Minimal quasi-ideals of near-rings

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1. Introduction

In ring theory, it is well known that each one of the intersection and the product of a minimal right ideal and a minimal left ideal of a ring is either $\{0\}$ or a minimal quasi-ideal of the ring (see [2]).

The purpose of this note is to extend the above result to a class of zero-symmetric near-rings. For the basic terminology and notation we refer to [1].

2. Preliminaries

Let N be a near-ring, which always means right one throughout this note.

If A and B are two non-empty subsets of N, then AB denotes the set of all finite sums of the form $\Sigma a_k b_k$ with $a_k \in A$, $b_k \in B$.

A *right N-subgroup* (*left N-subgroup*) of N is a subgroup H of $(N, +)$ such that $HN \subseteq H$ $(NH \subseteq H)$. Note that for every subgroup S of $(N, +)$, SN is a right N-subgroup of N, and that for every element n of N and every left N-subgroup L of N, Ln is a left N-subgroup of N.

A *quasi-ideal* of a zero-symmetric near-ring N is a subgroup Q of $(N, +)$ such that $NQ \cap QN \subseteq Q$ (see [3, Proposition 3]). Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

A *minimal left N-subgroup* of a zero-symmetric near-ring N is a left N-subgroup which is minimal in the set of all non-zero left N-subgroups. Similarly, one defines *minimal right N-subgroups* and *minimal quasi-ideals*.

Now we state here the known result which will be used later.

PROPOSITION 1 ([4, Lemma 1]). *Let e be an idempotent element of a near-ring N, and let R be a right N-subgroup of N. Then Re is a quasi-ideal of N such that $Re=R\cap Ne=R(Ne)$.*

3. Main results

We start with

PROPOSITION 2. *Every minimal left N-subgroup L of a zero-symmetric near-ring N is either a zero multiplication subnear-ring of N or it contains a non-zero idempotent element e such that $L=Ne$.*

PROOF. If $L^2\neq\{0\}$, then there exists a non-zero element l in L such that $Ll\neq\{0\}$. So, $\{0\}\neq Ll\subseteq L^2\subseteq L$, and Ll is a left N-subgroup of N. By the minimality of the left N-subgroup L, we get $Ll=L$. This implies that the existence of a non-zero element e in L with the property $el=l$. Hence $e^2l=el=l$. Therefore $(e^2-e)l=0$.

Let $M=\{m\in L \mid ml=0\}$. Then M is a left N-subgroup of N contained in L. By the minimality of the left N-subgroup L, either $M=\{0\}$ or $M=L$. In case of $M=L$, $L=Ll=Ml=\{0\}$, which contradicts $L\neq\{0\}$; so we get $M=\{0\}$.

Since e^2-e is an element of L such that $(e^2-e)l=0$, we get $e^2-e=0$, that is, $e^2=e$. Moreover, $\{0\}\neq Ne\subseteq NL\subseteq L$. By this and the minimality of the left N-subgroup L, we get $L=Ne$.

Now we are ready to state the main results of this note.

THEOREM 3. *The intersection of a minimal right N-subgroup R and a minimal left N-subgroup L of a zero-symmetric near-ring N is either $\{0\}$ or a minimal quasi-ideal of N.*

PROOF. The intersection $R\cap L=Q$ is a quasi-ideal of N. If $Q\neq\{0\}$, then we assume the existence of a non-zero quasi-ideal Q' such that $Q'\subset Q$. Hence $Q'\subset L$.

On the other hand, either $NQ'=\{0\}$ or $NQ'\neq\{0\}$. In case of $NQ'=$

$\{0\}$, Q' would be a left N-subgroup of N such that $\{0\}\subset Q'\subset L$, which contradicts the minimality of L; so we have $NQ'\neq\{0\}$. Then there exists an element q in Q' such that $Nq\neq\{0\}$. Hence $\{0\}\neq Nq\subseteq NL\subseteq L$. Since Nq is a left N-subgroup of N, by the minimality of L we have $Nq=L$.

Similarly, one can show that $Q'N=R$, since $Q'N$ is a right N-subgroup of N. Therefore

$$Q=R\cap L=Q'N\cap Nq\subseteq Q'N\cap NQ'\subseteq Q',$$

in contradiction with our assumption $Q'\subset Q$. Thus $Q=R\cap L(\neq\{0\})$ is a minimal quasi-ideal of N.

THEOREM 4. *The product RL of a minimal right N-subgroup R and a minimal left N-subgroup L of a zero-symmetric near-ring N is either $\{0\}$ or a minimal quasi-ideal of N.*

PROOF. Suppose that $RL\neq\{0\}$. Since $\{0\}\neq RL\subseteq R\cap L$ and $R\cap L$ is a minimal quasi-ideal of N by Theorem 3, we only have to show that RL is a quasi-ideal of N.

If $N(RL)=\{0\}$ or $(RL)N=\{0\}$, then

$$N(RL)\cap(RL)N=\{0\}\subseteq RL,$$

whence RL is a quasi-ideal of N.

If $N(RL)\neq\{0\}$ and $(RL)N\neq\{0\}$, then the second condition implies that $\{0\}\neq(RL)N\subseteq RN\subseteq R$. Since $(RL)N$ is a right N-subgroup of N, by the minimality of R we have $(RL)N=R$. On the other hand, the first condition implies that there exists an element m in RL such that $Nm\neq\{0\}$. So, $\{0\}\neq Nm\subseteq N(RL)\subseteq NL\subseteq L$. Since Nm is a left N-subgroup of N, by the minimality of L we have $Nm=L$. Hence

$$\{0\}\neq RL=((RL)N)(Nm)\subseteq(LN)(Nm)\subseteq L(Nm)=L^2,$$

that is, $L^2\neq\{0\}$. This and Proposition 2 imply that there exists a non-zero idempotent element e in L such that $L=Ne$, whence $RL=R(Ne)$. Thus, by

Proposition 1, $RL=R(Ne)$ is a quasi-ideal of N.

Now, it is natural to ask whether Theorems 3 and 4 are true or not respectively without the assumption that N is zero-symmetric. This question is still open.

References

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