N. Any elements a, b of eN have the form

$$a=\Sigma en_i \quad \text{and} \quad b=\Sigma em_j,$$

where n_i, $m_j \in N$. Since e is idempotent, by Lemma 3.1 we get

$$-b=\Sigma(-e)m_j=\Sigma(-ee)m_j=\Sigma(-e)em_j=\Sigma e(-e)m_i.$$

So $a-b=\Sigma een_i+\Sigma e(-e)m_j=e(\Sigma en_i+\Sigma(-e)m_j)\in eN$. Hence eN is a subgroup of $(N, +)$. Moreover, $(eN)N=eNN\subseteq eN$. Thus eN is a right N-subgroup of N.

Now, since the intersection of two quasi-ideals is a quasi-ideal, it suffices to prove the relation $eL=eN\cap L$. As $eL\subseteq eN\cap L$, we have to show only $eN\cap L\subseteq eL$. Any element a of $eN\cap L$ has the form $a=l=en$ with $l\in L$, $n\in N$, whence $a=en=een=el\in eL$.

4. Q-simple pseudo-distributive near-rings

A near-ring N is called *Q-simple* if it contains no quasi-ideals except $\{0\}$ and N.

In this section, we characterize Q-simple pseudo-distributive near-rings and generalize the result of Steinfeld [4].

We start with

PROPOSITION 4.1. *Let N be a pseudo-distributive near-ring such that $N^2\neq\{0\}$. Then N is Q-simple if and only if it is a division ring.*

PROOF. Assume that N is Q-simple. Since N^2 is a quasi-ideal of N and $N^2\neq\{0\}$, we have $N^2=N$. This implies that N is abelian and contains a non-zero distributive element. On the other hand, let $(a)_l$ be the principal left N-subgroup of N generated by a non-zero element a of N, then $(a)_l=N$. This and Proposition 3.2 imply $Na=N(a)_l=N^2=N$. From these and [2, Theorem 2.3], it follows that N is a division ring.

Conversely, assume that N is a division ring. Let $Q\neq\{0\}$ be a quasi-ideal of N and q a non-zero element of Q. Then $N=Nq=qN=Nq\cap qN\subseteq NQ\cap QN\subseteq Q$, whence $Q=N$.

In case that N is a Q-simple near-ring such that $N^2=\{0\}$, N is the zero multiplication near-ring and $(N, +)$ has no proper subgroup. So N is distributive and $(N, +)$ is of prime order. Hence N is a zero ring of prime order.

Thus we have

THEOREM 4.2. *A pseudo-distributive near-ring is Q-simple if and only if it is either a division ring or a zero ring of prime order.*

EXAMPLE 4.3. The following example shows that Theorem 4.2 can not be extended to arbitrary near-rings : Let $K=\{0, 1, 2\}$ be the near-ring due to [3, Near-rings of low order (C-2)] defined by the tables

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	0	1
2	0	0	2

Then K is Q-simple. But K is neither a division ring nor a zero ring of prime order.

5. Regular elements of pseudo-distributive near-rings

An element a of a near-ring N is said to be *regular* if $a=ana$ for some $n\in N$, and N is called *regular* if every element of N is regular.

Note that, by Lemma 3.1 (3), an element a of a pseudo-distributive near-ring N is regular if and only if $a\in[a]Na$.

LEMMA 5.1. *Let a be a regular element of a pseudo-distributive near-ring N.*

(1) $(a)_l=Na$ *and* $(a)_r=aN$.

(2) *There exist idempotent elements e, f in N such that* $(a)_l=(e)_l$ *and* $(a)_r=(f)_r$.

PROOF. It holds for arbitrary near-rings that $(a)_l=Na$, $(a)_r=[a]N$ and (2) (see [7, Lemma 2]). So we show only that $[a]N=aN$.

The inclusion $aN\subseteq[a]N$ is eviden. Any element x of $[a]N$ has the form $x=\Sigma an_i+\Sigma(-a)m_j$ with n_i, $m_j\in N$. As a is regular, $a=ana$ for some $n\in N$. Moreover, by Lemma 3.1, $-ana=a(-na)$. Hence

$$x=\Sigma(ana)n_i+\Sigma(-ana)m_j=a\Sigma nan_i+a\Sigma(-na)m_j\in aN,$$
whence $[a]N\subseteq aN$.

Lemma 5.1 (2) has a converse:

PROPOSITION 5.2. *An element a of a pseudo-distributive near-ring N is regular if and only if the principal left (right) N-subgroup of N generated by a has the form Ne with $e^2=e\in N$ (fN with $f^2=f\in N$).*

PROOF. In view of Lemma 5.1, it is enough to prove the sufficiency.

Assume that the principal left N-subgroup $(a)_l$ of N generated by a has the from Ne with $e^2=e\in N$. Then there exists an element n in N such that $a=ne$, whence $ae=ne^2=ne=a$.

On the other hand, $(a)_l=Ne$ implies $e\in(a)_l$. Moreover, since $a\in N^2$, by Proposition 3.2, $(a)_l=[a]+Na$. Hence $e=ka+ba$ for some integer k and for some $b\in N$. Since a, $ba\in N^2$ and N^2 is a subring of N, we get

$$e=e^2=k^2a^2+kba^2+kaba+baba=(k^2a+kba+kab+bab)a.$$

From this and $a=ae$, it follows that

$$a(k^2a+kba+kab+bab)a=ae=a,$$

that is, a is a regular element.

The statement concerning the principal right N-subgroup fN can be proved similarly.

Now we want to characterize the regular elements of a pseudo-distributive near-ring in terms of quasi-ideals and generalize the result of Steinfeld [5].

THEOREM 5.3. *The following assertions concerning an element a of a pseudo-distributive near-ring N are equivalent:*

(i) *a is regular.*

(ii) $(a)_r(a)_l=(a)_r\cap(a)_l$.

(iii) $(a)_r^2=(a)_r$, $(a)_l^2=(a)_l$ *and the product* $(a)_r(a)_l$ *is a quasi-ideal of* N.

(iv) $(a)_q=(a)_qN(a)_q$.

PROOF. (i) $\Rightarrow$ (iii) : By Lemma 5.1, we have $(a)_l=(e)_l$ with a suitable idempotent element e of N. Then $e=e^2\in(e)_l^2$, and $(e)_l^2$ is a left N-subgroup of N. Hence

$$(a)_l=(e)_l\subseteq(e)_l^2=(a)_l^2\subseteq(a)_l,$$

that is, $(a)_l=(a)_l^2$. Similarly, $(a)_r=(a)_r^2$.

Since $(a)_l=(e)_l=Ne$, we have $(a)_r(a)_l=(a)_rNe=((a)_rN)e$. Hence, by Proposition 3.3, the product $(a)_r(a)_l$ is a quasi-ideal of N.

(iii) $\Rightarrow$ (ii) : The condition (iii) implies that $a\in(a)_r^2\subseteq N^2$. So, by Proposition 3.2, we get

$$(a)_r=(a)_r^2\subseteq(a)_rN=[a]N,$$

whence $(a)_r=[a]N$. Similarly, $(a)_l=Na$.

Since the product $(a)_r(a)_l=([a]N)(Na)$ is a quasi-ideal of N, we get

$$([a]NNa)N\cap N([a]NNa)\subseteq[a]NNa.$$

Moreover, by Proposition 3.2 (i), we get $N[a]N=(N[a])N=NaN$. These results and the condition (iii) imply

$$(a)_r\cap(a)_l=[a]N\cap Na=([a]N)^3\cap(Na)^3\subseteq([a]N)N([a]N)\cap(Na)N(Na)$$
$$=([a]NNa)N\cap N([a]NNa)\subseteq[a]NNa=(a)_r(a)_l.$$

Since the inclusion $(a)_r(a)_l\subseteq(a)_r\cap(a)_l$ always holds, we have obtained

$$(a)_r\cap(a)_l=(a)_r(a)_l.$$

(ii) $\Rightarrow$ (iv) : By the definition of quasi-ideals, the inclusion $(a)_qN(a)_q\subseteq(a)_q$ always holds.

On the other hand, by the condition (ii) and Proposition 3.2, we have

$$a\in(a)_r\cap(a)_l=(a)_r(a)_l\subseteq(a)_rN=[a]N,$$

whence $(a)_r=[a]N$. Similarly, $(a)_l=Na$. These imply

$$(a)_q\subseteq(a)_r\cap(a)_l=(a)_r(a)_l=([a]N)(Na)\subseteq(a)_qN(a)_q.$$

(iv) $\Rightarrow$ (i) : From the condition (iv) and Proposition 3.2, it follows that

$$a\in(a)_qN(a)_q\subseteq(a)_rN(a)_l=[a]N(a)_l=[a]Na,$$

that is, a is a regular element.

From Lemma 5.1 and Theorem 5.3, we obtain immediately

COROLLARY 5.4. *Let a be a regular element of a pseudo-distributive near-ring N. Then*

$$(a)_q=(a)_r\cap(a)_l=(aN)(Na).$$

PROOF. Since $(a)_r\cap(a)_l$ is a quasi-ideal of N containing a, by Lemma 5.1 and Theorem 5.3, we get

$$(a)_q\subseteq(a)_r\cap(a)_l=(a)_r(a)_l=(aN)(Na)\subseteq(a)_qN(a)_q=(a)_q,$$

whence $(a)_q=(a)_r\cap(a)_l=(aN)(Na)$.

It is well known that a regular pseudo-distributive near-ring is a ring (see [**1**, Theorem 1]). From this and Theorem 5.3, we have

COROLLARY 5.5. *Let N be a pseudo-distributive near-ring. Then the following conditions are equivalent:*

(I) *N is a regular near-ring.*

(II) *N is a regular ring.*

(III) *Every element a of N has one of the properties* (i), (ii), (iii) *and* (iv) *of Theorem* 5.3.

Example 4.3 also shows that Theorem 5.3 can not be extended to arbitrary near-rings. In the near-ring K of Example 4.3, $(1)_r=(1)_l=(1)_q=K$. So, for the element 1, the properties (ii), (iii) and (iv) of Theorem 5.3 hold. But the element 1 is not regular, since $1x1=0$ for all $x\in K$.

6. Regular duo elements of pseudo-distributive near-rings

An element a of a near-ring N is said to be a *duo element* of N, if $(a)_r=(a)_l$.

We want to characterize the regular duo elements of pseudo-distributive near-ring in terms of quasi-ideals and generalize the results of Steinfeld [**5**].

For an element a of a near-ring N, $(a)_t$ denotes the two-sided N-subgroup of N generated by a.

THEOREM 6.1. *The following conditions on an element a of a pseudo-distributive near-ring N are equivalent:*

(1) *a is a regular duo element of N.*

(2) $(a)_t=(a)_q^2$

(3) $(a)_r^2=(a)_l$ *and* $(a)_l^2=(a)_r$

(4) $(a)_l(a)_r=(a)_l\cap(a)_r$

PROOF. (1) ⇒ (2) : Since a is a duo element, $(a)_r=(a)_l=(a)_t$. This and Corollary 5.4 imply

$$(a)_q=(a)_r\cap(a)_l=(a)_r=(a)_t.$$

Moreover, by Theorem 5.3, $(a)_r^2=(a)_r$. Hence

$$(a)_t=(a)_r=(a)_r^2=(a)_q^2.$$

(2) ⇒ (3) : The assumption (2) and the definitions imply

$$(a)_q^2\subseteq(a)_r^2\subseteq(a)_l^2\subseteq(a)_t=(a)_q^2\subseteq(a)_q\subseteq(a)_l\subseteq(a)_t=(a)_q^2$$

whence $(a)_r^2=(a)_l$. Similary, one can show $(a)_l^2=(a)_r$.

(3) ⇒ (4) : From the assumption (3), it follows

$$(a)_l\cap(a)_r=(a)_l\cap(a)_l^2=(a)_l^2=(a)_l(a)_l=(a)_l(a)_r^2\subseteq(a)_l(a)_r.$$

On the other hand,

$$(a)_l(a)_r=\begin{cases}(a)_l(a)_l^2\subseteq(a)_l,\\(a)_r^2(a)_r\subseteq(a)_r,\end{cases}$$

whence $(a)_l(a)_r\subseteq(a)_l\cap(a)_r$.

(4) ⇒ (1) : It is evident that $(a)_l(a)_r$ is a two-sided N-subgroup of N. From this and $a\in(a)_l\cap(a)_r=(a)_l(a)_r$, it follows that the principal left N-subgroup $(a)_l$ is contained in the two-sided N-subgroup $(a)_l(a)_r$, whence

$$(a)_l\subseteq(a)_l(a)_r=(a)_l\cap(a)_r\subseteq(a)_r.$$

Similarly, $(a)_r\subseteq(a)_l(a)_r=(a)_l\cap(a)_r\subseteq(a)_l$. Thus $(a)_l=(a)_r$ and a is a duo element. This and the condition (4) imply

$$(a)_r\cap(a)_l=(a)_r(a)_l.$$

So, by Theorem 5.3, a is a regular element.

A near-ring N is called a *duo near-ring* if every one-sided (right or left) N-subgroup of N is a two-sided N-subgroup of N.

From Theorems 5.3, 6.1 and [8, Proposition 2], one gets immediately

COROLLRAY 6.2. *Let N be a pseudo-distributive near-ring. Then the following conditions are equivalent:*

(i) *N is a regular duo near-ring.*

(ii) *N is a regular duo ring.*

(iii) *Every element a of N has one of the properties* (1), (2), (3) *and* (4) *of Theorem* 6.1.

Example 4.3 also shows that Theorem 6.1 can not be extended to arbitray near-rings. In the near-ring K of Example 4.3, $(1)_r=(1)_l=(1)_q=(1)_t=K$. So, for the element 1, the properties (2), (3) and (4) of Theorem 6.1 hold. But the element 1 is not regular.

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