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1. Introduction

Let Ω be a bounded domain in R^3 with smooth boundary $\partial\Omega$. We consider the initial-boundary value problem for the Navier-Stokes equations ;

$$(\mathrm{NS})\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u+(u\cdot\nabla)u=f-\nabla p & \text{in } \Omega\times(0,\ T),\\ \operatorname{div} u=0 & \text{in } \Omega\times(0,\ T),\\ u=0 & \text{on } \partial\Omega\times(0,\ T),\\ u(x,\ 0)=a(x) & \text{in } \Omega,\end{cases}$$

where the function $u=(u^1(x,\ t),\ u^2(x,\ t),\ u^3(x,\ t))$ represents the unknown velocity field of the fluid, $p=p(x,\ t)$ is the unknown pressure, $a=(a^1(x),\ a^2(x),\ a^3(x))$ is a given initial velocity and $f=(f^1(x,\ t),\ f^2(x,\ t),\ f^3(x,\ t))$ is a given external force. Here we use the notation

$$(u\cdot\nabla)v=\sum_{i=1}^{3}u^i\frac{\partial v}{\partial x_i},\qquad \operatorname{div} u=\sum_{i=1}^{3}\frac{\partial u^i}{\partial x_i}$$

for vector functions $u,\ v$. In this paper, let $f=0$ for simplicity. For this problem (NS), E. Hopf [9] proved the existence of a global (in time) weak solution (more generally, for an arbitrary domain Ω in R^n). Although various researches have been made by many mathematicians, the uniqueness and regularity of the weak solutions are still open problems for the space dimension $n\geqq 3$. If $n=2$, it has been proved that there exists a unique global strong solution. (See Ladyzhenskaia [12], Lions [13], Lions-Prodi [14], Serrin [16], and Temam [19]). On the uniqueness of the weak solutions J. Serrin [16] gave a uniqueness theorem under the assumption that a weak solution belongs to the space $L^q((0,\ T)\ ;\ L^p(\Omega))$ for some p and q satisfying $n/p+2/q\leqq 1$, $p>n$ and that Ω is an arbitrary domain of $R^n (n=2,\ 3,\ 4)$. C. Foias [2] proved a similar theorem for $\Omega=R^n$. K. Masuda [15] generalized the Foias-Serrin uniqueness theorem by removing the restriction on the dimension n imposed in the theorem of Serrin and moreover showed that if there is a weak solution u in $L^{\infty}((0, T)$; $L^{n}(\Omega))$ which is right continuous for t as an L^{n}-valued function, then u is the only weak solution.

We consider a bounded domain Ω in R^{3}. Let A be Stokes operator. In this paper we shall show that (1) if there is a weak solution u in $L^{1/\varepsilon}((0, T)$; $D(A^{\frac{1}{4}+\varepsilon}))$ for some $0<\varepsilon\leqq 1/2$, then u is the only weak solution and (2) if there is a weak solution u in $L^{\infty}((0, T)$; $D(A^{\frac{1}{4}}))$ which is right continuous for t as a $D(A^{\frac{1}{4}})$-valued function, then u is the only weak solution. In the proof of (1) we shall show directly the inequality ;

$$\|A^{-(\frac{1}{4}+\varepsilon)}P(u\cdot\nabla)v\|\leqq C\|\nabla u\|^{1-2\varepsilon}\|\nabla v\|\|u\|^{2\varepsilon} \quad \text{for } u,\ v\in D(A^{\frac{1}{2}}) \quad (0\leqq\varepsilon<1/2)$$

which is well known for $\varepsilon=0$. (See Sobolevskii [18], Fujita-T. Kato [3]). We shall define the projection P in section 2. For $\varepsilon=1/2$, we showed [10] that if a weak solution u belongs to the space $L^{2}((0, T)$; $D(A^{\frac{3}{4}}))$ then u is in $C^{\infty}(\bar{\Omega}\times(0, T))$. We shall prove (2) analogously to the uniqueness theorem due to Masuda [15].

We note the following.

D. Henry has discussed in his lecture notes [8] concerning the imbedding ;

$$D(B^{\alpha})\subset L^{p}(\Omega) \text{ for } 1/p>1/2-2\alpha/3,$$

where $B=-\Delta$ in $L^{2}(\Omega)$.

On the other hand, the imbedding in the case $1/p=1/2-2\alpha/3$ seems to need careful considerations. It has been established by introducing the complex interpolation space. (See Adams [1], Fujiwara [5], Giga [6][7]). Thus it follows that

$$L^{\frac{1}{\varepsilon}}((0, T)\ ;\ D(A^{\frac{1}{4}+\varepsilon}))\subset L^{q}((0, T)\ ;\ L^{p}(\Omega)), \quad (0\leqq\varepsilon<1/2)$$

for $1/p=1/2-2(1/4+\varepsilon)/3$ and $q=1/\varepsilon$ satisfying $3/p+2/q=1$ $(p\geqq 3)$.

However, since the questions on the uniqueness of the weak solutions and on the regularity of the weak solution belonging to the space $L^{\infty}((0, T)$; $D(A^{\frac{1}{4}}))$ or $L^{\infty}((0, T)$; $L^{3}(\Omega))$ are still open problems, it seems to be meaningful that we represent in this paper the Serrin-Masuda uniqueness theorem by using the space $L^{1/\varepsilon}((0, T)$; $D(A^{\frac{1}{4}+\varepsilon}))$.

2. Weak solution and inequality

Let $C_{0,\sigma}^{\infty}=\{\varphi\in C_0^{\infty}(\Omega)\ ;\ \mathrm{div}\ \varphi=0\}$,

$H_\sigma=$the closure of $C_{0,\sigma}^{\infty}$ in $L^2(\Omega)$,

$H_{0,\sigma}^1=$the closure of $C_{0,\sigma}^{\infty}$ in $H^1(\Omega)$,

and P be the orthogonal projection from L^2 onto H_σ. By Stokes operator A we denote the Friedrichs extension of the symmetric operator $-P\Delta$ in H_σ with domain $D(A)=H^2\cap H_0^1\cap H_\sigma$. It is well known that $D(A^{\frac{1}{2}})=H_{0,\sigma}^1$.

Now we give a definition of weak solutions of (NS). It has been proved that there exist weak solutions satisfying the following definition. (See Giga [6], Hopf [9], Ladyzhenskaia [12], Lions [13], Masuda [15], Sibagaki-Rikimaru [17], Temam [19] etc.).

DEFINITION 2.1.

Let the initial function $a=a(x)$ be in H_σ. A measurable function u on $\Omega\times(0,T)$ is called *a weak solution* of the problem (NS), if

(1) $u\in L^\infty((0,T)\ ;\ H_\sigma)\cap L^2((0,T)\ ;\ H_{0,\sigma}^1)$,

(2) u is weakly continuous from $[0,T)$ to H_σ,

(3) $\displaystyle\int_0^T\{-(u,\varphi_t)+(\nabla u,\nabla\varphi)+(u\cdot\nabla u,\varphi)\}dt$

$=(a,\varphi(0))$ for all $\varphi\in C^\infty(\Omega\times[0,T))$, $\mathrm{div}\ \varphi=0$

such that φ vanishes outside a compact set of $\Omega\times[0,T)$,

(3′) $\displaystyle\int_s^t\{-(u,\varphi_t)+(\nabla u,\nabla\varphi)+(u\cdot\nabla u,\varphi)\}dt$

$=(u(s),\varphi(s))-(u(t),\varphi(t))$

for any $\varphi\in C^1([s,t]\ ;\ H_{0,\sigma}^1)$, $0\leqq s\leqq t<T$,

and moreover the energy inequality holds for u ;

(4) $\displaystyle\|u(t)\|^2+2\int_0^t\|\nabla u\|^2dt\leqq\|a\|^2$ for any $t\in[0,T)$,

(4′) $\displaystyle\|u(t)\|^2+2\int_s^t\|\nabla u\|^2dt\leqq\|u(s)\|^2$ for almost all $s\geqq0$ including $s=0$ and for all $t>s$.

Firstly, we shall show the following inequality.

LEMMA 2.1. *Let B be the Friedrichs extension of the symmetric operator $-\Delta$ in $L^2(\Omega)$. If $0 \leqq \varepsilon < 1/2$, then*

(2.1) $\|B^{-(\frac{1}{4}+\varepsilon)}(u\cdot\nabla)v\| \leqq C_\varepsilon \|\nabla u\|^{1-2\varepsilon}\|u\|^{2\varepsilon}\|\nabla v\|$ *for any $u, v \in C_0^1(\Omega)$.*

PROOF. We begin by noticing that $B^{-\alpha}(0<\alpha<1)$ is an integral operator. By considering the maximum principle for the equation;

$$-\Delta v + \lambda v = f(x),\ x\in\Omega, \qquad v(x)=0 \text{ on } \partial\Omega\ (\lambda \geqq 0),$$

it holds the representation

(2.2) $v(x) = \int_\Omega G(x, y; \lambda) f(y)\,dy$ for any sufficiently smooth function f

and with Green function $G(x, y; \lambda)$ such that

$$0 \leqq G(x, y; \lambda) \leqq \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \quad (\lambda \geqq 0) \qquad \text{(See Fujita-T. Kato[3])}.$$

On the other hand, from the definition of fractional powers of closed operators it follows that

$$(2.3)\quad B^{-\alpha}f = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(B+\lambda)^{-1}f\,d\lambda \qquad (0<\alpha<1)$$

$$= \int_0^\infty \lambda^{-\alpha}dE(\lambda)f \quad . \qquad \text{(spectral representation)}$$

Put

$$(2.4)\quad G_\alpha(x, y) = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}G(x, y; \lambda)\,d\lambda,$$

then

$$(2.5)\quad (B^{-\alpha}f)(x) = \int_\Omega G_\alpha(x, y)f(y)\,dy$$

for any sufficiently smooth function f.

Here,

$$(2.6)\quad |G_\alpha(x, y)| \leqq C''_\alpha \int_0^\infty \lambda^{-\alpha}\frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|}d\lambda$$

$$= C'_\alpha \int_0^\infty t^{1-2\alpha}e^{-t}dt\cdot|x-y|^{2\alpha-3} = C_\alpha|x-y|^{2\alpha-3}.$$

To prove the inequality (2.1) we notice the following inequality

$$(2.7)\quad \int_\Omega \frac{1}{|x-y|^{3-2\alpha}|x-z|^{3-2\alpha}}dx \leqq C|y-z|^{4\alpha-3}, \qquad (x, y, z\in\Omega, \quad 3-4\alpha>0).$$

In fact, put $|y-z|=\delta$ and $x'=x/\delta$, $y'=y/\delta$, $z'=z/\delta$, then $|y'-z'|=1$ and the left-hand side of (2.7) $\leqq \delta^{4\alpha-3}\int_{R^3}\frac{dx'}{|x'-y'|^{3-2\alpha}|x'-z'|^{3-2\alpha}}$.

If we choose y' as an origin and take the first coordinate axis x_1' such that z' represents (1, 0, 0), the right-hand side of the above inequality is equal to

$$\delta^{4\alpha-3}\int_{R^3}\frac{dt}{|t|^{3-2\alpha}|t-t_0|^{3-2\alpha}} \qquad \text{with } t_0=(1,0,0),$$

where, since $(3-2\alpha)+(3-2\alpha)>3=$ space dimension and $3-2\alpha<3$, it holds $\int_{R^3}\frac{dt}{|t|^{3-2\alpha}|t-t_0|^{3-2\alpha}}\leqq C$.

Now, let u, $v\in C_0^1(\Omega)$, $\alpha=1/4+\varepsilon$ $(0<\varepsilon<1/2)$, then

$$\|B^{-\alpha}(u\cdot\nabla)v\|^2=(B^{-\alpha}(u\cdot\nabla)v,\ B^{-\alpha}(u\cdot\nabla)v)$$

$$=\left(\int_\Omega G_\alpha(x,y)(u\cdot\nabla)v(y)\,dy,\ \int_\Omega G_\alpha(x,z)(u\cdot\nabla)v(z)\,dz\right)$$

$$=\int_\Omega\left(\int_\Omega G_\alpha(x,y)(u\cdot\nabla)v(y)\,dy\right)\left(\int_\Omega G_\alpha(x,z)(u\cdot\nabla)v(z)\,dz\right)dx$$

$$\leqq C_\alpha\int_\Omega\int_\Omega\int_\Omega\frac{|u(y)||\nabla v(y)|}{|x-y|^{3-2\alpha}}\ \frac{|u(z)||\nabla v(z)|}{|x-z|^{3-2\alpha}}dxdydz \qquad \text{(by (2.6))}$$

$$=C_\alpha\int_\Omega\int_\Omega|u(y)||\nabla v(y)||u(z)||\nabla v(z)|\left(\int_\Omega\frac{dx}{|x-y|^{3-2\alpha}|x-z|^{3-2\alpha}}\right)dydz$$

$$\leqq C_\alpha\int_\Omega\int_\Omega\frac{|u(y)||\nabla v(y)||u(z)||\nabla v(z)|}{|y-z|^{3-4\alpha}}dydz \qquad \text{(by (2.7))}.$$

On the other hand, the inequality

$$(2.8)\qquad \int_\Omega\frac{|u(x)|^2}{|x-y|^2}dx\leqq 4\|\nabla u\|^2,\ u\in C_0^1(\Omega)$$

is well known. (See Ladyzhenskaia [12]).

Hence $\|B^{-\alpha}(u\cdot\nabla)v\|^2$

$$\leqq C_\alpha\int_\Omega\int_\Omega\left(\frac{|u(y)||\nabla v(z)|}{|y-z|}\right)^{\frac{3-4\alpha}{2}}\left(\frac{|u(z)||\nabla v(y)|}{|y-z|}\right)^{\frac{3-4\alpha}{2}}(|u(y)||u(z)||\nabla v(y)||\nabla v(z)|)^{\frac{4\alpha-1}{2}}dydz$$

$$\leqq C_\alpha\|\nabla u\|^{3-4\alpha}\|\nabla v\|^{3-4\alpha}\left(\int_\Omega|u(y)||\nabla v(y)|dy\right)^{\frac{4\alpha-1}{2}}\left(\int_\Omega|u(z)||\nabla v(z)|dz\right)^{\frac{4\alpha-1}{2}}$$

$$\leqq C_\alpha\|\nabla u\|^{3-4\alpha}\|\nabla v\|^{3-4\alpha}\|u\|^{4\alpha-1}\|\nabla v\|^{4\alpha-1}.$$

Consequently we have

$$\|B^{-(\frac{1}{4}+\varepsilon)}(u\cdot\nabla)v\|\leqq C_\varepsilon\|\nabla u\|^{1-2\varepsilon}\|u\|^{2\varepsilon}\|\nabla v\| \qquad \left(0<\varepsilon<\frac{1}{2}\right),$$

which implies (2.1).

For any u, $v \in D(B^{\frac{1}{2}}) = H_0^1$, there exist sequences u_n, $v_n \in C_0^\infty(\Omega)$ such that u_n tends to u and v_n tends to v in H_0^1. Therefore, since $B^{-(\frac{1}{4}+\varepsilon)}(u_n \cdot \nabla) v_n$ tends to an element $w \in L^2$, $B^{-(\frac{1}{4}+\varepsilon)}(u \cdot \nabla) v$ is defined by w. Thus we have

(2.1′) $\|B^{-(\frac{1}{4}+\varepsilon)}(u \cdot \nabla) v\| \leqq C_\varepsilon \|\nabla u\|^{1-2\varepsilon} \|u\|^{2\varepsilon} \|\nabla v\| \quad (0 \leqq \varepsilon < 1/2)$

for any u, $v \in D(B^{\frac{1}{2}}) = H_0^1$.

(For $\varepsilon = 0$ see Sobolevskii [18], Fujita-T. Kato[3]).

Moreover, we notice the following equalities ;

(2.9) $(B^{-\alpha}(u_n \cdot \nabla) v_n, B^\alpha \varphi) = ((u_n \cdot \nabla) v_n, \varphi)$ for any $\varphi \in D(B^\alpha)$, $n = 1, 2, 3, \cdots$.

Letting $n \to \infty$ in (2.9) it follows that

(2.10) $(B^{-\alpha}(u \cdot \nabla) v, B^\alpha \varphi) = ((u \cdot \nabla) v, \varphi)$ for any $\varphi \in D(B^\alpha) \cap H_0^1$ and u, $v \in H_0^1$, where $\alpha = 1/4 + \varepsilon$ $(0 \leqq \varepsilon < 1/2)$.

Secondly we shall show that the inequality (2.1) holds for Stokes operator A also.

LEMMA 2.2. *If* $0 \leqq \varepsilon < 1/2$, *then*

(2.11) $\|A^{-(\frac{1}{4}+\varepsilon)} P(u \cdot \nabla) v\| \leqq C_\varepsilon \|A^{\frac{1}{2}} u\|^{1-2\varepsilon} \|u\|^{2\varepsilon} \|A^{\frac{1}{2}} v\|$

for any u, $v \in D(A^{\frac{1}{2}}) = H_{0,\sigma}^1$.

PRROOF. By the inequality (See Fujita-Morimoto [4], Ladyzhenskaia [12]) ;

(2.12) $\|\Delta A^{-1} \psi\| \leqq C_1 \|A^{-1} \psi\|_{H^2} \leqq C_2 \|\psi\|$ for any $\psi \in H_\sigma$,

we have

(2.13) $\|B\varphi\| \leqq C_2 \|A\varphi\|$ for any $\varphi \in D(A)$.

This gives

(2.14) $\|B^\alpha \varphi\| \leqq C_\alpha \|A^\alpha \varphi\|$ for any $\varphi \in D(A^\alpha)$ $(0 \leqq \alpha \leqq 1)$

by Heinz inequality. (See T. Kato [11]).

Hence $B^\alpha A^{-\alpha}$ is a bounded operator from H_σ into L^2 with $\|B^\alpha A^{-\alpha}\| \leqq C_\alpha$. Therefore $A^{-\alpha} P B^\alpha$ admits of the bounded extension $\overline{A^{-\alpha} P B^\alpha}$ with $\|\overline{A^{-\alpha} P B^\alpha}\| \leqq C_\alpha$ as an operator from L^2 into H_σ. It follows that

$$\|A^{-(\frac{1}{4}+\varepsilon)} P(u \cdot \nabla) v\| \leqq \|\overline{A^{-(\frac{1}{4}+\varepsilon)} P B^{(\frac{1}{4}+\varepsilon)}}\| \|B^{-(\frac{1}{4}+\varepsilon)}(u \cdot \nabla) v\|$$
$$\leqq C_\varepsilon \|B^{-(\frac{1}{4}+\varepsilon)}(u \cdot \nabla) v\|,$$ which implies (2.11).

Lastly, we notice the equality

(2.15) $(A^{-\alpha} P(u \cdot \nabla) v, A^\alpha \varphi) = ((u \cdot \nabla) v, \varphi)$ for any $\varphi \in D(A^\alpha) \cap H_{0,\sigma}^1$

and for any $u, v \in D(A^{\frac{1}{2}}) = H^1_{0,\sigma}$ $(\alpha = 1/4+\varepsilon,\ 0 \leqq \varepsilon < 1/2)$.
This is established in the same way as (2.10).

3. Uniqueness theorem

We discuss our uniqueness results.

THEOREM 1. *Let u and v be weak solutions of the problem* (NS) *in* $\Omega \times (0, T)$ *taking the initial value $a \in H_\sigma$. Suppose also that u belongs to the space* $L^{\frac{1}{\varepsilon}}((0, T); D(A^{\frac{1}{4}+\varepsilon}))$ *for some ε satisfying* $0 < \varepsilon < 1/2$. *Then $u \equiv v$ on* $[0, T)$.

PROOF. We define the mollifier;

$$v_\rho(\tau) = \int_0^t \omega_\rho(\tau - \sigma) v(\sigma)\, d\sigma \qquad (\rho > 0)$$

for an arbitrarily fixed $t\,(0 < t < T)$ and the weak solution v. Here $\omega = \omega(t)$ is a C^∞ function in R^1 with support in $|t| \leqq 1$ such that $\omega(t) = \omega(-t)$, $\omega(t) \geqq 0$ and $\int_{-\infty}^{\infty} \omega(t)\, dt = 1$. We set $\omega_\rho(t) = \rho^{-1} \omega\left(\frac{t}{\rho}\right)$. Since it follows $v_\rho \in C^1([0, t]; H^1_{0,\sigma})$ from $v \in L^2((0, T); H^1_{0,\sigma})$, we can take the v_ρ as a test function in the weak equation (3′) of the definition 2.1. Hence

$$\text{(3.1)} \quad \int_0^t \{-(u, v_{\rho t}) + (\nabla u, \nabla v_\rho) + (u \cdot \nabla u, v_\rho)\}\, dt$$
$$= -(u(t), v_\rho(t)) + (a, v_\rho(0))$$

and similarly

$$\text{(3.2)} \quad \int_0^t \{-(v, u_{\rho t}) + (\nabla v, \nabla u_\rho) + (v \cdot \nabla v, u_\rho)\}\, dt$$
$$= -(v(t), u_\rho(t)) + (a, u_\rho(0)).$$

By means of Fubini's theorem it holds

$$\int_0^t (u, v_{\rho t})\, dt = -\int_0^t (u_{\rho t}, v)\, dt.$$

Thus, by addition of (3.1) and (3.2) it follows that

$$\text{(3.3)} \quad \int_0^t \{(\nabla u, \nabla v_\rho) + (\nabla v, \nabla u_\rho) + (u \cdot \nabla u, v_\rho) + (v \cdot \nabla v, u_\rho)\}\, dt$$
$$= -(u(t), v_\rho(t)) + (a, v_\rho(0)) - (v(t), u_\rho(t)) + (a, u_\rho(0)).$$

Here, it holds that

$$(3.4)\quad \left|\int_0^t (\nabla u, \nabla v_\rho)\,dt-\int_0^t (\nabla u, \nabla v)\,dt\right|$$

$$\leqq \left(\int_0^t \|\nabla u\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla v_\rho-\nabla v\|^2 dt\right)^{\frac{1}{2}}\longrightarrow 0 \quad \text{as } \rho\to 0$$

owing to the relation $v\in L^2((0, T)\,;H^1_{0,\sigma})$.

From $(u(t), v_\rho(t))=(u(t), \int_0^t \omega_\rho(t-\sigma)\,v(\sigma)\,d\sigma)$

$$=\int_0^\rho \omega_\rho(s)\,(u(t), v(t-s))\,ds,$$

it follows

$$(3.5)\quad \left|(u(t), v_\rho(t))-\frac{1}{2}(u(t), v(t))\right|$$

$$=\left|\int_0^\rho \omega_\rho(s)\,(u(t), v(t-s))\,ds-\int_0^\rho \omega_\rho(s)\,(u(t), v(t))\,ds\right|$$

$$\leqq \operatorname*{Max}_{0\leqq s\leqq \rho}|(u(t), v(t-s)-v(t))|\cdot\frac{1}{2}\longrightarrow 0 \quad \text{as } \rho\to 0,$$

where we used the weak continuity of v.

As for the non-linear term we have the following estimates.

$$(3.6)\quad \left|\int_0^t (u\cdot\nabla u, v_\rho)\,dt-\int_0^t (u\cdot\nabla u, v)\,dt\right|$$

$$=\left|\int_0^t (u\cdot\nabla v_\rho-u\cdot\nabla v, u)\,dt\right|$$

$$=\left|\int_0^t (A^{-(\frac{1}{4}+\varepsilon)}P(u\cdot\nabla)(v_\rho-v), A^{\frac{1}{4}+\varepsilon}u)\,dt\right| \qquad \text{(by (2.15))}$$

$$\leqq\int_0^t \|A^{-(\frac{1}{4}+\varepsilon)}P(u\cdot\nabla)(v_\rho-v)\|\,\|A^{\frac{1}{4}+\varepsilon}u\|\,dt$$

$$\leqq C\int_0^t \|\nabla v_\rho-\nabla v\|\,\|\nabla u\|^{1-2\varepsilon}\|u\|^{2\varepsilon}\|A^{\frac{1}{4}+\varepsilon}u\|\,dt \qquad \text{(by lemma 2.2)}$$

$$\leqq C\left(\int_0^t \|\nabla v_\rho-v\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla u\|^2 dt\right)^{\frac{1-2\varepsilon}{2}}\left(\int_0^t \|A^{\frac{1}{4}+\varepsilon}u\|^{\frac{1}{\varepsilon}} dt\right)^{\varepsilon}$$

$$\longrightarrow 0 \quad \text{as } \rho\to 0 \qquad \text{(by Hölder inequality).}$$

On the other hand

$$
\begin{aligned}
(3.7)\quad & \left|\int_0^t (v\cdot\nabla v,\ u_\rho)\,dt-\int_0^t (v\cdot\nabla v,\ u)\,dt\right| \\
&=\left|\int_0^t (A^{-(\frac{1}{4}+\varepsilon)}P(v\cdot\nabla)v,\ A^{\frac{1}{4}+\varepsilon}(u_\rho-u))\,dt\right| \\
&\leqq C\int_0^t \|\nabla v\|^{2-2\varepsilon}\|v\|^{2\varepsilon}\|A^{\frac{1}{4}+\varepsilon}(u_\rho-u)\|\,dt \\
&\leqq C\left(\int_0^t \|\nabla v\|^2 dt\right)^{1-\varepsilon}\left(\int_0^t \|A^{\frac{1}{4}+\varepsilon}u_\rho-A^{\frac{1}{4}+\varepsilon}u\|^{\frac{1}{\varepsilon}}dt\right)^{\varepsilon} \\
&\longrightarrow 0 \quad \text{as } \rho\longrightarrow 0,
\end{aligned}
$$

because that $A^\alpha(u_\rho)=A^\alpha\int_0^t \omega_\rho(\tau-\sigma)\,u(\sigma)\,d\sigma$

$$
=\int_0^t \omega_\rho(\tau-\sigma)A^\alpha u(\sigma)\,d\sigma=(A^\alpha u)_\rho
$$

owing to the closedness of the operator A^α. By (3.4)～(3.7) we have

$$
(3.8)\quad \int_0^t \{2(\nabla u,\ \nabla v)+(v\cdot\nabla v,\ u)+(u\cdot\nabla u,\ v)\}dt
$$

$$
=-(u(t),\ v(t))+(a,\ a).
$$

Now put $w=v(t)-u(t)$, then it follows that

$$
\begin{aligned}
(3.9)\quad & \|w\|^2+2\int_0^t \|\nabla w\|^2 dt \\
&=\|u\|^2+2\int_0^t \|\nabla u\|^2 dt+\|v\|^2+2\int_0^t \|\nabla v\|^2 dt-2\|a\|^2 \\
&+2\int_0^t \{(v\cdot\nabla v,\ u)+(u\cdot\nabla u,\ v)\}dt
\end{aligned}
$$

because of the above equality (3.8). Since

$$
(v\cdot\nabla v,\ u)+(u\cdot\nabla u,\ v)=(w\cdot\nabla w,\ u),
$$

we have

$$
(3.10)\quad \|w(t)\|^2+2\int_0^t \|\nabla w\|^2 dt\leqq 2\int_0^t (w\cdot\nabla w,\ u)\,dt
$$

in consideration of the energy inequality (4) of the definition 2.1. As for the right-hand side of (3.10)

$$(3.11)\quad \left|\int_0^t (w\cdot\nabla w,\ u)\,dt\right| = \left|\int_0^t (A^{-(\frac{1}{4}+\varepsilon)}P(w\cdot\nabla)w,\ A^{\frac{1}{4}+\varepsilon}u)\,dt\right| \qquad \text{(by (2.15))}$$

$$\leqq \int_0^t \|A^{-(\frac{1}{4}+\varepsilon)}P(w\cdot\nabla)w\|\|A^{\frac{1}{4}+\varepsilon}u\|\,dt$$

$$\leqq C\int_0^t \|\nabla w\|^{2-2\varepsilon}\|w\|^{2\varepsilon}\|A^{\frac{1}{4}+\varepsilon}u\|\,dt \qquad \text{(by lemma 2.2)}$$

$$\leqq C\left(\int_0^t \|\nabla w\|^2 dt\right)^{1-\varepsilon}\left(\int_0^t \|w\|^2\|A^{\frac{1}{4}+\varepsilon}u\|^{\frac{1}{\varepsilon}}dt\right)^{\varepsilon}.$$

Hence, it follows that for any $\sigma>0$

$$(3.12)\quad \left|\int_0^t (w\cdot\nabla w,\ u)\,dt\right| \leqq \sigma\int_0^t \|\nabla w\|^2 dt + C(\sigma)\int_0^t \|w\|^2\|A^{\frac{1}{4}+\varepsilon}u\|^{\frac{1}{\varepsilon}}dt,$$

owing to the Young inequality.
Thus, we can choose $\sigma>0$ such that it holds

$$(3.13)\quad \|w\|^2 \leqq C\int_0^t \|w\|^2\|A^{\frac{1}{4}+\varepsilon}u\|^{\frac{1}{\varepsilon}}dt.$$

Consequently, by means of the Gronwall inequality (See Henry [8]) and the relations $w\in L^\infty((0,\ T)\ ;\ H_\sigma)$, $u\in L^{1/\varepsilon}((0,\ T)\ ;\ D(A^{\frac{1}{4}+\varepsilon}))$, we have $w=0$ a.e. on $[0,\ T)$. On the other hand w is weakly continuous. It gives $w\equiv 0$ on $[0,\ T)$. This completes the proof of theorem 1.

THEOREM 2. *Let u and v be weak solutions of the problem* (NS) *with the initial data $a\in H_\sigma$. Suppose also that $u\in L^\infty((0,\ T)\ ;\ D(A^{\frac{1}{4}}))$ and that u is right continuous for all $t\in[0,\ T)$ in the norm of $D(A^{\frac{1}{4}})$, then $u\equiv v$ on $[0,T)$.*
PROOF. We can prove this theorem analogously to the proof of the Masuda uniqueness theorem [15] as follows. Let ζ be a monotone nonincreasing C^∞ function in $[0,\ \infty)$ such that $0\leqq\zeta(s)\leqq 1$, $|\zeta'(s)|\leqq 1$ and $\zeta(s)=1$ if $0\leqq s\leqq 1$, $\zeta(s)=0$ if $4\leqq s$. Set $\zeta_q(x)=\zeta\left(\frac{|x|}{q}\right)$ $q=1,\ 2,\ 3\cdots$. Firstly we can show the inequality (3.10) similarly to the proof of theorem 1. In fact it is enough to examine the convergency of the non-linear term ;

$$\int_0^t (v\cdot\nabla v,\ u_\rho)\,dt \longrightarrow \int_0^t (v\cdot\nabla v,\ u)\,dt \qquad \text{as } \rho\to 0,$$

owing to the estimates ;

$$(3.14)\quad \left|\int_0^t (v\cdot\nabla v,\ u_\rho - u)\,dt\right|$$

$$\leqq \int_0^t \|A^{-\frac{1}{4}}P(v\cdot\nabla)v\|\,\|A^{\frac{1}{4}}u_\rho - A^{\frac{1}{4}}u\|\,dt$$

$$\leqq C\int_0^t \|\nabla v\|^2\|A^{\frac{1}{4}}u_\rho - A^{\frac{1}{4}}u\|\,dt.$$

Here, because of $A^{\frac{1}{4}}u\in L^\infty(0,\ T)\ ;\ H_\sigma)\subset L^2((0,\ T)\ ;\ H_\sigma)$ it follows

$$(A^{\frac{1}{4}}u)_\rho \longrightarrow A^{\frac{1}{4}}u \qquad \text{in } L^2((0,\ T)\ ;\ H_\sigma) \text{ as } \rho\to 0,$$

and further there exists a sequence $\rho_n>0$ tending to zero such that

$$\|(A^{\frac{1}{4}}u)_{\rho_n}(\tau) - A^{\frac{1}{4}}u(\tau)\| \to 0 \qquad \text{a.e. } \tau\in(0,\ T) \text{ as } n\to\infty.$$

On the other hand $\|(A^{\frac{1}{4}}u)_\rho(\tau)\|^2 \leqq \int_0^t \omega_\rho(\tau-\sigma)\|A^{\frac{1}{4}}u(\sigma)\|^2 d\sigma$

$$\leqq \operatorname*{ess.sup}_{0\leqq\sigma\leqq T}\|A^{\frac{1}{4}}u(\sigma)\|^2 \equiv C \qquad (\text{by } u\in L^\infty((0,\ T)\ ;\ D(A^{\frac{1}{4}}))).$$

Thus, the right-hand side of (3.14) tends to zero. Consequently we have the inequality (3.10) ;

$$\|w(t)\|^2 + 2\int_0^t \|\nabla w\|^2 dt \leqq 2\int_0^t (w\cdot\nabla w,\ u)\,dt. \quad (w=v-u)$$

Secondly, we estimate the right-hand side of (3.10) in the following way. We set $u_1=(1-\zeta_q(|B^{\frac{1}{4}}u(x,\ t)|))B^{\frac{1}{4}}u(x,\ t)$,

$$u_2=\zeta_q(|B^{\frac{1}{4}}u(x,\ t)|)B^{\frac{1}{4}}u(x,\ t),$$

then $B^{\frac{1}{4}}u=u_1+u_2$. For $0\leqq s<t<T$

$$(3.15)\quad \left|\int_s^t (w\cdot\nabla w,\ u)\,dt\right| = \left|\int_s^t (A^{-\frac{1}{4}}P(w\cdot\nabla)w,\ A^{\frac{1}{4}}u)\,dt\right|$$

$$= \left|\int_s^t (B^{-\frac{1}{4}}w\cdot\nabla w,\ B^{\frac{1}{4}}u)\,dt\right| \qquad (\text{by } (2.10),\ (2.15))$$

$$\leqq \int_s^t |(B^{-\frac{1}{4}}w\cdot\nabla w,\ u_1)|\,dt + \int_s^t |(B^{-\frac{1}{4}}w\cdot\nabla w,\ u_2)|\,dt.$$

As for the first term of the right-hand side it holds

$$(3.16)\quad \int_s^t |(B^{-\frac{1}{4}}w\cdot\nabla w,\ u_1)|\,dt$$

$$\leqq C\int_s^t \|\nabla w\|^2 \|u_1\| dt \leqq C\int_s^t \|\nabla w\|^2 \|B^{\frac{1}{4}}u\| dt.$$

Here, we notice that for each fixed q, $\|u_1(t)\|$ is right continuous.
In fact,

$$(3.17)\quad \|u_1(t)-u_1(s)\| \leqq \|B^{\frac{1}{4}}u(t)-B^{\frac{1}{4}}u(s)\| + \|\zeta_q(|B^{\frac{1}{4}}u(t)|)B^{\frac{1}{4}}u(t)-\zeta_q(|B^{\frac{1}{4}}u(s)|)B^{\frac{1}{4}}u(s)\|.$$

By using the inequality

$$|\zeta_q(|\xi|)\xi-\zeta_q(|\xi'|)\xi'| \leqq C\sup_{0\leqq s}\{\zeta_q(s)+|\zeta_q'(s)|s\}|\xi-\xi'|$$

for two vectors ξ, ξ',

the right-hand side of (3.17) $\leqq C\|B^{\frac{1}{4}}u(t)-B^{\frac{1}{4}}u(s)\|$

$\leqq C\|A^{\frac{1}{4}}u(t)-A^{\frac{1}{4}}u(s)\|$ (by (2.14)),

which implies the right continuity of $\|u_1(t)\|$, owing to the right continuity of $A^{\frac{1}{4}}u(t)$. Further, we remark that for each fixed t, $\|u_1(t)\|$ decreases monotonously to zero as $q\to\infty$. As for the second term of the right-hand side of (3.15) it holds

$$\int_s^t |(B^{-\frac{1}{4}}w\cdot\nabla w,\ u_2)|dt$$

$$\leqq \operatorname*{ess.sup}_{\Omega\times[s,\ t]} |\zeta_q(|B^{\frac{1}{4}}u|)B^{\frac{1}{4}}u| \int_s^t\int_\Omega |B^{-\frac{1}{4}}w\cdot\nabla w|dxdt$$

$$\leqq 4q\int_s^t\int_\Omega \left|\int_\Omega G_{\frac{1}{4}}(x,\ y)w\cdot\nabla w(y)dy\right|dxdt$$

$$\leqq 4qC\int_s^t\int_\Omega\int_\Omega \frac{|w(y)||\nabla w(y)|}{|x-y|^{\frac{5}{2}}}dxdydt$$

$$\leqq 4qC\int_s^t \|w\|\|\nabla w\|dt.$$

Hence,

$$(3.18)\quad \int_s^t |(B^{-\frac{1}{4}}w\cdot\nabla w,\ u_2)|dt \leqq 4qC\int_s^t \|w\|\|\nabla w\|dt.$$

Lastly we notice that if $\int_s^t \|\nabla w\|^2 dt>0$ for any $t\in(s,\ T')$ $(0<T'<T)$

then for any $\sigma>0$ there exists a integer q_0 such that

$$\int_s^t \|\nabla w\|^2 \|u_1\| dt \leqq \sigma \int_s^t \|\nabla w\|^2 dt \qquad \text{for } q \geqq q_0 \text{ and } s \leqq t \leqq T'. \tag{3.19}$$

Really, we set $f_q(t)=\int_s^t \|\nabla w\|^2 \|u_1\| dt \Big/ \int_s^t \|\nabla w\|^2 dt, \qquad t>s$

and $\qquad f_q(s)=\|u_1(s)\|, \qquad q=1, 2, \cdots$

then $f_q(t)$ is continuous on $[s, T']$ owing to the inequality ;

$$|f_q(t)-\|u_1(s)\||\leqq \sup_{s\leqq\tau\leqq t} |\|u_1(\tau)\|-\|u_1(s)\||$$

and the right continuity of $\|u_1(t)\|$ at $t=s$. As we remarked in the above, for each fixed t, $\|u_1(t)\|$ decreases monotonously to zero as $q\to\infty$. From the definition, $f_q(t)$ also has the same property. Hence, by means of the Dini theorem, $f_q(t)$ converges to zero as $q\to\infty$, uniformly on $[s, T']$ which implies (3.19). Thus, in consideration of (3.15), (3.18) and (3.19) it holds that if $\int_s^t \|\nabla w\|^2 dt>0$ for any $t\in(s, T')$ then for any $\sigma>0$ there exists an integer q_0 such that

$$\left|\int_s^t (w\cdot\nabla w, u)\, dt\right| \leqq \sigma \int_s^t \|\nabla w\|^2 dt + C(\sigma)\int_s^t \|w\|^2 dt \qquad (s\leqq t\leqq T'). \tag{3.20}$$

Now, under the above preparation we shall show theorem 2. We set $\delta_0=\sup\{\delta|\ w(t)=0 \text{ on } [0, \delta)\}$, then it holds $\delta_0=T'$. In fact if we had $\delta_0<T'$, it follows for any $t\in(\delta_0, T')$

$\int_{\delta_0}^t \|\nabla w\|^2 dt \geqq C\int_{\delta_0}^t \|w\|^2 dt>0$. Hence, from the above preparation,

we have

$$\|w(t)\|^2+2\int_0^t \|\nabla w\|^2 dt \leqq 2\int_0^t (w\cdot\nabla w, u)\, dt = 2\int_{\delta_0}^t (w\cdot\nabla w, u)\, dt$$

$\leqq 2\sigma\int_{\delta_0}^t \|\nabla w\|^2 dt+2C\int_{\delta_0}^t \|w\|^2 dt$, where we take σ such that $2\text{-}2\sigma>0$.

Then,

$$\|w(t)\|^2 \leqq 2C\int_{\delta_0}^t \|w\|^2 dt \qquad (\delta_0\leqq t\leqq T'). \tag{3.21}$$

Here, $w(\delta_0)=0$ owing to the weak continuity of w in H_σ. From (3.21) and $w(\delta_0)=0$, we must have $w\equiv 0$ on $[\delta_0, T']$; a contradiction. It means $\delta_0=T'$ and further $w\equiv 0$ on $[0, T)$. This completes the proof of theorem 2.

REMARK. *Let u and v be weak solutions of the problem* (NS) *with the initial data* $a\in D(A^\alpha)$ $(1/4\leqq\alpha<1)$. *Suppose also that* $u\in L^2((0, T); D(A^{\frac{3}{4}}))$. *Then* $u\equiv v$ *on* $[0, T)$. In fact, we have showed in our paper [10] that if a weak solution u belongs to the space $L^2((0, T); D(A^{\frac{3}{4}}))$, then u is in $C^\infty(\overline{\Omega}\times(0, T))$. On the other hand, for the initial data $a\in D(A^\alpha)$ $(1/4\leqq\alpha<1)$ there is a unique strong solution $U\in C([0, T_0); D(A^\alpha))$, where T_0 is a positive number depending on a. (See Fujita-T. Kato [3], Giga [7], etc.). Moreover $u\equiv v\equiv U$ on $[0, T_0)$. By these discussion the above statement can be shown.

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